Pole skipping in a non-black-hole geometry

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Pole skipping has been discussed in black-hole backgrounds, but we point out that pole skipping exists even in a non-black-hole background, the anti–de Sitter soliton. For black holes, the pole-skipping points are typically located at imaginary Matsubara frequencies $\omega = -(2\pi T)ni$ with an integer n. The anti–de Sitter soliton is obtained by the double Wick rotation from a black hole. As a result, the pole-skipping points are located at $q_z = -(2\pi n)/l$, where l is the S^1 periodicity and q_z is the S^1 momentum. The "chaotic" and the "hydrodynamic" pole-skipping points lie in the physical region. We also propose a method to identify all pole-skipping points instead of the conventional method.

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I. INTRODUCTION AND SUMMARY

Retarded Green's functions play fundamental roles in physics, and considerable knowledge has been accumulated over the years. However, surprisingly, a new universal feature of Green's functions are found using the AdS/CFT duality or holographic duality [1–4] (see, e.g., Refs. [5–10]). This feature is known as *pole skipping* [11–15]. Since then, various aspects of pole skipping have been investigated (see, e.g., Refs. [16–39]).

Typical bulk perturbation problems are the scalar field, the Maxwell field, and the gravitational field. Dual Green's functions are not uniquely determined at "pole-skipping points" in the complex momentum space (ω, q) where ω is the frequency and q is the wave number. Near a pole-skipping point, Green's function typically takes the form

$$G^R \propto \frac{\delta\omega + \delta q}{\delta\omega - \delta q}. \tag{1.1}$$

In this sense, Green's function is not uniquely determined, and it depends on the slope $\delta q/\delta\omega$ how one approaches the pole-skipping point. The hydrodynamic pole is an elementary example. For example, the sound mode behaves as

$$G^R \propto \frac{q^2}{3\omega^2 - q^2} \to \frac{\delta(q^2)}{3\delta(\omega^2) - \delta(q^2)}. \tag{1.2}$$

From the bulk point of view, pole skipping occurs because the bulk solution is not uniquely determined there.

There is a universality for the pole-skipping points ω. In these examples, the pole-skipping points are located at Matsubara frequencies.[1] The pole-skipping points start from $\mathfrak{w} := \omega/(2\pi T) = (s-1)i$ (s is the spin of the bulk field) and continue to $\mathfrak{w}_n = (s-1-n)i$ for a non-negative integer n. Namely, for the scalar field, they start from $\mathfrak{w}_1 = -i$. For the Maxwell field, they start from $\mathfrak{w}_0 = 0$, which is the hydrodynamic pole. In the gravitational sound mode, they start from $\mathfrak{w}_{-1} = +i$. It is argued that the $\mathfrak{w}_{-1} = +i$ point is related to many-body quantum chaos [40–44].

Pole skipping was first found in the context of holographic chaos [11,12], so one naturally has studied pole skipping at a finite temperature or in a black-hole background. However, an analogous phenomenon was found even in an elementary quantum mechanics problem [33], so one may discuss the phenomena in a broader context.

In this paper, we study pole skipping in a non-black-hole geometry, the anti–de Sitter (AdS) soliton [45]. The AdS

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[1]A rotating black hole is an exception [28]. For the rotating Bañados-Teitelboim-Zanelli black hole, the pole-skipping points ω depend on the left-moving temperature T_L, the right-moving temperature T_R, and the conformal dimension Δ_+ of the dual boundary operator.

soliton is obtained by the double Wick rotation from the AdS black hole (Sec. III). The AdS soliton is not a black hole. The geometry has a compact S^1-direction z with periodicity l, and the geometry ends smoothly at the "horizon." The AdS black hole describes the plasma phase of a large-N_c gauge theory, whereas the AdS soliton describes the confining phase.

Our results are summarized as follows:

(1) Because of the double Wick rotation, the universality of the pole-skipping points ω is translated to the universality of the pole-skipping points $\mathfrak{q}_z := q_z/(2\pi/l)$, where q_z is the wave number in the S^1-direction z.

(2) The pole-skipping points start from $\mathfrak{q}_z = (s-1)$ and continue to $\mathfrak{q}_z = -n$. While some pole-skipping points lie in the physical region (Sec. IV), most pole-skipping points do not lie in the physical region. The former corresponds to the pole-skipping points in the upper-half ω-plane in the black-hole case, namely the "chaotic" mode and the "hydrodynamic" mode. Of course, the AdS soliton is not a black hole, so the chaotic mode does not imply a chaotic behavior.

(3) Because q_z is the S^1 momentum, there is also the mirror image of the pole-skipping tower. Namely, they start from $\mathfrak{q}_z = -(s-1)$ and continue to $\mathfrak{q}_z = n$.

We discuss the bulk scalar field, the Maxwell field, and the gravitational field. There is a conventional pole-skipping method, but we propose an alternative method in order to analyze the pole skipping systematically (Sec. II). The conventional method requires a separate treatment for the chaotic and hydrodynamic pole skippings, but our method covers them as well. The perturbation problems in the AdS soliton with $\mathfrak{q}_z \neq 0$ are little discussed in the literature, and our work is interesting from this point of view as well.

II. POLE SKIPPING

In this section, we briefly review pole skipping in the context of the Schwarzschild-AdS$_5$ (SAdS$_5$) black hole:

$$ds_5^2 = r^2(-f dt^2 + dx^2 + dy^2 + dz^2) + \frac{dr^2}{r^2 f} \tag{2.1a}$$

$$= \frac{r_0^2}{u}(-f dt^2 + dx^2 + dy^2 + dz^2) + \frac{du^2}{4u^2 f}, \tag{2.1b}$$

$$f = 1 - \left(\frac{r_0}{r}\right)^4 = 1 - u^2, \tag{2.1c}$$

where $u := r_0^2/r^2$. For simplicity, we set the AdS radius $L = 1$ and the horizon radius $r_0 = 1$. The Hawking temperature is given by $\pi T = r_0/L^2$.

