

Hodge duality and supergravity

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(Received 5 June 2023; accepted 7 August 2023; published 28 August 2023)

The Hodge dual operator, recently introduced for supermanifolds, is used to reformulate super Yang-Mills and supergravity in $d = 4$. We first recall the definition of the Hodge dual operator for flat and curved supermanifolds. Then we show how to recover the usual super-Yang-Mills equations of motion for $N = 1, 2$ supersymmetry, and the obstacles (as seen from the Hodge dual point of view) in the case $N \geq 3$. We reconsider several ingredients of supergeometry, relevant for a superspace formulation of supergravity, in terms of the Hodge dual operator. Finally we discuss how $d = 4$ and $N = 1$ supergravity is obtained in this framework.

DOI: [10.1103/PhysRevD.108.046018](https://doi.org/10.1103/PhysRevD.108.046018)

I. INTRODUCTION

Since its early days, supergravity [1–4] has been formulated using superspace [5,6] and supergroup manifold [7–13] techniques. In both these approaches local supersymmetry arises as invariance of the action under superdiffeomorphisms. While the superspace actions involve *superfields*, whose θ (anticommuting coordinates) expansions contain the physical and auxiliary fields, the group manifold actions are given directly in terms of these fields, each of them originating from the *supergroup manifold vielbein*.

In the superspace approach, supergravity actions are constructed as integrals on superspace, and actions on d -dimensional spacetime are obtained after Berezin integration on the θ coordinates.

In the supergroup manifold approach, the action is defined as an integral on a d -dimensional bosonic *submanifold* of the supergroup manifold. The Lagrangian is a d -superform, written in terms of the supergroup vielbein and its exterior derivative.

A theory of integration on supermanifolds in the language of differential forms has recently been developed in Refs. [14–18] and makes use of *integral forms*, the analog

of top forms for supermanifolds. Earlier papers with applications of integral and differential forms on superspaces to super Yang-Mills (SYM) and supergravity can be found in [19,20]. Integral forms are instrumental for a rigorous definition of an action principle and have been applied in the past years to supersymmetric and supergravity models [14–18]. Only integral forms can be integrated on supermanifolds, and therefore, the d -superform Lagrangian must be converted into an integral form. This can be done by multiplying it with the analog of the Poincaré dual (as in ordinary integration on submanifolds), called picture changing operator (PCO), a representative of a De Rahm cohomology class. It is characterized by a form number equal to 0 and a “picture number” equal to the fermionic dimension of the supermanifold, as explained in Appendix A. If there exists only one such cohomology class, one can still choose different representatives. For each choice, the final form of the action will have a different expression, but all these expressions are equivalent and correspond to the same field theory. In the case of more than one cohomology class, one can construct actions describing physically different theories, each corresponding to a particular cohomology class.

This procedure extends the group manifold approach and allows one to make contact with superspace formulations. One starts from the group manifold (“rheonomic”) action, determines the cohomology of the corresponding supergroup manifold, and chooses representatives in the cohomology classes, i.e., the PCOs. Multiplying the group manifold Lagrangian by these PCOs yields the action as an integral form. The choice of PCO within the same cohomology class is dictated essentially by the (super) symmetries one wishes to be manifest in the action. The

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superspace action is then obtained as described in Secs. 3 and 4.

Another way to construct integral forms is to combine pieces with different form and picture numbers, without reference to a d -superform Lagrangian. This second way often makes use of a generalization of the Hodge dual for supermanifolds, introduced in Refs. [16,21] by means of a generalized Fourier transform.

In this paper we obtain two main results: (i) the extension to $d = 4$ of the integral form construction for supergravity, and the proof of its equivalence with the superspace formulation, with the same logic used in Ref. [17] for the $d = 3$ case, i.e., with the use of appropriate PCOs; (ii) the reformulation of super Yang-Mills and supergravity theories in $d = 4$ by using the super Hodge dual of Refs. [16,21].

The paper is organized as follows. In Sec. II we use Cartan calculus and the super Hodge dual to translate superspace geometry [5,22] in the language of differential geometry of supermanifolds. We show how the Berezinian emerges naturally and what its relation is with chiral volume forms. Super Yang-Mills and supergravity in $d = 4$ are reformulated using the super Hodge dual. In Sec. III we give the group manifold treatment of $d = 4$ supergravity with the “old minimal” set [23] of auxiliary fields. The resulting 4-form Lagrangian is then multiplied by a suitably chosen PCO, thus obtaining an integral form that can be integrated on superspace. The result is shown to coincide with the superspace expression of the action as given in [5], i.e., the superspace integral of the super-determinant. A similar exercise is carried out in Sec. IV, where the “new minimal” set of auxiliary fields [24] is employed. Finally, in the appendixes we discuss in more detail aspects of supermanifold geometry and give the derivation of some useful formulas recalled in the text.

II. FIELD THEORIES AND HODGE DUAL OPERATOR

In this section, we translate some of the superspace equations in terms of the super Hodge dual introduced in [16,21]. We use here the following notation: indices A, B, \dots refer to the super tangent space: $A = (a, \alpha), B = (b, \beta)$, where $a, b, \dots = 0, \dots, 3$ are the tangent vector indices and $\alpha, \beta, \dots = 1, \dots, 4$ the spinorial tangent indices. We denote by M, N, \dots the curved indices of the supermanifold $M = (m, \mu), N = (n, \nu)$ where m, n, \dots are the indices of the bosonic coordinates and μ, ν, \dots the indices of the fermionic coordinates. Collectively, we denote by $Z^M = (x^m, \theta^\mu)$ the coordinates of the supermanifold. We denote by $\mathbb{R}^{(4|4)}$ the superspace (flat supermanifold). In Appendix A we collect the definitions for forms on supermanifolds and integration theory; for more details, we refer to the literature [15,25,26].

After fixing the geometrical setting and discussing some properties of the super Hodge dual operator \star , we present

in this section simple applications to scalar field theory, Abelian gauge theory, and supergravity. We will use uniquely the Hodge dual operator, without referring to the group manifold (or *rheonomic*) approach [10] (see [12,13] for recent reviews), where the action is constructed [14,15,17] from the rheonomic Lagrangian as follows:

$$S = \int_{\mathcal{SM}} \mathcal{L}_{\text{rheo}}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}. \quad (2.1)$$

The suffix $(4|0)$ stands for form number equal to four and picture number equal to zero (as defined in Appendix A). $\mathcal{L}_{\text{rheo}}^{(4|0)}$ is the rheonomic Lagrangian written in terms of the form fields and their differentials, while $\mathbb{Y}^{(0|4)}$ is the PCO converting the action from a $(4|0)$ form to a $(4|4)$ form, which can be integrated on the supermanifold \mathcal{SM} . The action S depends on the supergravity fields in the rheonomic Lagrangian $\mathcal{L}_{\text{rheo}}^{(4|0)}$ and in the PCO $\mathbb{Y}^{(0|4)}$, and being integrated on the entire supermanifold \mathcal{SM} is automatically invariant under super-reparametrizations. In this framework the action is built *without* using the Hodge dual operator \star .

In the present section we adopt a different strategy: we replace the factorized form of the action with a novel one,

$$S = \int_{\mathcal{SM}} \mathcal{L}_{\star}^{(4|4)}, \quad (2.2)$$

where a suitable Hodge dual operator \star is used. The Lagrangian is no longer factorized into the wedge product of a $(4|0)$ form with a $(0|4)$ form. Again, since $\mathcal{L}_{\star}^{(4|4)}$ is a $(4|4)$ form integrated on the entire supermanifold \mathcal{SM} , it is invariant under super-reparametrizations.

Consider the actions for (free) scalar field theory and Abelian gauge theory:

$$S[\phi] = \int_{\mathcal{M}} d\phi \wedge \star d\phi, \quad S[A] = \int_{\mathcal{M}} dA \wedge \star dA, \quad (2.3)$$

where \mathcal{M} is a bosonic manifold, ϕ is a scalar field, and A is the 1-form potential. The operator \star is the Hodge dual operator related to a metric g on the manifold \mathcal{M} and the actions (2.3), which are 4-forms integrated on a four-dimensional manifold, are manifestly reparametrization invariant. In the following, we replace the manifold \mathcal{M} with a supermanifold \mathcal{SM} , the \star operator with a corresponding super \star operator (see [16,21] for a general treatment); ϕ is replaced by a superfield at zero picture, and A is replaced by a 1-superform at zero picture (the picture number being defined in Appendix A).

A. Super Hodge dual operator

We define the super Hodge dual operator \star as follows. Given a form $\omega(x, \theta, V, \psi)$, considered as a generalized

function of the supervielbein (V^a, ψ^α) and of the coordinates (x^a, θ^α) (see Appendix A), its Hodge dual is written in terms of a Fourier transform

$$\star\omega(x, \theta, V, \psi) = i^{r^2-n^2} i^l \int_{\mathcal{SM}'} e^{i(\nu^a \eta_{ab} V^b + p^\alpha \lambda C_{\alpha\beta} \psi^\beta)} \times \omega(x, \theta, \nu, p) [d^4 \nu d^4 p], \quad (2.4)$$

where r is the number of V 's, l is the number of ψ 's, n is the bosonic dimension, and \mathcal{SM}' is the dual superspace whose fundamental coordinates are (ν^a, p^α) (respectively, anti-commuting and commuting). The symbol $[d^4 \nu d^4 p]$ denotes the Berezin integral over ν^a and the Riemann-Lebesgue integral over p^α . The metric g to which \star is related is given by the tensor¹

$$g = \eta_{ab} V^a \otimes V^b + \lambda C_{\alpha\beta} \psi^\alpha \otimes \psi^\beta, \quad (2.5)$$

$$\begin{aligned} V_2^{ab} &\equiv \epsilon^{abcd} \frac{1}{2!} \eta_{cc'} \eta_{dd'} \wedge V^{c'} \wedge V^{d'}, & V_3^a &\equiv \epsilon^{abcd} \frac{1}{3!} \eta_{bb'} \eta_{cc'} \eta_{dd'} V^{b'} \wedge V^{c'} \wedge V^{d'}, \\ V^4 &= V_4 \equiv \frac{1}{4!} \epsilon_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d, & \delta^4(\psi) &\equiv \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta} \delta(\psi^\alpha) \delta(\psi^\beta) \delta(\psi^\gamma) \delta(\psi^\delta), \\ \iota_\alpha \delta^4(\psi) &\equiv \frac{\partial}{\partial \psi^\alpha} \delta^4(\psi), & \text{Vol}^{(4|4)} &\equiv V^4 \delta^4(\psi), \end{aligned} \quad (2.7)$$

and the pseudoform $\delta(\psi^\alpha)$ (see Appendix A) naturally arises as the Fourier transform $\delta(\psi^\alpha) = \int dp e^{ip\psi^\alpha}$. Notice that the scaling dimensions of V^a and of its dual $\star V^a$ are the same, as well as for ψ^α and $\star\psi^\alpha$. Using the above definitions, we immediately find

$$\begin{aligned} V^a \wedge \star V^b &= \eta^{ab} \text{Vol}^{(4|4)}, & \psi^\alpha \wedge \star\psi^\beta &= \lambda^{-1} C^{\alpha\beta} \text{Vol}^{(4|4)}, \\ \star \text{Vol}^{(4|4)} &= 1, & \star 1 &= \text{Vol}^{(4|4)}. \end{aligned} \quad (2.8)$$

With our conventions, the dimension of $\text{Vol}^{(4|4)}$ is equal to 2.

As is well known, the Hodge dual operator for usual manifolds is a key ingredient for defining the Laplace-Beltrami operator acting on differential forms. In the present context we can consistently extend the definition of the Laplace-Beltrami operator to supermanifolds as

$$\Delta = d^\dagger d + d d^\dagger \equiv \star d \star d + d \star d \star \equiv d^\dagger d + d d^\dagger, \quad (2.9)$$

the conjugated differential operator being defined by $d^\dagger \equiv \star d \star$. Note that we can define a pairing between two forms on a supermanifold, with complementary form numbers $p, 4-p$ and complementary picture numbers $q, 4-q$, as

¹Note that, since V^a and ψ^α are, respectively, bosonic and fermionic 1-forms, $\psi^\alpha \wedge \psi^\beta = \psi^\beta \wedge \psi^\alpha$ and $V^a \wedge V^b = -V^b \wedge V^a$.

where λ is a dimensionful constant. This constant is needed in order to preserve the homogeneity of dimensions between the two terms since V^a scales as the square of ψ^α . For $\lambda \rightarrow 0$ the metric is degenerate. $C_{\alpha\beta}$ is the charge conjugation antisymmetric matrix (this holds specifically for the case $d=4$). The function $\omega(x, \theta, \nu, p)$ is the function obtained from $\omega(x, \theta, V, \psi)$ by substituting $V \rightarrow \nu$ and $\psi \rightarrow p$, leaving the supermanifold coordinates x, θ untouched.

We can easily compute the Hodge dual of the supervielbeins

$$\star V^a = V_3^a \delta^4(\psi), \quad \star \psi^\alpha = \lambda^{-1} C^{\alpha\beta} V^4 \iota_\beta \delta^4(\psi). \quad (2.6)$$

where

$$\langle \omega, \eta \rangle = \int_{\mathcal{SM}} \omega \wedge \star \eta \quad (2.10)$$

and consequently $\langle \omega, d\eta \rangle = \langle d^\dagger \omega, \eta \rangle$. This pairing does *not* define a positive definite scalar product on forms due to the symplectic nature of the scalar product for the fermionic components of a vector field. Because of that, even though we have the identity $\langle \omega, \Delta \omega \rangle = \langle d\omega, d\omega \rangle + \langle d^\dagger \omega, d^\dagger \omega \rangle$ based on Stokes' theorem for supermanifolds, we cannot conclude that $d\omega = 0$ and $d^\dagger \omega = 0$ if $\Delta \omega = 0$. The Hodge theory for supermanifolds will be discussed in [27].

We can compute the Laplacian (2.9) on the simplest examples of a superfield ϕ and of a top-integral form $\phi^{(4|4)}$. On a 0-form ϕ we find

$$\begin{aligned} d\phi &= V^a \nabla_a \phi + \psi^\alpha \nabla_\alpha \phi, \\ \star d\phi &= V_3^a \delta^4(\psi) \nabla_a \phi + \lambda^{-1} V^4 C^{\alpha\beta} \iota_\alpha \delta^4(\psi) \nabla_\beta \phi, \\ d(\star\phi) &= d(\phi \text{Vol}^{(4|4)}) = 0, \\ \Delta\phi &= (\star d \star d + d \star d \star) \phi \\ &= (\eta^{ab} \nabla_a \nabla_b + \lambda^{-1} C^{\alpha\beta} \nabla_\alpha \nabla_\beta) \phi, \end{aligned} \quad (2.11)$$

where $\nabla_a = E_a^M \partial_M$, $\nabla_\alpha = E_\alpha^M \partial_M$, E_A^M is the supervielbein, and $\partial_M = (\partial_m, \partial_\mu)$. Δ is the generalized Laplacian on superfields, and the two terms of the last equation in (2.11) scale with different powers of λ .

