Scattering amplitudes in high-energy limit of projectable Hořava gravity

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(Received 7 June 2023; accepted 10 July 2023; published 25 August 2023)

We study the high-energy limit of projectable Hořava gravity using on-shell graviton scattering amplitudes. We compute the tree-level amplitudes using symbolic computer algebra and analyze their properties in the case of collisions with zero total momentum. The amplitudes grow with collision energy in the way consistent with tree-level unitarity. We discuss their angular dependence and derive the expression for the differential cross section that happens to depend only on the essential combinations of the couplings. One of our key results is that the amplitudes for arbitrary kinematics are finite when the coupling λ in the kinetic Lagrangian is taken to infinity—the value corresponding to candidate asymptotically free ultraviolet fixed points of the theory. We formulate a modified action which reproduces the same amplitudes and is directly applicable at $\lambda = \infty$, thereby establishing that the limit $\lambda \to \infty$ of projectable Hořava gravity is regular. As an auxiliary result, we derive the generalized Ward identities for the amplitudes in nonrelativistic gauge theories.

DOI: 10.1103/PhysRevD.108.046017

I. INTRODUCTION

Hořava gravity (HG), proposed in [1], is a metric quantum theory of gravity realized as a power-counting renormalizable quantum field theory (see Refs. [2–6] for reviews). The power-counting renormalizability is achieved by separating spacetime into space *and* time: The theory at tree level and at high energies is taken to be invariant under anisotropic (Lifshitz) scaling

$$\mathbf{x} \to b^{-1}\mathbf{x}, \qquad t \to b^{-z}t, \tag{1.1}$$

where b is a scaling parameter and z is the Lifshitz exponent. In HG the latter is taken to be equal to the number of spatial dimensions, $z = d$. Such a symmetry implies that we can have a Lagrangian quadratic in first time derivatives, yet containing terms with more than two spatial derivatives of fields. The propagators then have more powers of momenta than of energy in the denominators, which makes them decay fast in the ultraviolet (UV) and improves convergence of the loop integrals in perturbation theory. Since the equations of motion contain only two time derivatives, we do not get any problematic extra degrees of freedom (ghosts), in contrast to the generally covariant higher curvature gravity [7–10].[1]

The price to pay is the violation of Lorentz invariance at high energies that propagates down to low energies in the form of a preferred spacelike foliation whose dynamics is described by a new scalar field called *scalar graviton* or *khronon* [2]. The violation of Lorentz invariance in the visible sector can be sufficiently small in the *nonprojectable* version of the theory [13] to reproduce the observed phenomenology, albeit with some degree of tuning [14]. From the theoretical perspective, the nonprojectable theory is complicated since it involves large (but still finite) number of marginal couplings that describe its behavior in the UV. Its renormalizability beyond power counting has not yet been established, though there has been important progress in this direction recently [15,16]. Further analysis of its UV properties, such as the renormalization group (RG) flow, is presently beyond reach.

In this paper we consider a simpler version of the theory: the *projectable* model, which has been proven to be perturbatively renormalizable [17,18] and whose one-loop RG flow has been computed in [19,20]. The flow possesses a number of UV fixed points with vanishing gravitational constant which indicates asymptotically free behavior. Some of these points, however, are characterized by a divergent dimensionless coefficient in the kinetic term of

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[1]See Refs. [11,12] and references therein for suggested interpretations of quantum theories with higher time derivatives.

the action conventionally denoted by λ. Since positive powers of λ appear in the interaction vertices, one may worry if this divergence jeopardizes the asymptotic freedom.

The purpose of the present paper is to address this concern by scrutinizing the projectable HG in the limit[2]

$$\lambda \to \infty, \qquad \text{other couplings fixed.} \tag{1.2}$$

Early work [21] studied cosmological perturbations in HG and showed that their power spectrum and cubic interactions remain well behaved in the limit (1.2), suggesting that it corresponds to a regular theory. More recently a similar limit in a supersymmetric version of HG has been connected to the Perelman-Ricci flows [22].

We take a different approach and use the scattering amplitudes as gauge-invariant probes of the theory. We compute the full set of tree-level amplitudes for $2 \to 2$ scattering of transverse and scalar gravitons in the projectable HG taking into account all marginal couplings. This calculation is of high algebraic complexity which we overcome by making use of computer algebra [23–26]. The resulting expressions for the amplitudes at general kinematics are too cumbersome to be analyzed explicitly,[3] so we focus in the paper on the simplest case of scattering with vanishing net momentum, to which we refer as "head-on collisions."[4] We discuss the energy and angular dependence of the amplitudes and observe that they are finite in the limit (1.2). We verify the latter property for an arbitrary kinematics using our code and encouraged by these results develop an analytic proof of cancellation between potentially divergent contributions. Further, we show that a reformulation of the theory with introduction of an additional auxiliary field allows one to take the limit (1.2) at the level of the action, implying that this limit is regular beyond the tree-level and $2 \to 2$ processes.

The complexity of the amplitudes calls for subjecting them to various consistency checks. An important class of such checks are requirements of gauge invariance. In relativistic theories and for relativistic gauges they imply two types of conditions. First, the on-shell amplitudes for physical states must be independent of the gauge-fixing parameters. Second, they must satisfy the Ward identities stating that an amplitude for scattering of a gauge boson vanishes whenever its polarization vector (for Yang–Mills theories) or tensor (for gravity) is replaced by a vector/tensor proportional to the boson's four-momentum. While the first condition translates without change to nonrelativistic theories, the second is less obvious since the four-momentum is no longer a useful object. To generalize the Ward identities to the case of HG, we go back to first principles and construct its Hilbert space using the Becchi-Rouet-Stora-Tyutin (BRST) quantization. The sought after conditions then arise from the requirement of the BRST invariance of the $\mathcal{S}$-matrix. This approach is not restricted to HG and applies to any nonrelativistic gauge theory, as we illustrate on an example of a Yang–Mills model with $z = 2$ Lifshitz scaling in $(4 + 1)$ dimensions.

It is worth noting that the phenomenological viability of projectable HG is problematic since it does not possess a stable perturbative Minkowski vacuum where gravitons would propagate with the speed of light [2,29] (see also [6] for recent discussion). References [3,30,31] suggested that it may still reproduce general relativity with an additional sector behaving as dark matter if the khronon field is strongly coupled. We do not attempt to add anything to this aspect of the model and focus on its properties at high energies where it is stable and weakly coupled.

The paper is organized as follows. In Sec. II we review the projectable HG and perform its BRST quantization. In Sec. III we derive the generalized Ward identities for scattering amplitudes in nonrelativistic gauge theories, illustrating the general framework on the Yang-Mills theory with Lifshitz scaling before applying it to HG. In Sec. IV we outline the calculation of amplitudes in projectable HG and present our results for scattering with zero total momentum. In Sec. V we consider the limit (1.2) and show that the amplitudes remain finite. We also present an alternative formulation of the theory which allows us to take the limit (1.2) directly at the level of the action. We conclude in Sec. VI. Lengthy formulas are relegated to the appendixes.

II. PROJECTABLE HOŘAVA GRAVITY

A. Formulating the theory

A theory symmetric under scaling (1.1) cannot be invariant under the full group of spacetime diffeomorphisms. However, it can still be invariant under its foliation-preserving subgroup (FDiffs):

$$\mathbf{x} \mapsto \tilde{\mathbf{x}}(\mathbf{x}, t), \qquad t \mapsto \tilde{t}(t), \tag{2.1}$$

with $\tilde{t}(t)$-monotonic function. Hořava gravity [1] is a metric theory with this symmetry, conventionally formulated using the Arnowitt-Deser-Misner (ADM) decomposition of the spacetime line element,

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad i, j = 1, 2, 3, \tag{2.2}$$

where we have specified to three spatial dimensions. The *lapse*, *shift*, and the spatial metric transform under FDiffs as

[2]The directionality of the limit, i.e. whether λ goes to $+\infty$ or $-\infty$ is unimportant, at least within the perturbation theory.

[3]They are available in the *Mathematica* [27] format at [28].

[4]In contrast to relativistic theories, this is a genuine restriction. Due to the absence of Lorentz invariance in HG one cannot set the net momentum to zero by boosting to the center-of-mass frame.

$$N \mapsto N\frac{dt}{d\tilde{t}}, \quad N^i \mapsto \left(N^j \frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{x}^i}{\partial t}\right)\frac{dt}{d\tilde{t}}, \quad \gamma_{ij} \mapsto \gamma_{kl}\frac{\partial x^k}{\partial \tilde{x}^i}\frac{\partial x^l}{\partial \tilde{x}^j}. \tag{2.3}$$

These transformations are compatible with the *projectability* condition which states that the lapse N is only a function of time, $N = N(t)$. In this case it can be set to 1 by an appropriate choice of the time coordinate. Equivalently, at least in perturbation theory, we can consider a model without time reparametrizations and with unit lapse from the start. This is the formulation we adopt in this paper. An alternative option—taking the lapse to be a function of both time and space—leads to the nonprojectable HG.

Using the remaining variables γ_{ij} and N^i we construct the most general action with two time derivatives which is invariant under FDiffs (2.3) and the Lifshitz scaling (1.1) with $z = 3$. To do this, we need to assign the *scaling dimensions* to the metric and the shift, which will determine how their quantum fluctuations scale in the UV. In more detail, we say that a field Φ has scaling dimension $\dim \Phi = r$ if under the symmetry (1.1) it transforms as

$$\Phi(\mathbf{x}, t) \mapsto \Phi'(b^{-1}\mathbf{x}, b^{-z}t) = b^r \Phi(\mathbf{x}, t). \tag{2.4}$$

The metric γ_{ij} enters into the action nonlinearly, while its time derivative enters through the extrinsic curvature of the constant-time slices which transforms covariantly under the FDiffs,[5]

$$K_{ij} = \frac{1}{2}(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i). \tag{2.5}$$

To preserve the homogeneous scaling of different terms in the action, we assign the dimensions 0 to γ_{ij} and 2 to N_i,

$$\dim \gamma_{ij} = 0, \qquad \dim N_i = 2. \tag{2.6}$$

This leads us to the action,

$$S = \frac{1}{2G}\int d^3x dt \sqrt{\gamma}(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}), \tag{2.7}$$

where G is the gravitational coupling controlling the overall strength of the interactions and $K \equiv \gamma^{ij}K_{ij}$ is trace of the extrinsic curvature. Note the free parameter λ which appears in the kinetic term of HG compared to general relativity, where it is fixed to be $\lambda = 1$ by the full spacetime diff-invariance.

The "potential" term $\mathcal{V}$ in Eq. (2.7) depends on three-dimensional curvature invariants constructed using the spatial metric γ_{ij}. To be compatible with the Lifshitz scaling, it must consist of operators with scaling dimension 6. The most general such potential reads,

$$\mathcal{V}^{\dim=6} = \nu_1 R^3 + \nu_2 R R_{ij}R^{ij} + \nu_3 R_{ij}R^{jk}R^i_k + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk}\nabla^i R^{jk}, \tag{2.8}$$

where R_{ij} and R are the three-dimensional Ricci tensor and the scalar curvature, respectively; ν_a, $a = 1, \ldots, 5$, are coupling constants. Note that there are no terms with the Riemann tensor, since in three dimensions it is not independent and can be expressed through R_{ij}.

One can also add to the potential the terms of lower scaling dimension which represent relevant deformations of the Lifshitz scaling,

$$\mathcal{V}^{\dim<6} = 2\Lambda - \eta R + \mu_1 R^2 + \mu_2 R_{ij}R^{ij}. \tag{2.9}$$

In fact, these terms are required for renormalizability since the Lifshitz scaling is broken by quantum corrections, as manifested by the RG running of the couplings [20]. In this paper we disregard the low-dimension terms because we are interested in the high-energy properties of the theory controlled by the marginal operators collected in (2.8).

B. BRST quantization

The flat static metric $\gamma_{ij} = \delta_{ij}$ with vanishing shift $N^i = 0$ is a solution of the classical equations following from the action (2.7) with the potential (2.8). We want to quantize the theory around this background, so we introduce the metric perturbation

$$h_{ij} \equiv \gamma_{ij} - \delta_{ij}. \tag{2.10}$$

Next we need to fix the gauge. This is done consistently within the BRST formalism [32,33]; we follow here [17,34]. We introduce the fermionic Faddeev–Popov ghosts c^i and antighosts $\bar{c}_i$, bosonic Nakanishi-Lautrup field b_i and the Slavnov operator $\mathbf{s}$ which implements the BRST transformations of the original and new fields,

$$\mathbf{s}h_{ij} = \partial_i c_j + \partial_j c_i + \partial_i c^k h_{jk} + \partial_j c^k h_{ik} + c^k \partial_k h_{ij},$$
$$\mathbf{s}N^i = \dot{c}^i - N^j \partial_j c^i + c^j \partial_j N^i, \tag{2.11a}$$

$$\mathbf{s}c^i = c^j \partial_j c^i, \qquad \mathbf{s}\bar{c}_i = b_i, \qquad \mathbf{s}b_i = 0. \tag{2.11b}$$

Note that from now on the indices are raised and lowered with flat background metric δ_{ij}. The first two expressions here are, of course, nothing but the infinitesimal gauge transformations of the metric and shift with the gauge parameters replaced by the ghosts. With these definitions it is straightforward to to show that the Slavnov

[5]The indices are raised and lowered using the spatial metric γ_{ij}.

operator is nilpotent, i.e. the action of $\mathbf{s}^2$ on any field vanishes.[6]

The quantum tree-level action is constructed as the sum of the original action (2.7) and the BRST variation of a gauge-fixing fermion Ψ,

$$S_q = S + \frac{1}{2G}\int d^3x dt \mathbf{s}\Psi. \tag{2.12}$$

Gauge invariance of the original action and the nilpotency of the Slavnov operator imply that S_q is BRST invariant, $\mathbf{s}S_q = 0$. The gauge-fixing fermion is conventionally taken in the form

$$\Psi = 2\bar{c}_i F^i - \bar{c}_i O^{ij} b_j, \tag{2.13}$$

where F^i are the gauge-fixing functions and O^{ij} is a nondegenerate operator.

Following [17] we adopt a family of gauges compatible with the Lifshitz scaling and possessing two free parameters σ, ξ:

$$F^i = \dot{N}^i + \frac{1}{2}O^{ij}(\partial_k h_j^k - \lambda\partial_j h),$$
$$O^{ij} = -\frac{1}{\sigma}(\delta^{ij}\Delta^2 + \xi\partial^i\Delta\partial^j), \tag{2.14}$$

where $h \equiv h_k^k$ is the trace of the metric perturbation and $\Delta \equiv \partial_k\partial^k$ is the spatial Laplacian.[7] Upon substituting these expressions into (2.12), it is convenient to integrate out the nondynamical Nakanishi-Lautrup field, and the action takes the form,[8]

$$S_q = S + \int d^3x dt \left(\frac{1}{2G}F^i O_{ij}^{-1} F^j - \frac{1}{G}\bar{c}_i \mathbf{s} F^i\right). \tag{2.15}$$

The first term in the brackets is the gauge-fixing Lagrangian, whereas the second term gives the Lagrangian for ghosts. Note that the operator

$$O_{ij}^{-1} = -\frac{\sigma}{\Delta^2} + \frac{\sigma\xi}{(1+\xi)}\frac{\partial_i\partial_j}{\Delta^3} \tag{2.16}$$

is nonlocal in space which, however, does not lead to any complications since it enters the action only at the quadratic order.

Integrating out the Nakanishi–Lautrup field modifies the BRST transformation of the antighosts which now reads

$$\mathbf{s}\bar{c}_i = O_{ij}^{-1} F^j. \tag{2.17}$$

In other words, it is proportional to the gauge-fixing functions. This fact will be exploited in the next section when discussing the BRST invariance of the scattering amplitudes. Note that the nilpotency of the transformation (2.17) requires $\mathbf{s}F^i = 0$ which is satisfied only on-shell. Indeed, this is precisely the equation of motion for ghosts, as one can see by varying the action (2.15) with respect to $\bar{c}_i$.

We are now ready to quantize the theory and define its Fock space. To this end, we focus on the quadratic part of the Lagrangian. From the action (2.15) we have

$$\begin{aligned}\mathcal{L}_q^{(2)} = \frac{1}{2G}\Bigg\{&\frac{\dot{h}_{ij}^2}{4} - \frac{\lambda\dot{h}^2}{4} + \frac{\nu_5}{4}h_{ij}\Delta^3 h_{ij} + \left(\frac{\nu_5}{2} - \frac{1}{4\sigma}\right)\partial_j h_{ji}\Delta^2\partial_k h_{ki} + \left(\nu_4 + \frac{\nu_5}{2} + \frac{\xi}{4\sigma}\right)\partial_i\partial_j h_{ij}\Delta\partial_k\partial_l h_{kl}\\ &- \left(2\nu_4 + \frac{\nu_5}{2} + \frac{\lambda(1+\xi)}{2\sigma}\right)\Delta^2 h\partial_i\partial_j h_{ij} + \left(\nu_4 + \frac{\nu_5}{4} + \frac{\lambda^2(1+\xi)}{4\sigma}\right)h\Delta^3 h\\ &- \dot{N}_i\frac{\sigma}{\Delta^2}\dot{N}_i - \partial_i\dot{N}_i\frac{\sigma\xi}{(1+\xi)\Delta^3}\partial_j\dot{N}_j - \frac{1}{2}N_i\Delta N_i + \left(\frac{1}{2} - \lambda\right)(\partial_i N_i)^2\Bigg\}\\ &+ \frac{1}{G}\left\{\dot{\bar{c}}_i\dot{c}_i + \frac{1}{2\sigma}\bar{c}_i\Delta^3 c_i + \frac{\xi + (1+\xi)(1-2\lambda)}{2\sigma}\bar{c}_i\Delta^2\partial_i\partial_j c_j\right\},\end{aligned} \tag{2.18}$$

where we have made various integrations by part and placed all indices downwards for simplicity. We see the advantage of the gauge (2.14): it decouples h_{ij} and N_i in the quadratic action which significantly simplifies the quantization. We next perform the helicity decomposition of the fields entering (2.18), diagonalize the Lagrangian and solve the respective equations of motion. This leads us to a set of positive-frequency modes which we label with the spatial momentum $\mathbf{k}$ and helicity α:

$$h_{\mathbf{k}\alpha}, \qquad \alpha = \pm 2, \pm 1, 0, 0', \tag{2.19a}$$

$$N_{\mathbf{k}\alpha}, c_{\mathbf{k}\alpha}, \bar{c}_{\mathbf{k}\alpha}, \qquad \alpha = \pm 1, 0. \tag{2.19b}$$

[6]In the proof one should recall that, since $\mathbf{s}$ is a fermionic operator, it obeys a graded Leibniz rule: $\mathbf{s}(AB) = (\mathbf{s}A)B + (-1)^{|A|}A(\mathbf{s}B)$, where $|A| = 0$ ($|A| = 1$) for a bosonic (fermionic) field A.

[7]The operator O^{ij} corresponds to $-\sigma^{-1}(\mathcal{O}^{-1})^{ij}$ in the notations of [17]. The sign difference is due to the fact that [17] works with the Euclidean version of the theory obtained upon the Wick rotation, whereas here we work in the physical time.

[8]This procedure produces a factor $(\det O^{ij})^{-1/2}$ in the path integral measure of the theory, which is irrelevant at tree level.

The details, including the expressions for the polarization vectors/tensors of the modes, are given in Appendix A.

The modes with helicities ± 2 are present only in the metric and correspond to transverse traceless (tensor) gravitons. Their on-shell dispersion relation is manifestly gauge invariant,

$$\omega_{tt}^2 = \nu_5 k^6. \tag{2.20}$$

The stability of the mode requires $\nu_5 > 0$. The modes with helicities ± 1 and 0 are pure gauge and have dispersion relations

$$\omega_1^2 = \frac{k^6}{2\sigma}, \qquad \omega_0^2 = \frac{(1-\lambda)(1+\xi)}{\sigma} k^6, \tag{2.21}$$

which clearly depend on the gauge parameters. We choose the latter in such a way that both ω_1^2 and ω_0^2 are positive. Finally, an additional scalar mode $0'$ is present in the metric. This is physical and corresponds to a scalar graviton of HG. Its dispersion relation is gauge invariant and reads

$$\omega_s^2 = \nu_s k^6, \qquad \nu_s = \frac{1-\lambda}{1-3\lambda}(8\nu_4 + 3\nu_5). \tag{2.22}$$

The mode is stable provided $\nu_s > 0$ which together with the positivity of the kinetic term (see Appendix A) implies $\lambda < 1/3$ *or* $\lambda > 1$ and $8\nu_4 + 3\nu_5 > 0$.