We consider the perturbation of the form

$$Z(u)e^{-i\omega t + iqx}. \tag{2.2}$$

Near the horizon, the field equation typically takes the form

$$0 \sim Z'' + \frac{1}{u-1}Z' + \frac{\mathfrak{w}^2}{4(u-1)^2}Z \qquad (u \to 1), \tag{2.3}$$

where $' = \partial_u$ and $\mathfrak{w} := \omega/(2\pi T) = \omega/2$, so the solution behaves as

$$Z \propto (u-1)^{\pm i\mathfrak{w}/2}. \tag{2.4}$$

As usual, we impose the "incoming-wave" boundary condition at the horizon, so we set the ansatz

$$Z = (u-1)^{-i\mathfrak{w}/2}\tilde{Z}. \tag{2.5}$$

As a typical example of pole skipping, we consider the field equation of the form

$$0 = \tilde{Z}'' + P(u)\tilde{Z}' + Q(u)\tilde{Z}. \tag{2.6}$$

The horizon $u = 1$ is a regular singularity, and P and Q are expanded as

$$P = \frac{P_{-1}}{u-1} + P_0 + P_1(u-1) + \cdots, \tag{2.7a}$$

$$Q = \frac{Q_{-1}}{u-1} + Q_0 + Q_1(u-1) + \cdots. \tag{2.7b}$$

One typically has $P_{-1} = 1 - i\mathfrak{w}$ and $Q_{-1} = Q_{-1}(\mathfrak{w}, \mathfrak{q}^2)$, where $\mathfrak{q} := q/(2\pi T)$.

We first review a conventional method [14] and then propose an alternative method.

A. Conventional method

The solution can be written as a power series:

$$\tilde{Z}(u) = \sum_{n=0} a_n (u-1)^{n+\lambda}. \tag{2.8}$$

Substituting this into the field equation, one obtains the indicial equation at the lowest order:

$$\lambda = 0, \qquad 1 - P_{-1}. \tag{2.9}$$

The coefficient a_n is obtained by a recursion relation. The mode with $\lambda = 0$ ($\lambda = 1 - P_{-1} = i\mathfrak{w}$) represents the incoming (outgoing) mode, and we choose the incoming mode $\lambda = 0$ hereafter.

To obtain pole-skipping points systematically, write the rest of the field equation in a matrix form [14]:

$$0 = M\tilde{Z} \tag{2.10a}$$

$$= \begin{pmatrix} M_{11} & M_{12} & 0 & 0 & \cdots \\ M_{21} & M_{22} & M_{23} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \end{pmatrix}. \tag{2.10b}$$

One can show that $M_{n,n+1} = n(n-1+P_{-1}) = n(n-i\mathfrak{w})$. The matrix $\mathcal{M}^{(n)}$ is obtained by keeping the first n rows and n columns of M. The pole-skipping points are obtained by

$$\boxed{M_{n,n+1}(\mathfrak{w}_n) = 0, \qquad \det \mathcal{M}^{(n)}(\mathfrak{w}_n, \mathfrak{q}_n) = 0.} \tag{2.11}$$

For example, consider the first row:

$$0 = M_{11}a_0 + M_{12}a_1 = Q_{-1}a_0 + P_{-1}a_1 = 0. \tag{2.12}$$

Normally, this equation determines a_1 from a_0. However, when $M_{12} = M_{11} = 0$ or $P_{-1} = Q_{-1} = 0$, both a_0 and a_1 are free parameters, and the bulk solution is not uniquely determined.

Similarly, when $M_{23} = \det \mathcal{M}^{(2)} = 0$, a_0 and a_2 become free parameters. One gets

$$M_{23} = 2(2 - i\mathfrak{w}) = 0, \tag{2.13a}$$

$$\det \mathcal{M}^{(2)} = Q_{-1}(Q_{-1} + P_0) - P_{-1}Q_0 = 0. \tag{2.13b}$$

As is clear from the construction, one can find pole-skipping points only in the lower-half ω-plane $\mathfrak{w}_n = -in(n > 0)$ in this method. The chaotic pole-skipping $n = -1$ and the hydrodynamic pole-skipping $n = 0$ need a separate treatment as pointed out in Ref. [14]. In the next subsection, we propose an alternative method where one can find all pole-skipping points $\mathfrak{w}_n = -in(n \geq -1)$.

B. Alternative method

1. The formalism

We start to write the field equation in a matrix form:

$$0 = \vec{X}' - M\vec{X}, \tag{2.14a}$$

$$\vec{X} := \begin{pmatrix} \tilde{Z} \\ \tilde{Z}' \end{pmatrix}, \tag{2.14b}$$

$$M := \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix}. \tag{2.14c}$$

The matrix M can be expanded as

$$M = \frac{M_{-1}}{u-1} + M_0 + M_1(u-1)\cdots. \tag{2.15}$$

The solution can be written as a power series:

$$\vec{X} = \sum_{n=0} \vec{a}_n (u-1)^{n+\lambda}. \tag{2.16}$$

Substituting this into the field equation, at the lowest order, one obtains

$$0 = (\lambda - M_{-1})\vec{a}_0. \tag{2.17}$$

This indicial equation is an eigenvalue equation for M_{-1}. The eigenvalue and the eigenvector of M_{-1} are

$$\lambda = 0, \qquad \vec{a}_0 = \begin{pmatrix} 1 \\ b_1 \end{pmatrix}, \qquad b_1 = -\frac{Q_{-1}}{P_{-1}}, \tag{2.18a}$$

$$\lambda = i\mathfrak{w} - 1, \qquad \vec{a}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.18b}$$

The mode with $\lambda = 0$ $(\lambda = i\mathfrak{w} - 1)$ represents the incoming (outgoing) mode, and we choose the incoming mode $\lambda = 0$ hereafter.

In this matrix formalism, one looks for the points where the coefficient $\vec{a}_n$ becomes ambiguous. The functions P_{-1} and Q_{-1} are polynomials in $(\mathfrak{w}, \mathfrak{q}^2)$, so b_1 is a rational function of $(\mathfrak{w}, \mathfrak{q}^2)$. Then, b_1 becomes ambiguous or becomes $0/0$ when $P_{-1} = Q_{-1} = 0$. Then, the bulk solution is not uniquely determined because the coefficient of the Frobenius series becomes ambiguous. This agrees with the first pole-skipping point in the conventional formalism in the last subsection.

Once one obtains $\vec{a}_0$, $\vec{a}_n$ is obtained recursively:

$$(\lambda + n - M_{-1})\vec{a}_n = \sum_{k=0}^{n-1} M_{n-1-k}\vec{a}_k, \tag{2.19}$$

with $\lambda = 0$.

For example, at the next order,

$$(1 - M_{-1})\vec{a}_1 = M_0\vec{a}_0, \tag{2.20a}$$

$$\vec{a}_1 = \begin{pmatrix} b_1 \\ 2b_2 \end{pmatrix}, \tag{2.20b}$$

$$2b_2 = \frac{Q_{-1}(Q_{-1} + P_0) - P_{-1}Q_0}{P_{-1}(P_{-1} + 1)}. \tag{2.20c}$$

Thus, b_2 becomes ambiguous when

$$P_{-1}+1=0,\qquad Q_{-1}(Q_{-1}+P_0)-P_{-1}Q_0=0. \tag{2.21}$$

This agrees with the second pole-skipping point in the conventional method (2.13).

