Dressing in AdS spacetime and a conformal Bethe-Salpeter equation

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We initiate the study of Dyson equations of perturbative quantum field theory in anti–de Sitter (AdS) and their consequences for large-N conformal field theory (CFT). We show that the dressed one-particle AdS propagator features wave function renormalization and operator mixing, giving rise to finite corrections to one-pion exchange data. We show how the resummation of $1/N$ effects in the CFT emerges from the dressing in AdS. When a boundary-to-bulk propagator is dressed by propagators whose sum of conformal dimensions is lower than the main dimension, it cannot map onto a CFT source; we relate this to an AdS/CFT version of particle instability. We investigate the dressing of the two-particle propagator and obtain a conformal Bethe-Salpeter equation for the conformal partial wave of a "bound state" operator. We provide a self-contained calculation for the case of a ladder kernel. We show that a bound state with conformal dimension equal to the sum of its constituents plus a $1/N^2$-suppressed "binding energy" emerges. Resummation of the Dyson equations is essential for deriving these results.

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I. INTRODUCTION

The AdS/CFT correspondence establishes a profound connection between two fields of physics: gravity and strongly-coupled gauge theories. In its most studied form, the duality implies that the boundary amplitudes of weakly-coupled theories of gravity in $d+1$-dimensional anti–de Sitter (AdS_{d+1}) spacetime correspond to correlators of a strongly-coupled d-dimensional conformal field theory (CFT_d) with many degrees of freedom (large-N). The $1/N$ expansion of the CFT correlators maps onto the perturbative expansion of the AdS quantum field theory (QFT) amplitudes [1–12].

The concepts and tools of flat space perturbative QFT have been gradually identified/generalized in the context of AdS/CFT. For example, the structure of the AdS boundary "S-matrix" has been identified in [13,14]. More recently, the CFT Froissard-Gribov formula for analytic continuation in spin was found in [15]. AdS/CFT is now studied at loop level [16–56], and AdS/CFT unitarity methods have been identified and explored in [50,54,57–59], both in the space of conformal dimensions and in momentum space.

One item of the pertubative QFT toolkit has not been deeply explored yet; the quantum dressing of AdS amplitudes induced by interactions as described by the Dyson equations. Its CFT counterpart corresponds to the resummation of $1/N$ effects. The aim of this note is to initiate a study of these dressing equations and its consequences in AdS/CFT.

Some aspects related to dressing and resummation have been addressed in the literature. Resummation in Mellin space has been discussed in [18]. A bubble resummation in the $O(N)$ and Gross-Neveu models has been made in [40]. Effects of 1PI insertions on 2pt functions have been discussed in [50] in the context of unitarity methods. A resummation in AdS has been done in [55], with a focus on propagation in timelike regime. Aspects of ladder diagrams have also been discussed in [56].

Our aim here is to initiate a systematic understanding of dressing in AdS/CFT. The focus is more conceptual than technical, although a detailed calculation is given in the Supplemental Material [60]. We will work with the one- and two-particle propagators.

II. PRELIMINARY OBSERVATIONS

At large but finite N, the conformal algebra can be viewed as the $N=\infty$ conformal algebra with small, $1/N$-suppressed conformal deformations to the generators [57,61]. Denoting the free CFT generators as $D_0, K_0^\mu, P_0^\mu, M^{\mu\nu}$, the $N=$ finite generators are $D=D_0+D_I$, $K^\mu=K_0^\mu+K_I^\mu$, $P^\mu=P_0^\mu+P_I^\mu$, with $D_I, K_I^\mu, P_I^\mu \propto 1/N$. [62].