Upon quantization, the coefficients of the positive-frequency modes (2.19) become the annihilation operators and together with their respective creation operators $h^+_{\mathbf{k}\alpha}$, $N^+_{\mathbf{k}\alpha}$, $\bar{c}^+_{\mathbf{k}\alpha}$, $c^+_{\mathbf{k}\alpha}$ generate the Fock space. The states with only the transverse traceless and scalar $0'$ gravitons have positive norm, whereas the gauge sector with helicities ± 1 and 0 contains both positive and negative-norm states. As usual, the negative norm states are eliminated by restricting to the cohomology of the BRST operator Q—the Noether charge associated with the BRST invariance. Importantly, we are dealing here with the action of the BRST transformations on the asymptotic states made of free particles, implying that the transformations are restricted to linear order. Accordingly, the operator Q is restricted to the quadratic part, which we will highlight with the superscript "(2)". Applying the Noether theorem to the quadratic Lagrangian (2.18) we obtain

$$Q^{(2)} = \frac{1}{2G}\int d^3x \bigg[\dot{c}_i(\partial_j h_{ji} - \lambda\partial_i h) - c_i(\partial_j \dot{h}_{ji} - \lambda\partial_i \dot{h}) - 2\dot{c}_i\left(\frac{\sigma}{\Delta^2}\dot{N}_i - \frac{\sigma\xi}{(1+\xi)\Delta^3}\partial_i\partial_j\dot{N}_j\right) + c_i\Delta N_i + (1-2\lambda)c_i\partial_i\partial_j N_j\bigg]. \tag{2.23}$$

Note that if we want the BRST charge to be Hermitian, we must choose the ghost field c_i to be Hermitian as well. Then the Hermiticity of the Lagrangian requires the antighost $\bar{c}_i$ to be anti-Hermitian. Using the commutation relations from Appendix A, one verifies that

$$i[Q^{(2)}, \Phi]_{\mp} = (\mathbf{s}\Phi)_{\rm lin}, \tag{2.24}$$

for any field Φ of the theory. Here the square brackets with subscript $\mp$ mean commutator (anticommutator) for bosonic (fermionic) fields, and $(\mathbf{s}\Phi)_{\rm lin}$ is the linear part of the BRST transformations (2.11), (2.17). Clearly, $Q^{(2)}$ is nilpotent since $\mathbf{s}$ is nilpotent on-shell.

Physical states $|\psi\rangle$ have zero ghost number[9] and are $Q^{(2)}$-closed. Besides, two states are equivalent if their difference is $Q^{(2)}$-exact. Thus, we have

$$Q^{(2)}|\psi\rangle = 0,$$
$$|\psi_1\rangle \sim |\psi_2\rangle \leftrightarrow |\psi_1\rangle = |\psi_2\rangle + Q^{(2)}|\chi\rangle. \tag{2.25}$$

Then using the standard arguments [35,36] one can show that each equivalence class contains a state made only of the physical tensor and scalar gravitons. The norm of all states in the equivalence class coincides with the norm of this state and is positive definite.

III. GENERALIZED WARD IDENTITIES

In this section we derive the constraints imposed on the scattering amplitudes by the BRST invariance of the $\mathcal{S}$-matrix. We then illustrate them on an example of nonrelativistic Yang-Mills theory and finally apply to the projectable HG.

A. General considerations

We consider a gauge theory that may or may not be relativistic, the latter case being of primary interest to us. We assume that there exists an $\mathcal{S}$-matrix which establishes a map between the asymptotic *in* and *out* states,

$$\langle q', \text{out}|q, \text{in}\rangle = \langle q', \text{in}|\mathcal{S}|q, \text{in}\rangle, \tag{3.1}$$

where q, q' stand for the collection of quantum numbers such as particle types, momenta and polarizations in the initial and final states. The space of asymptotic states is assumed to be isomorphic to the Fock space of non-interacting theory. In what follows we will omit the labels *in* when writing the $\mathcal{S}$-matrix elements.

It should be noted that in making these assumptions we disregard the infrared divergences plaguing the definition of the $\mathcal{S}$-matrix in theories with massless particles. In Lifshitz theories with $z > 1$ these problems can be further

[9] Defined as the number of ghosts minus the number of antighosts. It corresponds to the symmetry of the action (2.15) under opposite scaling of the ghost and antighost fields and is preserved by the evolution.

aggravated due to a softer scaling of particle energy with the momentum. Moreover, the dispersion relation $\omega \propto k^z$ with $z > 1$ kinematically allows a single particle to split into two particles, rendering all particles unstable and further complicating the definition of asymptotic states. Thus, our derivation below in this subsection should be considered as rather formal and strictly applicable only at tree level where the above problems do not arise. Still, we believe that, with a proper infrared regularization, the end result should also hold beyond the tree level. We leave its rigorous derivation for future study.

The BRST transformations constitute a symmetry of the gauge-fixed action implying that the $\mathcal{S}$-matrix commutes with the BRST charge. Since the $\mathcal{S}$-matrix acts on the asymptotic free-particle states, we have to restrict the charge to its quadratic part $Q^{(2)}$ which gives

$$[Q^{(2)}, \mathcal{S}] = 0. \tag{3.2}$$

The restriction to $Q^{(2)}$ here is nontrivial. Within the interaction picture one can think of $\mathcal{S}$ as the operator describing nonlinear evolution from $t = -\infty$ to $t = +\infty$. The full BRST charge Q commuting with the nonlinear Hamiltonian contains terms of higher order in the fields, so one may wonder if the higher-order terms in Q must be also kept in the commutator (3.2). This is not the case, as can be shown [37] using the Lehmann-Symanzik-Zimmermann (LSZ) representation for the $\mathcal{S}$-matrix. For completeness, we reproduce the argument in Appendix B.

The property (3.2) implies that the $\mathcal{S}$-matrix element between a physical state $|\psi'\rangle$ and any $Q^{(2)}$-exact state vanishes,

$$\langle\psi'|\mathcal{S}Q^{(2)}|\chi\rangle = 0. \tag{3.3}$$

In particular, for $|\chi\rangle$ we can take a state obtained by adding an antighost to another physical state,

$$|\chi\rangle = \bar{c}^{+}_{\mathbf{k}\alpha}|\psi\rangle. \tag{3.4}$$

In general, the BRST transformation of the antighost is proportional to the gauge-fixing function and can be written as the linear combination of gauge modes with the same momentum and helicity,

$$i[Q^{(2)}, \bar{c}^{+}_{\mathbf{k}\alpha}]_{+} = i\sum_a \mathcal{C}_a \Phi^{a+}_{\mathbf{k}\alpha}, \tag{3.5}$$

where Φ^a are various gauge fields in the theory and $\mathcal{C}_a$ are c-number coefficients that can depend on the momentum and helicity. Substituting this into Eq. (3.3) and using $Q^{(2)}|\psi\rangle = 0$ we obtain

$$\sum_a \mathcal{C}_a \langle\psi'|\mathcal{S}\Phi^{a+}_{\mathbf{k}\alpha}|\psi\rangle = 0. \tag{3.6a}$$

Similar arguments apply to the final state and give

$$\sum_a \mathcal{C}^{*}_a \langle\psi'|\Phi^{a}_{\mathbf{k}\alpha}\mathcal{S}|\psi\rangle = 0. \tag{3.6b}$$

These are linear constraints on the amplitudes involving the gauge modes. As we are going to see, in relativistic Yang-Mills theory they lead to the usual Ward identity implying that an amplitude vanishes when the polarization vector of a gluon is replaced by its four-momentum k_μ. In general, they are more complicated and do not reduce to a simple replacement of polarization vectors. Note, in particular, that in nonrelativistic theories, the dispersion relations of gauge modes entering (3.6) need not be the same as those of the physical particles.

We can continue the process and add another combination of gauge modes (3.5) to a state already containing one such combination. The $\mathcal{S}$-matrix element must again be zero due to the identity (3.3) and the nilpotency of $Q^{(2)}$. This gives us

$$\sum_{a,b} \mathcal{C}_a \mathcal{C}_b \langle\psi'|\mathcal{S}\Phi^{a+}_{\mathbf{k}_1\alpha}\Phi^{b+}_{\mathbf{k}_2\beta}|\psi\rangle = 0, \tag{3.7}$$

and so on.

Another condition that the amplitudes between physical states must satisfy is independence of the choice of gauge.[10] In concrete calculations, this is easily verified by making sure that the gauge parameters drop out from the answer. Since this condition is the same in relativistic and nonrelativistic theories, we are not going to discuss it any further.

B. Examples: Yang-Mills

1. Relativistic

Let us first see how Eqs. (3.6) work in the standard case of the relativistic Yang-Mills theory. For simplicity, we work in the Feynman gauge, so the gauge-fixed Lagrangian reads[11]

$$\mathcal{L}^{\rm YM}_q = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2}(\partial_\mu A^a_\mu)^2 + \bar{c}^a\partial_\mu D_\mu c^a, \tag{3.8}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$
$$D_\mu c^a = \partial_\mu c^a + g f^{abc} A^b_\mu c^c, \tag{3.9}$$

g is the coupling constant, and f^{abc} are the structure constants of the gauge group. The quadratic kinetic term

[10]This condition can also be derived from the LSZ representation of the $\mathcal{S}$-matrix, see Appendix B.

[11]The repeated Greek indices are summed with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

for the gauge fields diagonalizes and they are straightforwardly quantized with the result

$$A^a_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} A^a_{\mu\mathbf{k}} e^{-i\omega_{\mathbf{k}} t + i\mathbf{k}\mathbf{x}} + \text{H.c}, \tag{3.10a}$$

$$[A^a_{\mu\mathbf{k}}, A^{b+}_{\nu\mathbf{k}'}] = 2\omega_{\mathbf{k}} \eta_{\mu\nu} \delta^{ab} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), \tag{3.10b}$$

where $\omega_{\mathbf{k}} = k$. The BRST transformation of the antighost coincides, up to a sign, with the gauge-fixing function,

$$i[Q^{(2)}, \bar{c}^a]_+ = \mathbf{s}\bar{c}^a = -\partial_\mu A^a_\mu, \tag{3.11}$$

where we read off

$$i[Q^{(2)}, \bar{c}^{a+}_{\mathbf{k}}]_+ = -ik_\mu A^{a+}_{\mu\mathbf{k}}. \tag{3.12}$$

Substituting this expression into Eqs. (3.6) we find

$$k_\mu \langle \psi' | \mathcal{S} A^{a+}_{\mu\mathbf{k}} | \psi \rangle = k_\mu \langle \psi' | A^a_{\mu\mathbf{k}} \mathcal{S} | \psi \rangle = 0. \tag{3.13}$$

On the other hand, the scattering amplitudes involving a physical gluon with helicity ± 1 in the initial or final state are given by

$$e^{(\pm 1)}_\mu \langle \psi' | \mathcal{S} A^{a+}_{\mu\mathbf{k}} | \psi \rangle, \qquad e^{(\pm 1)*}_\mu \langle \psi' | A^a_{\mu\mathbf{k}} \mathcal{S} | \psi \rangle, \tag{3.14}$$

where the transverse polarization vectors are defined as in (A8), with their temporal components set to zero. Thus, we recover the standard Ward identity stating that the amplitudes in relativistic Yang-Mills vanish whenever a gluon polarization vector is replaced by k_μ.

2. Yang-Mills with Lifshitz scaling

As a new application of the conditions (3.6) we consider a nonrelativistic Yang-Mills theory with the Lagrangian

$$\mathcal{L}^{\text{YM}} = \frac{1}{2} F^a_{i0} F^a_{i0} - \frac{\kappa_1}{4} D_i F^a_{jk} D_i F^a_{jk} - \frac{\kappa_2}{2} D_i F^a_{ik} D_j F^a_{jk} - g \frac{\kappa_3}{3} f^{abc} F^a_{ij} F^b_{jk} F^c_{ki}, \tag{3.15}$$

where we use the notations (3.9) and $\kappa_{1,2,3}$ are new constant parameters. In what follows we will denote the zeroth component of the gauge field by the calligraphic letter,

$$\mathcal{A}^a \equiv A^a_0, \tag{3.16}$$

to avoid confusion with the helicity 0 polarization. The action built from this Lagrangian is invariant under Lifshitz scaling (1.1) with $z = 2$ in $(4+1)$-dimensional spacetime with the following assignment of the scaling dimensions:

$$\dim \mathcal{A}^a = 2, \qquad \dim A^a_i = 1. \tag{3.17}$$

When supplemented with a relevant operator $F^a_{ij} F^a_{ij}$, the model is renormalizable. A similar model with $U(1)$ gauge group and fermionic matter was studied in [38].

We take the gauge fixing function and the operator O^{-1} in the gauge fixing term in the form consistent with the Lifshitz scaling,

$$F^a = \dot{\mathcal{A}}^a + \xi \Delta \partial_i A^a_i, \qquad O^{-1}_{ab} = \frac{\delta_{ab}}{\xi \Delta}. \tag{3.18}$$

Here ξ is an arbitrary gauge fixing parameter. The tree-level quantum Lagrangian then reads

$$\mathcal{L}^{\text{YM}}_q = \mathcal{L}^{\text{YM}} + \frac{1}{2\xi} (\dot{\mathcal{A}}^a + \xi \Delta \partial_i A^a_i) \frac{1}{\Delta} (\dot{\mathcal{A}}^a + \xi \Delta \partial_j A^a_j) + \dot{\bar{c}}^a (\dot{c}^a + f^{abc} \mathcal{A}^b c^c) + \xi \partial_i \bar{c}^a \Delta (\partial_i c^a + f^{abc} A^b_i c^c). \tag{3.19}$$

The choice of the gauge ensures cancellation of the quadratic mixing terms between $\mathcal{A}^a$ and A^a_i. Diagonalization of the remaining quadratic Lagrangian is straightforward and yields the general linear solution:

$$\mathcal{A}^a(\mathbf{x}, t) = \int \frac{d^4k}{(2\pi)^4} \frac{\sqrt{\xi} k}{2\omega_{\mathbf{k}0}} \mathcal{A}^a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}0} t + i\mathbf{k}\mathbf{x}} + \text{H.c.}, \tag{3.20a}$$

$$A^a_i(\mathbf{x}, t) = \int \frac{d^4k}{(2\pi)^4} \sum_{\alpha=-1}^{+1} \frac{e^\alpha_i(\mathbf{k})}{2\omega_{\mathbf{k}\alpha}} A^a_{\mathbf{k}\alpha} e^{-i\omega_{\mathbf{k}\alpha} t + i\mathbf{k}\mathbf{x}} + \text{H.c.}, \tag{3.20b}$$

$$c^a(\mathbf{x}, t) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\omega_{\mathbf{k}0}} c^a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}0} t + i\mathbf{k}\mathbf{x}} + \text{H.c.}, \tag{3.20c}$$

$$\bar{c}^a(\mathbf{x}, t) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\omega_{\mathbf{k}0}} \bar{c}^a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}0} t + i\mathbf{k}\mathbf{x}} - \text{H.c.}, \tag{3.20d}$$

where the polarization vectors $e^\alpha_i(\mathbf{k})$ are defined in (A8). The dispersion relations are different for the transverse and longitudinal modes, as expected in theories without Lorentz invariance:

$$\omega^2_{\mathbf{k}1} = (\kappa_1 + \kappa_2) k^4, \qquad \omega^2_{\mathbf{k}0} = \xi k^4. \tag{3.21}$$

Canonically quantizing the fields we obtain the commutation relations,

$$[\mathcal{A}^a_{\mathbf{k}}, \mathcal{A}^{b+}_{\mathbf{k}'}] = -2\omega_{\mathbf{k}0} \delta^{ab} (2\pi)^4 \delta(\mathbf{k} - \mathbf{k}'), \tag{3.22a}$$

$$[A^a_{\mathbf{k}\alpha}, A^{b+}_{\mathbf{k}'\beta}] = 2\omega_{\mathbf{k}1} \delta^{ab} \delta_{\alpha\beta} (2\pi)^4 \delta(\mathbf{k} - \mathbf{k}'), \tag{3.22b}$$

$$[c^a_{\mathbf{k}}, \bar{c}^{b+}_{\mathbf{k}'}]_+ = [\bar{c}^a_{\mathbf{k}}, c^{b+}_{\mathbf{k}'}]_+ = -2\omega_{\mathbf{k}0} \delta^{ab} (2\pi)^4 \delta(\mathbf{k} - \mathbf{k}'), \tag{3.22c}$$

with all other (anti)commutators vanishing.

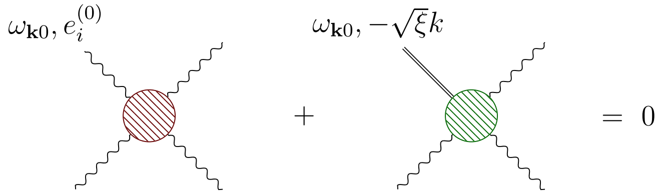


FIG. 1. Generalized Ward identity satisfied by the amplitudes in Yang-Mills theory with Lifshitz scaling. Wavy lines correspond to spatial gauge fields A_i^a, whereas the straight double line corresponds to the temporal component $\mathcal{A}^a$.

According to the general rules, the BRST transformation of the antighost is

$$i[Q^{(2)}, \bar{c}^a]_+ = \mathbf{s}\bar{c}^a = \frac{1}{\xi\Delta}(\dot{\mathcal{A}}^a + \xi\Delta\partial_i A_i^a). \tag{3.23}$$

Comparing the Fourier decomposition of the left- and right-hand sides we get

$$i[Q^{(2)}, \bar{c}_{\mathbf{k}}^{a+}]_+ = ik(\mathcal{A}_{\mathbf{k}}^{a+} + A_{\mathbf{k}0}^{a+}). \tag{3.24}$$

Incidentally, this has the same form as in the relativistic case, cf. Eq. (3.12). Hence, the constraint (3.6) becomes

$$\langle\psi'|\mathcal{S}A_{\mathbf{k}0}^{a+}|\psi\rangle + \langle\psi'|\mathcal{S}\mathcal{A}_{\mathbf{k}}^{a+}|\psi\rangle = 0, \tag{3.25}$$

and similarly for the outgoing mode. It can be represented graphically as shown in Fig. 1, where we explicitly indicate the energy and polarization factors carried by the external legs. Note that the factor for the $\mathcal{A}$-leg is negative due to the minus sign in the commutator (3.22a). We now observe an important difference from the relativistic case. The verification of the gauge invariance does not reduce to a mere substitution of the longitudinal polarization in the external leg of a diagram for transverse gluons—the first diagram in the figure. First, since the dispersion relations of the longitudinal modes is different from that of the transverse modes, the diagram must be reevaluated with a different incoming energy. Second, the interaction vertices for the spatial and temporal parts of the gauge field are essentially different, so the green blob in the second diagram is different from the red blob and must be evaluated separately. We have verified by an explicit calculation that the identity shown in Fig. 1 holds for tree-level $2 \to 2$ amplitudes in the theory (3.15).

C. Application to Hořava gravity

We return to the projectable HG. All preliminary work has been already done in Sec. II B. We can directly use the BRST transformation of the antighost (2.17) which we write explicitly:

$$i[Q^{(2)}, \bar{c}_i]_+ = -\frac{\sigma}{\Delta^2}\dot{N}_i + \frac{\sigma\xi}{(1+\xi)\Delta^3}\partial_i\partial_j\dot{N}_j + \frac{1}{2}(\partial_j h_{ij} - \lambda\partial_i h). \tag{3.26}$$

Expanding the left- and right-hand sides into Fourier modes according to Eqs. (A6) we obtain simple relations

$$i[Q^{(2)}, \bar{c}_{\mathbf{k}\alpha}^+]_+ = \frac{ik}{\sqrt{2}}(N_{\mathbf{k}\alpha}^+ + h_{\mathbf{k}\alpha}^+), \qquad \alpha = \pm 1, \tag{3.27a}$$

$$i[Q^{(2)}, \bar{c}_{\mathbf{k}\alpha}^+]_+ = ik\sqrt{|1-\lambda|}(N_{\mathbf{k}\alpha}^+ + h_{\mathbf{k}\alpha}^+), \qquad \alpha = 0. \tag{3.27b}$$

Substitution into Eqs. (3.6) yields the identities

$$\langle\psi'|\mathcal{S}h_{\mathbf{k}\alpha}^+|\psi\rangle + \langle\psi'|\mathcal{S}N_{\mathbf{k}\alpha}^+|\psi\rangle = 0, \qquad \alpha = 0, \pm 1, \tag{3.28}$$

which are depicted graphically in Fig. 2. The polarization tensors corresponding to the external legs of the diagrams in the figure are given in Eqs. (A9), (A10). Note that the shift polarization is multiplied by (-1) due to the different signs in the commutators of the h and N creation-annihilation operators, see Eqs. (A7a), (A7b).