In general, the coefficient $\vec{a}_n$ can be written as

$$\vec{a}_n=\begin{pmatrix} b_n \\ (n+1)b_{n+1}\end{pmatrix} \tag{2.22}$$

with $b_0=1$. The pole-skipping condition is

$$\boxed{b_n(\mathfrak{w}_n,\mathfrak{q}_n)=\frac{0}{0}.} \tag{2.23}$$

2. Advantages

The matrix formalism has two advantages compared with the conventional method.

First, as discussed above, this formalism gives the equivalent results to the conventional formalism, but there are important exceptions. When $Q_{-1}=0/0$, b_1 becomes ambiguous as well. This is the case for the chaotic pole skipping and the hydrodynamic pole skipping. We show this explicitly in the context of the AdS soliton (Sec. IV). Of course, to obtain all pole-skipping points, one would use the conventional formalism and impose the condition $Q_{-1}=0/0$ separately. But in this matrix formalism, all pole-skipping points are included naturally.

Namely, there are two kinds of pole skipping:

(1) The first one is the pole skipping in the lower-half ω-plane. It comes from the fact that field equations have a regular singularity at the horizon. In the conventional method, two roots of the indicial equation are $\lambda=0, i\mathfrak{w}$. λ depends on $\mathfrak{w}$ because of a regular singularity, and the pole skipping occurs when they differ by an integer.

(2) The second one is the pole skipping in the upper-half ω-plane, namely the chaotic and hydrodynamic pole skipping. In this case, the coefficients of the field equations themselves partly have the $0/0$ structure.

Second, the conventional method is applicable if one can find a master equation, but the matrix formalism is applicable even if one cannot find a master equation:

(1) It is often not easy to find a master field.

(2) Even if one finds a master field, the choice of a master field is not unique. This is particularly problematic for the pole-skipping analysis. As discussed in Appendix D, one is not able to find some pole-skipping points if one chooses a different master variable.

The matrix formalism does not have such disadvantages. In perturbation problems, one typically has coupled first-order differential equations with a multiple number of variables. For example, the Maxwell scalar mode (Sec. IV C 2) consists of two coupled first-order differential equations with two variables $(\mathfrak{A}_z,\mathfrak{A}_u)$. In this case, it is easy to find the master equation for the master field $\mathfrak{A}_z$. But if one chooses the master field $\mathfrak{A}_u$, one is not able to find the hydrodynamic pole-skipping (Appendix D).

In this case, it is not really necessary to use a master equation. It is enough to rewrite the coupled equations in a matrix form setting $\vec{X}$ as

$$\vec{X}\coloneqq\begin{pmatrix}\mathfrak{A}_z\\ \mathfrak{A}_u\end{pmatrix}. \tag{2.24}$$

3. A subtlety

There is a subtlety to use the matrix formalism for higher n. For example, consider b_2. In general, b_2 takes a complicated form, and its numerator may have an accidental zero at the first pole-skipping point $n=1$. Such points are not pole-skipping points and should be excluded. In fact, one can easily check that the eigenvector $\vec{a}_1$ does not have the slope dependence. $\vec{a}_1$ is given by

$$\vec{a}_1=\begin{pmatrix} b_1\\ 2b_2\end{pmatrix}, \tag{2.25}$$

and b_1 determines the first pole-skipping point. The accidental zero is not included in b_1, so b_1 diverges there. Also, such accidental ones in general do not appear at higher orders. We show this explicitly in the context of the AdS soliton (Sec. IV). To avoid them, a simple rule is

$$\text{Find } b_n=0/0 \quad \text{at } \mathfrak{w}_n,$$

and one should not consider smaller n because the fake ones appear.

We end this section with a list of first pole-skipping points:

(1) Massive scalar: $(\mathfrak{w},\mathfrak{q}^2)=(-i,-(6+m^2)/4)$.
(2) Maxwell vector: $(\mathfrak{w},\mathfrak{q}^2)=(-i,-1/2)$.
(3) Maxwell scalar: $(\mathfrak{w},\mathfrak{q}^2)=(0,0),(-i,+1/2)$.
(4) Gravitational vector (shear):
$(\mathfrak{w},\mathfrak{q}^2)=(0,0)$, $(-i,\ 3/2)$.
(5) Gravitational scalar (sound):
$(\mathfrak{w},\mathfrak{q}^2)=(+i,-3/2),(0,0),(-i,(1\pm 2\sqrt{2}i)/2)$.

The above results can be obtained from the master equations by Kovtun and Starinets [46] (Appendix A).

III. AdS SOLITON

The SAdS$_5$ black hole is given by Eq. (2.1). We now compactify the z-direction as $0\le z<l$. However, the compactified solution is not the only geometry with asymptotic geometry $\mathbb{R}^{1,2}\times S^1$. The AdS soliton is also the solution with the same asymptotic geometry. The AdS

soliton is obtained by the double Wick rotation from the $SAdS_5$ black hole:

$$t = i\hat{z}, \qquad \hat{t} = iz. \tag{3.1}$$

Then, the metric becomes

$$ds_5^2 = \frac{r_0^2}{u}(-d\hat{t}^2 + dx^2 + dy^2 + f d\hat{z}^2) + \frac{du^2}{4u^2 f}, \tag{3.2}$$

with $f = 1 - u^2$. We set $r_0 = 1$.

For the $SAdS_5$ black hole, the imaginary time direction has the periodicity $\beta = \pi/r_0$ to avoid a conical singularity. Similarly, for the AdS soliton, $\hat{z}$ has the periodicity $l = \pi/r_0$. The AdS soliton is not a black hole. Rather, it has a cigarlike geometry, and the geometry ends smoothly at $u = 1$ because of the factor f just as the Euclidean black hole. We focus on the asymptotically AdS_5 geometry, but the generalization to the other dimensions is straightforward.

The $SAdS_5$ black hole describes the plasma phase, whereas the AdS soliton describes the confining phase. Some evidences are

(1) The spectrum has a mass gap [3,47,48].
(2) The quark-antiquark potential is linear and describes the confining potential.
(3) The $SAdS_5$ black hole has entropy density $s \sim O(N_c^2)$, whereas the AdS soliton has $s \sim 0$ at leading order in N_c because it is not a black hole.

See Appendix B for more details.