In the same way, considering a top integral form $\phi^{(4|4)} = \phi \text{Vol}^{(4|4)}$, where ϕ is its section, we have

$$\begin{aligned} \Delta\phi^{(4|4)} &= (\star d\star d + d\star d\star)(\phi \text{Vol}^{(4|4)}) \\ &= \left[(\eta^{ab} \nabla_a \nabla_b + \lambda^{-1} C^{\alpha\beta} \nabla_\alpha \nabla_\beta) \phi \right] \text{Vol}^{(4|4)}, \end{aligned} \quad (2.12)$$

which is again the Laplacian operator on the section ϕ of the integral form $\phi^{(4|4)}$, multiplied by $\text{Vol}^{(4|4)}$.

B. Chiral superfields

The smallest irreducible representation of supersymmetry with scalar bosonic degrees of freedom is described by a *chiral superfield* (see the textbooks [5,6,22]). The components of an off-shell chiral superfield describe a complex scalar, its fermionic superpartner, and a complex auxiliary field, i.e., 4 (2) bosonic and 4 (2) fermionic degrees of freedom if off-shell (on-shell). A superfield ϕ is in general a reducible representation: imposing some invariant equations we can single out irreducible representations. This is done by setting $D_L\phi = 0$ and $D_R\bar{\phi} = 0$ for chiral ϕ and antichiral $\bar{\phi}$ superfields, where $(D_L)_\alpha = \frac{1}{2}(1 + \gamma^5)_\alpha{}^\beta \nabla_\beta$ and $(D_R)_\alpha = \frac{1}{2}(1 - \gamma^5)_\alpha{}^\beta \nabla_\beta$. In this subsection, we illustrate how a chiral superfield can be defined using the Hodge dual operator \star and its chiral/antichiral relatives $\star_C, \star_{\bar{C}}$.

In the case of flat $d = 4$ superspace, one can define a *chiral* and an *antichiral volume* form as follows:

$$\begin{aligned} \text{Vol}_L^{(4|2)} &= \epsilon_{abcd} V^a \dots V^d \delta^2(\psi_L), \\ \text{Vol}_R^{(4|2)} &= \epsilon_{abcd} V^a \dots V^d \delta^2(\psi_R), \end{aligned} \quad (2.13)$$

where $\psi = \psi_L + \psi_R$ and $\psi_{L/R} = (1 \pm \gamma^5)/2\psi$. The volume forms Vol_L and Vol_R have a form degree equal to 4, but a picture number equal to 2 unlike $\text{Vol}^{(4|4)}$. They transform under the chiral transformations as

$$\begin{aligned} \text{Vol}_L^{(4|2)} &= \text{Sdet}_L(J)^{-1} \text{Vol}_L^{(4|2)}, \\ \text{Vol}_R^{(4|2)} &= \text{Sdet}_R(\bar{J})^{-1} \text{Vol}_R^{(4|2)}, \end{aligned} \quad (2.14)$$

where J (\bar{J}) is a 6×6 supermatrix corresponding to coordinate transformations in chiral (antichiral) superspace. This means that the coordinate transformations are generated by chiral/antichiral superfields as we now describe. Notice that although $\text{Vol}_L^{(4|2)}$ and $\text{Vol}_R^{(4|2)}$ are not top integral forms, they are closed, i.e., $d\text{Vol}_L^{(4|2)} = 0$ and $d\text{Vol}_R^{(4|2)} = 0$, since $dV^a = \bar{\psi}_L \gamma^a \psi_R$ and either ψ_L or ψ_R are annihilated by $\delta^2(\psi_L)$ or $\delta^2(\psi_R)$. The volume form $\text{Vol}^{(4|4)}$ is related to the chiral volume forms as

$$\text{Vol}^{(4|4)} = \mathcal{R}_L \text{Vol}_L^{(4|2)} \wedge \mathbb{Y}_R + \mathcal{R}_R \text{Vol}_R^{(4|2)} \wedge \mathbb{Y}_L, \quad (2.15)$$

where \mathbb{Y}_L and \mathbb{Y}_R are suitable PCOs and where \mathcal{R}_L and \mathcal{R}_R are two superfields. We will discuss this point further in the context of a generic supermanifold; it leads to Siegel's formula [see formula (5.5.21) in [22]] for the volume of a $D = 4, N = 1$ supermanifold.

Now we can define *chiral* and *antichiral superfields* as follows:

$$d\phi \wedge \text{Vol}_L^{(4|2)} = 0, \quad d\bar{\phi} \wedge \text{Vol}_R^{(4|2)} = 0. \quad (2.16)$$

Notice that $d\phi = V^a \partial_a \phi + \bar{\psi}_R D_R \phi + \bar{\psi}_L D_L \phi$ is a 1-superform, and therefore, the above equations select only the following pieces:

$$\bar{\psi}_R D_R \phi \wedge \text{Vol}_L^{(4|2)} = 0, \quad \bar{\psi}_L D_L \bar{\phi} \wedge \text{Vol}_R^{(4|2)} = 0, \quad (2.17)$$

since the remaining terms are set to zero either by the product of V 's or by $\delta^2(\psi_{L/R})$. Therefore, by imposing (2.17), we obtain the usual chiral and antichiral constraints $D_R \phi = 0$ and $D_L \bar{\phi} = 0$.²

We introduce the chiral and antichiral Hodge dual operators [defined in the same way as the full Hodge dual operator (2.4), where the fermionic variables are replaced by their chiral or antichiral counterparts], denoted by \star_C and $\star_{\bar{C}}$. Then we have

$$\begin{aligned} \star_L \phi &= \phi \text{Vol}_L^{(4|2)}, & \star_R \bar{\phi} &= \bar{\phi} \text{Vol}_R^{(4|2)}, \\ \star_L 1 &= \text{Vol}_L^{(4|2)}, & \star_R 1 &= \text{Vol}_R^{(4|2)}. \end{aligned} \quad (2.18)$$

Acting with d yields

$$d(\star_L \phi) = d\phi \wedge \text{Vol}_L^{(4|2)}, \quad d(\star_R \bar{\phi}) = d\bar{\phi} \wedge \text{Vol}_R^{(4|2)}. \quad (2.19)$$

Acting again with \star_L and \star_R on both members of these equations, we find

$$\star_L d(\star_L \phi) = \bar{\psi}_R D_R \bar{\phi}, \quad \star_R d(\star_R \bar{\phi}) = \bar{\psi}_L D_L \bar{\phi}, \quad (2.20)$$

and therefore, the chirality conditions are equivalent to

$$d_L^\dagger \phi = 0, \quad d_R^\dagger \bar{\phi} = 0. \quad (2.21)$$

For a 0-form, we have $d^\dagger \phi = 0$ because $d^\dagger : \Omega^{(p|q)} \rightarrow \Omega^{(p-1|q)}$. This is due to the fact that $\star\phi$ is a top integral form, and therefore $d(\star\phi)$ automatically vanishes. When using chiral Hodge dual operators \star_L or \star_R this is no longer true in general, but remains true on

²In the group manifold framework this is obtained by choosing the differential of ϕ as $d\phi = V^a \nabla_a \phi + \psi_L W_L$ where W_R is set to zero from the beginning [10].

chiral/antichiral superfields. With this language the chirality conditions have a clear geometrical origin.

If we compute the Laplace-Beltrami differential for a chiral/antichiral superfield we get

$$\begin{aligned}\Delta_C\phi &= (d_C^\dagger d + dd_C^\dagger)\phi = d_C^\dagger d\phi = (\eta^{ab}\nabla_a\nabla_b + \lambda^{-1}D_L^2)\phi, \\ \Delta_{\bar{C}}\bar{\phi} &= (d_{\bar{C}}^\dagger d + dd_{\bar{C}}^\dagger)\bar{\phi} = d_{\bar{C}}^\dagger d\bar{\phi} = (\eta^{ab}\nabla_a\nabla_b + \lambda^{-1}D_R^2)\bar{\phi},\end{aligned}\quad (2.22)$$

where $D_L^2 = \bar{D}_L D_L$ and $D_R^2 = \bar{D}_R D_R$. The free equations of motion $\Delta_C\phi = 0$ and $\Delta_{\bar{C}}\bar{\phi} = 0$ for chiral/antichiral superfields are obtained in the limit $\lambda \rightarrow 0$. Otherwise, these equations are higher-derivative modifications of the free equations of motion.

Before discussing the action, we consider another type of multiplet. As is known, there is a second irreducible multiplet with scalar bosonic degrees of freedom: the *linear* superfield. It is defined to satisfy $D_R^2\phi = 0$ and $D_L^2\bar{\phi} = 0$, and the difference with the chiral/antichiral superfields is the different set of auxiliary fields for the off-shell multiplet. From the computation (2.11), we see that

$$\begin{aligned}(d^\dagger - d_C^\dagger)d\phi &= \lambda^{-1}D_R^2\phi, \\ (d^\dagger - d_{\bar{C}}^\dagger)d\bar{\phi} &= \lambda^{-1}D_L^2\bar{\phi},\end{aligned}\quad (2.23)$$

and therefore, the requirement $D_R^2\phi = 0$ and $D_L^2\bar{\phi} = 0$ implies that on $d\phi$ the two differentials d^\dagger and d_C^\dagger coincide. Since the linear superfields and the chiral superfields are related by duality transformations (see [5]), it would be interesting to verify whether indeed the geometric equation (2.23) emerges as dual to (2.21).

The action can be written using the Hodge dual operator \star for a chiral superfield $\text{Vol}_R \wedge d\phi = 0$ as follows:

$$S = \int_{S\mathcal{M}^{(4|4)}} \star\bar{\phi}\phi, \quad (2.24)$$

which reduces to the usual chiral superfield action by integrating on the cotangent directions V^a , ψ_L , and ψ_R . Notice that a geometrical action $S = \int_{S\mathcal{M}^{(4|4)}} d\bar{\phi} \wedge \star d\phi$ would yield a high derivative theory after Berezin integration. In superspace language higher derivative theories can be built in terms of prepotentials. The present approach based on the Hodge dual naturally provides a specific higher derivative extension.

C. Gauge fields

On a bosonic manifold \mathcal{M} , the Bianchi identity and the Maxwell equations for the Abelian potential A are given by

$$dF = 0, \quad d^\dagger F = J, \quad (2.25)$$

where J is the conserved ($d^\dagger J = 0$) electric current coupled to the gauge field A and $F = dA$ is its field strength.

Moving to a supermanifold, one considers a gauge superfield A and its field strength F which are superforms

$$A = A_a V^a + A_\alpha \psi^\alpha, \quad F = F_{ab} V^a V^b + F_{a\beta} V^a \psi^\beta + F_{\alpha\beta} \psi^\alpha \psi^\beta. \quad (2.26)$$

They have many components since A_a and A_α are superfields. To single out the physical ones, one imposes the condition that $F_{\alpha\beta} = 0$ and by means of the Bianchi identity $dF = 0$, we find that there are only one gauge vector field, the gaugino, and one auxiliary field. Thus, the field strength can be written as

$$F = f_{ab} V^a V^b + \bar{W}\gamma_a \psi V^a, \quad (2.27)$$

where W^α is the gaugino field strength and γ^a are the Dirac matrices in the Majorana representation. In the component expansion of W^α (as illustrated in [5] or [6]) one retrieves the physical degrees of freedom and the auxiliary fields. The gauge field strength f_{ab} is not an independent superfield since it satisfies $f_{\alpha\beta} = \frac{1}{4}\nabla\gamma_{ab}W$ as a consequence of the Bianchi identity, where ∇_a is the superderivative introduced in the previous section. Again, as a consequence of the Bianchi identities, we have $\nabla_a W^\alpha = 0$.

Computing the Hodge dual of F we obtain

$$\star F = f_{ab}\eta^{aa'}\eta^{bb'}\epsilon_{a'b'cd}V^cV^d\delta^4(\psi) + \lambda^{-1}(V_3)^a\bar{W}\gamma_a C\mathcal{I}\delta^4(\psi). \quad (2.28)$$

Notice that the charge conjugation matrix C enters because, due to the Hodge dual operator, indices are contracted covariantly. Then we compute the differential and finally the Hodge dual again, finding

$$\star d\star F \equiv d^\dagger F = (\nabla^a f_{ab} - \lambda^{-1}\bar{\nabla}\gamma_b CW)V^b + (\bar{\psi}\gamma^a\nabla_a W) = 0. \quad (2.29)$$

Using the parametrization (2.27), we find $\bar{\nabla}_a W^\beta = (\gamma_{ab})^\beta_\alpha f^{ab}$, so that $\bar{\nabla}\gamma_b CW = \text{tr}(\gamma_b C\gamma_{ac})f^{ac} = 0$ since the trace vanishes. We finally obtain the usual free equations of motion

$$\nabla^a f_{ab} = 0, \quad \gamma^a\nabla_a W = 0. \quad (2.30)$$

Since $\bar{\nabla}_a W^\beta = (\gamma_{ab})^\beta_\alpha f^{ab}$, the field strength f_{ab} is not independent of W ; therefore, the first equation (which looks like the Maxwell equations for the superfield f_{ab}) is a high-derivative equation $\nabla^b(\nabla\gamma_{ab}W) = \eta_{ab}\nabla^b(\nabla_a W^\beta) - \nabla_a(\gamma_a\gamma^b\nabla_b W) = 0$, but it is satisfied if the Dirac equation is satisfied (the first term vanishes because of the Bianchi identities).

Therefore, we have

$$F \wedge \star F = (f_{ab}f^{ab} + \lambda^{-1}\bar{W}CW)V^4\delta^4(\psi), \quad (2.31)$$

which can be naturally integrated on the supermanifold leading to the high-derivative action

$$\begin{aligned} \int_{\mathcal{SM}^{(4|4)}} F \wedge \star F &= \int (f_{ab}f^{ab} + \lambda^{-1}\bar{W}CW) \\ &= \int (\nabla\gamma_{ab}W\nabla\gamma^{ab}W + \lambda^{-1}\bar{W}CW). \end{aligned} \quad (2.32)$$

After integration on the integral forms we are left with the conventional superspace integral for a function. The action starts with the conventional $f_{ab}f^{ab}$ term. To compute the Berezin integral one has to expand $f_{ab}f^{ab}$ into powers of θ 's, and therefore, the final result will contain higher derivatives of the field strength. The second term, however, is the conventional superspace action, recovered in the limit $\lambda \rightarrow 0$. The equations of motion are easily retrieved by introducing the potentials A_a , A_α and varying the action with respect to them.

Let us consider now the $N = 2$ case. The field strength (after adopting the conventional constraints) is expanded as

$$F = f_{ab}V^aV^b + \bar{W}^A\gamma_a\psi_AV^a + \Phi^{AB}\bar{\psi}_A\gamma_5\psi_B, \quad (2.33)$$

and its Hodge dual is

$$\begin{aligned} \star F &= f_{ab}\eta^{aa'}\eta^{bb'}\epsilon_{a'b'cd}V^cV^d\delta^8(\psi) \\ &\quad + \lambda^{-1}(V_3)^a\epsilon_{AA'}(\bar{W}^A\gamma_a C t^{A'})\delta^8(\psi) \\ &\quad - \lambda^{-2}V_4\epsilon_{AA'}\epsilon_{BB'}\phi^{AB}\bar{t}^{A'}\gamma_5 t^{B'}\delta^8(\psi). \end{aligned} \quad (2.34)$$

To take into account the correct mass dimensions, we introduce again the scale λ . The action takes the form

$$\int_{\mathcal{SM}^{(4|4)}} F \wedge \star F = \int (f_{ab}f^{ab} + \lambda^{-1}\bar{W}^A C W_A - \lambda^{-2}\bar{\Phi}^{AB}\Phi_{AB}) \quad (2.35)$$

and is a high-derivative action. It is not possible to use the same formula for values of N greater than 3, as the mass dimensions are greater than the integration measure. This aligns with the belief that there is no superspace action for super Yang-Mills with $N = 4$ extended supersymmetry. However, for $N = 3$, $d = 4$ SYM, the harmonic superspace technique provides a clear and supersymmetric expression [28]. Connecting the current formalism with integral forms and the rheonomic action to the harmonic superspace formulation would be interesting. It is worth noting that in [29], a bridge between the pure spinor formulation of SYM $d = 10$, $N = 1$ has been established using PCO's and the group manifold approach. Additionally, in [30], the relationship between $d = 10$,

$N = 1$ SYM in the pure spinor approach has been used to derive the $N = 2$ and $N = 3$ harmonic superspace formulation. Therefore, we anticipate that a relation can eventually be found, to link the two frameworks by selecting a suitable class representative for the PCO.