We use the above identities to cross-check the validity of our calculation of $2 \to 2$ scattering amplitudes in the next section.

IV. CALCULATING THE AMPLITUDES

A. Algorithm and overview of the result

We have automated the computation of scattering amplitudes in HG using the *xAct* package [23–26] for *Mathematica* [27]. Our code [28] starts by extracting propagators and vertices from the action. For the propagators, we use the gauge-fixed Lagrangian (2.18).

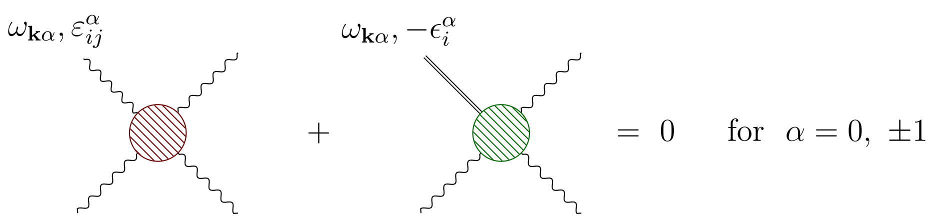


FIG. 2. Generalized Ward identities for the amplitudes in projectable Hořava gravity. Wavy lines and the straight double line represent the spatial metric h_{ij} and the shift N_i, respectively.

The gauge-fixing term is quadratic and thus does not affect the vertices, which we obtain directly from the original action (2.7) by taking variational derivatives with respect to the metric perturbation h_{ij} and the shift N_i. Since we restrict to the tree level, we do not need the propagators or vertices involving ghosts. Finally, the external lines are determined from the mode decomposition of the fields (A6) and their commutators (A7). More details on the Feynman rules used in the calculation are given in Appendix C.

We then follow the standard procedure to construct all diagrams contributing to a given scattering process. For example, the scattering amplitudes for two gravitons in the initial and final states is given by the sum of the diagrams shown in Fig. 3. We treat all momenta and energies as flowing into the diagram. The polarization tensors for incoming particles with negative energies are defined according to

$$\varepsilon^{\alpha}_{ij}(-\mathbf{k},-\omega) = \varepsilon^{\alpha}_{ij}(\mathbf{k},\omega) = [\varepsilon^{-\alpha}_{ij}(\mathbf{k},\omega)]^*. \tag{4.1}$$

This is consistent with the crossing rule that an incoming particle is equivalent to an outgoing particle with opposite momentum and helicity. The amplitude $\mathcal{M}$ is defined in the standard manner, as the $\mathcal{S}$-matrix element with unit operator subtracted and the energy-momentum conserving δ-function stripped off,

$$\mathcal{S} = \mathbb{1} + i\mathcal{M}(\mathbf{k}_I,\omega_I,\alpha_I)(2\pi)^4\delta\left(\sum_I \omega_I\right)\delta\left(\sum_I \mathbf{k}_I\right). \tag{4.2}$$

The scattering of physical states corresponds to choosing the helicities α_I in Fig. 3 equal to ± 2 or $0'$. We have checked that such amplitudes, evaluated on-shell, are independent of the gauge parameters σ, ξ. In addition, we have evaluated the amplitudes with one gauge mode having $\alpha = \pm 1$ or 0, as well as the amplitudes with the shift in the external line, and verified that on-shell they satisfy the generalized Ward identity shown in Fig. 2. Finally, we validated the code on the example of general relativity and reproduced the standard results [39]. The success of these tests makes us confident that the code works correctly.

The resulting expressions for the amplitudes are very long and are available in the form of a *Mathematica* file [28]. Similar to general relativity [39], they can be cast into a sum of terms representing various contractions of the polarization tensors with the external momenta, multiplied by scalar functions of momenta and energies. However, the variety of structures in our case is richer due to the presence of higher powers of momenta (higher spatial derivatives) in the vertices. In particular, we obtain terms containing six and eight momenta contracted with the polarizations, such as e.g.,

$$\begin{gathered}(\mathbf{k}_3\varepsilon_1\varepsilon_2\mathbf{k}_4)(\mathbf{k}_1\varepsilon_3\mathbf{k}_1)(\mathbf{k}_2\varepsilon_4\mathbf{k}_2),\\ (\mathbf{k}_2\varepsilon_1\mathbf{k}_2)(\mathbf{k}_1\varepsilon_2\mathbf{k}_1)(\mathbf{k}_4\varepsilon_3\mathbf{k}_4)(\mathbf{k}_3\varepsilon_4\mathbf{k}_3),\end{gathered} \tag{4.3}$$

where we have used condensed notations

$$\begin{gathered}(\mathbf{k}_1\varepsilon_3\mathbf{k}_1) = k_1^i\varepsilon_{3ij}k_1^j,\\ (\mathbf{k}_3\varepsilon_1\varepsilon_2\mathbf{k}_4) = k_3^i\varepsilon_{1ij}\varepsilon_{2jk}k_4^k, \text{ etc.}\end{gathered} \tag{4.4}$$

We have not been able to reduce the structures (4.3) to those with fewer momenta by using the momentum conservation or other identities.

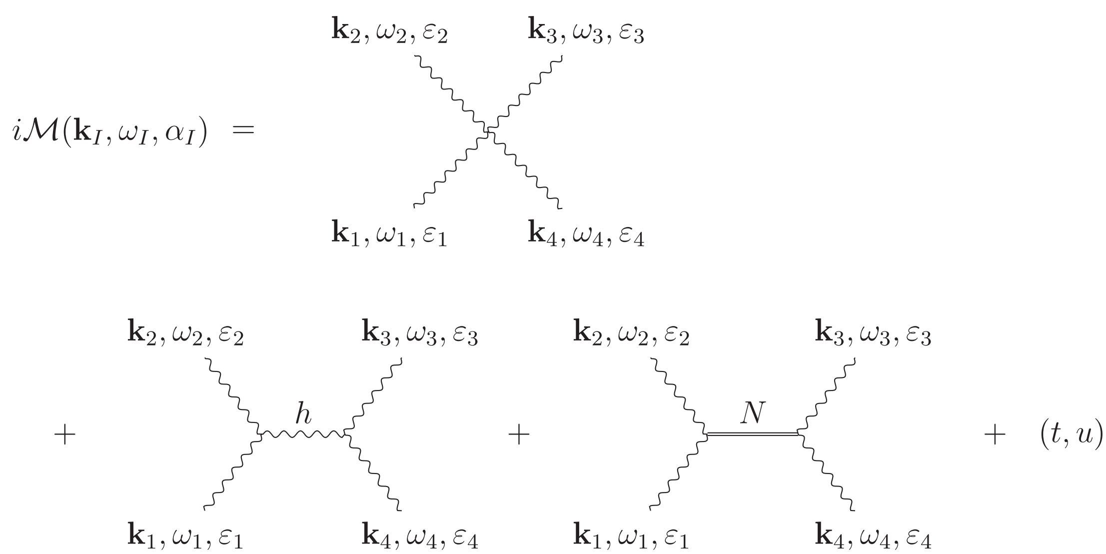


FIG. 3. Feynman diagrams for $2 \to 2$ scattering of gravitons at tree level. Wavy lines represent an external leg or propagator of the metric h_{ij}, and the double line is the propagator of the shift N_i. All momenta and energies are incoming; (t,u) stands for the diagrams with permutations $(\mathbf{k}_2,\omega_2,\varepsilon_2) \leftrightarrow (\mathbf{k}_3,\omega_3,\varepsilon_3)$ and $(\mathbf{k}_2,\omega_2,\varepsilon_2) \leftrightarrow (\mathbf{k}_4,\omega_4,\varepsilon_4)$.

The coefficient functions multiplying the aforementioned structures depend on the scalar invariants of the momenta—their absolute values k_I, $I = 1, 2, 3, 4$, and scalar products. We express the latter through the "Mandelstam-like" variables

$$S = (\mathbf{k}_1 + \mathbf{k}_2)^2, \quad T = (\mathbf{k}_1 + \mathbf{k}_3)^2, \quad U = (\mathbf{k}_1 + \mathbf{k}_4)^2. \quad (4.5)$$

Note that as a consequence of momentum conservation these variables obey the identity

$$S + T + U = k_1^2 + k_2^2 + k_3^2 + k_4^2. \quad (4.6)$$

The energy-conservation is implemented by using a set of three independent combinations,

$$\Omega_S = \omega_1 + \omega_2, \quad \Omega_U = \omega_1 + \omega_3, \quad \Omega_T = \omega_1 + \omega_4. \quad (4.7)$$

In this way we arrive at the coefficient functions depending on ten variables k_1, k_2, k_3, k_4, S, T, U, Ω_S, Ω_T, Ω_U related by the constraint (4.6). We keep this form and simplify the expressions as much as possible, without using the dispersion relations until the very last step. The reason for this strategy is twofold. First, it allows us to easily switch between physical and gauge modes in order to verify the cancellation (3.28). Second, the dispersion relations introduce nonanalyticity [square roots of the coefficients in Eqs. (2.20)–(2.22)] which complicate the manipulation of the formulas. The price to pay is that the off-shell amplitudes preserve the dependence on the gauge parameters σ, ξ. This dependence disappears once we put the amplitudes on-shell and assign physical polarizations to the particles.

B. Head-on collisions

The expressions for the amplitudes greatly simplify in the special case of head-on collisions when the momenta of two colliding particles are opposite in direction and equal in magnitude.[12] In more detail, we choose the particle momenta and energies to be

$$\mathbf{k}_1 = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}, \qquad \mathbf{k}_2 = \begin{pmatrix} 0 \\ 0 \\ -k \end{pmatrix},$$
$$\mathbf{k}_3 = \begin{pmatrix} -k' \sin\theta \\ 0 \\ -k' \cos\theta \end{pmatrix}, \qquad \mathbf{k}_4 = \begin{pmatrix} k' \sin\theta \\ 0 \\ k' \cos\theta \end{pmatrix}, \quad (4.8a)$$

$$\omega_1 = \sqrt{\nu_{(1)}} k^3, \qquad \omega_2 = \sqrt{\nu_{(2)}} k^3,$$
$$\omega_3 = -\sqrt{\nu_{(3)}} k'^3, \qquad \omega_4 = -\sqrt{\nu_{(4)}} k'^3, \quad (4.8b)$$

where $\nu_{(I)} = \nu_5$ or ν_s, depending on the type of the physical graviton—tensor or scalar. The final momentum k' is determined by the energy conservation,

$$(\sqrt{\nu_{(1)}} + \sqrt{\nu_{(2)}}) k^3 = (\sqrt{\nu_{(3)}} + \sqrt{\nu_{(4)}}) k'^3 \equiv E. \quad (4.9)$$

Note that the physical momenta of the final particles 3 and 4 are $-\mathbf{k}_3$ and $-\mathbf{k}_4$ and thus θ is the scattering angle defined in the usual way as the angle between the directions of incoming particle 1 and outgoing particle 3.

The amplitude depends on the polarizations of the particles $\alpha_I = \pm 2$ or $0'$ which we will write as $+, -, s$ for short.[13] We find it more convenient for the discussion of the physical properties of the amplitudes in this section to label them with the *physical* polarizations, i.e. upon performing the crossing for final particles. In these notations, the amplitude $\mathcal{M}_{++,++}$ stands for elastic scattering of two right-handed gravitons, the amplitude $\mathcal{M}_{++,+-}$ describes a process where one right-handed graviton flips helicity, etc.

We find that the helicity amplitudes have the form,

$$\mathcal{M}_{\alpha_1\alpha_2,\alpha_3\alpha_4} = G E^2 f_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\cos\theta; u_s, v_a, \lambda), \quad (4.10)$$

where

$$u_s = \sqrt{\frac{\nu_s}{\nu_5}}, \qquad v_a = \frac{\nu_a}{\nu_5}, \quad a = 1, 2, 3, \quad (4.11)$$

are the essential couplings of the theory introduced in [6]. They are singled out by the requirement that their RG running is independent of the gauge choice (this is not true for ν_a individually). Note that the gravitational coupling G multiplying the overall amplitude is not essential, implying that its RG improvement depends on the gauge. This is not a problem since the amplitude is not directly observable. We will say more about this shortly.

The functions $f_{\alpha_1\alpha_2,\alpha_3\alpha_4}$ describing the angular dependence of the amplitudes are listed in Appendix D. They are rational functions of $\cos\theta$. Many of them have singularities in the forward scattering limit, as is typical in theories with massless particles. The strongest singularity is featured by elastic amplitudes with $\alpha_1 = \alpha_3$, $\alpha_2 = \alpha_4$ which behave as $\sim\theta^{-6}$ at small θ. On the other hand, the helicity violating amplitude $f_{++,--}$ is regular at all angles and is much

[12]In relativistic theories any collision can be brought to the head-on kinematics by a boost to the center-of-mass frame. This is not possible in HG.

[13]A technical remark: The overall phases of the polarization vectors $e_i^{(\pm 1)}$ defined in (A8) and used to construct the graviton polarization tensors are ambiguous for particle 2 moving in the direction opposite the 3d axis. We set the phases to 0 which renders all amplitudes real.

simpler than the elastic amplitudes, though, in contrast to general relativity, it does not vanish completely.

Notably, when the dispersion relations of the transverse traceless and scalar gravitons do not coincide ($u_s \neq 1$), the amplitudes involving both types of particles have poles at nonzero angles. They arise in t- and u- channels and are a consequence of the fact that in HG a single graviton is kinematically allowed to decay into a pair of gravitons with lower energies. Thus, whenever, say, $\omega_3 \neq \omega_1$, the propagator in the t-channel diagram can go on-shell. For the head-on collisions this is possible only if particles participating in the process are of different types. For more general kinematics, we expect these resonant poles to occur also in $2 \to 2$ amplitudes for identical particles and in all three s, t, u channels.

One more peculiarity of amplitudes involving both tensors and scalars can be illustrated on the example of $f_{++,+s}$. When $u_s \neq 1$, this amplitude is finite in the forward and backward limits and in fact vanishes in the way consistent with the conservation of angular momentum. The incoming state has zero projection of the angular momentum on the 3d axis. On the other hand, for the final state the projection of the graviton spin becomes $+2$ or -2 for $\theta \to 0$ or π. This means that two units of angular momentum must be carried away by the orbital wave function implying a d-wave scattering. This leads to suppression θ^2 and $(\pi - \theta)^2$ in the two limits, respectively, which we indeed obtain from the direct computation, cf. Eq. (D8). By contrast, in the case $u_s = 1$ we recover the collinear singularities, which are only partially compensated by the d-wave factors, see Eq. (D9). A similar pattern emerges for other amplitudes. More details on their angular dependence can be found in Appendix D.

The quadratic growth of the amplitudes with energy is the same as in general relativity where it is known to contradict the tree-level unitarity. Nevertheless, it is consistent with unitarity in theories with the Lifshitz scaling [40]. It is instructive to derive the cross section corresponding to the amplitude (4.10). We define the cross section σ in the standard way, through the number of collisions happening in a unit of time and volume in the intersection of two beams of particles with number densities n_1, n_2:

$$\frac{dN_{\rm coll}}{dtdV} = \sigma n_1 n_2 \mathrm{v}_{\rm rel}, \tag{4.12}$$

where

$$\mathrm{v}_{\rm rel} = |\mathbf{v}_1 - \mathbf{v}_2| = \left|\frac{d\omega_1}{d\mathbf{k}_1} - \frac{d\omega_2}{d\mathbf{k}_2}\right| \tag{4.13}$$

is the relative group velocity of colliding particles. Following the usual steps, we obtain the standard expression

$$\sigma = \frac{1}{4\omega_1\omega_2\mathrm{v}_{\rm rel}} \int \frac{d^3k_3}{(2\pi)^3 2\omega_3} \frac{d^3k_4}{(2\pi)^3 2\omega_4} |\mathcal{M}|^2 (2\pi)^4 \times \delta\Big(\sum \omega_I\Big)\delta\Big(\sum \mathbf{k}_I\Big). \tag{4.14}$$

Let us for simplicity focus on the case when all particles participating in the scattering are transverse traceless gravitons—the results for other cases are similar. Performing integration over the phase space and expressing the energy and relative velocity through the absolute value of graviton's momentum, $E = 2\sqrt{\nu_5}k^3$, $\mathrm{v}_{\rm rel} = 6\sqrt{\nu_5}k^2$, we arrive at the differential cross section

$$\frac{d\sigma_{\alpha_1\alpha_2,\alpha_3\alpha_4}}{\sin\theta d\theta} = \frac{G^2}{72\pi\nu_5 k^2} |f_{\alpha_1\alpha_2,\alpha_3\alpha_4}|^2. \tag{4.15}$$

We observe that the cross section at fixed angle decreases as the square of the inverse momentum (de Broglie wavelength squared) which is a typical behavior in weakly coupled local theories compatible with unitarity. On the other hand, the total cross section diverges at small angles signaling the necessity of an infrared regulator.

The cross section (4.15) is proportional to the square of the essential coupling [6]

$$\mathcal{G} = \frac{G}{\sqrt{\nu_5}}. \tag{4.16}$$

Also, as already noted, $f_{\alpha_1\alpha_2,\alpha_3\alpha_4}$ depends only on essential couplings. This is reassuring. In contrast to the amplitude, the cross section is a physical observable and its RG improvement must be gauge invariant. We see that this is indeed the case.

V. THE LIMIT $\lambda \to \infty$

It was conjectured in [21] that projectable Hořava gravity can have a regular limit at $\lambda \to \infty$. This is supported by the regularity of the dispersion relation for physical transverse traceless and scalar gravitons, Eqs. (2.20), (2.22), and by the regularity of the one-loop β-functions for the essential couplings [20]. This limit is interesting since it corresponds to a likely behavior of the theory in the deep UV [20,21]. In this section we discuss evidence for its regularity from the scattering amplitudes' perspective. We then prove the above conjecture by recasting the $\lambda \to \infty$ theory in a manifestly regular form.

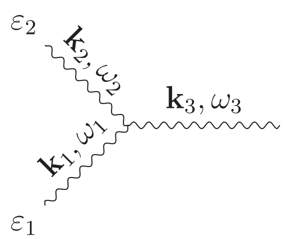
A. Cancellation of enhanced terms in σ, ξ gauge

A scrutiny of the expressions in Appendix D for the head-on scattering amplitudes between physical states shows that they are regular in the limit (1.2). Using our symbolic code, we have checked that this property holds also for arbitrary kinematics. This is nontrivial. Indeed, interaction vertices contain contributions proportional to λ. Thus, the amplitudes given by the diagrams in Fig. 3 could, *a priori*, contain terms as large as $O(\lambda^2)$. It is instructive to study how these large contributions cancel.

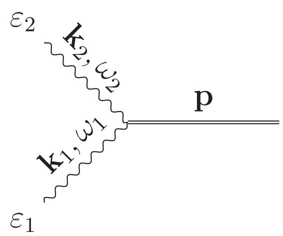
We start by observing that the polarizations of physical states are traceless in the limit (1.2). This is, of course, always true for the helicity ± 2 gravitons, whereas for the scalar graviton we obtain from Eqs. (A10)

$$\varepsilon^{0\prime}_{ij}(\mathbf{k}) = \sqrt{\frac{2}{3}}(\delta_{ij} - 3\hat{k}_i\hat{k}_j) + O(\lambda^{-1}). \tag{5.1}$$

This removes many terms in the contraction of the interaction vertices with the polarization tensors. Let us first consider the building blocks involving a cubic vertex and two graviton external legs. Using Eqs. (c6a), (c6b) we get



$$h_{mn} \quad = -i\frac{\lambda}{4}(\varepsilon_1\varepsilon_2)(\omega_1+\omega_2)\,\omega_3\delta_{mn} + O(\lambda^0)\,, \tag{5.2a}$$



$$N_m \quad = -i\frac{\lambda}{2}(\varepsilon_1\varepsilon_2)(\omega_1+\omega_2)\,p_m + O(\lambda^0)\,, \tag{5.2b}$$

where $(\varepsilon_1\varepsilon_2) \equiv \varepsilon_{1ij}\varepsilon_{2ij}$ and we have taken into account the symmetry factors 3! and 2! for the two diagrams, respectively, as well as a factor $\sqrt{G}$ for each external leg.