For the $SAdS_5$, there is a $SO(3)$ invariance for the boundary direction (x, y, z), so one can set the perturbation of the form

$$\phi(u)e^{-i\omega t + iqx} \tag{3.3}$$

without loss of generality. For the AdS soliton, the invariance is broken due to the S^1-direction z, so we consider the perturbation of the form

$$\phi(u)e^{-i\hat{\omega}\hat{t} + iq_x x + i\hat{q}_z\hat{z}}. \tag{3.4}$$

However, there is a remaining $SO(1,2)$ invariance for $(\hat{t}, x, y)$, so $\hat{\omega}$ and q_x appear only in the combination $p^2 \coloneqq -\hat{\omega}^2 + q_x^2$.

Because the z-direction is compact, $\hat{q}_z$ physically takes only a discrete value:

$$\hat{q}_z = \frac{2\pi n}{l} \quad \text{or} \quad \mathfrak{q}_z \coloneqq \frac{\hat{q}_z}{\frac{2\pi}{l}} = n \tag{3.5}$$

for an integer n. But we make it continuous. Such a treatment is often done in the pole-skipping literature. For example, the Bañados-Teitelboim-Zanelli black hole has a compact S^1-direction x, but one makes it noncompact for the pole-skipping analysis. It is similar in spirit to the S-matrix approach: even though the angular momentum l is discrete physically, one analytically continues to the whole complex l-plane.

A. Expected results

The double Wick rotation $t = i\hat{z}$, $\hat{t} = iz$ means that

$$i\hat{q}_z = \omega, \qquad iq_z = \hat{\omega}. \tag{3.6}$$

Namely, the role of ω and q_z is exchanged. Thus, one expects that

(1) There is no universality for $\hat{\omega}$ at pole-skipping points. Rather, there is a universality for $\hat{q}_z$.
(2) The pole-skipping points start from $\hat{\mathfrak{q}}_z = s - 1$, where s is the spin of the bulk field. The pole-skipping tower continues to $\hat{\mathfrak{q}}_z = s - 1 - n$ where n is a non-negative integer.
(3) Because $\hat{\mathfrak{q}}_z$ can be both positive and negative, there is also the mirror image of the pole-skipping tower. Namely, they start from $\hat{\mathfrak{q}}_z = -(s-1)$ and continue to $\hat{\mathfrak{q}}_z = -(s-1-n)$. For simplicity, we consider only the tower of item 2.
(4) For $SAdS_5$, the pole-skipping points depend on q. For the AdS soliton, the pole-skipping points depend on $p^2 \coloneqq -\hat{\omega}^2 + q_x^2$.

These results are easily expected, but we derive various field equations in the AdS soliton background explicitly and show that the above results indeed hold. Also, we have not discussed the boundary condition at the tip of the cigar $u = 1$. One should ask about the double Wick rotation including the boundary condition, so we discuss the boundary condition below.

To distinguish the $SAdS_5$ and the AdS soliton, we use variables such as $\hat{t}$, but we omit "$\hat{\ }$" in the rest of our paper.

B. Boundary condition at the tip of the cigar

As we see below, all master fields behave as

$$0 \sim Z'' + \frac{1}{u-1}Z' - \frac{\mathfrak{q}_z^2}{4(u-1)^2}Z \qquad (u \to 1), \tag{3.7}$$

near the tip of the cigar. Thus, there are two independent solutions:

$$Z \propto (u-1)^{\pm \mathfrak{q}_z/2}. \tag{3.8}$$

For simplicity, we set $\mathfrak{q}_z > 0$ below. The generic solution is a linear combination of these two solutions. The problem is how to choose the boundary condition at $u = 1$.

We follow the standard textbook treatment of quantum mechanics (Chap. 35 of [49]). By redefining $Z = G\varphi$

where $G = (u-1)^{-1/2}$, the master equation reduces to the Schrödinger problem near $u \to 1$:

$$0 \sim \varphi'' - \frac{\mathfrak{q}_z^2 - 1}{4x^2}\varphi \qquad (x \to 0), \tag{3.9}$$

where $x \coloneqq u - 1$. We impose the "UV cutoff" at small x:

$$V(x) = \begin{cases} V(x), & x > x_0, \\ V(x_0) = V_0, & x < x_0. \end{cases} \tag{3.10}$$

As usual, we impose the conditions that the solution and its derivative are continuous at $x = x_0$, and we take the $x_0 \to 0$ limit. When $x > x_0$, the generic solution is

$$\varphi \sim C_1 x^{s_1} + C_2 x^{s_2}, \tag{3.11a}$$

$$s_1 = \frac{1 + \mathfrak{q}_z}{2}, \qquad s_2 = \frac{1 - \mathfrak{q}_z}{2}, \tag{3.11b}$$

where $s_1 > s_2$. One can show

$$\frac{C_2}{C_1} \propto x_0^{s_1 - s_2} \sim x_0^{\mathfrak{q}_z} \to 0. \tag{3.12}$$

So, it is enough to consider the solution $\varphi \sim (u-1)^{s_1}$ which falls faster. This is equivalent to choose $Z \sim (u-1)^{\mathfrak{q}_z/2}$ as the boundary condition.

For a black hole, we impose the incoming-wave boundary condition $Z \sim (u-1)^{-i\mathfrak{w}/2}$, and the above choice is the analytic continuation from the black-hole case. Both the geometry and the boundary condition are obtained by the double Wick rotation, so one expects that the pole-skipping results are also obtained by the double Wick rotation.

For simplicity, we set $\mathfrak{q}_z > 0$ here; namely we set $\mathfrak{q}_z > 0$ as the physical region. But $\mathfrak{q}_z < 0$ should also be possible. In this case, one chooses $Z \sim (u-1)^{-\mathfrak{q}_z/2}$ as the boundary condition.

C. Relation to quantum mechanical pole skipping

Reference [33] finds a pole-skipping-like phenomenon in quantum mechanics. It studies various potential problems with angular momentum $l \eqqcolon \nu - 1/2$:

$$0 = -\partial_x^2\psi + V\psi - k^2\psi, \tag{3.13a}$$

$$V = \frac{\nu^2 - 1/4}{x^2} + V_1. \tag{3.13b}$$

It turns out that the S-matrix is not uniquely determined at

$$\nu = -\frac{n}{2} \qquad (n = 1, 2, \ldots) \tag{3.14}$$

with appropriate k. Reference [33] considers

(1) The Coulomb potential
(2) The Pöshl-Teller potentials

as V_1. In those examples, near $x \to 0$, the angular momentum part dominates:

$$0 \sim -\partial_x^2\psi + \frac{\nu^2 - 1/4}{x^2} \qquad (x \to 0), \tag{3.15}$$

so the solution is

$$\psi \sim x^{\lambda_\pm}, \qquad \lambda_\pm = \frac{1}{2} \pm \nu. \tag{3.16}$$

For physical momentum, $\nu \geq 1/2$, so one chooses λ_+. Comparing with the AdS soliton case, $\mathfrak{q}_z = 2\nu$, so this corresponds to choose s_1. The quantum mechanical pole skipping occurs at $\nu = -n/2$ ($n = 1, 2, \ldots$). This translates into the $\mathfrak{q}_z = -n$ pole skipping for the AdS soliton. As far as the $x \to 0$ behavior is concerned, the quantum mechanical pole skip and the AdS soliton pole skip reduce to the same problem.