Using radial quantization, the CFT correlators can in principle be computed perturbatively in $1/N$, in a way structurally similar to the flat space S-matrix. The free

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"multiparticle" states are CFT states built from single-trace operators of the $N = \infty$ algebra. These map onto free bulk fields in AdS. In the interaction picture, the interaction piece of the Hamiltonian (namely D_I) is exponentiated and generates the correlators in a perturbative expansion around $N = \infty$. The perturbative picture is explicitly realized in AdS, where one effectively has a weakly-coupled QFT in the bulk, and the free multiparticle states are formed by the free bulk fields ending on the boundary.

The irreducible representations of the $N = \infty$ and $N =$ finite algebras are related by $1/N$-suppressed deformations. For example, the conformal dimensions of operators in the $N =$ finite algebra take the form $\Delta = \Delta_0 + \gamma$ with the anomalous dimension $\gamma \propto 1/N^2$. [63] Similarly, since the operators in $N = \infty$ and $N =$ finite theories differ, their respective one-pion exchange (OPE) coefficients should differ and be related by a $1/N$-suppressed deformation $c = c_0 + \delta c$ with $\delta c \propto 1/N^2$.

We may also expect a notion of "operator mixing" induced by $1/N$ corrections. How may such a mixing appear? Consider flat space QFT. At the level of propagators, mixing between states appears from the resummation of 1PI insertions dressing the propagator (i.e., the Born series generated by the 2pt Dyson equation). Hence we should investigate the analogous dressing in AdS.

As another simple "invitation" to the question of dressing, consider a 2pt correlator of a CFT with $N =$ finite. Using $\Delta = \Delta_0 + \gamma$ we can always write

$$\frac{1}{x^{2\Delta}} = \frac{1}{x^{2\Delta_0}}\left(1 - \gamma \log x^2 + \frac{1}{2}\gamma^2 \log^2 x^2 + \ldots\right), \quad (1)$$

where the $1/N$-suppressed terms explicitly appear as corrections to the correlator from the $N = \infty$ CFT. The form of the series is totally fixed by the dilatation symmetry of the $N =$ finite conformal algebra. We can then wonder—How does this series emerge from the AdS dual? Since conceptually the series Eq. (1) is generated by repeated application of the dilatation operator of the $N =$ finite CFT, on the AdS side the corresponding series should be generated by repeated insertions of the interaction Hamiltonian. We can thus expect that the above exponential series should emerge from the dressed AdS propagator—we will show how this happens in the following. [64].

Having set the stage and motivations, we proceed to the AdS calculations.

III. THE DRESSED ONE-PARTICLE PROPAGATOR

We consider scalar fields in the Poincaré patch of AdS_{d+1}, with metric $ds^2 = (kz)^{-2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2)$, and Fourier transform along the constant z slices to work in "Poincaré momentum space" (p^μ, z). The free propagator between arbitrary points is denoted $G_p^{(0)}(z,z')$, with $G^{(0)}(X,X') = \int \frac{d^d p}{(2\pi)^d} G_p^{(0)}(z,z')$.

We use the "harmonic" (or conformal spectral) representation (see e.g., [65–67]), which takes the form

$$G^{(0)}_{p,\alpha}(z,z') = \int_{-i\infty}^{i\infty} d\hat{\alpha} P(\hat{\alpha},\alpha)\Omega_{\hat{\alpha}}(z,z'), \quad (2)$$

where

$$P(\hat{\alpha},\alpha) = \frac{1}{\hat{\alpha}^2 - \alpha^2}, \qquad \Omega_{\hat{\alpha}}(z,z') \propto \mathcal{K}^-_{\hat{\alpha}}(z)\mathcal{K}^+_{\hat{\alpha}}(z') \quad (3)$$

with $\mathcal{K}^{\pm}(z)$ the boundary-to-bulk propagators ($\Omega_{\hat{\alpha}}$ is the kernel of the harmonic transform). Here the overall constants are not needed, see e.g., [39,40,50,55,65–68] for detailed formalism and applications. The $\mathcal{K}^{\pm}_{\alpha}(z)$ map onto CFT operators with dimension $\Delta_{\pm} = d/2 \pm \alpha$.