D. Supergravity

Finally, we use the Hodge dual operator also for the construction of the supergravity action. It has been observed in [5] that the full supergravity action with auxiliary fields can be expressed in terms of the superdeterminant. To write the supergravity action in our framework we observe that, in the case of a curved supermanifold, the volume form can be written as

$$\text{Vol}^{(4|4)} = V^4\delta^4(\psi) = \text{Sdet}(E)d^4x\delta^4(d\theta), \quad (2.36)$$

where we express the vielbein V^a and the graviton ψ^α in terms $E^A = (V^a, \psi^\alpha)$ on a curved basis:

$$\begin{aligned} V^a &= E_m^a dx^m + E_\mu^a d\theta^\mu, \\ \psi^\alpha &= E_m^\alpha dx^m + E_\mu^\alpha d\theta^\mu. \end{aligned} \quad (2.37)$$

All components are superfields and

$$\text{Sdet}(E) = \frac{\det(E_m^a - E_\mu^a(E^{-1})^\mu_a E_m^a)}{\det(E_\mu^\alpha)}. \quad (2.38)$$

The $(4|4)$ form $\text{Vol}^{(4|4)}$ is trivially closed (being a top integral form) and, if it represents a cohomology class, it is not exact. Then

$$\int_{\mathcal{SM}^{(4|4)}} \star 1 = \int_{\mathcal{SM}^{(4|4)}} \text{Vol}^{(4|4)} = \int \text{Sdet}(E)(x, \theta)[d^4x d^4\theta], \quad (2.39)$$

where the second integration is performed on the superspace coordinates (x^m, θ^μ) . The symbol $[d^4x d^4\theta]$ denotes only on which coordinates the integral has to be performed, but it does not represent a measure. Since, as discussed in the literature [5], the Berezin integral of the superdeterminant in (2.39) yields the action of $d = 4$ supergravity, this action can be written as the integral of the Hodge dual of a constant (Newton's constant). In the next sections, we derive this formula from the group-manifold approach.

To make contact with other superspace formulations of supergravity, it is convenient to define also the chiral/antichiral volume forms (see also [18]) as follows:

$$\begin{aligned} \text{Vol}_L^{(4|2)} &= \epsilon_{abcd}V^a \wedge \dots \wedge V^d \delta^2(\psi_L) = V^4 \delta^2(\psi_L), \\ \text{Vol}_R^{(4|2)} &= \epsilon_{abcd}V^a \wedge \dots \wedge V^d \delta^2(\psi_R) = V^4 \delta^2(\psi_R), \end{aligned} \quad (2.40)$$

where $\psi_{L/R} = \frac{1}{2}(1 \pm \gamma_5)\psi$. Next, we separate in $V^a = V_L^a + E_\mu^\alpha d\theta_L^\mu$ the part of the vielbein along $d\theta_L^\alpha$ (in the flat space, this means $V_L^a = dx^a + \bar{\theta}_L \gamma^a \psi_R$), and, similarly, we write $\psi_L^\alpha = E_\mu^\alpha d\theta_L^\mu + \Lambda_a^\alpha V_L^a$. Notice that in general, $\psi_L^\alpha = E_\mu^\alpha d\theta_L^\mu + \Lambda_a^\alpha V_L^a + \Lambda_\beta^\alpha d\theta_R^\beta$ but the last term can be set to zero by choosing a suitable gauge (reached by a superdiffeomorphism) known as chiral/antichiral representation (see [5,22]). Then, using the flat basis $(dx^a, d\theta_L^\alpha, d\theta_R^\alpha)$, we have

$$\text{Vol}_L^{(4|2)} = \epsilon_{abcd}(V_L^a + E_\mu^\alpha d\theta_L^\mu) \wedge \dots \wedge (V_L^d + E_\mu^d d\theta_L^\mu) \delta^2(E_\mu^\alpha d\theta_L^\mu + \Lambda_a^\alpha V_L^a) \quad (2.41)$$

$$= \epsilon_{abcd} \frac{(V_L^a + E_\mu^\alpha d\theta_L^\mu) \wedge \dots \wedge (V_L^d + E_\mu^d d\theta_L^\mu)}{\det(E_\mu^\alpha)} \times \delta^2(d\theta_L^\mu + (E^{-1})_\mu^\alpha \Lambda_a^\alpha V_L^a) \quad (2.42)$$

$$= \frac{\det(\delta_b^a + E_\mu^\alpha (E^{-1})_\alpha^\mu \Lambda_b^\alpha)}{\det(E_\mu^\alpha)} \epsilon_{abcd} V_L^a \wedge \dots \wedge V_L^d \delta^2(d\theta_L^\mu) = \mathcal{E}_L \epsilon_{abcd} V_L^a \wedge \dots \wedge V_L^d \delta^2(d\theta_L^\mu) = \mathcal{E}_L V_L^4 \delta^2(d\theta_L^\mu), \quad (2.43)$$

$$\mathcal{E}_L = \frac{\det(\delta_b^a + E_\mu^\alpha (E^{-1})_\alpha^\mu \Lambda_b^\alpha)}{\det(E_\mu^\alpha)}, \quad (2.44)$$

where $\mathcal{E}_L(x, \theta_L)$ is a chiral superfield³ according to our definition. Indeed, we have

$$0 = d\text{Vol}_L^{(4|2)} = d\mathcal{E}_L \wedge V_L^4 \delta^2(\psi_L) = d \ln \mathcal{E}_L \wedge \text{Vol}_L^{(4|2)}, \quad (2.45)$$

implying that $\ln \mathcal{E}_L$ is a chiral superfield. In the same way, we can study $\text{Vol}_R^{(4|2)}$ and the chiral density \mathcal{E}_R .

Let us now relate the volume form $\text{Vol}^{(4|4)}$ to the chiral ones. The superdeterminant $E = \text{Sdet}(E)$ is a function of $(x, \theta, \bar{\theta})$ and reads

$$\begin{aligned} \text{Sdet}(E) &= \frac{\det(E_m^a - E_\mu^\alpha (E^{-1})_\alpha^\mu E_m^\beta - E_\mu^\alpha (\bar{E}^{-1})_\beta^\mu E_m^\beta)}{\det(E_L) \det(E_R)} \\ &= \frac{\text{Sdet}_R(\hat{E})}{\det(E_R)} = \frac{\text{Sdet}_L(\hat{E})}{\det(E_L)}, \end{aligned} \quad (2.46)$$

where $\text{Sdet}_L(\hat{E})$ is the chiral superdeterminant written in terms of a redefined vielbein $\hat{E}_m^a = E_m^a - E_{R\mu}^\alpha (E^{-1})_\alpha^\mu E_{Lm}^\beta$. In terms of the volume form we have

$$\begin{aligned} \text{Vol}^{(4|4)} &= \frac{1}{2} V_L^4 \delta^2(\psi_L) \frac{1}{\det(E_R)} \delta^2(d\theta_R) + \frac{1}{2} V_R^4 \delta^2(\psi_R) \frac{1}{\det(E_L)} \delta^2(d\theta_L) \\ &= \frac{1}{2} \text{Vol}_L^{(4|2)} \frac{1}{\det(E_R)} \delta^2(d\theta_R) + \frac{1}{2} \text{Vol}_R^{(4|2)} \frac{1}{\det(E_L)} \delta^2(d\theta_L), \end{aligned} \quad (2.47)$$

where V_L^a, ψ_L^α and V_R^a, ψ_R^α are given in terms of the redefined supervielbeins. Then, using the fact that $\delta^2(d\theta_R)$ and $\delta^2(d\theta_L)$ are not the PCOs, unless they are multiplied by θ_R^2 and θ_L^2 , we can expand $1/\det(E_R)$ and $1/\det(E_L)$ up to second order [we recall that $\delta^2(d\theta_R)$ and $\theta_R^2 \delta^2(d\theta_R)$ are cohomologically trivial], and since $\text{Vol}_L^{(4|2)}$ and $\text{Vol}_R^{(4|2)}$ are closed, we can always discard d -exact terms. Therefore, we are left with

$$\begin{aligned} \text{Vol}^{(4|4)} &= \text{Vol}_L^{(4|2)} D_R^2 \left(\frac{1}{2 \det(E_R)} \right) \mathbb{Y}_R \\ &+ \text{Vol}_R^{(4|2)} D_L^2 \left(\frac{1}{2 \det(E_L)} \right) \mathbb{Y}_L, \end{aligned} \quad (2.48)$$

where \mathbb{Y}_L and \mathbb{Y}_R are the PCOs. The two expressions $D_R^2(1/2\det(E_R))$ and $D_L^2(1/2\det(E_L))$ are computed

³Several textbooks use the chiral coordinates $z_L = (x_L, \theta_L)$ and identify the chiral measure with the Berezinian of the transformation $z'_L = z'_L(z_L)$.

in [5,22] and are identified with the superfields \mathcal{R}_R and \mathcal{R}_L . The superfields $\mathcal{R}_{R/L}$ contain the auxiliary fields and the Ricci scalar; they appear in the commutation relations $\{\nabla_\alpha, \nabla_\beta\} = -\bar{\mathcal{R}} \mathcal{M}_{\alpha\beta}$ and are components of the torsion T^A as will be discussed in the forthcoming section. Finally, we can write the volume form as

$$\text{Vol}^{(4|4)} = \text{Vol}_L^{(4|2)} \mathcal{R}_R \mathbb{Y}_R + \text{Vol}_R^{(4|2)} \mathcal{R}_L \mathbb{Y}_L. \quad (2.49)$$

The formula reproduces, in terms of integral forms, the Siegel chiral-integration formula [5]. In superspace language this formula reads

$$\int \text{Sdet}(E) = \int_L \text{Sdet}_L(\hat{E}) \mathcal{R}_L + \int_R \text{Sdet}_R(\hat{E}) \mathcal{R}_R, \quad (2.50)$$

which turns out to be crucial in comparing the geometric formulation of supergravity with its superspace formulation (a clear derivation and discussion in the context of conformal supergravity can be found in [22]).

III. OLD MINIMAL SUPERGRAVITY

In supergravity the supervielbeins E^a and E^α become dynamical and satisfy equations of motion. In the present section, we review the group manifold formulation of $d = 4$, $N = 1$ supergravity, we define a suitable PCO, and we finally show the equivalence with the superspace formulation. In the end, we write the action in terms of the Hodge dual operator.

A. Off-shell degrees of freedom

The theory contains a vielbein 1-form V^a with 6 off-shell degrees of freedom and a Majorana gravitino ψ^α with 12 off-shell degrees of freedom (d.o.f.). We can match off-shell d.o.f. by adding three 0-form auxiliary fields: an axial vector A^a with 4 d.o.f., a scalar S with one d.o.f., and a pseudoscalar P with 1 d.o.f. This set of auxiliary fields was first introduced in [23]. Here we reformulate the theory in the group manifold approach.

B. The algebra and Bianchi identities

We start from the super Poincaré algebra, extended with the three 0-forms A^a, S, P . The deformed Maurer-Cartan equations for this extended superalgebra are given by

$$R^a = dV^a - \omega^a{}_c V^c - \frac{i}{2} \bar{\psi} \gamma^a \psi \equiv \mathcal{D}V^a - \frac{i}{2} \bar{\psi} \gamma^a \psi, \quad (3.1)$$

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \omega^{cb}, \quad (3.2)$$

$$\rho = d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi \equiv \mathcal{D}\psi, \quad (3.3)$$

$$R(A)^a = dA^a - \omega^a{}_c A^c, \quad (3.4)$$

$$R(S) = dS, \quad (3.5)$$

$$R(P) = dP. \quad (3.6)$$

These equations can be seen as the definition of curvatures. Taking the exterior derivative of both sides yields the Bianchi identities:

$$\begin{aligned} dR^a - \omega^a{}_b R^b + R^a{}_b V^b - i \bar{\psi} \gamma^a \rho \\ \equiv \mathcal{D}R^a + R^a{}_b V^b - i \bar{\psi} \gamma^a \rho = 0, \end{aligned} \quad (3.7)$$

$$dR^{ab} - \omega^a{}_c R^{cb} + \omega^b{}_c R^{ca} \equiv \mathcal{D}R^{ab} = 0, \quad (3.8)$$

$$d\rho - \frac{1}{4} \omega^{ab} \gamma_{ab} \rho + \frac{1}{4} R^{ab} \gamma_{ab} \psi \equiv \mathcal{D}\rho + \frac{1}{4} R^{ab} \gamma_{ab} \psi = 0, \quad (3.9)$$

$$\mathcal{D}R(A)^a + R^{ab} A_b = 0, \quad (3.10)$$

$$dR(S) = 0, \quad (3.11)$$

$$dR(P) = 0. \quad (3.12)$$

The Maurer Cartan equations, and therefore, also the Bianchi identities, are invariant under the rescalings

$$\begin{aligned} \omega^{ab} &\rightarrow \lambda^0 \omega^{ab}, & V^a &\rightarrow \lambda V^a, & \psi &\rightarrow \lambda^{\frac{1}{2}} \psi, \\ A^a &\rightarrow \lambda^{-1} A^a, & S &\rightarrow \lambda^{-1} S, & P &\rightarrow \lambda^{-1} P. \end{aligned} \quad (3.13)$$