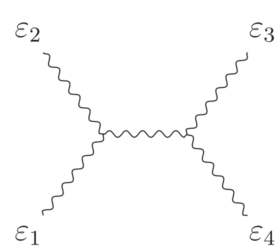
Next, the contraction of the graviton propagator (C3) with δ_{mn} yields

$$\delta_{mn}\cdot\ h_{mn} \overset{\mathbf{k},\omega}{\sim\!\sim\!\sim} h_{pq} \ = -\frac{4G}{3\lambda}\mathcal{P}_s\big(\delta_{pq} - 3\hat{k}_p\hat{k}_q\big) + O(\lambda^{-2})\,. \tag{5.3}$$

In deriving this expression we have used the limiting form of the longitudinal mode pole factor,

$$\mathcal{P}_0 = \frac{i}{\omega^2 - \frac{(1-\lambda)(1+\xi)}{\sigma}k^6} = i\frac{\sigma}{\lambda(1+\xi)k^6} + O(\lambda^{-2}). \tag{5.4}$$

Note that the first term in (5.3) is again traceless and vanishes when contracted with δ_{pq}. Combining this with Eq. (5.2a) we conclude that for the physical states the diagram

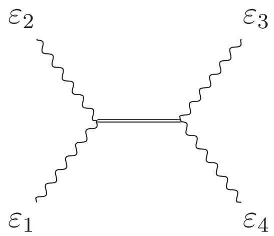


is $O(\lambda^0)$, i.e. it is finite in the limit (1.2).

Consider now the diagram with the exchange of the shift. Here we have from Eq. (C3a) for the propagator:

$$p_m \cdot \; N_m \overset{\mathbf{p}}{=\!=\!=\!=} N_n \; = -i\frac{G}{\lambda}\frac{\hat{p}_n}{p} + O(\lambda^{-2}) \,, \tag{5.5}$$

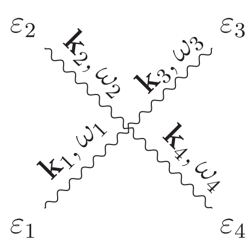
where we have again used the limiting form (5.4). Combining with Eq. (5.2b), we find a $O(\lambda)$ contribution,



$$= -i\frac{G\lambda}{4}(\varepsilon_1\varepsilon_2)(\varepsilon_3\varepsilon_4)(\omega_1+\omega_2)(\omega_3+\omega_4) + O(\lambda^0) \,. \tag{5.6}$$

Note the minus sign in this expression which comes from the fact that the momentum in the propagator is inflowing into one vertex and outflowing from the other. Similar contributions with exchange of particles $(2 \leftrightarrow 3)$ and $(2 \leftrightarrow 4)$ come from the t and u channels.

These $O(\lambda)$ contributions are precisely canceled by the diagram with the 4-point vertex. Indeed, contraction with the traceless polarizations leaves only terms in the third line in Eq. (c6c) which upon symmetrization read



$$\begin{aligned} &= i\frac{G\lambda}{4}\Big[(\varepsilon_1\varepsilon_2)(\varepsilon_3\varepsilon_4)(\omega_1+\omega_2)(\omega_3+\omega_4) \\ &\quad + (\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_4)(\omega_1+\omega_3)(\omega_2+\omega_4) \\ &\quad + (\varepsilon_1\varepsilon_4)(\varepsilon_3\varepsilon_2)(\omega_1+\omega_4)(\omega_3+\omega_2)\Big] + O(\lambda^0) \,. \end{aligned} \tag{5.7}$$

Thus, we have confirmed explicitly the cancellation of dangerous contributions to the amplitudes in the limit $\lambda \to \infty$. It relies on a rather delicate interplay between the tracelessness of the physical polarizations and the structure of the vertices and propagators.

B. Regular limit with an auxiliary field

Encouraged by the previous results, we look for a way to cast the action of HG in the form which would be manifestly regular at $\lambda \to \infty$. This is indeed possible to do by integrating in an auxiliary nondynamical scalar field χ and rewriting the λ-term in the Lagrangian as

$$-\frac{\lambda}{2G}\sqrt{\gamma}K^2 \to \frac{\sqrt{\gamma}}{G}\left[-\chi K + \frac{\chi^2}{2\lambda}\right]. \tag{5.8}$$

Clearly, at finite λ the two forms of the theory are equivalent, since we can always integrate out χ and restore the original action. On the other hand, in the new form we can easily take the limit (1.2) and get for the action of HG,

$$S \underset{\lambda\to\infty}{\longrightarrow} S' = \frac{1}{2G}\int d^3x dt \sqrt{\gamma}(K_{ij}K^{ij} - 2\chi K - \mathcal{V}). \tag{5.9}$$

We see that the field χ takes the role of a Lagrange multiplier constraining the extrinsic curvature to be traceless, $K = 0$. Note that the new action is still invariant under Lifshitz scaling (1.1) if we assign $\dim \chi = 3$.

Quantization of theories with Lagrange multipliers is in general subtle. We need to make sure that the propagators of all the fields, including χ, are well defined and the theory can be perturbatively quantized. We also want to preserve renormalizability. For this, it will suffice to have a gauge choice which renders all propagators *regular* [17,41]. In real-time signature adopted here the regularity condition is formulated as follows: A propagator $\langle\Phi_1\Phi_2\rangle$ of two fields Φ_1, Φ_2 with scaling dimensions r_1, r_2 is regular if it decomposes into a sum of terms of the form

$$\frac{P(\mathbf{k},\omega)}{D(\mathbf{k},\omega)}, \tag{5.10a}$$

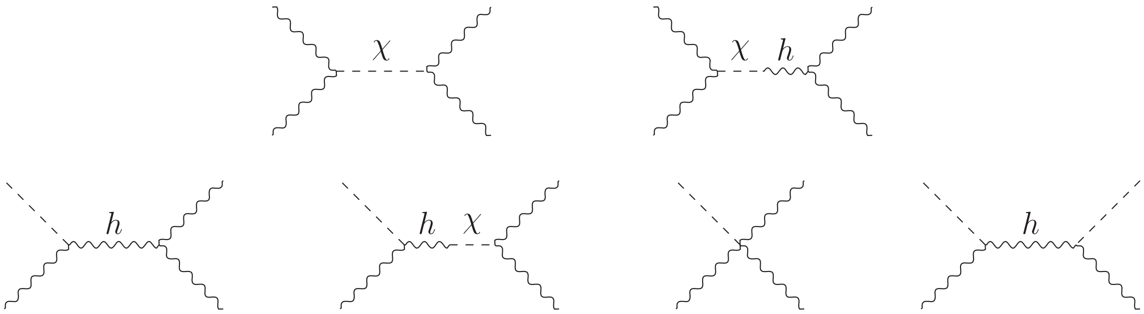


FIG. 4. Additional diagrams for the graviton scattering in the theory (5.9) describing the $\lambda = \infty$ limit of projectable Hořava gravity. The diagrams in the first row contribute to the amplitudes for the helicity ± 2 states, and the diagrams in the second row must be further added for scattering of scalar gravitons.

where D is a product of monomials,

$$D = \prod_{m=1}^{M} (A_m \omega^2 - B_m k^6 + i\epsilon) \qquad (5.10b)$$

with strictly positive coefficients A_m, B_m, and $P(\mathbf{k}, \omega)$ is a polynomial of scaling degree less than or equal to $r_1 + r_2 + 6(M - 1)$.

We cannot use the functions F^i from Eq. (2.14) for gauge fixing since they contain terms proportional to λ that preclude setting $\lambda = \infty$. Then it appears impossible to design a gauge that would eliminate the quadratic mixing between the scalar parts of the metric h_{ij}, the shift N_i and χ. Thus, we just pick up a gauge compatible with the scaling and disentangling at least the helicity ± 1 parts:

$$\tilde{F}^i = \dot{N}^i + \frac{1}{2} O^{ij} \partial_k h_j^k, \qquad (5.11)$$

with the same operator O^{ij}, as in Eq. (2.14). The propagators in this gauge are derived in Appendix E. We obtain many off-diagonal propagators between h_{ij}, N_i and χ which make practical calculations rather cumbersome. Most importantly, however, all these propagators are regular in the above sense guaranteeing the perturbative renormalizability of the theory with the χ field. In particular, this implies that no terms[14] with gradients or time derivatives of the field χ are generated by quantum corrections and χ remains nondynamical.

To check the equivalence between the $\lambda \to \infty$ limit of the original formulation of HG and the action (5.9), we have computed the graviton scattering amplitudes directly with the Feynman rules following from (5.9). To avoid, proliferation of diagrams, we fix one of the gauge parameters,[15] $\xi = -1$. This eliminates the off-diagonal propagators involving the shift N_i, as well as the overlap of the shift with the scalar graviton state (see Appendix E). On the other hand, the mixing between the metric and χ still remains, implying that we need to include diagrams with internal, and for scalar gravitons, external, χ-lines. This gives us the set of new diagrams shown in Fig. 4 which must be added to those of Fig. 3, with all possible permutations of the external states. Note that the h^3, h^4 and $h^2 N$ vertices for this new calculation can be obtained from the expressions used in Sec. IV by simply dropping the parts containing λ. At the same time we have new cubic and quartic vertices with a χ-line giving rise to diagrams in Fig. 4.

We have evaluated the amplitudes for the physical transverse-traceless and scalar gravitons in the χ-theory using our code and found that they exactly coincide with the $\lambda \to \infty$ limit of the amplitudes computed with the original HG action. This confirms that the action (5.9) correctly captures the dynamics of HG at $\lambda \to \infty$.

All in all, we conclude that the limit (1.2) of projectable HG is regular and is described by the action (5.9).

VI. CONCLUSIONS

In this paper we computed tree-level scattering amplitudes in projectable HG in $(3 + 1)$ dimensions. For this purpose, we developed a symbolic computer code which can be found at [28]. We focused on the high-energy behavior of the theory keeping only marginal interactions with respect to Lifshitz scaling with $z = 3$.

We started by deriving the Ward identities for the amplitudes which we used to cross-check our computation. Our approach is based on the BRST quantization and is not restricted to HG. We illustrated it on the case of a Yang–Mills theory with Lifshitz scaling. To the best of our knowledge, this is the first derivation of Ward identities in nonrelativistic gauge theories.

We next discussed the general structure of the HG scattering amplitudes and presented explicit results for the case of head-on collisions, i.e. collisions with vanishing total momentum. The amplitudes have peculiar dependence on the scattering angle. Their dependence on the collision

[14] Such terms would be irrelevant by Lifshitz power counting.

[15] This special choice spoils the regularity of the propagators in the above sense: The pole term corresponding to the helicity-0 gauge mode becomes $\tilde{\mathcal{P}}_0 = i(\omega^2 + i\epsilon)^{-1}$, i.e. it does not depend on the spatial momentum. This, however, is not a problem for the tree-level calculation where no possible divergences associated with this behavior can arise.

energy is compatible with tree-level unitarity. In particular, the differential cross section decreases as the square of the colliding particles' momentum, as it should be for a theory weakly coupled in UV.

We found that the amplitudes remain finite in the limit when the coupling constant λ in the kinetic term of the Lagrangian is taken to infinity. We have further reformulated the action of the theory in the form which is manifestly regular at $\lambda \to \infty$ and checked that it reproduces the same scattering amplitudes. This establishes the $\lambda \to \infty$ limit as a viable location for asymptotically free UV fixed points [20].

Our research opens several directions. The tree amplitudes that we computed have analytic properties quite similar to those in relativistic theories: They have poles corresponding to physical particles in the internal propagators, feature soft and collinear singularities, etc. It would be interesting to understand if these properties can be exploited in adapting to HG the powerful on-shell methods developed for relativistic gauge theories and gravity [42]. An obvious missing ingredient is the spinor-helicity formalism which relies on Lorentz invariance. Whether an adequate substitute for it exists in nonrelativistic theories is an open question.

Another possible extension of our is the study of amplitudes beyond tree level. On top of the usual issues associated with infrared divergences, which are also present in relativistic context, such study will have to face several new challenges. To see them consider a single tensor or scalar graviton with the dispersion relation (2.20) or (2.22). Energy and momentum conservation allow it to decay into two or more gravitons of lower energy. This implies absence of any stable asymptotic states, thus undermining the standard assumptions used in the definition of the $\mathcal{S}$-matrix. Hopefully, this problem can be overcome by adapting the methods used in relativistic theories to describe scattering of metastable particles. Another peculiarity of HG gravity and nonrelativistic theories in general is that the parameters entering into particles' dispersion relations receive loop corrections and exhibit RG running. The definition of the asymptotic states must take these corrections into account order by order in the loop expansion, which further challenges the standard construction of the $\mathcal{S}$-matrix.

Having established good behavior of the projectable HG in UV, our work motivates revisiting its low-energy properties. It is known [2,29] that Minkowski background in this theory suffers from a tachyon-like instability associated with the scalar graviton mode. It is important to understand the fate of this instability. Can it lead to a new phase of the theory which could be phenomenologically viable? We plan to address this question in the future.

Finally, it will be interesting to apply the amplitude-based approach developed in this work to the nonprojectable version of HG where it can provide valuable information about the UV properties of the theory.

ACKNOWLEDGMENTS

We thank Andrei Barvinsky, Diego Blas, Alexander Kurov, Maxim Pospelov and Oriol Pujolas for useful discussions. The work is supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Colleges and Universities.

APPENDIX A: HELICITY DECOMPOSITION

In this appendix we diagonalize the quadratic Lagrangian (2.18) and summarize the relations obeyed by particle creation-annihilation operators. We start by splitting the fields into tensor, vector and scalar parts,

$$h_{ij} = \zeta_{ij} + \partial_i v_j + \partial_j v_i + \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right)\psi + \frac{\partial_i \partial_j}{\Delta} E, \quad \text{(A1a)}$$

$$N_i = u_i + \partial_i B, \quad c_i = w_i + \partial_i C, \quad \bar{c}_i = \bar{w}_i + \partial_i \bar{C}, \quad \text{(A1b)}$$

where the components satisfy

$$\partial_i \zeta_{ij} = \zeta_{ii} = \partial_i v_i = \partial_i u_i = \partial_i w_i = \partial_i \bar{w}_i = 0. \quad \text{(A2)}$$

The Lagrangian separates into contributions of different sectors:

$$\mathcal{L}_q^{(2t)} = \frac{1}{2G}\left\{\frac{\dot{\zeta}_{ij}^2}{4} + \frac{\nu_5}{4}\zeta_{ij}\Delta^3\zeta_{ij}\right\}, \quad \text{(A3a)}$$

$$\mathcal{L}_q^{(2v)} = \frac{1}{2G}\left\{-\frac{1}{2}\dot{v}_i\Delta\dot{v}_i - \frac{1}{4\sigma}v_i\Delta^4 v_i - \dot{u}_i\frac{\sigma}{\Delta^2}\dot{u}_i - \frac{1}{2}u_i\Delta u_i + 2\dot{\bar{w}}_i\dot{w}_i + \frac{1}{\sigma}\bar{w}_i\Delta^3 w_i\right\}, \quad \text{(A3b)}$$

$$\mathcal{L}_q^{(2s)} = \frac{1}{2G}\left\{\frac{1-2\lambda}{2}\dot{\psi}^2 - \lambda\dot{E}\dot{\psi} + \frac{1-\lambda}{4}\dot{E}^2 + \left(\frac{8\nu_4 + 3\nu_5}{2} + \frac{\lambda^2(1+\xi)}{\sigma}\right)\psi\Delta^3\psi - \frac{\lambda(1-\lambda)(1+\xi)}{\sigma}E\Delta^3\psi + \frac{(1-\lambda)^2(1+\xi)}{4\sigma}E\Delta^3 E + \dot{B}\frac{\sigma}{(1+\xi)\Delta}\dot{B} + (1-\lambda)B\Delta^2 B - 2\dot{\bar{C}}\Delta\dot{C} - \frac{2(1-\lambda)(1+\xi)}{\sigma}\bar{C}\Delta^4 C\right\}. \quad \text{(A3c)}$$

The scalar part still contains mixing between the ψ and E components, which is removed by the change of variables,

$$E \mapsto \tilde{E} = E - \frac{2\lambda}{1-\lambda}\psi. \tag{A4}$$

The final Lagrangian in this sector reads

$$\mathcal{L}_q^{(2\psi\tilde{E})} = \frac{1}{2G}\left\{\frac{1-3\lambda}{2(1-\lambda)}\dot{\psi}^2 + \frac{8\nu_4+3\nu_5}{2}\psi\Delta^3\psi + \frac{1-\lambda}{4}\dot{\tilde{E}}^2 + \frac{(1-\lambda)^2(1+\xi)}{4\sigma}\tilde{E}\Delta^3\tilde{E}\right\}. \tag{A5}$$

Note that the positivity of the kinetic term for the gauge invariant scalar ψ requires λ to be outside the range $1/3 \leq \lambda \leq 1$. From Eqs. (A3), (A5) we read off the dispersion relations (2.20), (2.21), (2.22) quoted in the main text.

Collecting the helicity modes together, we obtain the expressions for the local fields which we write in the form

$$h_{ij}(\mathbf{x},t) = \sqrt{G}\int\frac{d^3k}{(2\pi)^3}\sum_\alpha \frac{\varepsilon_{ij}^\alpha(\mathbf{k})}{2\omega_{\mathbf{k}\alpha}}h_{\mathbf{k}\alpha}e^{-i\omega_{\mathbf{k}\alpha}t+i\mathbf{kx}} + \text{H.c.}, \tag{A6a}$$

$$N_i(\mathbf{x},t) = \sqrt{G}\int\frac{d^3k}{(2\pi)^3}\sum_\alpha \frac{\epsilon_i^\alpha(\mathbf{k})}{2\omega_{\mathbf{k}\alpha}}N_{\mathbf{k}\alpha}e^{-i\omega_{\mathbf{k}\alpha}t+i\mathbf{kx}} + \text{H.c.}, \tag{A6b}$$

$$c_i(\mathbf{x},t) = \sqrt{G}\int\frac{d^3k}{(2\pi)^3}\sum_\alpha \frac{e_i^\alpha(\mathbf{k})}{2\omega_{\mathbf{k}\alpha}}c_{\mathbf{k}\alpha}e^{-i\omega_{\mathbf{k}\alpha}t+i\mathbf{kx}} + \text{H.c.}, \tag{A6c}$$

$$\bar{c}_i(\mathbf{x},t) = \sqrt{G}\int\frac{d^3k}{(2\pi)^3}\sum_\alpha \frac{e_i^\alpha(\mathbf{k})}{2\omega_{\mathbf{k}\alpha}}\bar{c}_{\mathbf{k}\alpha}e^{-i\omega_{\mathbf{k}\alpha}t+i\mathbf{kx}} - \text{H.c.}, \tag{A6d}$$

where the sum runs over the helicities α contained in the corresponding field [see Eq. (2.19)]. Note that the ghosts c_i are taken to be Hermitian, whereas the antighosts $\bar{c}_i$ are anti-Hermitian. The former property is needed for the Hermiticity of the BRST operator, whereas the latter then follows from the Hermiticity of the Lagrangian.

We normalize the mode coefficients in such a way that upon quantization they become the annihilation-creation operators with the commutation relations:

$$[h_{\mathbf{k}\alpha}, h^+_{\mathbf{k}'\beta}] = 2\omega_{\mathbf{k}\alpha}\delta_{\alpha\beta}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')[\text{sign}(1-\lambda)]^{\delta_{\alpha 0}}, \tag{A7a}$$

$$[N_{\mathbf{k}\alpha}, N^+_{\mathbf{k}'\beta}] = -2\omega_{\mathbf{k}\alpha}\delta_{\alpha\beta}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')[\text{sign}(1-\lambda)]^{\delta_{\alpha 0}}, \tag{A7b}$$

$$[c_{\mathbf{k}\alpha}, \bar{c}^+_{\mathbf{k}'\beta}]_+ = [\bar{c}_{\mathbf{k}\alpha}, c^+_{\mathbf{k}'\beta}]_+ = -2\omega_{\mathbf{k}\alpha}\delta_{\alpha\beta}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'). \tag{A7c}$$

Two comments are in order. First note that we use the "relativistic" normalization including a factor 2ω for the operators and corresponding scattering states. Though in our case it is not connected with Lorentz invariance, it is still convenient since it results in dimensionless $2\to 2$ scattering amplitudes. Second, the helicity ± 1 modes of the shift N_i clearly have negative norm. In the helicity 0 sector the situation is subtler. Here the negative-norm state is in N_i or h_{ij}, depending on whether λ is less or bigger than 1, as reflected by the last factor in Eqs. (A7a), (A7b).