IV. POLE SKIPPING IN THE AdS SOLITON GEOMETRY

A. Massive scalar

We start with a massive scalar field. The field equation is given by

$$0 = (\nabla^2 - m^2)\phi \tag{4.1a}$$

$$\propto \phi'' + \left(\frac{f'}{f} - \frac{1}{u}\right)\phi' - \frac{4\mathfrak{q}_z^2 + (4\mathfrak{p}^2 + m^2/u)f}{4uf^2}\phi. \tag{4.1b}$$

where $\mathfrak{p}^2 \coloneqq -\mathfrak{w}^2 + \mathfrak{q}_x^2$ with $\mathfrak{w} \coloneqq \omega/(2\pi/l), \mathfrak{q}_x \coloneqq q_x/(2\pi/l)$. Near the tip $u = 1$, the field equation takes the form of Eq. (3.7):

$$0 \sim \phi'' + \frac{1}{u-1}\phi' - \frac{\mathfrak{q}_z^2}{4(u-1)^2}\phi. \tag{4.2}$$

So, set the ansatz

$$\phi = (1 - u^2)^{\mathfrak{q}_z/2}\tilde{Z}. \tag{4.3}$$

Then,

$$0 \sim \tilde{Z}'' + \frac{1 + \mathfrak{q}_z}{u-1}\tilde{Z}' + O\left(\frac{1}{u-1}\right)\tilde{Z}. \tag{4.4}$$

Using the matrix formalism, one obtains the eigenvalue and the eigenvector $\vec{a}_0$:

$$\lambda = 0, \qquad \vec{a}_0 = \begin{pmatrix} 1 \\ b_1 \end{pmatrix}, \qquad b_1 = -\frac{4\mathfrak{p}^2 + 6\mathfrak{q}_z^2 + m^2}{8(\mathfrak{q}_z + 1)}. \tag{4.5}$$

Thus, the first pole-skipping point is given by

$$(\mathfrak{q}_z, \mathfrak{p}^2) = \left(-1, -\frac{6+m^2}{4}\right). \tag{4.6}$$

The pole-skipping point lies outside the physical region $\mathfrak{q}_z \geq 0$. Now, move away from the pole-skipping point: $\mathfrak{q}_z = \mathfrak{q}_* + \delta\mathfrak{q}_z, \mathfrak{p}^2 = \mathfrak{p}_*^2 + \delta(\mathfrak{p}^2)$, where $(\mathfrak{q}_*, \mathfrak{p}_*^2)$ is the pole-skipping point. Then, $\vec{a}_0$ actually has the slope dependence:

$$b_1 = \frac{3}{2} - \frac{\delta(\mathfrak{p}^2)}{2\delta\mathfrak{q}_z}. \tag{4.7}$$

At the next order, b_2 has a complicated form, so we give the $m = 0$ result for simplicity:

$$2b_2 = \frac{4\mathfrak{p}^4 + 3\mathfrak{q}_z^2(2 + 6\mathfrak{q}_z + 3\mathfrak{q}_z^2) + 4\mathfrak{p}^2(2 + 4\mathfrak{q}_z + 3\mathfrak{q}_z^2)}{16(\mathfrak{q}_z + 1)(\mathfrak{q}_z + 2)}, \tag{4.8}$$

and b_2 is ambiguous at

$$(\mathfrak{q}_z, \mathfrak{p}^2) = \left(-1, -\frac{3}{2}\right), \left(-1, \frac{1}{2}\right), (-2, -3 \pm \sqrt{3}). \tag{4.9}$$

It includes the first pole-skipping point $(-1, -3/2)$. On the other hand, a new first pole-skipping point $(-1, 1/2)$ seems to appear. But this is a fake one as discussed in Sec. II B. In fact, it does not have the slope dependence. Near the point,

$$\vec{a}_1 = \begin{pmatrix} b_1 \\ 2b_2 \end{pmatrix}, \tag{4.10a}$$

$$b_1 = \frac{-2 - \delta(\mathfrak{p}^2) + 3\delta\mathfrak{q}_z}{2\delta\mathfrak{q}_z} \to \infty, \qquad 2b_2 = \frac{4\delta(\mathfrak{p}^2) + \delta\mathfrak{q}_z}{8\delta\mathfrak{q}_z}. \tag{4.10b}$$

Finally, b_3 is ambiguous at

$$\begin{aligned}(\mathfrak{q}_z, \mathfrak{p}^2) = (-3, -3/2), &\qquad \left(-3, \frac{-15 \pm 2\sqrt{6}}{2}\right), \\ (-2, -3 \pm \sqrt{3}), &\qquad (-2, 0), \\ (-1, -3/2), &\qquad \left(-1, \frac{-3 \pm \sqrt{3}}{2}\right). \end{aligned} \tag{4.11}$$

Note that the fake one from b_2 disappears. On the other hand, new fake ones appear: $(-2, 0)$, $(-1, (-3 \pm \sqrt{3})/2)$.

B. Gauge-invariant variables and master equations

To discuss the Maxwell and gravitational perturbations, we first decompose the background spacetime into two parts $x^M = (x^a, y^i)$, where $x^a = (z, u)$ and $y^i = (t, x, y)$:

$$ds_5^2 = g_{ab}(u) dx^a dx^b + \frac{1}{u}\eta_{ij} dy^i dy^j. \tag{4.12}$$

Here, the decomposition is chosen so that the y^i-part remains maximally symmetric. We decompose perturbations under the transformation of y^i. The scalar (vector) mode transforms as a scalar (vector) under the transformation.

As an example, consider the Maxwell perturbations $A_M = (A_a, A_i)$. One normally fixes the gauge $A_u = 0$. We do not fix the gauge and carry out analysis in a fully gauge-invariant manner (Appendix C). This is essentially the formalism by Kodama and Ishibashi [50].

For the Maxwell case, gauge-invariant variables are

(1) Scalar mode: $\mathfrak{A}_z$ and $\mathfrak{A}_u$.

(2) Vector mode: A_{Tx} and $A_{Ty} = A_y$.