C. Parametrizations of the curvatures

According to the group manifold approach, we parametrize the curvatures so that their outer components (i.e., components along at least one fermionic direction) are related to the inner components (i.e., components along bosonic directions) and to the auxiliary fields. A parametrization compatible with the scalings (3.13) and Lorentz invariance is given by

$$R^a = 0, \quad (3.14)$$

$$R^{ab} = R^{ab}{}_{cd} V^c V^d + \bar{\theta}_c^{ab} \psi V^c + \frac{c_2}{2} \bar{\psi} (\gamma^{ab} \eta + \eta \gamma^{ab}) \psi, \quad (3.15)$$

$$\rho = \rho_{ab} V^a V^b + i c_3 \gamma_5 \psi V^a A_a + i c_4 \gamma_a \eta' \psi V^a, \quad (3.16)$$

$$R(A)^a = (\mathcal{D}A^a)_b V^b + \bar{\psi} \zeta^a, \quad (3.17)$$

$$R(S) = (\partial_a S) V^a + \bar{\psi} \xi, \quad (3.18)$$

$$R(P) = (\partial_a P) V^a + \bar{\psi} \chi, \quad (3.19)$$

with real c_2, c_3, c_4 . The only *a priori* choice is $R^a = 0$, i.e., vanishing (super)torsion. It can be shown that $R^a = R^a{}_{bc} V^b V^c$ would only lead to a redefinition of the spin connection, in terms of V and ψ , while outer components of R^a cannot be found using inner components and auxiliary fields, due to scaling and Lorentz index structure. The Bianchi identities will fix

$$\begin{aligned} \bar{\theta}_c^{ab} \equiv 2i \bar{\rho}_c^{[a} \gamma^{b]} - i \bar{\rho}^{ab} \gamma_c, & \quad \eta = \eta' \equiv S - i \gamma_5 P + A^a \gamma_a \gamma_5, \end{aligned} \quad (3.20)$$

and c_2, c_3 in terms of c_4 , which remains the only free parameter. It is convenient to write the gravitino field strength ρ as follows:

$$\rho^\alpha \equiv \mathcal{D}\psi^\alpha = \rho^\alpha{}_{ab} V^a V^b + (\rho_a \psi)^\alpha V^a, \quad (3.21)$$

where $(\rho_a \psi)^\alpha = \rho^\alpha{}_{ab} \psi^b$ and the general form of the matrix $\rho^\alpha{}_{ab}$ is read off from the parametrization of the curvatures

$$\rho^\alpha{}_{ab} = i c_4 (\gamma_{ab}^\alpha S - i (\gamma_a \gamma_5)^\alpha{}_\beta P + (2(\gamma_5)^\alpha{}_\beta \delta_a^b - (\gamma_5 \gamma_a^b)^\alpha{}_\beta) A_b)$$

or, suppressing the spinorial indices

$$\begin{aligned}
 \rho_a &= ic_4 \left(\gamma_a S - i\gamma_a \gamma_5 P + \gamma_5 (2\delta_a^b - \gamma_a^b) A_b \right) \\
 &= ic_4 \left(\frac{1}{2} \gamma_a (1 - \gamma_5) (S + iP) + \frac{1}{2} \gamma_a (1 + \gamma_5) (S - iP) \right. \\
 &\quad \left. + \gamma_5 (2\delta_a^b - \gamma_a^b) A_b \right) \\
 &= ic_4 \left(\gamma_a P_L \mathcal{R} + \gamma_a P_R \bar{\mathcal{R}} + \gamma_5 (2\delta_a^b - \gamma_a^b) A_b \right) \quad (3.22)
 \end{aligned}$$

with $P_L, P_R, \mathcal{R}, \bar{\mathcal{R}}$ defined by the last equality. It is a function of the auxiliary fields S, P, A_a . Notice that if we give a vacuum expectation value to S denoted by s , it reduces to

$$\rho_a(S = s, P = 0, A_a = 0) = ic_4 \gamma_a s.$$

D. The action

With the usual methods of the group manifold approach [10], the Lagrangian is found to be

$$\begin{aligned}
 \mathcal{L}^{(4|0)} &= R^{ab} V^c V^d \epsilon_{abcd} + 4\bar{\psi} \gamma_5 \gamma_a \rho^a V^a \\
 &\quad + \alpha (S^2 + P^2 + A^a A_a) \epsilon_{abcd} V^a V^b V^c V^d. \quad (3.23)
 \end{aligned}$$

The action is obtained by considering the most general $SO(3, 1)$ scalar 4-form, invariant under the rescalings in Eq. (3.13) and then requiring that the equations of motion admit the vanishing curvature solution

$$R^{ab} = R^a = \rho = R(A)^a = R(S) = R(P) = 0. \quad (3.24)$$

The remaining parameter α is fixed by requiring the closure of $\mathcal{L}^{(4|0)}$, i.e., $d\mathcal{L}^{(4|0)} = 0$. This yields $\alpha = -2(c_4)^2$ and ensures off-shell closure of the supersymmetry transformations.

There are some remarks to be considered. Even though the auxiliary fields appear in the new term in (3.23), they are also hidden in the Lorentz curvature term and the Rarita-Schwinger term. Then, if we define auxiliary field-independent curvatures R_*^{ab}, ρ_* as

$$\begin{aligned}
 R_*^{ab} &= R^{ab} - \frac{c_2}{2} \bar{\psi} (\gamma^{ab} \eta + \eta \gamma^{ab}) \psi, \\
 \rho_* &= \rho - ic_3 \gamma_5 \psi V^a A_a - ic_4 \gamma_a \eta \psi V^a, \quad (3.25)
 \end{aligned}$$

we can rewrite the action as

$$\begin{aligned}
 \mathcal{L}^{(4|0)} &= \epsilon_{abcd} R_*^{ab} V^c V^d + 4\bar{\psi} \gamma_5 \gamma_a \rho_*^a V^a \\
 &\quad + \epsilon_{abcd} c_4 \left(S \bar{\psi} \gamma^{ab} \psi V^c V^d - iP \bar{\psi} \gamma^{ab} \gamma_5 \psi V^c V^d \right) \\
 &\quad + 16ic_4 A_a \bar{\psi} \gamma_b \psi V^c V^d \\
 &\quad - 2(c_4)^2 (S^2 + P^2 + A^a A_a) \epsilon_{abcd} V^a V^b V^c V^d. \quad (3.26)
 \end{aligned}$$

In the second and third lines, we have made explicit the auxiliary fields. The first line is the usual rheonomic $N = 1$ supergravity action. Notice that the S field could acquire a vacuum expectation value (which can be achieved also by shifting $S \rightarrow S + \frac{1}{2l}$), and setting $S = A^a = P = 0$ we obtain the supergravity action with a cosmological term

$$\begin{aligned}
 \mathcal{L}_{\text{AdS}}^{(4|0)} &= \epsilon_{abcd} R_*^{ab} V^c V^d + 4\bar{\psi} \gamma_5 \gamma_a \rho_*^a V^a \\
 &\quad + \epsilon_{abcd} \frac{c_4}{2l} \bar{\psi} \gamma^{ab} \psi V^c V^d - \frac{2(c_4)^2}{(2l)^2} \epsilon_{abcd} V^a V^b V^c V^d. \quad (3.27)
 \end{aligned}$$

The third term is the usual gravitino mass term needed for supersymmetry with the cosmological term.⁴ The resulting action describes $N = 1$ supergravity on AdS_4 . The positive constant l is the radius of AdS_4 . The action turns out to be invariant under the isometry supergroup $OSp(1|4)$, and gauge invariant under local Lorentz symmetry and local supersymmetry transformations. The vielbein V^a , the gravitino ψ^α , and the spin connection ω^{ab} (fixed in terms of V^a, ψ^α) are the Maurer-Cartan forms of the $\mathfrak{osp}(1|4)$ superalgebra. The spin connection is the gauge connection of the Lorentz group, and therefore, we can consider V^a, ψ^α as the vielbeins of the coset manifold $OSp(1|4)/SO(1, 3)$. While $\mathcal{L}^{(4|0)}$ in (3.23) is d -closed in the presence of auxiliary fields (see the forthcoming subsection), the new $\mathcal{L}_{\text{AdS}}^{(4|0)}$ is no longer d -closed.

E. Fixing coefficients

From Bianchi identity (3.7) for R^a , after substituting the parametrizations, one finds in the $\psi V V$ sector the expression for $\bar{\theta}$ given in (3.20). In the $\psi \psi V$ sector one obtains the relation

$$c_2 = c_4. \quad (3.28)$$

Considering then the Bianchi identity for ρ , the $\psi \psi \psi$ sector yields

$$c_3 = 3c_4, \quad (3.29)$$

while the $\psi \psi V$ sector fixes ζ^a, ξ, χ in the parametrizations:

$$\zeta^a = \frac{1}{c_4} \left(\frac{1}{3} \gamma_5 \gamma_b \rho^{ab} - \frac{i}{12} \epsilon^{abcd} \gamma_b \rho_{cd} \right), \quad (3.30)$$

$$\xi = -\frac{1}{6c_4} \gamma^{ab} \rho_{ab}, \quad (3.31)$$

$$\chi = \frac{i}{6c_4} \gamma_5 \gamma^{ab} \rho_{ab}. \quad (3.32)$$

⁴By using the relation $\epsilon_{abcd} \gamma^{cd} = 2i\gamma^5 \gamma_{ab}$, and $c_4 = -1/2$, we match the conventional AdS_4 expression.

As a consequence of Bianchi's identities, the relations

$$\gamma_a \zeta^a - 2i\chi = 0, \quad \gamma_5 \xi + i\chi = 0, \quad (3.33)$$

follow. In superspace language [6] there are the identifications $\mathcal{R}_L = S + iP$ and $\mathcal{R}_R = S - iP$. The two superfields \mathcal{R}_L and \mathcal{R}_R are chiral and antichiral; i.e., $\nabla_R \mathcal{R}_L = \nabla_L \mathcal{R}_R = 0$ as a consequence of the Bianchi identities. Using the decomposition $\nabla_L = \frac{1}{2}(1 + \gamma^5)\nabla$ and $\nabla_R = \frac{1}{2}(1 - \gamma^5)\nabla$, where ∇ is the covariant spinorial derivative, we finally find

$$\nabla S - i\gamma^5 \nabla P = 0, \quad (3.34)$$

and with the identifications (3.14) we obtain Eqs. (3.33). In addition, another set of equations is reported in [5]:

$$\nabla^\alpha G_{\alpha\dot{\alpha}} + \bar{\nabla}_{\dot{\alpha}} \mathcal{R}_R = 0, \quad \bar{\nabla}^{\dot{\alpha}} G_{\alpha\dot{\alpha}} + \nabla_\alpha \mathcal{R}_L = 0, \quad (3.35)$$

with $G_{\alpha\dot{\alpha}}$ real superfield, again as a consequence of the Bianchi identities. Identifying $G_{\alpha\dot{\alpha}}$ with the vector field $A_a \gamma_{\alpha\dot{\alpha}}^a$ reproduces the first of Eqs. (3.33).

$$d\mathcal{L}^{(4|0)} = 4\bar{\rho}\gamma_5\gamma_a\rho V^a - 4i(c_4)^2\epsilon_{abcd}\bar{\psi}\gamma^a\psi V^b V^c V^d (S^2 + P^2 + A^a A_a) - 4(c_4)^2(SdS + PdP + A^a \mathcal{D}A_a)\epsilon_{abcd}V^a V^b V^c V^d. \quad (3.39)$$

Finally substituting into (3.39) the parametrizations for ρ and $dS, dP, \mathcal{D}A_a$ we can check that all terms cancel, and therefore,

$$d\mathcal{L}^{(4|0)} = 0. \quad (3.40)$$

The only remaining free parameter c_4 essentially sets the scale of the auxiliary fields (changing its value amounts to rescale η) and can be chosen as

$$c_4 = \frac{1}{6} \quad (3.41)$$

to make contact with the notations of Ref. [23].

G. Picture changing operators

To compute the superspace action starting from the rheonomic Lagrangian (3.23), we have to introduce a new picture changing operator. From the analysis performed in [18], we know that in $d = 4, N = 1$ the nontrivial PCO must have the form

F. Closure of the Lagrangian

Using the Bianchi identities (3.8) and (3.9), and the definition of the torsion R^a in (3.1) we find

$$d\mathcal{L}^{(4|0)} = 2R^{ab}R^c V^d \epsilon_{abcd} + iR^{ab}\bar{\psi}\gamma^c\psi V^d \epsilon_{abcd} + 4\bar{\rho}\gamma_5\gamma_a\rho V^a + \bar{\psi}\gamma_5\gamma_c\gamma_{ab}\psi R^{ab}V^c - 4\bar{\psi}\gamma_5\gamma_a\rho R^a - 2i\bar{\psi}\gamma_5\gamma_a\rho\bar{\psi}\gamma^a\psi - 4i(c_4)^2\epsilon_{abcd}\bar{\psi}\gamma^a\psi V^b V^c V^d (S^2 + P^2 + A^a A_a) - 4(c_4)^2(SdS + PdP + A^a \mathcal{D}A_a)\epsilon_{abcd}V^a V^b V^c V^d. \quad (3.36)$$

The gamma matrix identity

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a + i\epsilon_{abcd}\gamma_5\gamma^d \quad (3.37)$$

implies $\bar{\psi}\gamma_5\gamma_c\gamma_{ab}\psi = i\epsilon_{abcd}\bar{\psi}\gamma^d\psi$, so that the second and the fourth terms cancel in (3.36). Moreover the Fierz identity

$$\gamma_a\psi\bar{\psi}\gamma^a\psi = 0 \quad (3.38)$$

and $\bar{\psi}\gamma_5\gamma_a\rho = \bar{\rho}\gamma_5\gamma_a\psi$ imply that also the sixth term in (3.36) vanishes. Using then the parametrization $R^a = 0$ leads to

$$\mathbb{Y}^{(0|4)} \sim \theta^2 V_0^2 \iota_D^2 \delta^4(\psi_0), \quad (3.42)$$

with an explicit dependence on the θ coordinates. We used the flat supervielbeins

$$V_0^a = dx^a + \bar{\theta}\gamma^a d\theta, \quad \psi_0^\alpha = d\theta^\alpha. \quad (3.43)$$

Notice that the form number carried by the vielbeins V_0^a is compensated by the negative form number carried by the contractions along odd vector fields ι_D . This structure turns out to be the right expression to translate the group manifold actions for $N = 1, 2$ Wess-Zumino and SYM into the corresponding well-known superspace actions [18]. Notice that although the explicit dependence on the θ 's might indicate a supersymmetry breaking, it turns out that it corresponds to the chiral/antichiral projections to sub-superspaces which are supersymmetric invariant. The PCO can be built out of the following terms:

$$\mathbb{Y}_S^{(0|4)} = \bar{\theta}\theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \iota \delta^4(\psi_0), \quad (3.44)$$

$$\mathbb{Y}_P^{(0|4)} = \bar{\theta}\gamma_5 \theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \gamma_5 \iota \delta^4(\psi_0), \quad (3.45)$$

$$\mathbb{Y}_A^{(0|4)} = \epsilon_{abcd} \bar{\theta}\gamma^a \gamma_5 \theta V_0^b \wedge V_0^c \bar{\gamma}^d \iota \delta^4(\psi_0). \quad (3.46)$$

$$\mathbb{Y}_{\hat{S}}^{(0|4)} = \bar{\theta}\theta \epsilon_{abcd} V_0^a \wedge V_0^b \bar{\gamma}^c \gamma^d \iota \delta^4(\psi_0), \quad (3.47)$$

$$\mathbb{Y}_{\hat{P}}^{(0|4)} = \bar{\theta}\gamma_5 \theta \epsilon_{abcd} V_0^a \wedge V_0^b \bar{\gamma}^c \gamma^d \gamma_5 \iota \delta^4(\psi_0), \quad (3.48)$$

$$\mathbb{Y}_{\hat{A}}^{(0|4)} = \bar{\theta}\gamma_a \gamma_5 \theta V_0^a \wedge V_0^b \bar{\gamma}_b \iota \delta^4(\psi_0). \quad (3.49)$$