Let us consider the propagator dressed by generic 1PI insertions $i\Pi(z,z')$, including coupling constants, as pictured in Fig. 1. We first establish its general form, using the harmonic representation for the 1PI blobs

$$i\Pi(z,z') = \int_{-i\infty}^{i\infty} d\hat{\alpha}\mathcal{B}(\hat{\alpha})\Omega_{\hat{\alpha}}(z,z'). \quad (4)$$

We have

$$\int dzdz' \mathcal{K}^-_{\alpha}(z) i\Pi(z,z')\mathcal{K}^-_{\alpha'}(z') \propto \delta(\alpha-\alpha')i\mathcal{B}(\alpha) + s.t. \quad (5)$$

because the lhs amounts to a conformal 2pt function (where $s.t.$ stands for shadow transform, see e.g., [69]). This allows us to resum the Born series, giving the dressed propagator

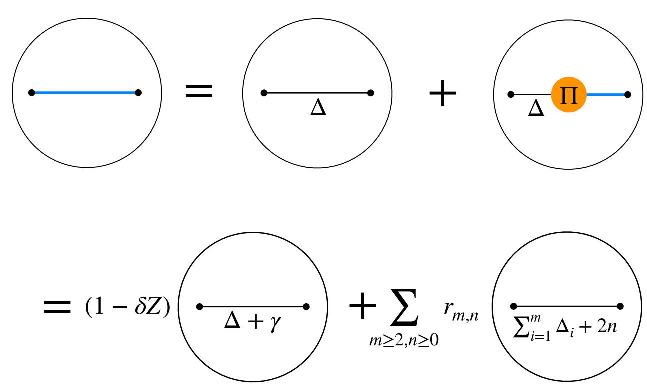


FIG. 1. The dressed one-particle propagator (blue line). *Top*: The 2pt Dyson equation with 1PI insertion $i\Pi$. *Bottom*: The dressed propagator as a combination of free propagators (dark lines) with different conformal dimensions.

$$G_{p,\alpha}(z,z') = \int_{-i\infty}^{i\infty} d\hat{\alpha} \frac{1}{P(\hat{\alpha},\alpha)^{-1} + \mathcal{B}(\hat{\alpha})} \Omega_{\hat{\alpha}}^{(0)}(z,z'). \qquad (6)$$

The loop insertion is, in general, challenging to evaluate, see [32,40,55] for the bubble case. However, we can establish its general form by using the AdS Kallen-Lehmann (KL) representation. In the AdS viewpoint one inserts a complete set of multiparticle states [70], expressing the $i\Pi(z,z')$ as an infinite sum over propagators with dimension of multitrace operators (primaries and descendants). Focusing for simplicity on nonderivative interactions, the exchanged operators are scalar. A generic m-tuple trace operator takes the form $\mathcal{O}_1\partial^{2n_1}\mathcal{O}_2\ldots\partial^{2n_{m-1}}\mathcal{O}_m$ with total dimension $\sum_{i=1}^m \Delta_i + 2n$ where n counts the derivatives. Introducing $\alpha_m = \sum_{i=1}^m \Delta_i - d/2$, the KL form of the 1PI insertion reads

$$i\Pi_p(z,z') = \sum_{m\geq 2, n\geq 0} g_m^2 a_{m,n} G^{(0)}_{p,\alpha_m+2n}(z,z'), \qquad (7)$$

where $a_{m,n}$ is the spectral function. The $m=1$ case is excluded from the sum by 1PI requirement. A bubble diagram, for example, gives a series of double-trace propagators $G^{(0)}_{\Delta_1+\Delta_2+2n-d/2}$. The $a_{2,n}$ coefficients for the bubble were found in various ways in [18,55]. Finally, the g_m contain the coupling constants from bulk vertices, scaling as $g_m \sim 1/N^{m-1}$ as dictated by the mapping onto CFT correlators.