However, gauge-invariant variables are not independent, and they are related by the Maxwell equation. Both scalar variables obey first-order differential equations, so one gets a second-order differential equation for a single variable. They are referred to as the master equation and the master field.

Thus, there is only 1 degree of freedom, but the choice of a master field is not unique. One can choose any linear combination of gauge-invariant variables as a master field. However, from the holographic point of view, it is natural to choose a master variable that does not involve u-derivatives of perturbations. This is because one imposes the Dirichlet boundary condition on the boundary. We choose such master fields below (see Appendix D for more comments). The master equations we derive below coincide with the master equations for the SAdS$_5$ black hole [46] after the double Wick rotation.

C. Maxwell field

1. Maxwell vector mode

The vector mode A_y is gauge invariant by itself. The Maxwell equation becomes

$$0 = A_y'' + \frac{f'}{f} A_y' - \frac{\mathfrak{q}_z^2 + \mathfrak{p}^2 f}{uf^2} A_y. \tag{4.13}$$

Asymptotically, $A_y \sim A + Bu$. Near the tip $u = 1$, the field equation takes the form of Eq. (3.7), so set the ansatz $A_y = (1 - u^2)^{\mathfrak{q}_z/2} \tilde{Z}$. Then,

$$0 \sim \tilde{Z}'' + \frac{1 + \mathfrak{q}_z}{u - 1} \tilde{Z}' + O\left(\frac{1}{u-1}\right) \tilde{Z}. \tag{4.14}$$

Using the matrix formalism, one obtains

$$b_1 = -\frac{2\mathfrak{p}^2 + 2\mathfrak{q}_z + 3\mathfrak{q}_z^2}{4(\mathfrak{q}_z + 1)}. \tag{4.15}$$

Thus, the first pole-skipping point is given by

$$(\mathfrak{q}_z, \mathfrak{p}^2) = \left(-1, -\frac{1}{2}\right). \tag{4.16}$$

The pole-skipping point lies outside the physical region $\mathfrak{q}_z \geq 0$.

2. Maxwell scalar mode

This mode corresponds to the "diffusive mode" in the SAdS_5 case. The gauge-invariant variables for the scalar mode are given by

$$\mathfrak{A}_z = A_z - iq_z A_L, \tag{4.17a}$$

$$\mathfrak{A}_u = A_u - A'_L \tag{4.17b}$$

(Appendix C). The Maxwell equation becomes

$$0 = \frac{i\mathfrak{q}_z}{2uf}\mathfrak{A}_z + (f\mathfrak{A}_u)', \tag{4.18a}$$

$$0 = (2\mathfrak{q}_z^2 + \mathfrak{p}^2 f)\mathfrak{A}_u + i\mathfrak{q}_z\mathfrak{A}'_z. \tag{4.18b}$$

We choose $\mathfrak{A}_z$ as the master variable because $\mathfrak{A}_u$ contains the u-derivative of the perturbation, A'_L. Then, the master equation is given by

$$0 = \mathfrak{A}_z'' + \frac{\mathfrak{q}_z^2 f'}{(\mathfrak{q}_z^2 + \mathfrak{p}^2 f)f}\mathfrak{A}'_z - \frac{\mathfrak{q}_z^2 + \mathfrak{p}^2 f}{uf^2}\mathfrak{A}_z. \tag{4.19}$$

Asymptotically, $\mathfrak{A}_z \sim A + Bu$. Near the tip $u = 1$, the field equation takes the form of Eq. (3.7), so set the ansatz $\mathfrak{A}_z = (1-u^2)^{\mathfrak{q}_z/2}\tilde{Z}$. Then,

$$0 \sim \tilde{Z}'' + \frac{1+\mathfrak{q}_z}{u-1}\tilde{Z}' + O\left(\frac{1}{u-1}\right)\tilde{Z}. \tag{4.20}$$

Using the matrix formalism, one obtains

$$b_1 = -\frac{2\mathfrak{p}^2(\mathfrak{q}_z+2) + \mathfrak{q}_z^2(3\mathfrak{q}_z+2)}{4\mathfrak{q}_z(\mathfrak{q}_z+1)}. \tag{4.21}$$

Thus, the first pole-skipping point is given by

$$(\mathfrak{q}_z, \mathfrak{p}^2) = (0,0), \left(-1, \frac{1}{2}\right). \tag{4.22}$$

The point $\mathfrak{q}_z = 0$ corresponds to the hydrodynamic mode in the SAdS_5 case. While the point $\mathfrak{q}_z = -1$ lies outside the physical region $\mathfrak{q}_z \geq 0$, the $\mathfrak{q}_z = 0$ point lies in the physical region.

D. Gravitational field

1. Gravitational vector mode

This mode corresponds to the "shear mode" in the SAdS_5 case. The gauge-invariant variables for the vector mode are given by

$$\mathfrak{h}_{zy} = h_{zy}^{(1)} - iq_z h_y^{(1)}, \tag{4.23a}$$

$$\mathfrak{h}_{uy} = h_{uy}^{(1)} - \frac{1}{u}(uh_y^{(1)})'. \tag{4.23b}$$

The Einstein equation becomes

$$0 = \frac{i\mathfrak{q}_z}{2uf}\mathfrak{h}_{zy} + (f\mathfrak{h}_{uy})', \tag{4.24a}$$

$$0 = -\frac{2iu}{\mathfrak{q}_z}(\mathfrak{q}_z^2 + \mathfrak{p}^2 f)\mathfrak{h}_{uy} + (u\mathfrak{h}_{zy})'. \tag{4.24b}$$

We choose $\mathfrak{h}_{zy}$ as the master variable because $\mathfrak{h}_{uy}$ contains the u-derivative of the perturbation. Then, the master equation is given by

$$0 = Z'' - \frac{(\mathfrak{q}_z^2 + \mathfrak{p}^2 f)f - \mathfrak{q}_z^2 uf'}{uf(\mathfrak{q}_z^2 + \mathfrak{p}^2 f)}Z' - \frac{\mathfrak{q}_z^2 + \mathfrak{p}^2 f}{uf^2}Z, \tag{4.25}$$

where $Z = u\mathfrak{h}_{zy}$. Asymptotically, $Z \sim A + Bu^2$. Near the tip $u = 1$, the field equation takes the form of Eq. (3.7), so set the ansatz $Z = (1-u^2)^{\mathfrak{q}_z/2}\tilde{Z}$. Then,

$$0 \sim \tilde{Z}'' + \frac{1+\mathfrak{q}_z}{u-1}\tilde{Z}' + O\left(\frac{1}{u-1}\right)\tilde{Z}. \tag{4.26}$$

Using the matrix formalism, one obtains

$$b_1 = -\frac{2\mathfrak{p}^2(\mathfrak{q}_z+2) + 3\mathfrak{q}_z^3}{4\mathfrak{q}_z(\mathfrak{q}_z+1)}. \tag{4.27}$$

Thus, the first pole-skipping points are given by

$$(\mathfrak{q}_z, \mathfrak{p}^2) = (0,0), \left(-1, \frac{3}{2}\right). \tag{4.28}$$

The point $\mathfrak{q}_z = 0$ corresponds to the hydrodynamic mode in the SAdS_5 case. While the point $\mathfrak{q}_z = -1$ lies outside the physical region $\mathfrak{q}_z \geq 0$, the $\mathfrak{q}_z = 0$ point lies in the physical region.