Let us check which combination is closed. We compute their exterior derivatives:

$$\begin{aligned} d\mathbb{Y}_S^{(0|4)} &= 2\bar{\psi}\theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \iota \delta^4(\psi_0) + i\bar{\theta}\theta \bar{\psi}\gamma^a \psi_0 \wedge V_0^b \bar{\gamma}_{ab} \iota \delta^4(\psi_0) \\ &= -4V_0^a \wedge V_0^b \bar{\theta}\gamma_{ab} \iota \delta^4(\psi_0) + 2i\bar{\theta}\theta \text{tr}(\gamma^a \gamma_{ab}) \delta^4(\psi_0) = -4V_0^a \wedge V_0^b \bar{\theta}\gamma_{ab} \iota \delta^4(\psi_0), \\ d\mathbb{Y}_P^{(0|4)} &= 2\bar{\psi}\gamma_5 \theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \gamma_5 \iota \delta^4(\psi_0) + i\bar{\theta}\gamma_5 \theta \bar{\psi}\gamma^a \psi_0 \wedge V_0^b \bar{\gamma}_{ab} \gamma_5 \iota \delta^4(\psi_0) \\ &= -4V_0^a \wedge V_0^b \bar{\theta}\gamma_{ab} \iota \delta^4(\psi_0) + 2i\bar{\theta}\gamma_5 \theta \text{tr}(\gamma^a \gamma_{ab} \gamma_5) \delta^4(\psi_0) = -4V_0^a \wedge V_0^b \bar{\theta}\gamma_{ab} \iota \delta^4(\psi_0), \end{aligned} \quad (3.50)$$

$$\begin{aligned} d\mathbb{Y}_A^{(0|4)} &= 2V_0^a \wedge V_0^b \bar{\theta}\gamma_{ab} \iota \delta^4(\psi_0) + 2i\epsilon_{abcd} \bar{\theta}\gamma^{ab} \gamma_5 \theta \text{tr}(\gamma^c \gamma^d) \delta^4(\psi_0) \\ &= 2V_0^a \wedge V_0^b \bar{\theta}\gamma_{ab} \iota \delta^4(\psi_0), \\ d\mathbb{Y}_{\hat{S}}^{(0|4)} &= 2\bar{\psi}\theta \epsilon_{abcd} V_0^a \wedge V_0^b \bar{\gamma}^c \gamma^d \iota \delta^4(\psi_0) + i\bar{\theta}\theta \epsilon_{abcd} \bar{\psi}\gamma^a \psi_0 \wedge V_0^b \bar{\gamma}^c \gamma^d \iota \delta^4(\psi_0) \\ &= -4\epsilon_{abcd} V_0^a \wedge V_0^b \bar{\theta}\gamma^c \gamma^d \iota \delta^4(\psi_0), \\ d\mathbb{Y}_{\hat{P}}^{(0|4)} &= 2\bar{\psi}\gamma_5 \theta \epsilon_{abcd} V_0^a \wedge V_0^b \bar{\gamma}^c \gamma^d \gamma_5 \iota \delta^4(\psi_0) + i\bar{\theta}\gamma_5 \theta \epsilon_{abcd} \bar{\psi}\gamma^a \psi_0 \wedge V_0^b \bar{\gamma}^c \gamma^d \gamma_5 \iota \delta^4(\psi_0) \\ &= -4\epsilon_{abcd} V_0^a \wedge V_0^b \bar{\theta}\gamma^c \gamma^d \iota \delta^4(\psi_0), \\ d\mathbb{Y}_{\hat{A}}^{(0|4)} &= 2V_0^a \wedge V_0^b \bar{\theta}\gamma_{ab} \gamma_5 \iota \delta^4(\psi_0) + 2i\bar{\theta}\gamma_a \gamma_5 \theta V_0^a \delta^4(\psi_0) \\ &= 2V_0^a \wedge V_0^b \epsilon_{abcd} \bar{\theta}\gamma^c \gamma^d \iota \delta^4(\psi_0) + 2i\bar{\theta}\gamma_a \gamma_5 \theta V_0^a \delta^4(\psi_0), \end{aligned} \quad (3.51)$$

and find that there are three independent closed $\mathbb{Y}^{(0|4)}$'s. Namely the most general solution of the equation $d\mathbb{Y}^{(0|4)} = 0$ is

$$\mathbb{Y}^{(0|4)} = \alpha \mathbb{Y}_S^{(0|4)} + \beta \mathbb{Y}_P^{(0|4)} + 2(\alpha + \beta) \mathbb{Y}_A^{(0|4)} + \sigma (\mathbb{Y}_{\hat{S}}^{(0|4)} - \mathbb{Y}_{\hat{P}}^{(0|4)}), \quad (3.52)$$

where α, β, σ are arbitrary constants. However, to compute the cohomology we have to check whether (3.52) is exact. We observe that there are two possible candidates $\eta_S^{(-1|4)}, \eta_P^{(-1|4)}$,

$$\begin{aligned} \eta_S^{(-1|4)} &= V_0^a \wedge V_0^b (\bar{\theta}\theta \bar{\gamma}_{5t} + \bar{\theta}\gamma_5 \theta \bar{\theta} \iota) \bar{\gamma}_{ab} \iota \delta^4(\psi_0), \\ \eta_P^{(-1|4)} &= V_0^a \wedge V_0^b (\bar{\theta}\theta \bar{\gamma}_{5t} + \bar{\theta}\gamma_5 \theta \bar{\theta} \iota) \bar{\gamma}_{ab} \gamma_5 \iota \delta^4(\psi_0), \end{aligned} \quad (3.53)$$

whose differentials can be added to $\mathbb{Y}^{(0|4)}$ as $\mathbb{Y}^{(0|4)} + \tau d\eta_S^{(-1|4)} + \rho d\eta_P^{(-1|4)}$. This implies that there is only one cohomology class of $H^{(0|4)}(d)$ and we can choose the representative of the class by choosing τ and ρ . We find it convenient to adopt the following representative:

$$\begin{aligned} \mathbb{Y}^{(0|4)} &= \mathbb{Y}_S^{(0|4)} + \mathbb{Y}_P^{(0|4)} + 4\mathbb{Y}_A^{(0|4)} \\ &= \left(\bar{\theta}\theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \iota + \bar{\theta}\gamma_5 \theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \gamma_5 \iota \right. \\ &\quad \left. + \epsilon_{abcd} \bar{\theta}\gamma^a \gamma_5 \theta V_0^b \wedge V_0^c \bar{\gamma}^d \iota \right) \delta^4(\psi_0). \end{aligned} \quad (3.54)$$

Note that we can combine the first two terms $\mathbb{Y}_S^{(0|4)} + \mathbb{Y}_P^{(0|4)}$ into a chiral and an antichiral expression. Another convenient choice is $\alpha = -\beta$ and $\sigma = 0$. This leads to

$$\begin{aligned} \mathbb{Y}^{(0|4)} &= \mathbb{Y}_S^{(0|4)} - \mathbb{Y}_P^{(0|4)} \\ &= \left(\bar{\theta}\theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \iota - \bar{\theta}\gamma_5 \theta V_0^a \wedge V_0^b \bar{\gamma}_{ab} \gamma_5 \iota \right) \delta^4(\psi_0). \end{aligned} \quad (3.55)$$

Finally, note that all representatives are related also to

$$\mathbb{Y}_{\text{comp}}^{(0|4)} = \bar{\theta}\theta \bar{\theta}\gamma_5 \theta \delta^4(\psi_0), \quad (3.56)$$

which is closed and not exact, but completely breaks supersymmetry and, inserted into the action, yields the component action.

One may wonder whether it may be possible to build a PCO $\mathbb{Y}^{(0|4)}$ which is manifestly supersymmetric invariant, namely written in terms of V_0^a, ψ_0^α , and $\delta^4(\psi_0)$ and contractions along the ψ_0^α 's of the Dirac deltas. The requirement to be a 0-form restricts the possible choices to

$$\delta^4(\psi_0), \quad V_0^a \wedge V_0^b \bar{\gamma}_{ab} \delta^4(\psi_0), \quad V_0^a \wedge V_0^b \bar{\gamma}_{ab} \gamma_5 \delta^4(\psi_0). \quad (3.57)$$

These three terms turn out to be exact. Therefore, in the present case, there is no manifestly supersymmetric PCO. The same situation occurs also for extended supersymmetric models.

We want now to extend flat PCO (constructed out of flat V_0^a and ψ_0^α) to a PCO for a dynamical curved manifold. We can deform (3.54) without changing its cohomology class by acting with a super-diffeomorphism. This can be done by acting infinitesimally with a Lie derivative \mathcal{L}_X along a vector field X since

$$\delta \mathbb{Y}^{(0|4)} = \mathcal{L}_X \mathbb{Y}^{(0|4)} = d\left(\iota_X \mathbb{Y}^{(0|4)}\right) \quad (3.58)$$

or, in the same way, by performing a finite transformation generated by X such that

$$e^{\mathcal{L}_X} V_0^a = V^a, \quad e^{\mathcal{L}_X} \psi_0^\alpha = \psi^\alpha, \quad e^{\mathcal{L}_X} \theta^\alpha = \Theta^\alpha, \quad (3.59)$$

where Θ^α is the curved fermionic coordinate. Then we have

$$\begin{aligned} \mathbb{Y}^{(0|4)} &\rightarrow e^{\mathcal{L}_X} \mathbb{Y}^{(0|4)} \\ &= \left(\bar{\Theta} \Theta V^a \wedge V^b \bar{\gamma}_{ab} \iota + \bar{\Theta} \gamma_5 \Theta V^a \wedge V^b \bar{\gamma}_{ab} \gamma_5 \iota \right. \\ &\quad \left. + \epsilon_{abcd} \bar{\Theta} \gamma^a \gamma_5 \Theta V^b \wedge V^c \bar{\gamma}^d \iota \right) \delta^4(\psi) \end{aligned} \quad (3.60)$$

or, for the second choice (3.55),

$$\begin{aligned} \mathbb{Y}^{(0|4)} &\rightarrow e^{\mathcal{L}_X} \mathbb{Y}^{(0|4)} \\ &= \left(\bar{\Theta} \Theta V^a \wedge V^b \bar{\gamma}_{ab} \iota - \bar{\Theta} \gamma_5 \Theta V^a \wedge V^b \bar{\gamma}_{ab} \gamma_5 \iota \right) \delta^4(\psi), \end{aligned} \quad (3.61)$$

where V^a and ψ^α are the dynamical vielbeins. These curved $\mathbb{Y}^{(0|4)}$ belong to the same cohomology class of the original $\mathbb{Y}^{(0|4)}$.⁵

⁵The topology of the manifold is unchanged by local diffeomorphisms, and therefore, we cannot study in this way the case of curved rigid supermanifolds not connected by infinitesimal diffeomorphisms to the flat space.

H. Superspace action

Having found the group manifold Lagrangian using geometric means, we use now the PCOs to construct the action. On one side, using the flat nonsupersymmetric PCO (3.56), namely $\mathbb{Y}^{(0|4)} = \bar{\theta} \bar{\theta} \gamma^5 \theta \delta^4(\psi)$, the action reduces to the component supergravity action with auxiliary fields on spacetime. On the other side, we would like to use the supersymmetric PCO (3.61) to obtain the superspace action as given in [5,6], which is manifestly invariant under local supersymmetry. For this purpose, we use (3.61) to project the action to only a few terms, and then we use the parametrizations (3.14)–(3.19) to simplify the result and discover that the complete superspace action is encoded in the auxiliary superfields terms of (3.23). This allows the comparison with the result given in [6]. Finally, using the relation between chiral volume forms and the superdeterminant (discussed in Sec. II D), we show the equivalence with the superspace formulation of [5].

By inserting $\mathbb{Y}^{(0|4)}$ into the supermanifold integral, we have

$$S = \int_{SM^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}. \quad (3.62)$$

The action $\mathcal{L}^{(4|0)}$ is a superform and therefore can be expanded in powers of V and ψ as follows:

$$\begin{aligned} \mathcal{L}^{(4|0)} &= \mathcal{L} \epsilon_{abcd} V^a V^b V^c V^d + \mathcal{L}_{\alpha,abc} \psi^\alpha V^a V^b V^c \\ &\quad + \mathcal{L}^I(\bar{\psi} \mathcal{M}_{I,ab} \psi) V^a V^b \\ &\quad + \mathcal{L}_{abc}^I \psi^\alpha (\bar{\psi} \mathcal{M}_I^{ab} \psi) V^c \\ &\quad + \mathcal{L}_{ab,cd}^{IJ} (\bar{\psi} \mathcal{M}_I^{ab} \psi) (\bar{\psi} \mathcal{M}_J^{cd} \psi). \end{aligned} \quad (3.63)$$

The indices I and J denote the different Dirac matrix structures for bilinears $(\bar{\psi} \mathcal{M}_I^{ab} \psi)$. Inserting these expressions in (3.62) and using the PCO (3.61) we are left with

$$\begin{aligned} S &= \int_{SM^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)} \\ &= \int_{SM^{(4|4)}} (\mathcal{L}^I(\bar{\psi} \mathcal{M}_{I,ab} \psi) V^a V^b) \wedge \mathbb{Y}^{(0|4)} \\ &= \int_{SM^{(4|4)}} (-S \bar{\Theta} \Theta + iP \bar{\Theta} \gamma_5 \Theta) V^4 \delta^4(\psi). \end{aligned} \quad (3.64)$$

A crucial role is played by the derivative of Dirac delta functions of ψ 's in the PCO: with integration by parts, only the piece (3.64) is selected. To compare the result with superspace literature [5,6,22] we use the chiral notation $\Theta_{L/R} = \frac{1}{2}(1 \pm \gamma_5)\Theta$, and rewrite the above expression as

$$S = \int_{SM^{(4|4)}} (\mathcal{R}_R \bar{\Theta}_L \Theta_L + \mathcal{R}_L \bar{\Theta}_R \Theta_R) V^4 \delta^4(\psi), \quad (3.65)$$

where \mathcal{R}_L and \mathcal{R}_R are the auxiliary fields as discussed in Sec. III E, Eq. (3.33). In [5,6] a complete description of \mathcal{R}_L , \mathcal{R}_R , and their component expansion is given. We notice that, using the off-shell parametrization given in (3.14)–(3.19), the complete action (3.63) reduces to the terms containing the auxiliary superfields S , P , or equivalently to the \mathcal{R}_L , \mathcal{R}_R terms. This is a well-known phenomenon (see [18]), and the full superspace action is obtained by the superspace expansion of the auxiliary field terms.

The expression (3.66) can be conveniently rewritten as

$$S = \int_{\mathcal{SM}^{(4|4)}} (\mathcal{R}_R \text{Vol}_R^{(4|2)} \wedge \mathbb{Y}_L + \mathcal{R}_L \text{Vol}_L^{(4|2)} \wedge \mathbb{Y}_R), \quad (3.66)$$

where we used the notation $\mathbb{Y}_R = \Theta_R \Theta_R \delta^2(\psi_R)$ and $\mathbb{Y}_L = \Theta_L \Theta_L \delta^2(\psi_L)$ to denote the PCO projecting on the constraints $\Theta_R = 0$ and $\Theta_L = 0$. In the above equation, we used $\text{Vol}_R^{(4|2)}$ and $\text{Vol}_L^{(4|2)}$ to denote the chiral densities.