In the harmonic representation this general form of the 1PI insertion becomes

$$\mathcal{B}(\hat{\alpha}) = \sum_{m\geq 2, n\geq 0} g_m^2 a_{m,n} P(\hat{\alpha}, \alpha_m + 2n). \qquad (8)$$

Plugging Eq. (8) into Eq. (6) gives the general form of the AdS propagator dressed by arbitrary 1PI insertions.

We now extract information about the CFT from the dressed propagator $G_{p,\alpha}$.

A. Anomalous dimension

The first bit of information we extract is the anomalous dimension at the $\hat{\alpha} \sim \pm\alpha$ pole. Since $\mathcal{B}(\alpha)$ is a perturbative correction, we obtain the correction to α by expanding $\mathcal{B}(\alpha)$ near the pole, $\mathcal{B}(\hat{\alpha}) = \mathcal{B}(\alpha) + \cdots$. The corresponding CFT operator given by that pole acquires an anomalous dimension γ, with

$$\gamma = -\frac{\mathcal{B}(\alpha)}{2\alpha}, \qquad \Delta_\pm = \Delta_\pm^0 \pm \gamma = \frac{d}{2} \pm \alpha \pm \gamma, \qquad (9)$$

where $\Delta_\pm^0$ is the conformal dimension in the $N=\infty$ theory. This method is explicitly verified for the bubble via [32,55].

B. Wave function/one-pion exchange renormalization

We consider the next-to-leading term in the expansion of $\mathcal{B}(\hat{\alpha})$,

$$\mathcal{B}(\hat{\alpha}) = \mathcal{B}(\alpha) + \frac{\partial}{\partial\hat{\alpha}^2}\mathcal{B}|_{\alpha=\hat{\alpha}}(\hat{\alpha}^2 - \alpha^2) + \ldots. \qquad (10)$$

Introducing $\delta Z = \frac{\partial}{\partial\hat{\alpha}^2}\mathcal{B}|_{\alpha=\hat{\alpha}}$, we get that the residue of the pole is corrected by $1-\delta Z$. This is the spectral equivalent of "wave function renormalization". From the CFT viewpoint, upon unit-normalizing the 2pt functions, the factor becomes a correction to the OPE coefficients of the $N=\infty$ theory. For e.g., a 3pt function, the correction takes the form $\delta c_{123} = -\frac{1}{2}(\delta Z_1 + \delta Z_2 + \delta Z_3)c_{123}$.

One can notice a general relation between γ and δc,

$$\frac{\delta c}{c} = \frac{1}{2}\left(\frac{\gamma}{\alpha} + \frac{\partial\gamma}{\partial\alpha}\right). \qquad (11)$$

Hence for a known γ we can readily deduce the associated correction to the OPE coefficient.

C. Operator mixing

There is also an infinite sequence of new simple poles arising in $G_{p,\alpha}$ because of the poles in $\mathcal{B}(\alpha)$, see Eq. (8). For simplicity, consider poles far from the main one, $|\hat{\alpha}| \sim |\alpha_m + 2n| \gg |\alpha|$. The new poles are

$$\hat{\alpha} \approx \pm\left(\alpha_m + 2n - \frac{g_m^2 a_{m,n}}{2(\alpha_m+2n)^3}\right), \qquad (12)$$

where the last term is $1/N$-suppressed. The associated residues in this limit are

$$r_{m,n} \approx \mp \frac{g_m^2 a_{m,n}}{2(\alpha_m+2n)^5} \qquad (13)$$

and are thus $1/N$-suppressed. Using the definition Eq. (2) these poles contribute to the dressed propagator as

$$G_{p,\alpha}(z,z') \supset -\sum_{m\geq 2, n\geq 0} \frac{g_m^2 a_{m,n}}{(\alpha_m+2n)^5} G^{(0)}_{p,\alpha_m+2n}(z,z'). \qquad (14)$$

That is, an infinite sequence of multitrace propagators arises in the dressed propagator with $1/N$-suppressed coefficients. The complete form of $G_{p,\alpha}(z,z')$ is obtained similarly and is summarized in Fig. 1.