2. Gravitational scalar mode

This mode corresponds to the "sound mode" in the SAdS_5 case. The gauge-invariant variables are $\mathfrak{h}_{zz}$, $\mathfrak{h}_{zu}$, $\mathfrak{h}_{uu}$, and $\mathfrak{h}_L$. Their field equations are given in Eq. (C28).

From Eq. (C27), the gauge-invariant perturbations contain the parameters η_u, η_z, and their derivatives. η_u contains a derivative of a metric. So, choose the combination of gauge-invariant variables which do not involve η_u. Such a master field is given by

$$Z = u\{\mathfrak{h}_{zz} - (f - uf')\mathfrak{h}_L\} \tag{4.29a}$$

$$= u\left[h_{zz} - 2iq_z h_z - \left\{q_z^2 + \frac{p_i^2}{3}(1+u^2)\right\}h_T - (1+u^2)h_L\right]. \tag{4.29b}$$

This is the master field, e.g., by Kovtun and Starinets [46].

One can obtain the master equation for Z from $\mathfrak{h}_{zz}$ and $\mathfrak{h}_L$ equations. The master equation is given by

$$0 = Z'' - \frac{-3\mathfrak{q}_z^2(1+u^2) + \mathfrak{p}^2(-3+2u^2-3u^4)}{uf\{-3\mathfrak{q}_z^2 + \mathfrak{p}^2(-3+u^2)\}}Z' + \frac{3\mathfrak{q}_z^4 + \mathfrak{p}^4(3-4u^2+u^4) + \mathfrak{p}^2\{\mathfrak{q}_z^2(6-4u^2) - 4u^3 f\}}{uf^2\{-3\mathfrak{q}_z^2 + \mathfrak{p}^2(-3+u^2)\}}Z. \tag{4.30}$$

Asymptotically, $Z \sim A + Bu^2$. Near the tip $u = 1$, the field equation takes the form of Eq. (3.7), so set the ansatz $Z = (1-u^2)^{\mathfrak{q}_z/2}\tilde{Z}$. Then,

$$0 \sim \tilde{Z}'' + \frac{1+\mathfrak{q}_z}{u-1}\tilde{Z}' + O\left(\frac{1}{u-1}\right)\tilde{Z}. \tag{4.31}$$

Using the matrix formalism, one obtains

$$b_1 = -\frac{4\mathfrak{p}^4 + 9\mathfrak{q}_z^4 + 4\mathfrak{p}^2(3\mathfrak{q}_z^2 + 2\mathfrak{q}_z - 2)}{4(\mathfrak{q}_z+1)(3\mathfrak{q}_z^2 + 2\mathfrak{p}^2)}. \tag{4.32}$$

Thus, the first pole-skipping points are given by

$$(\mathfrak{q}_z, \mathfrak{p}^2) = \left(1, -\frac{3}{2}\right), (0,0), \left(-1, \frac{1}{2}(1 \pm 2\sqrt{2}i)\right). \tag{4.33}$$

The points $\mathfrak{q}_z = 1$ and $\mathfrak{q}_z = 0$ correspond to the chaotic mode and the hydrodynamic mode, respectively, in the SAdS_5 case. In the matrix formalism, one obtains all pole-skipping points including the chaotic and the hydrodynamic points as promised. While the point $\mathfrak{q}_z = -1$ lies outside the physical region $\mathfrak{q}_z \geq 0$, the $\mathfrak{q}_z = 1, 0$ points lie in the physical region.

V. DISCUSSION

A. Physical implication

In this paper, we study the pole skipping in the AdS soliton geometry. Even though the AdS soliton is not a black hole, the field equations have regular singularities at the tip of the cigar, so the pole skipping occurs.

It is interesting to explore the physical implications. However, even in the black-hole case, the physical implications of the pole skipping are unclear in general. This is because many pole-skipping points do not lie in the physical region (the wave number q is complex in general). This makes the physical interpretations difficult in general. The exceptions are the chaotic and the hydrodynamic pole skippings.

In the AdS soliton case, the pole-skipping points in general do not lie in the physical region at large n. However, the chaotic and hydrodynamic pole-skipping points lie in the physical region, so it is interesting to explore physical implications.

The pole skipping itself occurs even in the AdS soliton, but the physical interpretation is different from the black-hole case. First, for a black hole, the chaotic pole-skipping point lies in the upper-half ω-plane, which suggests a chaotic behavior. But the AdS soliton is not a black hole, and one does not expect the chaotic behavior. In fact, both $\mathfrak{q}_z$ and $\mathfrak{p}^2$ are real there.

Second, we make $\mathfrak{q}_z$ continuous following the conventional pole-skipping analysis, but $\mathfrak{q}_z$ is physically discrete $\mathfrak{q}_z \in \mathbb{Z}$. One may wonder if the pole skipping has any physical relevance for the AdS soliton.

Actually, there is an evident physical interpretation for the chaotic pole skipping in the AdS soliton. The chaotic pole-skipping point is located at $(\mathfrak{q}_z, \mathfrak{p}^2) = (1, -3/2)$. This is a pole-skipping point, so

(1) It would be a pole.
(2) But the residue of the pole vanishes.

The former implies that the dual field theory would have a normal mode with $(\mathfrak{q}_z, m_3^2) = (1, 6)$, where $m_3^2 = -p^2 = -4\mathfrak{p}^2$ is the dual $(2+1)$-dimensional mass. However, the latter implies that the state is actually *missing* due to the pole skipping.

In the black-hole case, near a pole-skipping point, Green's function typically takes the form

$$G^R \propto \frac{\delta\omega + \delta q}{\delta\omega - \delta q}, \tag{5.1}$$

so it depends on the slope $\delta q/\delta\omega$ how one approaches the pole-skipping point. However, if one first fixes $\delta q = 0$, one gets $G^R = (\text{const})$, so the pole would disappear.