Finally, integrating on the ‘‘cotangent’’ coordinates $(V^a, \psi_L^\alpha, \psi_R^\alpha)$, we arrive at the simplified expression [6]

$$S = \int_L \mathcal{R}_L \mathcal{E}_L + \int_R \mathcal{R}_R \mathcal{E}_R = \int E = \int_{\mathcal{SM}^{(4|4)}} \star 1, \quad (3.67)$$

where we used a chiral integration formula by Siegel [5,22] giving a relation between the chiral measures \mathcal{E}_L , \mathcal{E}_R and the superdeterminant E . The integrals in the above formula $\int_{L/R}$ are over the chiral superspaces, and the integral $\int E$ is extended to the entire superspace. The last expression finally shows the relation of the superspace action with the Hodge dual of a constant. Reinstalling the appropriate dimensions, this constant coincides with Newton’s constant.

IV. NEW MINIMAL SUPERGRAVITY

A. Off-shell degrees of freedom

We can match off-shell d.o.f. by adding an auxiliary bosonic 1-form A (3 d.o.f.) and an auxiliary bosonic 2-form T (3 d.o.f.). The theory with these auxiliary fields was first constructed in Ref. [24], and recast in the group manifold formulation in Ref. [9].

B. The extended super-Poincaré algebra

The starting superalgebra is the super-Poincaré algebra, extended with a 1-form A and a 2-form T . The deformed Cartan-Maurer equations for the extended soft super-Poincaré manifold are

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \omega^{cb}, \quad (4.1)$$

$$R^a = dV^a - \omega^a{}_b V^b - \frac{i}{2} \bar{\psi} \gamma^a \psi \equiv \mathcal{D}V^a - \frac{i}{2} \bar{\psi} \gamma^a \psi, \quad (4.2)$$

$$\rho = d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi - \frac{i}{2} \gamma_5 \psi A \equiv \mathcal{D}\psi - \frac{i}{2} \gamma_5 \psi A, \quad (4.3)$$

$$R^\square = dA, \quad (4.4)$$

$$R^\otimes = dT - \frac{i}{2} \bar{\psi} \gamma_a \psi V^a, \quad (4.5)$$

where \mathcal{D} is the Lorentz covariant derivative. These equations can be considered *definitions* for the Lorentz curvature, the (super)torsion, the gravitino field strength, and the 1-form and 2-form field strengths, respectively. The Cartan-Maurer equations are invariant under rescalings

$$\begin{aligned} \omega^{ab} &\rightarrow \lambda^0 \omega^{ab}, & V^a &\rightarrow \lambda V^a, & \psi &\rightarrow \lambda^{\frac{1}{2}} \psi, \\ A &\rightarrow \lambda^0 A, & T &\rightarrow \lambda^2 T. \end{aligned} \quad (4.6)$$

Taking exterior derivatives of both sides yields the Bianchi identities:

$$\mathcal{D}R^{ab} = 0, \quad (4.7)$$

$$\mathcal{D}R^a + R^a{}_b V^b - i \bar{\psi} \gamma^a \rho = 0, \quad (4.8)$$

$$\mathcal{D}\rho + \frac{1}{2} \gamma_5 \rho A + \frac{1}{4} R^{ab} \gamma_{ab} \psi - \frac{i}{2} \gamma_5 \psi R^\square = 0, \quad (4.9)$$

$$dR^\square = 0, \quad (4.10)$$

$$dR^\otimes - i \bar{\psi} \gamma_a \rho V^a + \frac{i}{2} \bar{\psi} \gamma_a \psi R^a = 0, \quad (4.11)$$

invariant under the rescalings (4.6).

C. Curvature parametrizations

According to the group manifold approach, we again parametrize the curvatures so that ‘‘outer’’ components (i.e., components along at least one fermionic direction) are related to inner components (i.e., components on bosonic directions). The most general parametrization compatible with the scalings (4.6) and $SO(3,1) \times U(1)$ gauge invariance is the following:

$$R^{ab} = R^{ab}{}_{cd} V^c V^d + \bar{\theta}^{ab}{}_c \psi V^c + ic_1 \epsilon^{abcd} \bar{\psi} \gamma_c \psi f_d, \quad (4.12)$$

$$R^a = 0, \quad (4.13)$$

$$\rho = \rho_{ab} V^a V^b + ia \gamma_5 \psi f_a V^a - ic_2 \gamma_5 \gamma_{ab} \psi V^a f^b, \quad (4.14)$$

$$R^\square = F_{ab} V^a V^b + \bar{\psi} \chi_a V^a + ic_3 \bar{\psi} \gamma_a \psi f^a, \quad (4.15)$$

$$R^\otimes = f^a V^b V^c V^d \epsilon_{abcd}, \quad (4.16)$$

$$\mathcal{D}f_a = (\mathcal{D}_b f_a) V^b + \bar{\psi} \Xi_a. \quad (4.17)$$

The VV component F_{ab} of F and the VVV component f_a of R^\otimes scale, respectively, as $F_{ab} \rightarrow \lambda^{-2} F_{ab}$ and $f_a \rightarrow \lambda^{-1} f_a$. The Bianchi identities require that

$$c_1 = c_2 = \frac{3}{2}, \quad c_3 = 3 - a, \quad (4.18)$$

and

$$\bar{\theta}^{ab}{}_c = 2i\bar{\rho}_c^{[a}\gamma^{b]} - i\bar{\rho}^{ab}\gamma_c, \quad (4.19)$$

$$\Xi^a = -\frac{i}{3!}\epsilon^{abcd}\gamma_b\rho_{cd}, \quad (4.20)$$

$$\chi_a = 2\left(\gamma_5\gamma^b\rho_{ab} + \frac{ia}{3!}\epsilon_{abcd}\gamma^b\rho^{cd}\right). \quad (4.21)$$

Note that thanks to the presence of the auxiliary fields, the Bianchi identities do not imply equations of motion for the spacetime components of the curvatures.

D. The group manifold action

With the usual group manifold methods, the action is determined to be

$$S_{d=4SG} = \int_{M^4} R^{ab}V^cV^d\epsilon_{abcd} + 4\bar{\psi}\gamma_5\gamma_a\rho V^a - 4R^\square T + \alpha\left(f_a R^\otimes V^a + \frac{1}{8}f_e f^e V^a V^b V^c V^d \epsilon_{abcd}\right). \quad (4.22)$$

This action is obtained by taking for the Lagrangian $\mathcal{L}^{(4|0)}$ the most general $SO(3, 1) \times U(1)$ scalar 4-form, invariant under the rescalings discussed above, and then requiring that the variational equations admit the vanishing curvatures solution

$$R^{ab} = R^a = \rho = R^\square = R^\otimes = f_a = 0. \quad (4.23)$$

The remaining parameter α is fixed by requiring the closure of $\mathcal{L}^{(4|0)}$, i.e., $d\mathcal{L}^{(4|0)} = 0$. This yields $\alpha = 4(4a - 3)$ and ensures off-shell closure of the supersymmetry transformations given below. Notice that a is essentially free, since the term $ia\gamma_5\psi f_a V^a$ in the parametrization of the gravitino curvature ρ can be reabsorbed into the definition of the $SO(3, 1) \times U(1)$ -covariant derivative on ψ , by redefining $A' = A + 2af_a V^a$. Choosing $a = \frac{3}{4}$ simplifies the action, reducing it to the first three terms, so that the 0-forms f_a do not appear.

E. Field equations

Varying ω^{ab} , V^a , ψ , A , T , and f_a in the action (4.22) leads to the equations of motion:

$$2\epsilon_{abcd}R^cV^d = 0 \Rightarrow R^a = 0, \quad (4.24)$$

$$2R^{bc}V^d\epsilon_{abcd} - 4\bar{\psi}\gamma_5\gamma_a\rho + \alpha\left(-f_a R^\otimes + \frac{1}{2}f_e f^e \epsilon_{abcd}V^bV^cV^d\right) = 0, \quad (4.25)$$

$$8\gamma_5\gamma_a\rho V^a - 4\gamma_5\gamma_a\psi R^a - i\alpha\gamma_a\psi V^a f_b V^b = 0, \quad (4.26)$$

$$R^\otimes = 0, \quad (4.27)$$

$$-4R^\square + \alpha\left(V^a\mathcal{D}f_a - \frac{i}{2}f_a\bar{\psi}\gamma^a\psi - f_a R^a\right) = 0, \quad (4.28)$$

$$R^\otimes = f^a V^b V^c V^d \epsilon_{abcd}. \quad (4.29)$$

These equations are satisfied by the curvatures parametrized as in Sec. IV C and also imply

$$R^a = R^\square = R^\otimes = f_a = 0, \quad (4.30)$$

$$R^{ac}{}_{bc} - \frac{1}{2}\delta_b^a R^{cd}{}_{cd} = 0 \quad (\text{Einstein eq.}), \quad (4.31)$$

$$\gamma^a\rho_{ab} = 0 \quad (\text{Rarita - Schwinger eq.}). \quad (4.32)$$

The theory has therefore the same dynamical content as the usual $N = 1$, $d = 4$ supergravity without auxiliary fields.

F. Off-shell supersymmetry transformations

Supersymmetry transformations are obtained by applying the Lie derivative along the fermionic directions (i.e., along tangent vectors dual to ψ):

$$\delta_\epsilon V^a = -i\bar{\psi}\gamma^a\epsilon, \quad (4.33)$$

$$\delta_\epsilon\psi = \mathcal{D}\epsilon + \frac{i}{2}\gamma_5 A\epsilon + ia\gamma_5\epsilon f_a V^a - \frac{3i}{2}\gamma_5\gamma_{ab}\epsilon V^a f^b, \quad (4.34)$$

$$\delta_\epsilon A = \bar{\epsilon}\left(\frac{ia}{3}\epsilon^{abcd}\gamma_b\rho_{cd} - 2\gamma_5\gamma_b\rho^{ba}\right) + 2i(3-a)\bar{\epsilon}\gamma_a\psi f^a V_a, \quad (4.35)$$

$$\delta\omega^{ab} = \bar{\theta}^{ab}{}_c\epsilon V^c - 3ie^{abcd}\bar{\psi}\gamma_c\epsilon f_d, \quad (4.36)$$

$$\delta_\epsilon T = -i\bar{\psi}\gamma_a\epsilon V^a, \quad (4.37)$$

$$\delta_\epsilon f^a = \bar{\epsilon}\Xi^a, \quad (4.38)$$

and close on all the fields without the need of imposing the field equations (4.31) and (4.32).

G. Superspace action

Having found the rheonomic Lagrangian for the new minimal set of auxiliary fields, by using the geometric means of the group manifold approach, we would again like to use the PCO to define its variational principle and

write the corresponding action. On one side, it is easy to check that using the flat nonsupersymmetric PCO $\mathbb{Y}^{(0|4)} = \bar{\theta}\theta\bar{\theta}\gamma^5\theta\delta^4(\psi)$, we obtain the usual component action with auxiliary fields. On another side, we would like to verify that using the supersymmetric PCO (3.61) leads to a superspace action for the new minimal $d = 4$ supergravity. The latter has been discussed in [31–33] and appears to be of a BF-type action. Therefore, the plan of the present section is to insert the curved PCO which selects some pieces of the Lagrangian and, using the parametrizations, to compare the result with the superspace action given in [33].

We begin by inserting $\mathbb{Y}^{(0|4)}$ into the supermanifold integral and consider the Lagrangian with $\alpha = 0$ for simplicity:

$$S = \int_{\mathcal{SM}^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)}. \quad (4.39)$$

The action $\mathcal{L}^{(4|0)}$ is a superform and therefore can be expanded in powers of V , ψ as follows:

$$\begin{aligned} \mathcal{L}^{(4|0)} &= \mathcal{L}\epsilon_{abcd}V^aV^bV^cV^d + \mathcal{L}_{\alpha,abc}\psi^\alpha V^aV^bV^c \\ &+ \mathcal{L}^I(\bar{\psi}\mathcal{M}_{I,ab}\psi)V^aV^b + \mathcal{L}_{\alpha abc}^I\psi^\alpha(\bar{\psi}\mathcal{M}_I^{ab}\psi)V^c \\ &+ \mathcal{L}_{\alpha,cd}^{IJ}(\bar{\psi}\mathcal{M}_I^{ab}\psi)(\bar{\psi}\mathcal{M}_J^{cd}\psi). \end{aligned} \quad (4.40)$$

The indices I and J denote the different Dirac matrix structures for bilinears $(\bar{\psi}\mathcal{M}_I^{ab}\psi)$. Inserting these expressions in (3.62) and using the curved PCO (3.61) we are left with

$$\begin{aligned} S &= \int_{\mathcal{SM}^{(4|4)}} \mathcal{L}^{(4|0)} \wedge \mathbb{Y}^{(0|4)} \\ &= \int_{\mathcal{SM}^{(4|4)}} \left(\mathcal{L}^I(\bar{\psi}\mathcal{M}_{I,ab}\psi)V^aV^b \right) \wedge \mathbb{Y}^{(4|0)} \\ &= \int_{\mathcal{SM}^{(4|4)}} \left(\epsilon^{abcd}(F_{ab}\bar{\gamma}_{cd}iT + \chi_a\gamma_{bc}iT_a)\bar{\Theta}\Theta \right. \\ &\quad \left. + i(F_{ab}\bar{\gamma}_5\gamma_{cd}iT + \chi_a\gamma_5\gamma_{bc}iT_a)\bar{\Theta}\gamma_5\Theta \right) V^4\delta^4(\psi). \end{aligned} \quad (4.41)$$

The additional superfields $\bar{\gamma}_{cd}iT$ and $i_\alpha T_a$ are defined in terms of the 2-form potential $T = T_{ab}V^aV^b + T_{aa}V^a\psi^\alpha + T_{\alpha\beta}\psi^\alpha\psi^\beta$,

$$\bar{\gamma}_{cd}iT = \gamma_{cd}^{\alpha\beta}T_{\alpha\beta}, \quad i_\alpha T_a = T_{aa}. \quad (4.42)$$

Integrating on the cotangent space we are left with the integral on the bosonic x^α and on the fermionic θ^α coordinates:

$$\begin{aligned} S &= \int \left(E\epsilon^{abcd}(F_{ab}\bar{\gamma}_{cd}iT + \chi_a\gamma_{bc}iT_a)\bar{\Theta}\Theta \right. \\ &\quad \left. + iE\epsilon^{abcd}(F_{ab}\bar{\gamma}_5\gamma_{cd}iT + \chi_a\gamma_5\gamma_{bc}iT_a)\bar{\Theta}\gamma_5\Theta \right), \end{aligned} \quad (4.43)$$

where F_{ab} and χ_a are the superfields appearing in the parametrization of F [see (4.12)]. The superfield f_a is absent due to the Dirac matrix structure of the PCO that projects out the Dirac structure corresponding to f_a . Notice that at the end we have to compute the Berezin integral over the fermionic coordinates, and this selects inside the superfields $T_{\alpha\beta}$, T_{aa} , F_{ab} , and χ_a all the needed pieces for reconstructing the supergravity in the new minimal formulation. Again, we notice that it is the auxiliary field term that reproduces the entire supergravity action.

To compare the final action with the superspace action, one needs to identify all components of the superfields in (4.42) and then perform the integration. We do not present this computation here; for this we refer to [31,32,34] and to the review [33]. This concludes the proof of the equivalence between the group-manifold approach and the superspace approach to new minimal $d = 4$ supergravity.

ACKNOWLEDGMENTS

We thank our friends and collaborators: L. Andrianopoli, R. Catenacci, C. A. Cremonini, R. D’Auria, O. Hulik, S. Noja, R. Norris, L. Ravera, G. Tartaglino-Mazzucchelli, and M. Trigiante, for useful discussions and comments. The work is partially funded by the University of Eastern Piedmont with FAR-2019 projects. P. A. G. thanks the Simons Center for Geometry and Physics (Stony Brook University, NY), where the present work has been completed.