These contributions to $G_{p,\alpha}(z,z')$ introduce a notion of mixing between the original CFT operator and the sequence of multitrace operators in the following sense: If the main operator appears in a given OPE, then the whole sequence of multitrace operators also appear in the OPE with $1/N$-suppressed coefficients. Our AdS result Eq. (14) dictates what are precisely the coefficients.

This can be seen explicitly at the level of a 4pt exchange diagram. When cutting on $G_{p,\alpha}(z,z')$—in either α-space [50] or momentum space [54]—one obtains a sum of squared 3pt correlators (or transition amplitudes) over the sequence of multitrace operators weighted by the $r_{i,n}$ residues.

D. Boundary propagators and (in)stability in AdS/CFT

Let us consider the dressed boundary-to-bulk propagator (i.e., $\mathcal{K}$), obtained from the dressed $G_{p,\alpha}(z,z')$ by sending e.g., z to the boundary and rescaling by $z^{d/2-\alpha-\gamma}$. If the dimensions in Eq. (14) satisfy $\alpha_m > \alpha$, all multitrace contributions drop faster than the main, single-trace term when approaching the boundary, leaving only this main operator with dimension $\Delta_0 + \gamma$ and wave function corrected by δZ. Hence, in spite of dressing, the dressed bulk field still maps onto a source for the single-trace operator in the CFT, and thus the standard AdS/CFT prescription remains unaffected.

In contrast, if $\alpha_m < \alpha$, the multitrace contribution grows faster than the single-trace one near the boundary. Thus the usual rescaling from the standard AdS/CFT prescription does *not* give a finite result. We understand this apparent failure of the AdS/CFT prescription for $\alpha_m < \alpha$ as follows. When $\alpha_m < \alpha$, rather than using $\mathcal{K}_\alpha$ as external leg, one should use the $\mathcal{K}_i$ from the fields inside the 1PI insertion defined at some given order in perturbation theory. The $\mathcal{K}_i$ have dimension lower than the main operator, and a subset of them satisfies the AdS/CFT prescription at the given order in perturbation theory. These $\mathcal{K}_i$ can thus be used as external legs.

This resolution matches the notion of particle instability in flat space QFT [71–73]; if a heavy particle with mass M can decay into particles of mass $\sum_{i=1}^n m_i < M$ at some order, then the space of final states is built from these lighter offspring particles rather than the original particle, which is seen as unstable at this order of perturbation theory. We have essentially obtained the analogous picture for bulk fields/sources in AdS/CFT, where the analog of the "kinematic threshold" is given by

$$\sum_{i=1}^{m} \Delta_i < \Delta. \qquad (15)$$

This condition was also found in [14]. This notion of instability in AdS/CFT may deserve further study.

E. Emergence of the $1/N$ resummation in CFT

Finally, we show how the conformal series Eq. (1) appears from the AdS side. We write the dressed propagator as a geometric series in α space, $G_{p,\alpha} = \sum_n G^{(n)}_{p,\alpha}$, with

$$G^{(n)}_{p,\alpha}(z,z') = \int_{-i\infty}^{i\infty} d\hat{\alpha} P(\hat{\alpha},\alpha)[-P(\hat{\alpha},\alpha)\mathcal{B}(\hat{\alpha})]^n \Omega^{(0)}_{\hat{\alpha}}(z,z'), \qquad (16)$$

Closing the contour, there is a $n+1$-tuple pole at $\hat{\alpha} = \pm\alpha$. Ignoring the other poles which correspond to the operator mixing discussed above, we have

$$G^{(n)}_{p,\alpha}(z,z') \supset \frac{(-1)^n}{n!} \frac{\partial^n}{\partial\hat{\alpha}^n} \left[\left(\frac{\mathcal{B}(\hat{\alpha})}{2\hat{\alpha}} \right)^n G^{(0)}_{p,\hat{\alpha}}(z,z') \right]_{\hat{\alpha}=\alpha}. \qquad (17)$$