It remains to specify the polarization vectors and tensors entering Eqs. (A7). Let us start with the ghosts. Their polarization vectors are given by the standard orthonormal triad which for the momentum with polar and azimuthal angles θ, ϕ has the form

$$e_i^{(0)} \equiv \hat{k}_i = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix},$$
$$e_i^{(\pm1)} = \mp\frac{e^{\pm i\phi}}{\sqrt{2}}\begin{pmatrix}\cos\theta\cos\phi \mp i\sin\phi\\ \cos\theta\sin\phi \pm i\cos\phi\\ -\sin\theta\end{pmatrix}. \tag{A8}$$

The polarizations in N_i differ by normalizations that can be read out of the Lagrangians (A3b), (A3c):

$$\epsilon_i^{(\pm1)} = \frac{k^2}{\sqrt{\sigma}}e_i^{(\pm1)}, \qquad \epsilon_i^{(0)} = k^2\sqrt{\frac{|1+\xi|}{\sigma}}\hat{k}_i. \tag{A9}$$

Finally, the polarization tensors in h_{ij} are constructed from the triad as follows:

$$\varepsilon_{ij}^{(\pm2)} = 2e_i^{(\pm1)}e_j^{(\pm1)},$$
$$\varepsilon_{ij}^{(\pm1)} = \sqrt{2}(e_i^{(\pm1)}\hat{k}_j + \hat{k}_i e_j^{(\pm1)}), \tag{A10a}$$

$$\varepsilon_{ij}^{(0)} = \frac{2}{\sqrt{|1-\lambda|}}\hat{k}_i\hat{k}_j,$$
$$\varepsilon_{ij}^{(0')} = \sqrt{\frac{2(1-\lambda)}{1-3\lambda}}\left(\delta_{ij} - \frac{1-3\lambda}{1-\lambda}\hat{k}_i\hat{k}_j\right). \tag{A10b}$$

APPENDIX B: BRST-INVARIANCE OF THE $\mathcal{S}$-MATRIX

In this appendix we review the derivation of Eq. (3.2) stating that the $\mathcal{S}$-matrix of a gauge theory commutes with the *asymptotic* quadratic BRST operator $Q^{(2)}$. We follow Ref. [37] generalizing the analysis to an abstract gauge theory which need not enjoy Lorentz invariance. We adopt the conventions and notations of [18] [except for (anti) ghosts which we denote with c, instead of ω].

Consider a gauge theory with local gauge-invariant action S built out of gauge and matter fields φ^a, where the label a collectively denotes all field indices and coordinates. The fields linearly transform under the action of the gauge group via

$$\delta_\varepsilon \varphi^a = \varepsilon^\alpha (P^a{}_\alpha + R^a{}_{b\alpha}\varphi^b), \tag{B1}$$

where ε^α is the transformation parameter. The gauge fields are supplemented by the Faddeev-Popov ghosts c^α, antighosts $\bar{c}_\alpha$, and the Nakanishi–Lautrup field b_α, related by the BRST transformations,

$$\mathbf{s}\varphi^a = c^\alpha (P^a{}_\alpha + R^a{}_{b\alpha}\varphi^b), \qquad \mathbf{s}c^\alpha = \frac{1}{2} C^\alpha_{\beta\gamma} c^\beta c^\gamma,$$
$$\mathbf{s}\bar{c}_\alpha = b_\alpha, \qquad \mathbf{s}b_\alpha = 0, \tag{B2}$$

where $C^\alpha{}_{\beta\gamma}$ are the structure constants of the gauge group. Implementing the BRST quantization procedure we obtain the quantum tree-level action S_q invariant under (B2),

$$S_q = S[\varphi] + b_\alpha \chi^\alpha_a \varphi^a - \frac{1}{2} b_\alpha O^{\alpha\beta} b_\beta - \bar{c}_\alpha \chi^\alpha_a (P^a{}_\beta + R^a{}_{b\beta}\varphi^b) c^\beta, \tag{B3}$$

where we have chosen linear gauge-fixing functions $\chi^\alpha_a \varphi^a$. Note that since the transformations (B2) are nonlinear, the conserved BRST charge Q generating them in the Heisenberg picture is nonlinear as well. However, instead of pursuing the operator quantization, we use the path integral approach.

We define the generating functional with sources for all the fields and their BRST variations:

$$Z[J,\bar{\xi},\xi,y,\gamma,\zeta] = \int D\Phi^A \exp\{i(S_q[\varphi,c,\bar{c},b] + J_a\varphi^a + \bar{\xi}_\alpha c^\alpha + \xi^\alpha \bar{c}_\alpha + y^\alpha b_\alpha + \gamma_a \mathbf{s}\varphi^a + \zeta_\alpha \mathbf{s}c^\alpha)\}, \tag{B4}$$

where Φ^A stands collectively for all the fields φ^a, c^α, $\bar{c}_\alpha$ and b_α. We further define the partition function (generating functional for the connected diagrams):

$$W = -i \log Z, \tag{B5}$$

and its Legendre transform—the effective action

$$\Gamma[\langle\varphi\rangle, \langle c\rangle, \langle\bar{c}\rangle, \langle b\rangle, \gamma, \zeta] = W - J_a\langle\varphi^a\rangle - \bar{\xi}_\alpha\langle c^\alpha\rangle - \xi^\alpha\langle c_\alpha\rangle - y^\alpha\langle b_\alpha\rangle, \tag{B6}$$

with the quantities in angular brackets denoting the *mean* fields. By definition, the latter are variational derivatives of W with respect to the sources,[16]

$$\langle\varphi^a\rangle = \frac{\delta W}{\delta J_a}, \qquad \langle c^\alpha\rangle = \frac{\delta W}{\delta \bar{\xi}_\alpha},$$
$$\langle\bar{c}_\alpha\rangle = \frac{\delta W}{\delta \xi^\alpha}, \qquad \langle b_\alpha\rangle = \frac{\delta W}{\delta y^\alpha}. \tag{B7}$$

Note that at tree level the effective action is

$$\Gamma^{\text{tree}} = S_q + \gamma_a \mathbf{s}\varphi^a + \zeta_\alpha \mathbf{s}c^\alpha. \tag{B8}$$

The relation (B6) implies the equality of the variational derivatives,

$$\frac{\delta\Gamma}{\delta\gamma_a} = \frac{\delta W}{\delta\gamma_a}. \tag{B9}$$

Importantly, the partition function satisfies the identities (see e.g., [18] for the derivation),

$$\mathbf{D}W \equiv \left(-J_a \frac{\delta}{\delta\gamma_a} + \bar{\xi}_\alpha \frac{\delta}{\delta\zeta_\alpha} + \xi^\alpha \frac{\delta}{\delta y^\alpha}\right) W = 0, \tag{B10a}$$

$$\left(\chi^\alpha_a \frac{\delta}{\delta J_a} - O^{\alpha\beta}\frac{\delta}{\delta y^\beta} + y^\alpha\right) W = 0. \tag{B10b}$$

The first equation here is the Slavnov-Taylor identity following from the BRST symmetry (B2), whereas the second is the equation of motion for the Nakanishi–Lautrup field.

We now use the Lehmann-Symanzik-Zimmermann (LSZ) reduction (see Ref. [43] for a recent discussion) to define the $\mathcal{S}$-matrix from the correlation functions. In a compact form, it can be written as (see e.g., [44])

$$\mathcal{S} =: \exp\left(-\Phi^A_{\text{as}} \mathcal{K}_{AB} \frac{\delta}{\delta J_B}\right): Z[\mathcal{J}]|_{\mathcal{J}=0} \equiv \mathbf{K}Z[\mathcal{J}]|_{\mathcal{J}=0}. \tag{B11}$$

Here $\Phi^A_{\text{as}} = \{\varphi^a_{\text{as}}, c^\alpha_{\text{as}}, \bar{c}_{\text{as}\,\alpha}\}$ are the asymptotic gauge and (anti)ghost field operators,[17] and $J_A = \{J_a, \bar{\xi}_\alpha, \xi^\alpha\}$ are the corresponding currents; we also denote by $\mathcal{J} = \{J_a, \bar{\xi}_\alpha, \xi^\alpha, \gamma_a, \zeta_\alpha, y^\alpha\}$ the currents supplemented with the BRST sources. Colon around the exponent stand for the normal ordering with respect to particle creation-annihilation operators contained inside Φ^A_{as}. The differential operator $\mathcal{K}_{AB}$ is taken from the wave equations satisfied by the asymptotic fields,

[16] We define the derivatives with respect to anticommuting variables as acting from the left, i.e. the differential of a function $f(\theta)$ of a Grassmann variable θ is $df = d\theta f'(\theta)$.

[17] We consider the asymptotic states as being generated by the free fields. This may not be true for a variety of reasons, such as infrared divergences or particle instability, see discussion in Sec. VI. We proceed under the assumption that these issues can be properly handled on the case-by-case basis.

In principle, one could also introduce the asymptotic Nakanishi–Lautrup field, but we choose not to do it since b_α is not an independent variable on-shell, being expressed through the gauge-fixing function.

$$\mathcal{K}_{AB}\Phi^B_{\rm as} = 0. \tag{B12}$$

Despite these equations, the exponent in (B11) is nontrivial because the operator $\mathcal{K}_{AB}$ in it acts to the right and cancels with the on-shell poles of the Green's functions produced by the variational derivatives with respect to the currents. The vertical line with subscript "$\mathcal{J}=0$" means that all sources must be set to zero *after* taking the variational derivatives.

In the second equality in (B11) we have introduced the notation $\mathbf{K}$ for the exponential factor acting on $Z[\mathcal{J}]$. This object is a "double operator": it is a variational operator acting on functionals of the currents, and a quantum-mechanical operator in the asymptotic Fock space. We observe that

$$[\mathbf{K},\mathbf{D}]W[\mathcal{J}]|_{\mathcal{J}=0} = \mathbf{K}\mathbf{D}W[\mathcal{J}]|_{\mathcal{J}=0} - \mathbf{D}\mathbf{K}W[\mathcal{J}]|_{\mathcal{J}=0} = 0. \tag{B13}$$

Indeed, the first term vanishes due to the Slavnov-Taylor identity (B10a), whereas the second term is zero because $\mathbf{D}$ is proportional to the sources. Evaluating $[\mathbf{K},\mathbf{D}]$ on the left-hand side as the commutator of two variational operators we obtain[18]

$$\begin{aligned}[\mathbf{K},\mathbf{D}]W[\mathcal{J}]|_{\mathcal{J}=0} &=: \mathbf{K}\cdot\left(\varphi^a_{\rm as}\mathcal{K}^\varphi_{ab}\frac{\delta}{\delta\gamma_b} - \bar{c}_{{\rm as}\,\alpha}(\mathcal{K}^c)^\alpha{}_\beta\frac{\delta}{\delta\zeta_\beta} + c^\alpha_{\rm as}(\mathcal{K}^c)^\beta{}_\alpha\frac{\delta}{\delta y^\beta}\right): \\ &\quad W[\mathcal{J}]|_{\mathcal{J}=0}.\end{aligned} \tag{B14}$$

Let us discuss the terms in brackets one by one, starting from the last. Using the relation (B10b) it can be transformed as

$$\frac{\delta W}{\delta y^\beta} = O^{-1}_{\beta\alpha}\chi^\alpha_a\frac{\delta W}{\delta J_a} + O^{-1}_{\beta\alpha}y^\alpha. \tag{B15}$$

The second term on the right-hand side does not contribute because upon acting with $\mathbf{K}$ it either leaves something proportional to y^α which is zero when we take currents to be zero, or, if the derivatives from $\mathbf{K}$ hit y^α instead of the generating functional, we are not getting poles from the Green's functions to compensate the action of $(\mathcal{K}^c)^\beta{}_\alpha$.

The second term in (B14) amounts to

$$\frac{\delta W}{\delta\zeta_\beta} = \langle \mathbf{s}c^\beta\rangle = \left\langle \frac{1}{2}C^\beta_{\gamma\delta}c^\gamma c^\delta + \dots\right\rangle \tag{B16}$$

with the dots representing corrections coming from renormalization. The diagrams contributing to this matrix element

[18] Note the different signs of the ghost and the antighost terms stemming from their anticommutativity: $\bar{c}_\alpha(\mathcal{K}^c)^\alpha{}_\beta c^\beta = -c^\beta(\mathcal{K}^c)^\alpha{}_\beta\bar{c}_\alpha$.

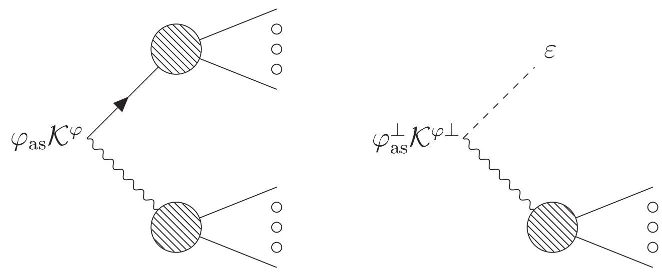


FIG. 5. Diagrams arising from the second terms in Eq. (B17) (left) and Eq. (B26) (right). They do not have on-shell poles to cancel the vanishing vertex factor $\varphi_{\rm as}\mathcal{K}^\varphi$.

do not have poles since there are no one-particle states with ghost number 2. Hence they vanish once we act on them by $\mathcal{K}^c$ and restrict on-shell.

The first term in Eq. (B14) requires a bit more work. Using Eqs. (B7), (B9) we can write a Taylor expansion,

$$\begin{aligned}\frac{\delta W}{\delta\gamma_b} &= \frac{\delta\Gamma}{\delta\gamma_b} \\ &= \frac{\delta^2\Gamma}{\delta\langle c^\alpha\rangle\delta\gamma_b}\bigg|_{\langle\Phi\rangle=0}\langle c^\alpha\rangle \\ &\quad + \frac{\delta^3\Gamma}{\delta\langle\varphi^a\rangle\delta\langle c^\alpha\rangle\delta\gamma_b}\bigg|_{\langle\Phi\rangle=0}\langle c^\alpha\rangle\langle\varphi^a\rangle + \dots \\ &= \frac{\delta^2\Gamma}{\delta\langle c^\alpha\rangle\delta\gamma_b}\bigg|_{\langle\Phi\rangle=0}\frac{\delta W}{\delta\bar{\xi}_\alpha} + \frac{\delta^3\Gamma}{\delta\langle\varphi^a\rangle\delta\langle c^\alpha\rangle\delta\gamma_b}\bigg|_{\langle\Phi\rangle=0}\frac{\delta W}{\delta\bar{\xi}_\alpha}\frac{\delta W}{\delta J_a} \\ &\quad + \dots.\end{aligned} \tag{B17}$$

The expansion starts with the term linear in the ghost field since the left-hand side has unit ghost number[19] and thus vanishes at $\langle c^\alpha\rangle = 0$. The second and subsequent terms lead to the diagrams of the form shown on the left of Fig. 5 which do not have poles. Thus, the only pole contribution comes from the first term. We notice that at tree level the second variational derivative entering it coincides with the generator of linear gauge transformations,

$$\frac{\delta^2\Gamma}{\delta\langle c^\alpha\rangle\delta\gamma_b}\bigg|_{\langle\Phi\rangle=0} = P^b{}_\alpha. \tag{B18}$$

In fact, this relation remains valid also after taking into account loop corrections, with P^b_α understood as the generator acting on properly normalized asymptotic fields [37].

We further have the identity

$$\mathcal{K}^\varphi_{ab}P^b{}_\alpha = (\mathcal{K}^{\varphi\perp}_{ab} + \chi^\beta_a O^{-1}_{\beta\gamma}\chi^\gamma_b)P^b{}_\alpha = \chi^\beta_a O^{-1}_{\beta\gamma}\chi^\gamma_b P^b{}_\alpha, \tag{B19}$$

[19] The source γ_a has ghost number -1 from the way it enters the partition function (B4) in combination with the BRST variation of the gauge field.

where we have split the wave operator for the asymptotic gauge fields into the "transverse" and "longitudinal" parts and used that the former is gauge invariant. By "transverse" part here we mean the operator coming from the original action S, whereas the "longitudinal" part arises upon eliminating from the action (B3) the nondynamical field b_α. Finally, we recall the structure of the ghost wave operator which is again read off from (B3),

$$(\mathcal{K}^c)^\alpha{}_\beta = -\chi^\alpha_a P^a{}_\beta. \tag{B20}$$

Combining together the above results gives

$$[\mathbf{K},\mathbf{D}]W[\mathcal{J}]|_{\mathcal{J}=0} =: \mathbf{K}\cdot\left(-\varphi^a_{\rm as}\chi^\alpha_a O^{-1}_{\alpha\beta}(\mathcal{K}^c)^\beta{}_\gamma\frac{\delta}{\delta\bar{\xi}_\gamma} - c^\alpha_{\rm as}P^a{}_\alpha\mathcal{K}^\varphi_{ab}\frac{\delta}{\delta J_b}\right): W[\mathcal{J}]|_{\mathcal{J}=0}. \tag{B21}$$

We recognize here the linear BRST variations of the asymptotic fields generated by $Q^{(2)}$,

$$i[Q^{(2)},\bar{c}_{{\rm as}\,\alpha}]_+ = O^{-1}_{\alpha\beta}\chi^\beta_a\varphi^a_{\rm as}, \qquad i[Q^{(2)},\varphi^a_{\rm as}] = P^a{}_\alpha c^\alpha_{\rm as}. \tag{B22}$$

Recall also that the linear BRST variation of the ghost field vanishes, $i[Q^{(2)}, c^\alpha_{\rm as}]_+ = 0$. This allows us to write

$$[\mathbf{K},\mathbf{D}]W[\mathcal{J}]|_{\mathcal{J}=0} = i[Q^{(2)},\mathbf{K}]W[\mathcal{J}]|_{\mathcal{J}=0}, \tag{B23}$$

where on the right-hand side we have the commutator of operators acting on the asymptotic Fock space. Together with Eq. (B13) and the definition of the $\mathcal{S}$-matrix (B11) it implies Eq. (3.2).

For completeness, let us also show that the elements of the $\mathcal{S}$-matrix (B11) between the states containing only physical particles do not depend on the choice of gauge. The physical particle states are interpolated by transverse components of the asymptotic fields satisfying $\chi^\alpha_a\varphi^{a\perp}_{\rm as} = 0$. Thus, the restriction of the $\mathcal{S}$-matrix to the physical states can be written as

$$\mathcal{S}^{\rm phys} =: \exp\left(-\varphi^{a\perp}_{\rm as}\mathcal{K}^{\varphi\perp}_{ab}\frac{\delta}{\delta J_b}\right): Z[\mathcal{J}]|_{\mathcal{J}=0}. \tag{B24}$$

An infinitesimal change of the gauge-fixing functions $\delta\chi^\alpha_a$ can be compensated by a properly chosen gauge transformation $\delta_\varepsilon\varphi^a$ of the integration variables in the path integral (B4), so that we have

$$\delta Z[\mathcal{J}] = \langle iJ_a\delta_\varepsilon\varphi^a\rangle Z[\mathcal{J}] = J_a\varepsilon^\alpha\left(iP^a{}_\alpha + R^a{}_{b\alpha}\frac{\delta}{\delta J_b}\right)Z[\mathcal{J}]. \tag{B25}$$

Substituting this into Eq. (B24) we obtain

$$\delta\mathcal{S}^{\rm phys} =: \mathbf{K}\cdot\left(-i\varphi^{a\perp}_{\rm as}\mathcal{K}^{\varphi\perp}_{ab}\varepsilon^\alpha P^b{}_\alpha - \varphi^{a\perp}_{\rm as}\mathcal{K}^{\varphi\perp}_{ab}\varepsilon^\alpha R^b{}_{c\alpha}\frac{\delta}{\delta J_c}\right): Z[\mathcal{J}]|_{\mathcal{J}=0}. \tag{B26}$$

The first term in brackets vanishes due to the gauge invariance of $\mathcal{K}^{\varphi\perp}_{ab}$, whereas the second term leads to the diagrams shown on the right of Fig. 5 and does not have on-shell poles. This implies $\delta\mathcal{S}^{\rm phys} = 0$, as expected.