APPENDIX A: FORMS ON SUPERMANIFOLDS

We collect here some basic definitions and facts about integration on supermanifolds and on *integral forms*. For exhaustive introductions to integral forms, we refer the reader to [25,26], while for their use in physics, we refer to [14,15,35].

Given a (smooth) supermanifold $\mathcal{M}^{(4|4)}$, the cotangent space $\mathcal{T}_P^*\mathcal{M}^{(4|4)}$ at a given point $P \in \mathcal{M}^{(4|4)}$ has both an even and an odd part, generated, in a given system of local coordinates $(x^\alpha, \theta^\alpha)$, $i = a, \dots, 4$, $\alpha = 1, \dots, 4$, by the $(1|0)$ forms $\{dx^\alpha, d\theta^\alpha\}$, called *superforms*, which are, respectively, odd and even. They have the following (super) commuting properties:

$$\begin{aligned} dx^a \wedge dx^b &= -dx^b \wedge dx^a, & d\theta^\alpha \wedge d\theta^\beta &= d\theta^\beta \wedge d\theta^\alpha, \\ dx^a \wedge d\theta^\alpha &= -d\theta^\alpha \wedge dx^a. \end{aligned} \quad (A1)$$

A generic $(p|0)$ form is an object of the (graded) symmetric power of $\mathcal{T}_P^*\mathcal{M}^{(4|4)}$, and it locally reads as

$$\omega^{(p|0)} = \omega_{[a_1 \dots a_r](\alpha_1 \dots \alpha_s)}(x, \theta) dx^{a_1} \wedge \dots \wedge dx^{a_r} \wedge d\theta^{\alpha_1} \wedge \dots \wedge d\theta^{\alpha_s}, \quad p = r + s, \quad (\text{A2})$$

where the coefficients $\omega_{[a_1 \dots a_r](\alpha_1 \dots \alpha_s)}(x, \theta)$ are a set of superfields and the indices $a_1 \dots a_r, \alpha_1 \dots \alpha_s$ are antisymmetrized and symmetrized, respectively, due to (A1). The analog of the determinant bundle can be found in a different form complex, the complex of *integral forms*. One can introduce the *Berezinian bundle* $Ber(\mathcal{M}^{(4|4)})$, i.e., the space of objects which transform as the Berezinian (i.e., the *superdeterminant*) under coordinate transformations.

Integral forms are then constructed on open sets starting from this space and tensoring with (graded) symmetric powers of the parity-changed tangent space (see the recent review [26] for a rigorous introduction to the subject). A practical and computationally powerful realization of the Berezinian and integral forms is given in terms of (formal) Dirac distributions on the cotangent space; a generic $(p|4)$ -integral form can be locally described as

$$\omega^{(p|N)} = \omega_{[a_1 \dots a_r]}^{(\alpha_1 \dots \alpha_s)}(x, \theta) dx^{a_1} \wedge \dots \wedge dx^{a_r} \wedge \iota_{\alpha_1} \dots \iota_{\alpha_s} \delta(d\theta^1) \wedge \dots \wedge \delta(d\theta^4), \quad p = r - s, \quad (\text{A3})$$

and the second number of the $(p|4)$ form keeps track of the number of Dirac deltas and is called *Picture number*. The contraction ι_α is defined as $\iota_\alpha \equiv \partial/\partial\psi^\alpha$. The formal Dirac deltas satisfy the following properties:

$$\begin{aligned} \int_{d\theta} \delta(d\theta) &= 1, & d\theta \wedge \delta(d\theta) &= 0, & \delta(d\theta^\alpha) \wedge \delta(d\theta^\beta) &= -\delta(d\theta^\beta) \wedge \delta(d\theta^\alpha), \\ dx \wedge \delta(d\theta) &= +\delta(d\theta) \wedge dx, & \delta(\lambda d\theta) &= \frac{1}{\lambda} \delta(d\theta), & d\theta \wedge \iota^p \delta(d\theta) &= -p \iota^{p-1} \delta(d\theta). \end{aligned} \quad (\text{A4})$$

The first property defines how $\delta(d\theta)$'s have to be used to perform form integration along the commuting directions $d\theta$'s; the second property reflects the usual property of the support of the Dirac distribution; the third and fourth properties imply $\delta(d\theta)$'s are odd objects and together with the fifth property they indicate that actually, these are not distributions, but rather *de Rham currents*, i.e., they define an *oriented* integration; and the last property allows the usual integration by parts of the Dirac delta.

A top form reads as

$$\omega_{\text{top}}^{(4|4)} \equiv \omega^{(4|4)} = \omega(x, \theta) \epsilon_{a_1 \dots a_4} dx^{a_1} \wedge \dots \wedge dx^{a_4} \wedge \epsilon_{\alpha_1 \dots \alpha_4} \delta(d\theta^{\alpha_1}) \wedge \dots \wedge \delta(d\theta^{\alpha_4}), \quad (\text{A5})$$

where $\omega(x, \theta)$ is a superfield. Any integral form of any form degree p can be obtained by acting with $4 - p$ contractions on (A5).

One can also consider other classes of forms, called *pseudoforms*, with a nonmaximal and nonzero number of deltas. A general pseudoform with q deltas is locally given by

$$\omega^{(p|q)} = \omega_{[a_1 \dots a_r](\alpha_1 \dots \alpha_s)[\beta_1 \dots \beta_q]}(x, \theta) dx^{a_1} \wedge \dots \wedge dx^{a_r} \wedge d\theta^{\alpha_1} \wedge \dots \wedge d\theta^{\alpha_s} \wedge \delta^{(t_1)}(d\theta^{\beta_1}) \wedge \dots \wedge \delta^{(t_q)}(d\theta^{\beta_q}), \quad (\text{A6})$$

where we used the compact notation $\delta^{(t)}(d\theta) \equiv (t)^t \delta(d\theta)$. The form number is obtained as

$$p = r + s - \sum_{i=1}^q t_i. \quad (\text{A7})$$

If $q = 0$, we have superforms; if $q = 4$, we have integral forms; and if $0 < q < 4$, we have pseudoforms. These kinds of forms are to be used, for example, in [35] to construct objects which implement naturally the self-duality condition on supermanifolds. This is a consequence of the fact that the Hodge operator [14,15] on supermanifolds changes not only the form number but also the picture number:

$$\star : \Omega^{(p|q)}(\mathcal{M}^{(4|4)}) \rightarrow \Omega^{(4-p|4-q)}(\mathcal{M}^{(4|4)}). \quad (\text{A8})$$

A notable example of an integral form is the *picture changing operator*: it is a $(0|4)$ form, in the cohomology of the operator d . It is used to lift a superform to an integral form by multiplication:

$$\begin{aligned} \mathbb{Y}^{(0|4)} : \Omega^{(p|0)}(\mathcal{M}^{(4|4)}) &\rightarrow \Omega^{(p|4)}(\mathcal{M}^{(4|4)}), \\ \omega^{(p|0)} &\mapsto \omega^{(p|4)} = \omega^{(p|0)} \wedge \mathbb{Y}^{(0|4)}. \end{aligned} \quad (\text{A9})$$

APPENDIX B: CURVED SUPERSPACE

One interesting formula in [5,22] used in the context of superspace supergravity gives the right action of an even vector field K on a superfield. It is a complicated formula, albeit useful and important, and can be translated into our language using the Hodge dual operator. We hope that our reformulation might add some new insights to the original work in [5,22] from a different perspective.

We consider the following definition:

$$\phi \cdot \bar{K} = (-1)^M \partial_M (\phi K^M) \quad (\text{B1})$$

for an even superfield ϕ and an even vector K . This also implies

$$\phi \cdot e^{\bar{K}} = (1 \cdot e^{\bar{K}}) e^K \phi, \quad (\text{B2})$$

where ϕ is a superfield.

We would like to translate (B1) using our Hodge dual operator. For that we observe that (B1) corresponds to the action of a vector field on a density, which can be obtained by acting with the Hodge dual of a Lie derivative on a top form. Then, using the Hodge dual operator \star we set

$$\phi \cdot \bar{K} = \mathcal{L}_K^\dagger \phi \equiv \star \mathcal{L}_K (\star \phi), \quad (\text{B3})$$

where $\star \phi = \phi \text{Vol}^{(4|4)}$. The equation can easily be verified using the properties of $\text{Vol}^{(4|4)}$ and of the Lie derivative \mathcal{L}_K . The derivation of (B2) is now straightforward:

$$\begin{aligned} \phi \cdot e^{\bar{K}} &= \left(e^{\mathcal{L}_K} \right)^\dagger \phi = \star e^{\mathcal{L}_K} (\star \phi) \\ &= \star e^{\mathcal{L}_K} (\star 1 \phi) = \star \left(e^{\mathcal{L}_K} (\star 1) \right) e^{\mathcal{L}_K} \phi \\ &= \left(\star e^{\mathcal{L}_K} \text{Vol}^{(4|4)} \right) e^{\mathcal{L}_K} = (1 \cdot e^{\bar{K}}) e^K \phi, \end{aligned} \quad (\text{B4})$$

where $\star 1 = \text{Vol}^{(4|4)}$ and

$$\begin{aligned} (1 \cdot e^{\bar{K}}) &= (e^{\mathcal{L}_K})^\dagger 1 = \star e^{\mathcal{L}_K} \omega^{(4|4)} \\ &= \star \left(\text{Sdet}(J) \omega^{(4|4)} \right) = \text{Sdet}(J). \end{aligned} \quad (\text{B5})$$

Another interesting formula is

$$(1 \cdot e^{\bar{K}})^{-1} = e^K (1 \cdot e^{-\bar{K}}), \quad (\text{B6})$$

which can be proven as follows:

$$(1 \cdot e^{-\bar{K}}) \cdot e^{\bar{K}} = \left(e^{\mathcal{L}_K} \right)^\dagger \left(e^{-\mathcal{L}_K} \right)^\dagger 1 = 1; \quad (\text{B7})$$

therefore, the superfield $(1 \cdot e^{-\bar{K}})$ is the inverse of the operator $\cdot e^{\bar{K}}$, and therefore, it is the inverse of $(1 \cdot e^{\bar{K}}) e^K$. Analogously, $(1 \cdot e^{\bar{K}})$ is the inverse of $e^K (1 \cdot e^{-\bar{K}})$.

This is only an example of how the use of the Hodge dual operator simplifies several complicated computations in superspace. We believe that this new point of view might prove to be useful in the superspace formulation of supergravity.

APPENDIX C: THE CHIRAL PROJECTORS USING PICTURE LOWERING OPERATORS

Here we want to show the correspondence between the geometric formulation of $d = 4$, $N = 1$ old minimal supergravity, via the super-Hodge operator, with its superspace formulation using the picture lowering operators. In particular, we have seen that the action can be written as

$$\begin{aligned} \int_{\mathcal{SM}^{(4|4)}} \mathcal{L}^{(4|4)} &= \int_{\mathcal{SM}^{(4|4)}} \star 1 \\ &= \int_{\mathcal{SM}_{\text{chiral}}^{(4|2)}} \mathcal{R}_L \text{Vol}_L^{(4|2)} + \int_{\mathcal{SM}_{\text{antichiral}}^{(4|2)}} \mathcal{R}_R \text{Vol}_R^{(4|2)}. \end{aligned} \quad (\text{C1})$$

We can obtain a $(4|2)$ form out of a $(4|4)$ form by decreasing the picture number by two. The geometrical operator that decreases the picture number is given by

$$\mathbb{Z}_X^{(0|-1)} = \{d, -i\Theta(\iota_X)\}, \quad (\text{C2})$$

where X is an odd vector field and the (odd) operator Θ is defined via its Fourier integral representation:

$$\Theta(\iota_X) \equiv -i \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{dt}{t + i\epsilon} e^{itX}. \quad (\text{C3})$$

The key point about the operator \mathbb{Z} is that it is a quasi-isomorphism; i.e., it maps cohomology classes into cohomology classes. This means that it allows one to obtain a form with the same degrees of freedom but with a different picture number out of a known form. This is strictly related to the same properties of the picture-changing operator \mathbb{Y} described in the previous sections. Indeed, \mathbb{Y} represents a quasi-isomorphism and is related to \mathbb{Z} via a quasi-inverse relation (i.e., they are inverse on cohomology classes), which can be schematically represented as

$$\mathbb{Z}\mathbb{Y}\mathbb{Z} = \mathbb{Z}, \quad \mathbb{Y}\mathbb{Z}\mathbb{Y} = \mathbb{Y}. \quad (\text{C4})$$

In the case under consideration, we can project from the whole supermanifold to the chiral/antichiral submanifolds using the following operator:

$$\mathbb{Z}^{(0|-2)} = \mathbb{Z}_C^{(0|-2)} + \bar{\mathbb{Z}}_C^{(0|-2)} = \bar{\mathbb{Z}}_{\bar{\nabla}_2} \bar{\mathbb{Z}}_{\bar{\nabla}_1} + \mathbb{Z}_{\nabla_2} \mathbb{Z}_{\nabla_1}. \quad (\text{C5})$$

Thus, $\mathbb{Z}_c^{(0|-2)}$ is made out of the two vector fields $\bar{\nabla}_\alpha$ and $\bar{\mathbb{Z}}_{ac}^{(0|-2)}$ is made out of the two vector fields ∇_α .⁶ We will discuss the action of $\mathbb{Z}_c^{(0|-2)}$ only, the antichiral calculation being similar. Let us start by applying the first operator $\bar{\mathbb{Z}}_{\bar{\nabla}_i}$ to the Lagrangian $\mathcal{L}^{(4|4)} = \star 1 = V^a V^b V^c V^d \epsilon_{abcd} \delta(\psi^\alpha) \epsilon_{\alpha\beta} \delta(\bar{\psi}^\beta) \delta(\bar{\psi}^\alpha) \epsilon_{\dot{\alpha}\dot{\beta}} \delta(\bar{\psi}^{\dot{\beta}})$; first of all, we notice that since $\mathcal{L}^{(4|4)}$ is a top form, its exterior derivative is trivially zero; hence, the action of the operator $\bar{\mathbb{Z}}_{\bar{\nabla}_i}$ reads

$$\begin{aligned} \bar{\mathbb{Z}}_{\bar{\nabla}_i} \mathcal{L}^{(4|4)} &= d\Theta \left(\bar{i}_{\bar{\nabla}_i} \right) V^a V^b V^c V^d \epsilon_{abcd} \delta(\psi^\alpha) \\ &\quad \times \epsilon_{\alpha\beta} \delta(\bar{\psi}^\beta) \delta(\bar{\psi}^\alpha) \epsilon_{\dot{\alpha}\dot{\beta}} \delta(\bar{\psi}^{\dot{\beta}}) \\ &= d \left[\text{Vol}_L^{(4|2)} \frac{1}{\bar{\psi}^i} \delta(\bar{\psi}^{\dot{2}}) \right]. \end{aligned} \quad (\text{C6})$$