We then focus only on the derivatives acting on G, because we already know that derivatives on $\mathcal{B}$ lead to wave function renormalization or to effects neglected here [55]. The anomalous dimension defined in Eq. (9) appears. The sequence of derivatives exponentiates, giving

$$G_{p,\alpha}(z,z') \supset e^{\gamma\partial_\alpha} G^{(0)}_{p,\alpha}(z,z'). \qquad (18)$$

Finally, we take the boundary-to-bulk limit and trade α for the conformal dimension using $\Delta^0_\pm = d/2 \pm \alpha$. We obtain an exponentiated operator acting on the $N = \infty$2pt CFT correlator,

$$e^{\pm\gamma\partial_{\Delta_0}} \frac{1}{x^{2\Delta_0}} \qquad (19)$$

whose action is to shift the conformal dimension $\Delta^0_\pm$ by $\pm\gamma$. We have therefore reproduced the conformal series of Eq. (1) from the AdS side, both for dimension larger and lower than $d/2$.

Reproducing the above steps by acting with one derivative on the bubble function in Eq. (16), and using the definition Eq. (10), we similarly obtain the $1-\delta Z$ correction to the normalization of the 2pt correlator.

IV. THE DRESSED TWO-PARTICLE PROPAGATOR

In flat space the 4pt Dyson equation and associated Bethe-Salpeter equation (BSE) for a weakly coupled bound state give rise to rich physics and challenging problems [74,75]. Here we present a conformal version of the BSE and show the existence of a "bound state" in spectral space.

The general AdS 4pt Dyson equation is shown in Fig. 2 (top). One could study it for arbitrary bulk points, but our focus is on sending the endpoints to the boundary. Rescaling appropriately the legs, one obtains the Witten diagram version of the Dyson equation. The diagrams map onto 4pt CFT correlators, and can be generically described using a decomposition over conformal partial waves (CPW) [76–79]

$$\mathcal{A}^{1234}(x_i) = \int_{-i\infty}^{i\infty} d\hat{\alpha}_{\mathcal{O}} \rho(\hat{\alpha}_{\mathcal{O}}) \Psi^{1234}_{\mathcal{O}}(x_i), \qquad (20)$$

where $\rho(\alpha)$ is the OPE function. In momentum space the CPW $\Psi^{1234}_{\mathcal{O}}$ is simply the product of 3pt functions,

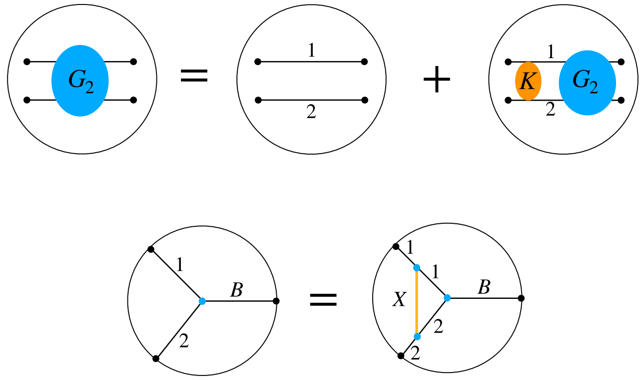


FIG. 2. The dressed two-particle propagator (blue). *Top*: The 4pt Dyson equation with 2PI insertion K. *Bottom*: The boundary Bethe–Salpeter equation with a ladder kernel.

$$\Psi^{1234}_{\mathcal{O}}(p_i) = \Gamma^{12\mathcal{O}}(p_{1,2})\Gamma^{34\tilde{\mathcal{O}}}(p_{3,4})(2\pi)^d\delta^{(d)}\left(\sum_i p_i\right) \tag{21}$$

with

$$\Gamma^{12\mathcal{O}}(p_{1,2}) = \langle\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}(-p_1-p_2)\rangle\rangle \tag{22}$$

and dimensions $[\mathcal{O}_i] = d/2 + \alpha_i$, $[\tilde{\mathcal{O}}_i] = d/2 - \alpha_i$.