APPENDIX C: FEYNMAN RULES IN σ, ξ GAUGE

Here we summarize the Feynman rules used in the computation of graviton $2 \to 2$ scattering amplitudes in the gauge of Sec. II B. We also include the ingredients entering diagrams with an external shift N_i which are used for verification of the gauge consistency relation (3.28).

External lines:

N_i 

$= -\sqrt{G}\,\epsilon^\alpha_i(\mathbf{k},\omega)\,,$ (C1a)

h_{ij} 

$= \sqrt{G}\,\varepsilon^\alpha_{ij}(\mathbf{k},\omega)\,,$ (C1b)

with

$$\epsilon^\alpha_i(\mathbf{k},\omega) = \begin{cases}\epsilon^\alpha_i(\mathbf{k}), & \omega > 0\\ -\epsilon^\alpha_i(-\mathbf{k}), & \omega < 0\end{cases},$$
$$\varepsilon^\alpha_{ij}(\mathbf{k},\omega) = \begin{cases}\varepsilon^\alpha_{ij}(\mathbf{k}), & \omega > 0\\ \varepsilon^\alpha_{ij}(-\mathbf{k}), & \omega < 0\end{cases}. \tag{C2}$$

The positive-frequency polarization factors $\epsilon^\alpha_i(\mathbf{k})$, $\varepsilon^\alpha_{ij}(\mathbf{k})$ are given by Eqs. (A9), (A10). Note that we treat all momenta and energies as flowing into the diagram.

Propagators:

$$N_i \overset{\mathbf{k},\omega}{=\!=\!=\!=} N_j \;=\; -G\left[\frac{k^4}{\sigma}(\delta_{ij}-\hat{k}_i\hat{k}_j)\mathcal{P}_1+\frac{(1+\xi)k^4}{\sigma}\hat{k}_i\hat{k}_j\mathcal{P}_0\right], \tag{C3a}$$

$$\begin{aligned} h_{ij} \overset{\mathbf{k},\omega}{\sim\!\sim\!\sim} h_{kl} \;=\; & 2G\Bigg\{(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\mathcal{P}_{tt}-\delta_{ij}\delta_{kl}\left[\mathcal{P}_{tt}-\frac{1-\lambda}{1-3\lambda}\mathcal{P}_s\right] \\ & -(\delta_{ik}\hat{k}_j\hat{k}_l+\delta_{il}\hat{k}_j\hat{k}_k+\delta_{jk}\hat{k}_i\hat{k}_l+\delta_{jl}\hat{k}_i\hat{k}_k)\left[\mathcal{P}_{tt}-\mathcal{P}_1\right] \\ & +(\delta_{ij}\hat{k}_k\hat{k}_l+\delta_{kl}\hat{k}_i\hat{k}_j)\left[\mathcal{P}_{tt}-\mathcal{P}_s\right] \\ & +\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l\left[\mathcal{P}_{tt}+\frac{1-3\lambda}{1-\lambda}\mathcal{P}_s-4\mathcal{P}_1+\frac{2\mathcal{P}_0}{1-\lambda}\right]\Bigg\}. \end{aligned} \tag{C3b}$$

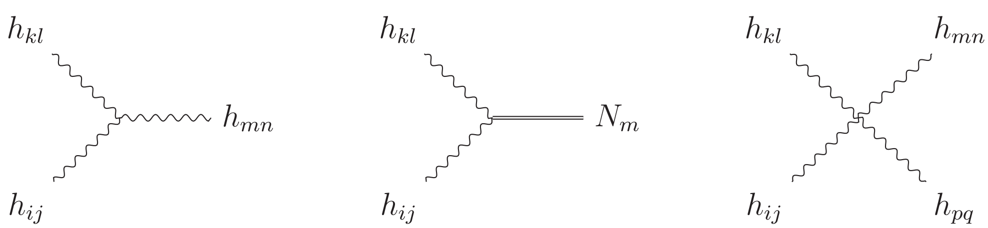
Here $\hat{\mathbf{k}}$ is the unit vector along the momentum, and the pole factors are

$$\mathcal{P}_1=\frac{i}{\omega^2-\omega_1^2(k)+i\epsilon},\quad \mathcal{P}_0=\frac{i}{\omega^2-\omega_0^2(k)+i\epsilon}, \tag{C4a}$$

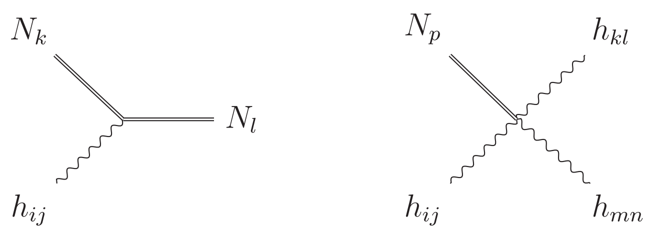
$$\mathcal{P}_{tt}=\frac{i}{\omega^2-\omega_{tt}^2(k)+i\epsilon},\quad \mathcal{P}_s=\frac{i}{\omega^2-\omega_s^2(k)+i\epsilon}, \tag{C4b}$$

with the dispersion relations (2.20)–(2.22). The Euclidean version of these propagators was derived in [17].

Vertices: In our calculation we use the following vertices:



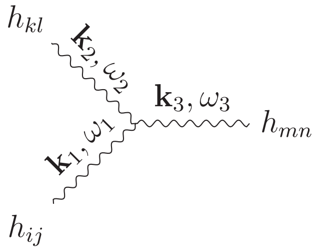
(C5a)



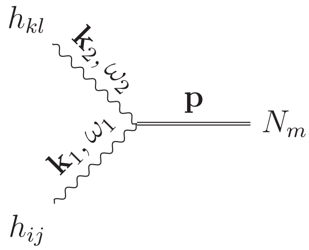
(C5b)

The vertices in the first line enter the graviton scattering amplitude, see Fig. 3, whereas the vertices in the second line are used to verify the identity (3.28).

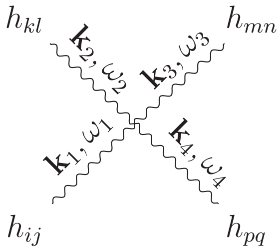
The full expressions for the vertices are lengthy and not illuminating. We present explicitly only the parts of (C5a) which are proportional to the coupling constant λ and could lead to large contributions to the graviton amplitudes in the limit $\lambda \to \infty$. These are used in the proof of Sec. V A that the divergent contributions actually cancel.



$$= i\frac{\lambda}{48G}\Big[\delta_{ij}\delta_{kl}\delta_{mn}(\omega_1\omega_2+\omega_2\omega_3+\omega_3\omega_1) - \delta_{ij}(\delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm})\omega_1(\omega_2+\omega_3) - \delta_{kl}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})\omega_2(\omega_3+\omega_1) - \delta_{mn}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\omega_3(\omega_1+\omega_2)\Big] + O(\lambda^0)\,, \tag{C6a}$$



$$= -i\frac{\lambda}{8G}\Big[\delta_{ij}\delta_{kl}(\omega_1 k_{1m}+\omega_2 k_{2m}) + (\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})(\omega_1+\omega_2)p_m - \delta_{ij}\omega_1(\delta_{lm}k_{1k}+\delta_{km}k_{1l}) - \delta_{kl}\omega_2(\delta_{jm}k_{2i}+\delta_{im}k_{2j})\Big] + O(\lambda^0)\,, \tag{C6b}$$



$$= i\frac{\lambda}{64G}\,\mathrm{sym}\Big\{\omega_1\omega_2\Big[2\delta_{ij}(\delta_{kq}\delta_{lm}\delta_{np}+\delta_{lq}\delta_{km}\delta_{np}+\delta_{kq}\delta_{ln}\delta_{mp}+\delta_{lq}\delta_{kn}\delta_{mp} +\delta_{kp}\delta_{lm}\delta_{nq}+\delta_{lp}\delta_{km}\delta_{nq}+\delta_{kp}\delta_{ln}\delta_{mq}+\delta_{lp}\delta_{kn}\delta_{mq}) +2(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})(\delta_{kp}\delta_{lq}+\delta_{kq}\delta_{lp}) -4\delta_{ij}\delta_{mn}(\delta_{kp}\delta_{lq}+\delta_{kq}\delta_{lp})-\delta_{ij}\delta_{kl}(\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np}) + \delta_{ij}\delta_{kl}\delta_{mn}\delta_{pq}\Big]\Big\} + O(\lambda^0)\,. \tag{C6c}$$

In the last expression "sym" stands for symmetrization over the graviton lines.

APPENDIX D: ANGULAR DEPENDENCE OF HEAD-ON AMPLITUDES

Throughout this appendix we denote $x = \cos\theta$. The subscripts $+, -, s$ stand for the ± 2, and $0'$-helicity gravitons. Here we use the *physical* helicities to label the incoming and outgoing particles: For example, the subscript $++, ++$ means that both gravitons in the initial and final states are right-handed. For the relation of the angular functions $f_{\alpha_1\alpha_2,\alpha_3\alpha_4}$ to the full amplitude see Eq. (4.10).

1. Processes without scalar gravitons

Using the notation $\hat{u}_s^2 = \frac{1-3\lambda}{1-\lambda}u_s^2 = \frac{8\nu_4}{\nu_5} + 3$, we have

$$\begin{aligned} f_{++,++} &= f_{--,--} \\ &= \frac{1}{512\hat{u}_s^2(1-x^2)^3}[x^8(-161-320v_2^2+v_2(464-720v_3)+39\hat{u}_s^2-9v_3^2(45-11\hat{u}_s^2)+6v_3(87-85\hat{u}_s^2)) \\ &\quad + 4x^6(231+443\hat{u}_s^2-72v_3^2\hat{u}_s^2-16v_2(21-8\hat{u}_s^2)+6v_3(63-53\hat{u}_s^2)) \\ &\quad + 2x^4(-287+448v_2^2-4783\hat{u}_s^2-16v_2(49-63v_3+48\hat{u}_s^2)+63v_3^2(9+\hat{u}_s^2)-6v_3(147+295\hat{u}_s^2)) \\ &\quad - 4x^2(581+128v_2^2-6343\hat{u}_s^2-16v_2(35-18v_3+24\hat{u}_s^2)+54v_3^2(3-\hat{u}_s^2)-6v_3(105+269\hat{u}_s^2)) \\ &\quad - 169-64v_2^2-19921\hat{u}_s^2-9v_3^2(9+17\hat{u}_s^2)+16v_2(13-9v_3+32\hat{u}_s^2)+6v_3(39-613\hat{u}_s^2)]; \end{aligned} \tag{D1}$$

$$
\begin{aligned}
f_{++,+-} &= f_{--,-+} = f_{+-,++} = f_{-+,--} \\
&= \frac{1}{512\hat{u}_s^2(1-x^2)}[x^4(133+64v_2^2-16v_2(13-12v_3)-243\hat{u}_s^2+9v_3^2(15-\hat{u}_s^2) \\
&\quad -12v_3(23-13\hat{u}_s^2))-2x^2(211+64v_2^2-16v_2(7-9v_3)-285\hat{u}_s^2+9v_3^2(9+\hat{u}_s^2) \\
&\quad -12v_3(15-11\hat{u}_s^2))+64v_2^2-16v_2(1-6v_3)+27v_3^2(1+\hat{u}_s^2)+12v_3(5+21\hat{u}_s^2) \\
&\quad -11(13+69\hat{u}_s^2)];
\end{aligned} \tag{D2}
$$

$$
\begin{aligned}
f_{++,--} &= f_{--,++} \\
&= \frac{1}{512\hat{u}_s^2}[3x^2(-35+64v_2^2-16v_2(1-7v_3)+501\hat{u}_s^2+3v_3^2(15-\hat{u}_s^2)+2v_3(5-79\hat{u}_s^2)) \\
&\quad +121+64v_2^2-1375\hat{u}_s^2+9v_3^2(1+\hat{u}_s^2)+66v_3(1+13\hat{u}_s^2)+16v_2(11+3v_3+32\hat{u}_s^2)];
\end{aligned} \tag{D3}
$$

$$
\begin{aligned}
f_{+-,+-} &= \frac{1+x}{512\hat{u}_s^2(1-x)^3}[-x^4(64v_2^2-16v_2(13-12v_3)+27v_3^2(5+\hat{u}_s^2)-12v_3(23+15\hat{u}_s^2)+7(19+59\hat{u}_s^2)) \\
&\quad -6x^3(4-v_3)(16v_2+3v_3(7+3\hat{u}_s^2)-4(5+7\hat{u}_s^2)) \\
&\quad +2x^2(-221+64v_2^2-16v_2(7-9v_3)-205\hat{u}_s^2+18v_3^2(3-\hat{u}_s^2)+12v_3(3+13\hat{u}_s^2)) \\
&\quad +6x(4-v_3)(16v_2+3v_3(3-\hat{u}_s^2)+4(1-\hat{u}_s^2)) \\
&\quad -145-64v_2^2+v_2(16-96v_3)+103\hat{u}_s^2+v_3(84-60\hat{u}_s^2)-9v_3^2(5+\hat{u}_s^2)].
\end{aligned} \tag{D4}
$$

2. Processes with one scalar graviton

Due to different dispersion relations of scalar and tensor modes the structure of the amplitudes involving both types of particles is more complicated. Let us consider the case when the scalar graviton is in the final state. Then the momentum of outgoing particles is related to the incoming momentum k as

$$k' = \varkappa k, \qquad \varkappa = \left(\frac{2}{1+u_s}\right)^{1/3}. \tag{D5}$$

Using this notation, we can write

$$f_{\alpha_1\alpha_2,\alpha_3 s} = \sqrt{\frac{2(1-\lambda)}{1-3\lambda}}\frac{P_{\alpha_1\alpha_2,\alpha_3 s}(x)}{g_1(x)}, \qquad \alpha_I = +,-, \tag{D6}$$

where

$$
\begin{aligned}
g_1(x) &= ((1-2x\varkappa+\varkappa^2)^3-(1-u_s\varkappa^3)^2) \\
&\quad \times (u_s^2(1-2x\varkappa+\varkappa^2)^3-(1-u_s\varkappa^3)^2) \\
&\quad \times ((1+2x\varkappa+\varkappa^2)^3-(1-u_s\varkappa^3)^2) \\
&\quad \times (u_s^2(1+2x\varkappa+\varkappa^2)^3-(1-u_s\varkappa^3)^2),
\end{aligned} \tag{D7}
$$

and $P_{\alpha_1\alpha_2,\alpha_3 s}(x)$ are polynomials of 14th degree in x that are too cumbersome to present explicitly. Note that for $u_s \neq 1$ the denominator (D7) has roots at nonzero scattering angles. As discussed in Sec. IV B, this corresponds to resonant poles in the amplitude due to on-shell graviton decays. On the other hand, $g_1(x)$ is regular in the forward and backward limits $x = \pm 1$. In fact, the amplitude vanishes in these limits since the polynomials in the numerator can be factorized as

$$
\begin{aligned}
P_{++,+s} &= (1-x^2)\tilde{P}_{++,+s}, \qquad P_{++,-s} = (1-x^2)\tilde{P}_{++,-s}, \\
P_{+-,-s} &= (1-x)^3(1+x)\tilde{P}_{+-,-s},
\end{aligned} \tag{D8}
$$

and similarly for the channels obtained by parity and time inversion. This is consistent with conservation of angular momentum (see Sec. IV B).

The amplitudes greatly simplify if the dispersion relations of the tensor and scalar gravitons coincide: $u_s = 1$, $\varkappa = 1$. Then we have

$$f_{++,+s} = f_{--,-s} = f_{+s,++} = f_{-s,--}$$
$$= \frac{1}{128\sqrt{2(1-\lambda)(1-3\lambda)^3}(1-x^2)^2}[x^6(81+80v_2^2(1-\lambda)^2-245\lambda+230\lambda^2$$
$$+18v_3^2(5-8\lambda+3\lambda^2)-3v_3(31-21\lambda-8\lambda^2)-4v_2(1-\lambda)(49-80\lambda-v_3(48-51\lambda)))$$
$$-x^4(447+240v_2^2(1-\lambda)^2-1175\lambda+402\lambda^2+18v_3^2(17-38\lambda+21\lambda^2)$$
$$-3v_3(111-165\lambda+76\lambda^2)-4v_2(1-\lambda)(85-60\lambda-3v_3(48-59\lambda)))$$
$$+x^2(-1221+112v_2^2(1-\lambda)^2+5729\lambda-5870\lambda^2+54v_3^2(5-16\lambda+11\lambda^2)$$
$$+3v_3(159-821\lambda+608\lambda^2)+4v_2(1-\lambda)(121-536\lambda+3v_3(32-59\lambda)))$$
$$+(1587+48v_2^2(1-\lambda)^2-6851\lambda+6426\lambda^2-4v_2(1-\lambda)(253-(708+51v_3)\lambda)$$
$$-54v_3^2(1-6\lambda+5\lambda^2)-3v_3(335-1253\lambda+884\lambda^2))]; \tag{D9}$$

$$f_{++,-s} = f_{--,+s} = f_{-s,++} = f_{+s,--}$$
$$= \frac{1}{128\sqrt{2(1-\lambda)(1-3\lambda)^3}}[x^2(299+18v_3^2(1-\lambda)+48v_2^2(1-\lambda)^2-1247\lambda+882\lambda^2$$
$$-3v_3(7-49\lambda+40\lambda^2)+4v_2(1-\lambda)(17-48\lambda+3v_3(6-5\lambda)))-299-48v_2^2(1-\lambda)^2$$
$$+1211\lambda-36v_3^2(1-\lambda)\lambda-810\lambda^2-4v_2(1-\lambda)(35+3v_3(4-\lambda)-84\lambda)$$
$$-3v_3(11-9\lambda+4\lambda^2)]; \tag{D10}$$

$$f_{+-,-s} = f_{-+,+s} = f_{-s,+-} = f_{+s,-+}$$
$$= \frac{1}{128\sqrt{2(1-\lambda)(1-3\lambda)^3}(1+x)^2}[x^4(187+16v_2^2(1-\lambda)^2-651\lambda+466\lambda^2$$
$$+27v_3^2(2-5\lambda+3\lambda^2)-6v_3(34-105\lambda+72\lambda^2)-4v_2(1-\lambda)(33-64\lambda-3v_3(5-6\lambda)))$$
$$+3x^3(4-v_3)(18-4v_2(1-\lambda)-56\lambda+36\lambda^2-3v_3(3-7\lambda+4\lambda^2))$$
$$-x^2(64v_2^2(1-\lambda)^2+9v_3^2(13-31\lambda+18\lambda^2)+2(25-69\lambda+62\lambda^2)$$
$$-6v_3(47-140\lambda+96\lambda^2)-4v_2(1-\lambda)(78-160\lambda-9v_3(5-6\lambda)))$$
$$+3x(4-v_3)(-18+64\lambda-52\lambda^2+4v_2(5-13\lambda+8\lambda^2)+3v_3(7-19\lambda+12\lambda^2))$$
$$-137+48v_2^2(1-\lambda)^2+561\lambda-438\lambda^2+9v_3^2(3-4\lambda+\lambda^2)$$
$$-6v_3(1+5\lambda-8\lambda^2)-12v_2(1-\lambda)(7-16\lambda-v_3(6-4\lambda))]. \tag{D11}$$

3. Processes with two scalar gravitons

a. Two scalars in the final state

Here the relation between the outgoing and incoming momenta is

$$k' = \varkappa k, \qquad \varkappa = u_s^{-1/3}, \tag{D12}$$

and the angular functions have the form

$$f_{\alpha_1\alpha_2,ss} = \frac{2(1-\lambda)}{(1-3\lambda)}\frac{P_{\alpha_1\alpha_2,ss}(x)}{g_2(x)}, \qquad \alpha_I = +,-, \tag{D13}$$

with the denominator

$$g_2(x) = (1+(2-4x^2)\varkappa^2+\varkappa^4)^6. \tag{D14}$$

We observe that this denominator does not have any zeros for $u_s \neq 1$. The absence of resonances is explained as follows. For the processes at hand the energies of initial and final particles are the same. So the energy flowing in the propagators of intermediate states in t- and u-channels vanishes, whereas the momentum does not. For the s-channel the situation is opposite. Thus these propagators never become on-shell. For the case of different helicities $\alpha_1 \neq \alpha_2$ the amplitude vanishes in the collinear limits since

$$P_{+-,ss} = (1-x^2)^2 \tilde{P}_{+-,ss}, \tag{D15}$$

as required by the angular momentum conservation.