At this point, it is worth observing that the intermediate step of calculation involves *inverse forms*, i.e., forms of the type $1/\psi$. For details of their geometrical interpretation and string-theoretic counterpart, we refer the reader to [36]. Here, we only note that inverse forms emerge in intermediate

passages, but disappear at the end of calculations. In (C6) we collected the V 's and the undotted ψ 's in the chiral volume form $\text{Vol}_L^{(4|2)}$ [see (2.13)]; notice that this volume form is d -closed (the same holds for the anti-chiral one): since the expression is covariant, we can substitute the action of the differential d with the action of the covariant one, defined in (3.1)–(3.3). The closure is directly verified: the covariant derivative on V 's either involves the torsion R^a , which is zero, or involves $\psi\bar{\psi}$, hence, giving zero because of the presence of the two undotted delta's $\delta(\psi^\alpha)\epsilon_{\alpha\beta}\delta(\psi^\beta)$; the covariant derivative on ψ 's is the curvature ρ , which is proportional to at least one V , which is then annihilated by the presence of the other four V 's. Hence, (C6) becomes

$$\bar{\mathbb{Z}}_{\bar{\nabla}_i} \mathcal{L}^{(4|4)} = \text{Vol}_L^{(4|2)} d \left[\frac{1}{\bar{\psi}^i} \delta(\bar{\psi}^{\dot{2}}) \right]. \quad (\text{C7})$$

Now the key point is that the expression between square brackets is not covariant; hence, the action of the exterior derivative involves the spin connection. In particular, we have

$$\begin{aligned} \text{Vol}_L^{(4|2)} d \left[\frac{1}{\bar{\psi}^i} \delta(\bar{\psi}^{\dot{2}}) \right] &= \text{Vol}_L^{(4|2)} \left[-\frac{1}{(\bar{\psi}^i)^2} \left(\bar{\rho}^i + \frac{1}{4} \omega^{ab} (\sigma_{ab} \bar{\psi})^i \right) \delta(\bar{\psi}^{\dot{2}}) + \frac{1}{\bar{\psi}^i} \left(\bar{\rho}^{\dot{2}} + \frac{1}{4} \omega^{ab} (\sigma_{ab} \bar{\psi})^{\dot{2}} \right) \delta^{(1)}(\bar{\psi}^{\dot{2}}) \right] \\ &= \text{Vol}_L^{(4|2)} \left[\frac{-1}{4(\bar{\psi}^i)^2} \omega^{ab} (\sigma_{ab} \bar{\psi})^i \delta(\bar{\psi}^{\dot{2}}) + \frac{1}{4\bar{\psi}^i} \omega^{ab} (\sigma_{ab} \bar{\psi})^{\dot{2}} \delta^{(1)}(\bar{\psi}^{\dot{2}}) \right], \end{aligned} \quad (\text{C8})$$

where the terms involving ρ have been dropped, since they contain at least one vielbein V . We use the notation

$$\begin{aligned} -(\sigma_{ab} \bar{\psi})^{\dot{\alpha}} &= [P_- (\gamma_{ab} \psi)]^{\dot{\alpha}} = [(\gamma_{ab} P_- \psi)]^{\dot{\alpha}}, \quad P_- = \frac{1 - \gamma_5}{2}, \\ \sigma_a^\alpha{}_\beta &= (1, \sigma_i)^\alpha{}_\beta, \quad \sigma_a^{\dot{\alpha}}{}_{\dot{\beta}} = (1, -\sigma_i)^{\dot{\alpha}}{}_{\dot{\beta}}, \quad \gamma_a = \begin{pmatrix} 0 & \sigma_a^\alpha{}_\beta \\ -\sigma_a^{\dot{\alpha}}{}_{\dot{\beta}} & 0 \end{pmatrix}, \end{aligned} \quad (\text{C9})$$

σ_i being the Pauli matrices. We could move to the more convenient chiral notation for the action of Lorentz generators:

$$\omega^{ab} (\sigma_{ab} \psi)^\alpha = \omega^\alpha{}_\beta \psi^\beta, \quad \omega^{ab} (\sigma_{ab} \bar{\psi})^{\dot{\alpha}} = \omega^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}, \quad \omega^{\alpha\beta} = \omega^{\beta\alpha}, \quad \omega^{\dot{\alpha}\dot{\beta}} = \omega^{\dot{\beta}\dot{\alpha}}. \quad (\text{C10})$$

Equation (C8) then becomes

$$\begin{aligned} \bar{\mathbb{Z}}_{\bar{\nabla}_i} \mathcal{L}^{(4|4)} &= \text{Vol}_L^{(4|2)} \left[\frac{-1}{4(\bar{\psi}^i)^2} \left(\omega_1^i \bar{\psi}^i + \omega_2^i \bar{\psi}^{\dot{2}} \right) \delta(\bar{\psi}^{\dot{2}}) + \frac{1}{4\bar{\psi}^i} \left(\omega_1^{\dot{2}} \bar{\psi}^i + \omega_2^{\dot{2}} \bar{\psi}^{\dot{2}} \right) \right] \delta^{(1)}(\bar{\psi}^{\dot{2}}) \\ &= -\text{Vol}_L^{(4|2)} \left[\frac{\omega_1^i}{4\bar{\psi}^i} + \frac{\omega_2^{\dot{2}}}{4\bar{\psi}^i} \right] \delta(\bar{\psi}^{\dot{2}}) + \frac{1}{4} \omega_c^{(4|2)} \omega_1^{\dot{2}} \delta^{(1)}(\bar{\psi}^{\dot{2}}) \\ &= \frac{1}{4} \text{Vol}_L^{(4|2)} \omega_1^{\dot{2}} \delta^{(1)}(\bar{\psi}^{\dot{2}}). \end{aligned} \quad (\text{C11})$$

⁶In the present section we use the Weyl/anti-Weyl notation to simplify the computations; see [6] for the conventions.

Notice that the terms containing inverse forms have canceled, as expected, because of the relation $\omega_1^{\dot{1}} = -\omega_2^{\dot{2}}$. We can now act with the second PCO \bar{Z}_{∇_2} ; since \bar{Z} is a cohomology operator, its action can be simplified as

$$\bar{Z}_{\nabla_2} \bar{Z}_{\nabla_1} \mathcal{L}^{(4|4)} = d\Theta(t_{\nabla_2}) \left[\frac{1}{4} \text{Vol}_L^{(4|2)} \omega_1^{\dot{2}} \delta^{(1)}(\bar{\psi}^{\dot{2}}) \right] = -\frac{1}{4} \text{Vol}_L^{(4|2)} d \left[\omega_1^{\dot{2}} \frac{1}{(\bar{\psi}^{\dot{2}})^2} \right]. \quad (\text{C12})$$

Again, inverse forms arise, but they cancel after the action of the exterior derivative. In particular, we have

$$\begin{aligned} -\frac{1}{4} \text{Vol}_L^{(4|2)} d \left[\omega_1^{\dot{2}} \frac{1}{(\bar{\psi}^{\dot{2}})^2} \right] &= -\frac{1}{4} \text{Vol}_L^{(4|2)} \left[d\omega_1^{\dot{2}} \frac{1}{(\bar{\psi}^{\dot{2}})^2} + 2\omega_1^{\dot{2}} \frac{1}{(\bar{\psi}^{\dot{2}})^3} \left(\omega_1^{\dot{2}} \bar{\psi}^{\dot{1}} + \omega_2^{\dot{2}} \bar{\psi}^{\dot{2}} \right) \right] \\ &= -\frac{1}{4} \text{Vol}_L^{(4|2)} \frac{1}{(\bar{\psi}^{\dot{2}})^2} \left[d\omega_1^{\dot{2}} + 2\omega_1^{\dot{2}} \omega_2^{\dot{2}} \right]. \end{aligned} \quad (\text{C13})$$

The expression in square brackets represents exactly a component of the curvature (3.2) in the chiral/antichiral notation. The terms containing V 's and ψ 's are automatically annihilated by the presence of $\omega_c^{(4|2)}$, while only after some algebra is it possible to verify that only the $(\bar{\psi}^{\dot{2}})^2$ term survives. In particular, we arrive at the final expression (up to an overall factor)

$$\bar{Z}_{\nabla_2} \bar{Z}_{\nabla_1} \mathcal{L}^{(4|4)} = \text{Vol}_L^{(4|2)} (S + iP) = \text{Vol}_L^{(4|2)} \mathcal{R}_L. \quad (\text{C14})$$

Analogous computations can be performed by acting on $\mathcal{L}^{(4|4)}$ with the picture lowering operators along the ∇_α directions, so that finally one obtains

$$(\bar{Z}_{\nabla_2} \bar{Z}_{\nabla_1} + Z_{\nabla_2} Z_{\nabla_1}) \mathcal{L}^{(4|4)} = \text{Vol}_L^{(4|2)} \mathcal{R}_L + \text{Vol}_R^{(4|2)} \mathcal{R}_R, \quad (\text{C15})$$

reproducing the usual action (C1).

APPENDIX D: SOME FORMULAS FOR SUPERGRAVITY

In this appendix, we give some explicit formulas used in the text and their Hodge dual. Using supergravity parametrizations we have

$$\begin{aligned} T^a &= 0, \\ \rho^\alpha &= \rho_{bc}^\alpha V^b \wedge V^c + \rho_{\beta c}^\alpha \psi^\beta \wedge V^c, \end{aligned} \quad (\text{D1})$$

with $\rho_{\beta c}^\alpha = i(\gamma_a S - i\gamma_a \gamma_5 P + \gamma_5(2\delta_a^b - \gamma_a^b) A_b)^\alpha_\beta$ (where I set $c_4 = 1$). Then,

$$\begin{aligned} \star T^a &= 0, \\ \star \rho^\alpha &= \rho_{bc}^\alpha V_2^{de} \delta^4(\psi) + \rho_{c\beta}^\alpha C^{\beta\gamma} V_3^\zeta l_\gamma \delta^4(\psi). \end{aligned} \quad (\text{D2})$$

Now acting with the differential d , we finally get

$$\begin{aligned} \nabla \star T^a &= 0, \\ \nabla \star \rho^\alpha &= \nabla_b \rho_{cd}^\alpha \epsilon^{bcd a'} \eta_{a'a} V_3^a \delta^4(\psi) \\ &\quad + (\eta^{bc} \nabla_b \rho_{c\beta}^\alpha + \rho_{ab}^\alpha \rho_{cd}^\gamma \epsilon^{abcd} + \rho_{c\beta}^\alpha C^{\beta\gamma} \rho_{d\gamma}^\alpha \eta^{cb}) \\ &\quad \times V_4 l_\gamma \delta^4(\psi), \end{aligned} \quad (\text{D3})$$

and

$$\begin{aligned} \star \nabla \star \rho^\alpha &= \nabla_b \rho_{cd}^\alpha \epsilon^{bcd a'} \eta_{a'a} V^a \\ &\quad + (\eta^{bc} \nabla_b \rho_{c\beta}^\alpha + \rho_{ab}^\alpha \rho_{cd}^\gamma \epsilon^{abcd} + \rho_{c\beta}^\alpha C^{\beta\gamma} \rho_{d\gamma}^\alpha \eta^{cb}) \\ &\quad \times C_{\gamma\sigma} \psi^\sigma. \end{aligned} \quad (\text{D4})$$

It is very instructive to compute $d\star d\phi$ explicitly using a generic supergravity parametrization for the super vielbeins⁷

$$\begin{aligned} \nabla V^a &= T^a = T_{bc}^a V^b \wedge V^c + T_{\beta c}^a \psi^\beta \wedge V^c + T_{\beta\gamma}^a \psi^\beta \wedge \psi^\gamma, \\ \nabla \psi^\alpha &= \rho^\alpha = \rho_{bc}^\alpha V^b \wedge V^c + \rho_{\beta c}^\alpha \psi^\beta \wedge V^c + \rho_{\beta\gamma}^\alpha \psi^\beta \wedge \psi^\gamma, \end{aligned} \quad (\text{D5})$$

where the superfields $T_{bc}^a, \dots, \rho_{\beta\gamma}^\alpha$ are defined in terms of the superspace constraints (also known as *rheonomic constraints*) and using the Bianchi identities. Then, we get

⁷Following the conventions of [10] we use the notation $T^a = \rho^\alpha$.

$$\begin{aligned}
d\star d\phi &= d(V_3^a \delta^4(\psi) \nabla_a \phi) + d(C^{\alpha\beta} V^4 \iota_\alpha \delta^4(\psi) \nabla_\alpha \phi) \\
&= \frac{1}{2} \epsilon^{abcd} \nabla V_b V_c V_d \delta^4(\psi) \nabla_a \phi - V_3^a \nabla \psi^\alpha \iota_\alpha \delta^4(\psi) \nabla_a \phi - V_3^a \delta^4(\psi) (V^d \nabla_d \nabla_a \phi + \psi^\delta \nabla_\delta \nabla_a \phi) \\
&\quad + \frac{C^{\alpha\beta}}{3!} \epsilon^{abcd} \nabla V_a V_b V_c V_d \iota_\alpha \delta^4(\psi) \nabla_\beta \phi + C^{\alpha\beta} V^4 \nabla \psi^\gamma \iota_\gamma \iota_\alpha \delta^4(\psi) \nabla_\beta \phi + C^{\alpha\beta} V^4 \iota_\alpha \delta^4(\psi) (V^d \nabla_d \nabla_\beta \phi + \psi^\delta \nabla_\delta \nabla_\beta \phi) \\
&= \frac{1}{2} \epsilon^{abcd} T_{brs} V^r V^s V_c V_d \delta^4(\psi) \nabla_a \phi - V_3^a (\rho_{\beta c}^\alpha \psi^\beta V^c) \iota_\alpha \delta^4(\psi) \nabla_a \phi - V_3^a \delta^4(\psi) V^d \nabla_d \nabla_a \phi \\
&\quad + \frac{C^{\alpha\beta}}{3!} \epsilon^{abcd} T_{a\beta r} \psi^\beta V^r V_b V_c V_d \iota_\alpha \delta^4(\psi) \nabla_\beta \phi + C^{\alpha\beta} V^4 \rho_{\beta\gamma}^\alpha \psi^\beta \psi^\gamma \iota_\gamma \iota_\alpha \delta^4(\psi) \nabla_\beta \phi + C^{\alpha\beta} V^4 \iota_\alpha \delta^4(\psi) \psi^\delta \nabla_\delta \nabla_\beta \phi \\
&= V^4 \delta^4(\psi) [(T_a^{ab} - \rho_a^{ab}) \nabla_b \phi + (T_a^{\alpha\beta} + \rho_a^{\alpha\beta}) \nabla_\beta \phi - (\eta^{ab} \nabla_a \nabla_b \phi + C^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi)], \tag{D6}
\end{aligned}$$

where the coefficients in front of $\nabla_a \phi$ and $\nabla_\alpha \phi$ are obtained by a suitable contraction of the supertorsion T , ρ .

Applying the Hodge dual directly on the torsion, we find

$$\begin{aligned}
\star T^a &= T_{bc}^a V_2^{bc} \delta^4(\psi) + T_{\beta c}^a V_3^c \iota_\alpha \delta^4(\psi) + T_{\beta\gamma}^a V^4 \iota^\beta \iota^\gamma \delta^4(\psi), \\
\star \rho^\alpha &= \rho_{bc}^\alpha V_2^{de} \delta^4(\psi) + \rho_{\beta c}^\alpha V_3^c \iota_\alpha \delta^4(\psi) + \rho_{\beta\gamma}^\alpha V^4 \iota^\beta \iota^\gamma \delta^4(\psi). \tag{D7}
\end{aligned}$$

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