Then, in analogy with the flat-space BSE approach, we *assume* the existence of a simple pole with dimension Δ_B in the OPE function of G_2,

$$\rho_{G_2}(\hat{\alpha}) \propto P(\hat{\alpha}, \alpha_B) \tag{23}$$

for $\hat{\alpha}$ near α_B, with $\Delta_B = d/2 + \alpha_B$.

Finally we project the Dyson equation to focus on the exchange of operator with dimension near Δ_B. This can be done using e.g., a CPW, or more directly by contracting the 3,4 legs with a $\Gamma^{\tilde{3}\tilde{4}\mathcal{O}'}$, giving a relation between 3pt correlators in the $\alpha \sim \alpha_B$ region. Because of the nearby pole in the connected diagrams, the diagram with free propagators is negligible. We end up with a self-consistent equation between 3pt Witten diagrams, which amounts to the AdS/CFT version of the BSE, as shown in Fig. 2 (bottom).

We see that the role of the "vertex function" of flat space BSE is here played by a 3pt diagram (i.e., a 3pt CFT correlator), which is fully constrained by conformal symmetry. Thus, the only unknown of our BSE is the dimension of the "bound state" operator Δ_B.

For a given interaction kernel, the value(s) of Δ_B can be extracted from the BSE at large N as follows. The vertices from the kernel are $1/N$-suppressed. Thus each side of the BSE can match only at values of Δ_B for which an N-enhancement cancels the $1/N$-suppression from the vertices.

The existence of such an enhancement due to the interaction kernel is *a priori* nontrivial. In next section we will see how it happens for a ladder kernel.

V. A CONFORMAL BETHE-SALPETER EQUATION

We study the case of a ladder kernel induced by a mediator X with cubic coupling to the 1 and 2 fields. The BSE is shown diagrammatically in Fig. 2 (bottom).

This BSE involves a triangle diagram, $\mathcal{A}_{3,\triangle}$. Our most technical task is to reduce it to $\mathcal{A}_{3,\triangle} = b_{12XB}\mathcal{A}_{3,\text{tree}}$ where b_{12XB} is an overall factor that encodes the nontrivial information about $\mathcal{A}_{3,\triangle}$. The BSE amounts to the equation

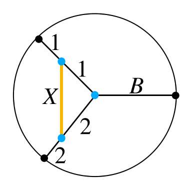
$$b_{12XB} = 1. \tag{24}$$

By studying Eq. (24) we can then determine whether a solution Δ_B exists and its value as a function of $\Delta_{1,2,X}$.

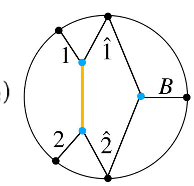
A. Computing the BSE

We provide the detailed computation of our triangle diagram in a self-contained Supplemental Material [60]. Here we describe schematically the steps and emphasize a few key points.

Starting from the triangle diagram, we split the internal lines using the harmonic representation Eq. (2),

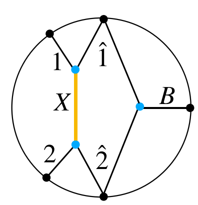


$$\sim \int d\hat{\alpha}_{1,2} P(\hat{\alpha}_1, \alpha_1) P(\hat{\alpha}_2, \alpha_2)$$

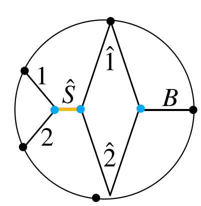


(25)