The 14th order polynomials $P_{\alpha_1\alpha_2,ss}(x)$ are again too lengthy in general. We explicitly present the amplitude for the case $u_s = 1$:

$$\begin{aligned} f_{++,ss} &= f_{--,ss} = f_{ss,++} = f_{ss,--} \\ &= \frac{1}{128(1-\lambda)^2(1-3\lambda)^2(1-x^2)}[-x^4(87-527\lambda+887\lambda^2-421\lambda^3-42\lambda^4-9v_3^2(1-\lambda)^3(2-3\lambda) \\ &\quad + 12v_3(1-\lambda)^2(8-24\lambda+17\lambda^2)+8v_2(1-\lambda)^3(20-51\lambda-v_3(3-6\lambda))) \\ &\quad + x^2(1-\lambda)(270+36v_3-207v_3^2-1416\lambda-36v_3\lambda+900v_3^2\lambda+1874\lambda^2+216v_3\lambda^2 \\ &\quad - 1179v_3^2\lambda^2-548\lambda^3-216v_3\lambda^3+486v_3^2\lambda^3-1536v_1(1-\lambda)^2(1-3\lambda) \\ &\quad - 16v_2^2(1-\lambda)^2(9-22\lambda)-8v_2(1-\lambda)(26-122\lambda+70\lambda^2+9v_3(5-17\lambda+12\lambda^2))) \\ &\quad - 183+60v_3+225v_3^2+1159\lambda-360v_3\lambda-1206v_3^2\lambda-2387\lambda^2+432v_3\lambda^2+2268v_3^2\lambda^2 \\ &\quad + 1953\lambda^3-24v_3\lambda^3-1818v_3^2\lambda^3-558\lambda^4-108v_3\lambda^4+531v_3^2\lambda^4+1536v_1(1-\lambda)^3(1-3\lambda) \\ &\quad + 16v_2^2(1-\lambda)^3(13-30\lambda)+8v_2(1-\lambda)^2(46-185\lambda+105\lambda^2+18v_3(3-10\lambda+7\lambda^2))]; \end{aligned} \tag{D16}$$

$$\begin{aligned} f_{+-,ss} &= f_{ss,+-} \\ &= \frac{-1}{128(1-\lambda)^2(1-3\lambda)^2(1-x^2)}[x^4(273-1633\lambda+3433\lambda^2-3179\lambda^3+1122\lambda^4 \\ &\quad + 9v_3^2(1-\lambda)^3(4-9\lambda)-12v_3(1-\lambda)^2(22-80\lambda+59\lambda^2) \\ &\quad - 8v_2(1-\lambda)^3(20-51\lambda-v_3(3-6\lambda)))-x^2(1-\lambda)(242-1240\lambda+2126\lambda^2-1372\lambda^3 \\ &\quad + 16v_2^2(1-\lambda)^2(1+2\lambda)+9v_3^2(1-\lambda)^2(7-6\lambda)-8v_2(1-\lambda)(46-9v_3(1-\lambda)-150\lambda+98\lambda^2) \\ &\quad + 12v_3(-35+151\lambda-194\lambda^2+78\lambda^3))-31+151\lambda-51\lambda^2-367\lambda^3+282\lambda^4 \\ &\quad + 9v_3^2(1-\lambda)^3(7-5\lambda)+16v_2^2(1-\lambda)^3(5-6\lambda)-12v_3(1-\lambda)^3(13-27\lambda) \\ &\quad + 8v_2(1-\lambda)^2(-26+18v_3(1-\lambda)^2+87\lambda-63\lambda^2)]. \end{aligned} \tag{D17}$$

b. Tensor-scalar scattering

In this case the absolute value of the initial and final momenta is the same, $k' = k$. The amplitude has the form

$$f_{\alpha_1 s,\quad \alpha_2 s} = \frac{2(1-\lambda)}{(1-3\lambda)} \frac{P_{\alpha_1 s,\quad \alpha_2 s}(x)}{g_3(x)}, \quad \alpha_I = +,-, \tag{D18}$$

with

$$g_3(x) = 64u_s^2(1-x)^3((1-u_s)^2 - 8(1+x)^3)) \times ((1-u_s)^2 - 8u_s^2(1+x)^3), \tag{D19}$$

and $P_{\alpha_1 s,\alpha_2 s}(x)$ an 11th order polynomial. The amplitude has resonant poles at nonzero values of x and diverges in the forward limit. The divergence is alleviated if the helicities of tensor gravitons in the initial and final states are different,

$$P_{\alpha_1 s,\alpha_2 s} = (1-x)^2 \tilde{P}_{\alpha_1 s,\alpha_2 s} \quad \text{for } \alpha_1 \neq \alpha_3, \tag{D20}$$

consistently with the angular momentum conservation. For the same initial and final helicities and $u_s \neq 1$ the amplitude vanishes in the backward limit:

$$P_{\alpha_1 s,\alpha_2 s} = (1+x)^2 \tilde{P}_{\alpha_1 s,\alpha_2 s} \quad \text{for } \alpha_1 = \alpha_3. \tag{D21}$$

The case of identical tensor and scalar dispersion relations, $u_s = 1$, leads to

$$
\begin{aligned}
f_{+s,+s} &= f_{-s,-s} \\
&= \frac{-1}{256(1-\lambda)^2(1-3\lambda)^2(1-x)^3(1+x)}[x^6(64v_2^2(1-\lambda)^4+27v_3^2(1-\lambda)^3(4-5\lambda) \\
&\quad -3v_3(119-605\lambda+1081\lambda^2-819\lambda^3+224\lambda^4)+4(93-533\lambda+1052\lambda^2-878\lambda^3+262\lambda^4) \\
&\quad +4v_2(1-\lambda)^2(3v_3(15-34\lambda+19\lambda^2)-4(21-67\lambda+43\lambda^2))) \\
&\quad +x^5(-342+198v_3+153v_3^2-192v_1(1-\lambda)^4(7-8v_2-9v_3)+1618\lambda-1014v_3\lambda \\
&\quad -657v_3^2\lambda-2450\lambda^2+1854v_3\lambda^2+1053v_3^2\lambda^2+1178\lambda^3-1434v_3\lambda^3-747v_3^2\lambda^3 \\
&\quad +60\lambda^4+396v_3\lambda^4+198v_3^2\lambda^4+16v_2^2(1-\lambda)^3(31-30\lambda) \\
&\quad +8v_2(1-\lambda)^2(-20+8\lambda+18\lambda^2+3v_3(29-59\lambda+30\lambda^2))) \\
&\quad +x^4(-738+1461v_3-441v_3^2-192v_1(1-8v_2-9v_3)(1-\lambda)^4+4086\lambda-8187v_3\lambda \\
&\quad +2394v_3^2\lambda-7522\lambda^2+15903v_3\lambda^2-4536v_3^2\lambda^2+6178\lambda^3-13029v_3\lambda^3+3654v_3^2\lambda^3 \\
&\quad -2084\lambda^4+3852v_3\lambda^4-1071v_3^2\lambda^4+16v_2^2(1-\lambda)^3(9+2\lambda)-4v_2(1-\lambda)^2(-324+69v_3 \\
&\quad +1156\lambda-348v_3\lambda-848\lambda^2+279v_3\lambda^2))+2x^3(1-\lambda)(770-918v_3+117v_3^2 \\
&\quad +192v_1(13-8v_2-9v_3)(1-\lambda)^3-3708\lambda+4704v_3\lambda-684v_3^2\lambda+5250\lambda^2-6990v_3\lambda^2 \\
&\quad +1017v_3^2\lambda^2-2132\lambda^3+3180v_3\lambda^3-450v_3^2\lambda^3-16v_2^2(1-\lambda)^2(21-10\lambda) \\
&\quad +8v_2(1-\lambda)(28-264\lambda+262\lambda^2+3v_3(11-\lambda-10\lambda^2)))+x^2(-1072+453v_3+90v_3^2 \\
&\quad +384v_1(1-\lambda)^4(7-8v_2-9v_3)+7688\lambda-2223v_3\lambda-855v_3^2\lambda-18724\lambda^2+3843v_3\lambda^2 \\
&\quad +2025v_3^2\lambda^2+17692\lambda^3-2913v_3\lambda^3-1845v_3^2\lambda^3-5504\lambda^4+840v_3\lambda^4+585v_3^2\lambda^4 \\
&\quad -32v_2^2(1-\lambda)^3(21-16\lambda)+4v_2(1-\lambda)^2(276-700\lambda+404\lambda^2-3v_3(55-54\lambda-\lambda^2))) \\
&\quad +x(-910+1638v_3-531v_3^2-192v_1(1-\lambda)^4(19-8v_2-9v_3)+5322\lambda-10422v_3\lambda \\
&\quad +2979v_3^2\lambda-10634\lambda^2+22302v_3\lambda^2-5751v_3^2\lambda^2+8882\lambda^3-19866v_3\lambda^3+4689v_3^2\lambda^3 \\
&\quad -2724\lambda^4+6348v_3\lambda^4-1386v_3^2\lambda^4-16v_2^2(1-\lambda)^3(5-42\lambda)+8v_2(1-\lambda)^2(76-568\lambda \\
&\quad +570\lambda^2-3v_3(23-105\lambda+82\lambda^2)))-192v_1(1-\lambda)^4(13-8v_2-9v_3)+1726-1557v_3 \\
&\quad +387v_3^2-11658\lambda+8787v_3\lambda-1800v_3^2\lambda+26998\lambda^2-17271v_3\lambda^2+3078v_3^2\lambda^2 \\
&\quad -25446\lambda^3+14445v_3\lambda^3-2304v_3^2\lambda^3+16v_2^2(1-\lambda)^3(45-62\lambda)+8396\lambda^4-4404v_3\lambda^4 \\
&\quad +639v_3^2\lambda^4-4v_2(1-\lambda)^2(516-285v_3+1652\lambda-696v_3\lambda-1208\lambda^2+411v_3\lambda^2)];
\end{aligned} \tag{D22}
$$

$$
\begin{aligned}
f_{+s,-s} &= f_{-s,+s} \\
&= \frac{1}{256(1-\lambda)^2(1-3\lambda)^2(1-x^2)}[-x^4(64v_2^2(1-\lambda)^4+81v_3^2(1-\lambda)^3\lambda \\
&\quad +4(2-30\lambda+67\lambda^2-51\lambda^3+16\lambda^4)+3v_3(81-459\lambda+871\lambda^2-685\lambda^3+192\lambda^4) \\
&\quad +4v_2(1-\lambda)^2(3v_3(7-10\lambda+3\lambda^2)+4(13-51\lambda+35\lambda^2))) \\
&\quad -x^3(78-108v_3+279v_3^2-192v_1(1-\lambda)^4(7-8v_2-9v_3)-194\lambda+240v_3\lambda \\
&\quad -1089v_3^2\lambda-694\lambda^2+1593v_3^2\lambda^2+1878\lambda^3-312v_3\lambda^3-1035v_3^2\lambda^3-1036\lambda^4+180v_3\lambda^4 \\
&\quad +252v_3^2\lambda^4+16v_2^2(1-\lambda)^3(39-38\lambda)+8v_2(1-\lambda)^2(-48+64\lambda-22\lambda^2+3v_3(40-79\lambda+39\lambda^2))) \\
&\quad +x^2(1-\lambda)(10+54v_3-81v_3^2-192v_1(1-\lambda)^3(11+8v_2+3v_3) \\
&\quad -84\lambda-528v_3\lambda+252v_3^2\lambda-46\lambda^2+918v_3\lambda^2-261v_3^2\lambda^2+44\lambda^3-444v_3\lambda^3+90v_3^2\lambda^3 \\
&\quad +16v_2^2(1-\lambda)^2(-29+26\lambda)-8v_2(1-\lambda)(56-72\lambda+26\lambda^2+v_3(39-69\lambda+30\lambda^2)))
\end{aligned}
$$

$$+ x(294 - 180v_3 + 315v_3^2 - 192v_1(7 - 8v_2 - 9v_3)(1-\lambda)^4 - 1706\lambda + 792v_3\lambda - 1269v_3^2\lambda + 2994\lambda^2 - 1416v_3\lambda^2 + 1917v_3^2\lambda^2 - 1842\lambda^3 + 1152v_3\lambda^3 - 1287v_3^2\lambda^3 + 292\lambda^4 - 348v_3\lambda^4 + 324v_3^2\lambda^4 + 16v_2^2(1-\lambda)^3(43 - 46\lambda) + 8v_2(1-\lambda)^2(-48 + 72\lambda - 38\lambda^2 + 3v_3(44 - 91\lambda + 47\lambda^2))) - 218 + 192v_1(1-\lambda)^4(11 + 8v_2 + 3v_3) + 261v_3 + 117v_3^2 + 1486\lambda - 1251v_3\lambda - 432v_3^2\lambda - 3426\lambda^2 + 2199v_3\lambda^2 + 594v_3^2\lambda^2 + 3330\lambda^3 - 1677v_3\lambda^3 - 360v_3^2\lambda^3 - 1156\lambda^4 + 468v_3\lambda^4 + 81v_3^2\lambda^4 + 16v_2^2(1-\lambda)^3(37 - 38\lambda) + 4v_2(1-\lambda)^2(3v_3(41 - 80\lambda + 39\lambda^2) + 4(41 - 83\lambda + 40\lambda^2)))]. \tag{D23}$$

4. Processes with three and four scalar gravitons

For scattering with three scalar gravitons—one in the beginning and two in the end—the final and initial momenta are related by

$$k' = \varkappa k, \qquad \varkappa = \left(\frac{1+u_s}{2u_s}\right)^{1/3}. \tag{D24}$$

The angular dependence of the amplitude reads

$$f_{\alpha s,ss} = \left(\frac{2(1-\lambda)}{1-3\lambda}\right)^{3/2} \frac{(1-x^2)P_{\alpha s,ss}(x)}{g_4(x)}, \quad \alpha = +,-, \tag{D25}$$

where

$$g_4(x) = ((1 + 2x\varkappa + \varkappa^2)^3 - (1 - u_s\varkappa^3)^2) \times (u_s^2(1 + 2x\varkappa + \varkappa^2)^3 - (1 - u_s\varkappa^3)^2) \times ((1 - 2x\varkappa + \varkappa^2)^3 - (u_s - u_s\varkappa^3)^2) \times (u_s^2(1 - 2x\varkappa + \varkappa^2)^3 - (u_s - u_s\varkappa^3)^2), \tag{D26}$$

and $P_{\alpha s,ss}(x)$ is a 12th order polynomial. For $u_s \neq 1$ the denominator has zeros at nonzero angles, and the amplitude vanishes in the forward and backward limits.

For $u_s = 1$ the amplitude simplifies (though it still remains quite lengthy):

$$f_{+s,ss} = f_{-s,ss} = f_{ss,+s} = f_{ss,-s} = \frac{1}{64\sqrt{2(1-\lambda)^3(1-3\lambda)^5}(1-x^2)^2}[x^6(1-\lambda)(119 - 16v_2^2(1-\lambda)^3 - 582\lambda + 825\lambda^2 - 422\lambda^3 + 9v_3^2(1-\lambda)^2(1-6\lambda) - 6v_3(30 - 147\lambda + 193\lambda^2 - 74\lambda^3) - 4v_2(1-\lambda)(35 - 139\lambda + 92\lambda^2 - v_3(3 + 3\lambda - 6\lambda^2))) + x^4(-275 + 102v_3 + 90v_3^2 - 192v_1(1-\lambda)^3(15 - 4v_2(1-\lambda) - 6v_3(1-\lambda) - 29\lambda) + 1717\lambda - 966v_3\lambda - 234v_3^2\lambda - 3811\lambda^2 + 2286v_3\lambda^2 + 162v_3^2\lambda^2 + 3799\lambda^3 - 2058v_3\lambda^3 + 18v_3^2\lambda^3 - 1422\lambda^4 + 636v_3\lambda^4 - 36v_3^2\lambda^4 + 16v_2^2(1-\lambda)^3(17 - 15\lambda) - 4v_2(1-\lambda)(161 - 554\lambda + 653\lambda^2 - 264\lambda^3 - 6v_3(1-\lambda)^2(19 + 13\lambda))) - x^2(167 - 480v_3 + 387v_3^2 - 384v_1(1-\lambda)^3(15 - 8v_2(1-\lambda) - 9v_3(1-\lambda) - 31\lambda) - 1381\lambda + 2214v_3\lambda - 1539v_3^2\lambda + 3615\lambda^2 - 3540v_3\lambda^2 + 2295v_3^2\lambda^2 - 3319\lambda^3 + 2322v_3\lambda^3 - 1521v_3^2\lambda^3 + 902\lambda^4 - 516v_3\lambda^4 + 378v_3^2\lambda^4 + 16v_2^2(1-\lambda)^3(67 - 71\lambda) - 4v_2(1-\lambda)(451 - 1790\lambda + 2183\lambda^2 - 836\lambda^3 - 3v_3(1-\lambda)^2(129 - 134\lambda))) + 323 - 546v_3 + 288v_3^2 - 576v_1(1-\lambda)^3(5 - 4v_2(1-\lambda) - 4v_3(1-\lambda) - 11\lambda) - 2493\lambda + 3126v_3\lambda - 1224v_3^2\lambda + 6595\lambda^2 - 6234v_3\lambda^2 + 1944v_3^2\lambda^2 - 6927\lambda^3 + 5226v_3\lambda^3 - 1368v_3^2\lambda^3 + 2478\lambda^4 - 1572v_3\lambda^4 + 360v_3^2\lambda^4 + 48v_2^2(1-\lambda)^3(17 - 19\lambda) - 12v_2(1-\lambda)(101 - 450\lambda + 609\lambda^2 - 256\lambda^3 - 2v_3(1-\lambda)^2(46 - 53\lambda))]. \tag{D27}$$

Finally, we consider the amplitude with four scalar gravitons. The kinematics in this case is simple, $k' = k$. Still, the amplitude is rather lengthy since it involves all vertices in an intricate way. In general it has the form

$$f_{ss,ss} = \left(\frac{2(1-\lambda)}{1-3\lambda}\right)^2 \frac{P_{ss,ss}(x)}{(1-x^2)^3}, \tag{D28}$$

where $P_{ss,ss}(x)$ is an even polynomial of degree 8. The amplitude has only forward singularities. For $u_s = 1$ we have

$$\begin{aligned}
f_{ss,ss} = {} & \frac{1}{64(1-\lambda)^2(1-3\lambda)^3(1-x^2)^3}[x^8(1-\lambda)^2(185 - 32v_2^2(1-\lambda)^3 - 992\lambda + 1525\lambda^2 - 838\lambda^3 - 9v_3^2(1-\lambda)^2(1+6\lambda) \\
& - 8v_2(1-\lambda)(15 + 6v_3(1-\lambda) - 68\lambda + 41\lambda^2) + 12v_3(-15 + 80\lambda - 103\lambda^2 + 36\lambda^3)) \\
& + x^6(-500 + 408v_3 + 171v_3^2 + 3592\lambda - 3624v_3\lambda - 954v_3^2\lambda - 9152\lambda^2 + 11280v_3\lambda^2 + 2106v_3^2\lambda^2 + 10584\lambda^3 \\
& - 16344v_3\lambda^3 - 2304v_3^2\lambda^3 - 5340\lambda^4 + 11304v_3\lambda^4 + 1251v_3^2\lambda^4 + 768\lambda^5 - 3024v_3\lambda^5 - 270v_3^2\lambda^5 \\
& + 192v_2^2(1-\lambda)^4(1-2\lambda) - 384v_1(1-\lambda)^3(1+2\lambda-15\lambda^2) - 48v_2(1-\lambda)^2(-2 + 34\lambda - 98\lambda^2 + 70\lambda^3 \\
& - v_3(1-\lambda)^2(8-15\lambda))) + x^4(458 + 585v_3^2 + 73728v_1^2(1-\lambda)^5 - 3844\lambda + 792v_3\lambda - 2574v_3^2\lambda + 12580\lambda^2 \\
& - 3840v_3\lambda^2 + 4446v_3^2\lambda^2 - 20984\lambda^3 + 7128v_3\lambda^3 - 3744v_3^2\lambda^3 + 17554\lambda^4 - 5904v_3\lambda^4 + 1521v_3^2\lambda^4 - 5684\lambda^5 \\
& + 1824v_3\lambda^5 + 64v_2^2(1-\lambda)^4(124 - 119\lambda) - 234v_3^2\lambda^5 + 384v_1(1-\lambda)^3(-3 + 128v_2(1-\lambda)^2 + 42v_3(1-\lambda)^2 \\
& + 30\lambda - 55\lambda^2) - 16v_2(1-\lambda)^2(3 - 151\lambda + 469\lambda^2 - 341\lambda^3 + 15v_3(1-\lambda)^2(-20 + 17\lambda))) \\
& + x^2(-300 + 936v_3 - 1647v_3^2 - 147456v_1^2(1-\lambda)^5 + 2744\lambda - 6936v_3\lambda + 8082v_3^2\lambda - 9120 + 18288v_3\lambda^2 \\
& - 15858v_3^2\lambda^2 + 13992\lambda^3 - 22728v_3\lambda^3 + 15552v_3^2\lambda^3 - 10532\lambda^4 + 13656v_3\lambda^4 - 7623v_3^2\lambda^4 + 3200\lambda^5 - 3216v_3\lambda^5 \\
& + 1494v_3^2\lambda^5 - 64v_2^2(1-\lambda)^4(255 - 254\lambda) - 384v_1(1-\lambda)^3(-33 + 256v_2(1-\lambda)^2 + 84v_3(1-\lambda)^2 + 150\lambda - 137\lambda^2) \\
& + 16v_2(1-\lambda)^2(234 - 1290\lambda + 1978\lambda^2 - 926\lambda^3 - 15v_3(1-\lambda)^2(44 - 43\lambda))) \\
& + 733 - 1164v_3 + 900v_3^2 + 73728v_1^2(1-\lambda)^5 - 6890\lambda + 8448v_3\lambda - 4536v_3^2\lambda + 23886\lambda^2 - 22392v_3\lambda^2 \\
& + 9144v_3^2\lambda^2 - 37880\lambda^3 + 28080v_3\lambda^3 - 9216v_3^2\lambda^3 + 27949\lambda^4 - 16956v_3\lambda^4 + 4644v_3^2\lambda^4 - 7814\lambda^5 + 3984v_3\lambda^5 \\
& - 936v_3^2\lambda^5 + 32v_2^2(1-\lambda)^4(257 - 259\lambda) + 384v_1(1-\lambda)^3(-29 + 128v_2(1-\lambda)^2 + 42v_3(1-\lambda)^2 + 122\lambda - 97\lambda^2) \\
& - 8v_2(1-\lambda)^2(459 - 2399\lambda + 3497\lambda^2 - 1549\lambda^3 - 6v_3(1-\lambda)^2(113 - 115\lambda))].
\end{aligned} \tag{D29}$$

APPENDIX E: MODES AND PROPAGATORS WITH AUXILIARY FIELD

The tensor and vector parts of the quadratic Lagrangian following from the action (5.9) with the gauge fixing (5.11) are the same as in the original HG action, see Eqs. (A3a), (A3b). The difference, however, occurs in the scalar sector. Using the same decomposition as in Eqs. (A1) we obtain

$$\begin{aligned}
\tilde{\mathcal{L}}_g^{(2s)} = {} & \frac{1}{2G}\left\{\frac{\dot{\psi}^2}{2} + \frac{\dot{E}^2}{4} - 2\chi\dot{\psi} - \chi\dot{E} + \frac{8\nu_4 + 3\nu_5}{2}\psi\Delta^3\psi \right. \\
& + \frac{1+\xi}{4\sigma}E\Delta^3 E + \dot{B}\frac{\sigma}{(1+\xi)\Delta}\dot{B} + B\Delta^2 B + 2\chi\Delta B \\
& \left. - 2\dot{\bar{C}}\Delta\dot{C} - \frac{2(1+\xi)}{\sigma}\bar{C}\Delta^4 C\right\}.
\end{aligned} \tag{E1}$$

The ghost part, of course, decouples and leads to a simple propagator which, combined with the vector contribution, gives

$$c_i \longrightarrow \bar{c}_j = G\Big[\delta_{ij}\mathcal{P}_1 + \hat{k}_i\hat{k}_j(\tilde{\mathcal{P}}_0 - \mathcal{P}_1)\Big], \tag{E2}$$

where $\mathcal{P}_1$ is given in (C4) and

$$\tilde{\mathcal{P}}_0 = \frac{i}{\omega^2 - \tilde{\nu}_0 k^6 + i\epsilon}, \qquad \tilde{\nu}_0 \equiv \frac{1+\xi}{\sigma}. \tag{E3}$$

The other components ψ, E, B, χ all mix with each other. To find their propagators, we switch to the Fourier space and invert the mixing matrix. Combining with the propagators of tensor and vector components we arrive at

$$\chi \text{ - - - - - - - } \chi = G\left\{-\frac{i}{3} + \frac{2\nu_s}{3}k^6\mathcal{P}_s\right\}, \tag{E4a}$$

$$\chi \text{ - - - ═══ } N_i \;= iG\frac{\tilde{\nu}_0 k^5 \hat{k}_i}{3(\nu_s - \tilde{\nu}_0)}\Big\{2\nu_s\mathcal{P}_s - (3\nu_s - \tilde{\nu}_0)\tilde{\mathcal{P}}_0\Big\}, \tag{E4b}$$

$$\chi \text{ - - - ∿∿ } h_{ij} \;= iG\frac{2\omega}{3}\Big\{-\delta_{ij}\mathcal{P}_s + \hat{k}_i\hat{k}_j\frac{3\nu_s - \tilde{\nu}_0}{\nu_s - \tilde{\nu}_0}\big[\mathcal{P}_s - \tilde{\mathcal{P}}_0\big]\Big\}, \tag{E4c}$$

$$N_i \text{ ═══ } N_j \;= -G\bigg\{\frac{k^4}{\sigma}(\delta_{ij} - \hat{k}_i\hat{k}_j)\mathcal{P}_1 - \frac{\tilde{\nu}_0 k^4 \hat{k}_i\hat{k}_j}{3(\nu_s - \tilde{\nu}_0)}\bigg[\frac{2\tilde{\nu}_0\nu_s}{\nu_s - \tilde{\nu}_0}\mathcal{P}_s - \frac{2\tilde{\nu}_0^2}{\nu_s - \tilde{\nu}_0}\tilde{\mathcal{P}}_0 + i(3\nu_s - \tilde{\nu}_0)\omega^2\tilde{\mathcal{P}}_0^2\bigg]\bigg\}, \tag{E4d}$$

$$N_i \text{ ═∿∿ } h_{jk} \;= \frac{2G\omega\tilde{\nu}_0}{3(\nu_s - \tilde{\nu}_0)k}\bigg\{-\hat{k}_i\delta_{jk}\big[\mathcal{P}_s - \tilde{\mathcal{P}}_0\big] + \hat{k}_i\hat{k}_j\hat{k}_k(3\nu_s - \tilde{\nu}_0)\bigg[\frac{\mathcal{P}_s - \tilde{\mathcal{P}}_0}{\nu_s - \tilde{\nu}_0} + ik^6\tilde{\mathcal{P}}_0^2\bigg]\bigg\}, \tag{E4e}$$

$$\begin{aligned} h_{ij} \text{ ∿∿∿ } h_{kl} \;= 2G\bigg\{&(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathcal{P}_{tt} - \delta_{ij}\delta_{kl}\bigg[\mathcal{P}_{tt} - \frac{\mathcal{P}_s}{3}\bigg] \\ &- (\delta_{ik}\hat{k}_j\hat{k}_l + \delta_{il}\hat{k}_j\hat{k}_k + \delta_{jk}\hat{k}_i\hat{k}_l + \delta_{jl}\hat{k}_i\hat{k}_k)\big[\mathcal{P}_{tt} - \mathcal{P}_1\big] \\ &+ (\delta_{ij}\hat{k}_k\hat{k}_l + \delta_{kl}\hat{k}_i\hat{k}_j)\bigg[\mathcal{P}_{tt} - \frac{3\nu_s - \tilde{\nu}_0}{3(\nu_s - \tilde{\nu}_0)}\mathcal{P}_s + \frac{2\tilde{\nu}_0}{3(\nu_s - \tilde{\nu}_0)}\tilde{\mathcal{P}}_0\bigg] \\ &+ \hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l\bigg[\mathcal{P}_{tt} + \frac{(3\nu_s - \tilde{\nu}_0)^2}{3(\nu_s - \tilde{\nu}_0)^2}\mathcal{P}_s - 4\mathcal{P}_1 \\ &- \frac{4\tilde{\nu}_0(3\nu_s - 2\tilde{\nu}_0)}{3(\nu_s - \tilde{\nu}_0)^2}\tilde{\mathcal{P}}_0 + i\frac{2\tilde{\nu}_0(3\nu_s - \tilde{\nu}_0)k^6}{3(\nu_s - \tilde{\nu}_0)}\tilde{\mathcal{P}}_0^2\bigg]\bigg\}. \end{aligned} \tag{E4f}$$

Here $\mathcal{P}_{tt}$ and $\mathcal{P}_s$ are the same as in (C4), and ν_s is given by the $\lambda \to \infty$ limit of Eq. (2.22), $\nu_s = (8/3)\nu_4 + \nu_5$.

We observe that all propagators (E2), (E4) are regular in the sense defined in Sec. V B. This follows from three properties. First, the pole factors $\mathcal{P}_{tt}$, $\mathcal{P}_s$, $\mathcal{P}_1$, $\tilde{\mathcal{P}}_0$ are regular. Second, the propagators scale homogeneously under the Lifshitz transformations, in the way compatible with the scaling dimensions of the corresponding fields. And third, the inverse powers of the spatial momentum k contained in the unit vector $\hat{k}_i$ cancel when we bring the combinations in the square brackets to the common denominator. As shown in Ref. [17], the regularity of the propagators is sufficient for the renormalizability of the theory.

Let us also note the presence of double poles $\tilde{\mathcal{P}}_0^2$. They signal presence of a linearly growing gauge mode, similarly as it happens in the Maxwell theory in general covariant gauges (see e.g., Sec. 18 of [45]).

Mixing between different components in the Lagrangian (E1) implies that the scalar graviton state has overlap not only with the metric h_{ij}, but also the shift N_i and the field χ. To see this, let us write the eigenmode equations following from (E1):

$$\omega^2\psi - 2i\omega\chi - 3\nu_s k^6\psi = 0, \tag{E5a}$$

$$\omega^2 E - 2i\omega\chi - \tilde{\nu}_0 k^6 E = 0, \tag{E5b}$$

$$-\omega^2 B + \tilde{\nu}_0 k^6 B - \tilde{\nu}_0 k^4\chi = 0, \tag{E5c}$$

$$2i\omega\psi + i\omega E - 2k^2 B = 0. \tag{E5d}$$

Substituting here the dispersion relation of the scalar graviton, $\omega = \sqrt{\nu_s} k^3$, and expressing all fields in terms of ψ we obtain

$$E = -\frac{2\nu_s}{\nu_s - \tilde{\nu}_0}\psi, \quad B = -\frac{i\tilde{\nu}_0\sqrt{\nu_s}k}{\nu_s - \tilde{\nu}_0}\psi, \quad \chi = i\sqrt{\nu_s}k^3\psi. \tag{E6}$$

So indeed, for the scalar graviton eigenmode all fields are in general nonvanishing.

The situation simplifies considerably if we choose the gauge $\xi = -1$, entailing $\tilde{\nu}_0 = 0$. Then the admixture of the scalar graviton to the shift vanishes, which also eliminates the mixed propagators (E4b), (E4e). The normalization of the scalar graviton mode is deduced by imposing the canonical commutations relations on ψ and its conjugate momentum

$$\pi_\psi = \frac{\dot{\psi} - 2\chi}{2G}.$$

Collecting everything together, we find the scalar graviton contribution to the metric and the field χ in the $\xi = -1$ gauge,

$$h_{ij}(\mathbf{x}, t) \ni \sqrt{G}\int \frac{d^3k}{(2\pi)^3 2\omega_s}\varepsilon_{ij}^{(0')} h_{\mathbf{k}0'} \mathrm{e}^{-i\omega_s t + i\mathbf{k}\mathbf{x}} + \mathrm{H.c.},$$

$$\varepsilon_{ij}^{(0')} = \sqrt{\frac{2}{3}}(\delta_{ij} - 3\hat{k}_i\hat{k}_j), \tag{E7a}$$

$$\chi(\mathbf{x}, t) = \sqrt{G}\int \frac{d^3k}{(2\pi)^3 2\omega_s} i\omega_s \sqrt{\frac{2}{3}} h_{\mathbf{k}0'} \mathrm{e}^{-i\omega_s t + i\mathbf{k}\mathbf{x}} + \mathrm{H.c.}, \tag{E7b}$$

where $h_{\mathbf{k}0'}$ is the scalar graviton annihilation operator satisfying

$$[h_{\mathbf{k}0'}, h^{+}_{\mathbf{k}'0'}] = 2\omega_s(2\pi)^3\delta(\mathbf{k} - \mathbf{k}'). \tag{E8}$$

This provides us with the expressions for the external lines of the scalar diagrams for the scalar graviton scattering. The form of the h-line is unchanged, see Eq. (C1b), with the polarization tensor from (E7a). Whereas the χ-line reads

$$\chi \text{ - - - - - - - - } = i\sqrt{\frac{2G}{3}}\,\omega\,. \tag{E9}$$

[1] P. Horava, Quantum gravity at a Lifshitz point, Phys. Rev. D **79**, 084008 (2009).

[2] D. Blas, O. Pujolas, and S. Sibiryakov, Models of non-relativistic quantum gravity: The good, the bad and the healthy, J. High Energy Phys. 04 (2011) 018.

[3] S. Mukohyama, Horava-Lifshitz cosmology: A review, Classical Quantum Gravity **27**, 223101 (2010).

[4] T. P. Sotiriou, Horava-Lifshitz gravity: A status report, J. Phys. Conf. Ser. **283**, 012034 (2011).

[5] A. Wang, Hořava gravity at a Lifshitz point: A progress report, Int. J. Mod. Phys. D **26**, 1730014 (2017).

[6] A. O. Barvinsky, Hořava models as palladium of unitarity and renormalizability in quantum gravity, arXiv:2301.13580.

[7] K. S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D **16**, 953 (1977).

[8] K. S. Stelle, Classical gravity with higher derivatives, Gen. Relativ. Gravit. **9**, 353 (1978).

[9] A. Salvio and A. Strumia, Agravity, J. High Energy Phys. 06 (2014) 080.

[10] M. B. Einhorn and D. R. T. Jones, Naturalness and dimensional transmutation in classically scale-invariant gravity, J. High Energy Phys. 03 (**2015**) 047.

[11] A. Salvio and A. Strumia, Quantum mechanics of 4-derivative theories, Eur. Phys. J. C **76**, 227 (2016).

[12] A. Strumia, Interpretation of quantum mechanics with indefinite norm, MDPI Phys. **1**, 17 (2019).

[13] D. Blas, O. Pujolas, and S. Sibiryakov, Consistent Extension of Horava Gravity, Phys. Rev. Lett. **104**, 181302 (2010).

[14] A. Emir Gümrükçüoğlu, M. Saravani, and T. P. Sotiriou, Hořava gravity after GW170817, Phys. Rev. D **97**, 024032 (2018).

[15] J. Bellorin, C. Borquez, and B. Droguett, Cancellation of divergences in the nonprojectable Hořava theory, Phys. Rev. D **106**, 044055 (2022).

[16] J. Bellorin, C. Borquez, and B. Droguett, BRST symmetry and unitarity of the Hořava theory, Phys. Rev. D **107**, 044059 (2023).

[17] A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov, and C. F. Steinwachs, Renormalization of Hořava gravity, Phys. Rev. D **93**, 064022 (2016).

[18] A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov, and C. F. Steinwachs, Renormalization of gauge theories in the background-field approach, J. High Energy Phys. 07 (2018) 035.

[19] A. O. Barvinsky, M. Herrero-Valea, and S. M. Sibiryakov, Towards the renormalization group flow of Horava gravity in (3 + 1) dimensions, Phys. Rev. D **100**, 026012 (2019).

[20] A. O. Barvinsky, A. V. Kurov, and S. M. Sibiryakov, Beta functions of (3 + 1)-dimensional projectable Hořava gravity, Phys. Rev. D **105**, 044009 (2022).

[21] A. E. Gumrukcuoglu and S. Mukohyama, Horava-Lifshitz gravity with $\lambda \to \infty$, Phys. Rev. D **83**, 124033 (2011).

[22] A. Frenkel, P. Horava, and S. Randall, Perelman's Ricci flow in topological quantum gravity, arXiv:2011.11914.
[23] J. M. Martin-Garcia, xAct: Efficient tensor computer algebra for the Wolfram Language, http://www.xact.es/.
[24] J. M. Martin-Garcia, xPerm: Fast index canonicalization for tensor computer algebra, Comput. Phys. Commun. **179**, 597 (2008).
[25] D. Brizuela, J. M. Martin-Garcia, and G. A. Mena Marugan, xPert: Computer algebra for metric perturbation theory, Gen. Relativ. Gravit. **41**, 2415 (2009).
[26] T. Nutma, xTras: A field-theory inspired xAct package for mathematica, Comput. Phys. Commun. **185**, 1719 (2014).
[27] Wolfram Research, Inc., Mathematica, Version 12.2, Champaign, IL, 2020.
[28] J. I. Radkovski, https://github.com/JIRadkovski.
[29] K. Koyama and F. Arroja, Pathological behaviour of the scalar graviton in Hořava-Lifshitz gravity, J. High Energy Phys. 03 (2010) 061.
[30] K. Izumi and S. Mukohyama, Nonlinear superhorizon perturbations in Horava-Lifshitz gravity, Phys. Rev. D **84**, 064025 (2011).
[31] A. E. Gumrukcuoglu, S. Mukohyama, and A. Wang, General relativity limit of Horava-Lifshitz gravity with a scalar field in gradient expansion, Phys. Rev. D **85**, 064042 (2012).
[32] C. Becchi, A. Rouet, and R. Stora, Renormalization of gauge theories, Ann. Phys. (N.Y.) **98**, 287 (1976).
[33] I. V. Tyutin, Gauge invariance in field theory and statistical physics in operator formalism, arXiv:0812.0580.
[34] S. Weinberg, *The Quantum Theory of Fields. Vol. 2: Modern Applications* (Cambridge University Press, Cambridge, England, 2013).
[35] T. Kugo and I. Ojima, Manifestly covariant canonical formulation of Yang-Mills field theories. 1, Prog. Theor. Phys. **60**, 1869 (1978).
[36] T. Kugo and I. Ojima, Local covariant operator formalism of nonabelian gauge theories and quark confinement problem, Prog. Theor. Phys. Suppl. **66**, 1 (1979).
[37] C. Becchi, Introduction to BRS symmetry, arXiv:hep-th/9607181.
[38] R. Iengo and M. Serone, A simple UV-completion of QED in 5D, Phys. Rev. D **81**, 125005 (2010).
[39] S. Sannan, Gravity as the limit of the type II superstring theory, Phys. Rev. D **34**, 1749 (1986).
[40] D. Blas, O. Pujolas, and S. Sibiryakov, Comment on "Strong coupling in extended Horava-Lifshitz gravity", Phys. Lett. B **688**, 350 (2010).
[41] D. Anselmi, Weighted power counting and Lorentz violating gauge theories. I. General properties, Ann. Phys. (Amsterdam) **324**, 874 (2009).
[42] H. Elvang and Y.-t. Huang, Scattering amplitudes, arXiv:1308.1697.
[43] J. Collins, A new approach to the LSZ reduction formula, arXiv:1904.10923.
[44] C. Itzykson and J. Zuber, *Quantum Field Theory*, Dover Books on Physics (Dover Publications, New York, 2012).
[45] J. Strathdee, QED Lecture Notes, ICTP, 1995, http://streaming.ictp.it/preprints/P/95/315.pdf.