The result amounts to a t-channel subdiagram glued to 3pt contact subdiagram. We then decompose the t-channel subdiagram onto a basis of s-channel CPWs by using the $6j$ symbol with pairwise equal dimensions $\Delta_{1,2}$ [39], here denoted by $\mathcal{J}^{1,2}_{A,B}$. This involves summing over an additional conformal dimension $\hat{\alpha}_S$. The relevant $6j$ symbol is given explicitly in the Supplemental Material [60]. We obtain

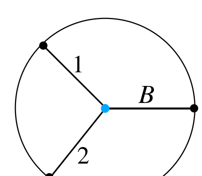


$$\sim \int d\hat{\alpha}_S \mathcal{J}^{1,2}_{X,\hat{S}}$$



(26)

$$\sim \mathcal{J}^{1,2}_{X,B}\,\mathcal{B}_B$$



The rhs of the first line involves a bubble topology. The conformal bubble integral is well-known and is, by conformal symmetry, proportional to $\delta(\hat{\alpha}_S - \alpha_B) + s.t.$ This readily eliminates the $d\hat{\alpha}_S$ integral, reducing the diagram to a nontrivial overall factor times $\mathcal{A}_{3,\mathrm{tree}}$ with conformal dimensions $\Delta_{1,2,X}$. We emphasize that the dependence on the "off shell" $\hat{\alpha}_{1,2}$ only remains in the overall factor, not in the $\mathcal{A}_{3,\mathrm{tree}}$. Thus, at that point we have reached the expected BSE form Eq. (24).

Performing the remaining $d\hat{\alpha}_{1,2}$ integrals inside the c_{12BX} factor requires to know the asymptotics of the integrand, including of the $6j$ symbol which involves ${}_4F_3$ functions. The relevant asymptotic formulas are given in the Supplemental Material [60]. The integrand at e.g., large $|\hat{\alpha}_1|$ turns out to be dominated by the 3pt coefficients which behave exponentially while the other factors behave as powers. The upshot is that we can close both contours towards the physical poles, $\hat{\alpha}_{1,2} = \Delta_{1,2} - \frac{d}{2}$.

When closing the $\hat{\alpha}_{1,2}$ contours, other poles may be picked by the contour integration. However, such contributions are irrelevant when solving the BSE because theory are not enhanced near the $6j$ pole, see details below.

B. Solving the BSE

We search for solutions of the BSE in Δ_B at fixed $\Delta_{1,2,X}$. Due to the overall $1/N^2$ suppression from the $11X$ and $22X$ vertices, such solutions can appear only for values of Δ_B for which a N^2 enhancement occurs, if they exist.

Interestingly, such an enhancement does happen due to the behavior of the $6j$ symbol. For legs with pairwise equal dimension, $\mathcal{J}$ contains double poles of the form $1/(\Delta_B - \Delta_1 - \Delta_2 - 2n)^2$ [39]. Possible contributions from other residues at shadow locations are irrelevant to our near-pole analysis since they are not N^2-enhanced in the region of interest.

At the level of the BSE, the $6j$ double pole behavior reduces to a single pole due to simplification with a single pole in $\mathcal{A}_{3,\mathrm{tree}}$ (see Supplemental Material [60]). It follows that the values of Δ_B computed by the BSE must take the form $\Delta_B = \Delta_1 + \Delta_2 + 2n + \delta_{B,n}$, where the "binding energy"

$$\delta_{B,n} \propto \frac{1}{N^2} \tag{27}$$

depends on $\Delta_{1,2,X}$ and n. We leave a detailed numerical analysis of the conformal BSE for future work.

In a nutshell, the BSE is an important perturbative tool whose implications for AdS/CFT remain to be analyzed in detail, including the $12B$ OPE coefficient, the spectator equation (i.e., large Δ_1), and cross-ladder diagrams. More broadly, it is about the resummation of $1/N$ corrections in 4pt CFT correlators, which would also be interesting to study directly from CFT methods such as the conformal bootstrap.

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