This is the situation that happens in the AdS soliton case. In the AdS soliton case, $\mathfrak{q}_z$ is actually discrete, and one first fixes $\mathfrak{q}_z = 1$, and so on, so one cannot choose the slope. Instead, the pole skipping appears as the "missing state."

Even though the black hole and the AdS soliton are related by a double Wick rotation, the physical interpretations of the chaotic pole skipping are very different. The black hole and the AdS soliton have very different physical interpretations in general. For the chaotic pole skipping,

(1) The black-hole case implies a chaotic behavior.
(2) The AdS soliton case implies a missing state.

We will explore this issue further in a separate paper, in particular the mass spectrum in details [51].

B. Other non-black-hole backgrounds

We focus on the AdS soliton, but a similar analysis should be possible for the other non-black-hole geometries with S^1. The Witten geometry is an example [3]. This geometry is obtained by the double Wick rotation from the D4-brane geometry.

Of course, there are many non-black-hole geometries without S^1. For a non-black-hole background, field equations in general would not have regular singularities (except $u = \infty$) and have only ordinary points. However, as mentioned in Sec. II B, there are two kinds of pole skipping. In the black-hole case, the pole skipping in the lower-half ω-plane comes from regular singularities. But the pole skipping in the upper-half ω-plane has a different origin. In this case, the coefficients of field equations themselves partly have the structure $0/0$. In particular, the hydrodynamic mode should survive as a pole-skipping point.

As a simple example, consider the cutoff SAdS where we impose the IR cutoff at $u = u_0 < 1$. Because the cutoff is an ordinary point, one can expand the solution as a Taylor series:

$$0 = Z'' + PZ' + QZ, \tag{5.2a}$$

$$P = \sum_{n=0} P_n (u - u_0)^n, \tag{5.2b}$$

$$Q = \sum_{n=0} Q_n (u - u_0)^n, \tag{5.2c}$$

$$Z = \sum_{n=0} a_n (u - u_0)^n. \tag{5.2d}$$

At the lowest order, one gets

$$0 = Q_0 a_0 + P_0 a_1 + 2a_2. \tag{5.3}$$

One needs to impose a boundary condition at the cutoff. If one imposes the Dirichlet boundary condition, $a_0 = 0$. Then, Eq. (5.3) determines a_2 from a_1. However, if $P_0 = 0/0$, both a_1 and a_2 are free parameters, and the solution is not uniquely determined. For example, for the Maxwell scalar mode, $P_0 = 0/0$ at $\omega = q = 0$ [see Eq. (4.19) for the AdS soliton counterpart].

It is unclear if there are any other pole-skipping points for non-black-hole geometries. It would be interesting to explore the issue further.

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APPENDIX A: THE SAdS$_5$ BLACK-HOLE MASTER EQUATIONS

Below we summarize the master equations for the SAdS$_5$ black hole for the reader's convenience [46]:

(1) Maxwell vector mode:

$$0 = Z'' + \frac{f'}{f} Z' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{u f^2} Z, \tag{A1}$$

where $' = \partial_u$ and $\mathfrak{w} := \omega/(2\pi T)$, $\mathfrak{q} := q/(2\pi T)$.

(2) Maxwell scalar mode:

$$0 = Z'' + \frac{\mathfrak{w}^2 f'}{(\mathfrak{w}^2 - \mathfrak{q}^2 f) f} Z' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{u f^2} Z. \tag{A2}$$

(3) Gravitational tensor mode (massless scalar):

$$0 = Z'' + \left(\frac{f'}{f} - \frac{1}{u}\right) Z' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{u f^2} Z. \tag{A3}$$

(4) Gravitational vector mode:

$$0 = Z'' - \frac{(\mathfrak{w}^2 - \mathfrak{q}^2 f) f - \mathfrak{w}^2 u f'}{u f (\mathfrak{w}^2 - \mathfrak{q}^2 f)} Z' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{u f^2} Z. \tag{A4}$$

(5) Gravitational scalar mode:

$$0 = Z'' - \frac{3\mathfrak{w}^2(1+u^2) + \mathfrak{q}^2(-3+2u^2-3u^4)}{uf\{3\mathfrak{w}^2 + \mathfrak{q}^2(-3+u^2)\}} Z' + \frac{3\mathfrak{w}^4 + \mathfrak{q}^4(3-4u^2+u^4) - \mathfrak{q}^2\{\mathfrak{w}^2(6-4u^2) + 4u^3 f\}}{uf^2\{3\mathfrak{w}^2 + \mathfrak{q}^2(-3+u^2)\}} Z. \tag{A5}$$

All master fields behave as

$$0 \sim Z'' + \frac{1}{u-1} Z' + \frac{\mathfrak{w}^2}{4(u-1)^2} Z \qquad (u \to 1), \tag{A6}$$

near the horizon, and one imposes the incoming-wave boundary condition

$$Z \propto (u-1)^{-i\mathfrak{w}/2}. \tag{A7}$$

APPENDIX B: PROPERTIES OF THE AdS SOLITON

1. Phase transition

At high temperature, the AdS soliton undergoes a first-order phase transition (the Hawking-Page transition) to the SAdS black hole. This describes the confinement/deconfinement transition in the dual gauge theory.

Because the (uncompactified) SAdS black hole is scale invariant, the only dimensionful quantity is T. Thus, the free energy should take the form

$$F_{\mathrm{BH}} = -cT^4 V_3, \tag{B1}$$

where $c = \pi^4/(16\pi G_5) = \pi^2 N_c^2/8$, and V_3 is the gauge theory volume. The S^1-periodicity l appears only in V_3. The AdS soliton has the same Euclidean geometry, so the free energy takes the same form when expressed in terms of the "horizon" radius r_0. But, for the AdS soliton, r_0 is related to the S^1 periodicity l, so

$$F_{\mathrm{soliton}} = -\frac{c}{l^4} V_3. \tag{B2}$$

Then, the free energy difference is

$$\Delta F = F_{\mathrm{BH}} - F_{\mathrm{soliton}} \propto -\left(T^4 - \frac{1}{l^4}\right) V_3. \tag{B3}$$

So, at low temperature $T < 1/l$, the stable solution is the AdS soliton. At high temperature $T > 1/l$, the stable solution is the black hole. Because the black hole has $O(N_c^2)$ entropy, the entropy is discontinuous at $Tl = 1$. The first derivative of free energy, $S = -\partial_T F$, is discontinuous there, so this is a first-order phase transition.

2. Energy-momentum tensor

The SAdS_5 black hole has the following energy-momentum tensor:

$$T_{\mu\nu} = \frac{r_0^4}{16\pi G_5} \mathrm{diag}(3,1,1,1) = \frac{\pi^2}{8} N_c^2 T^4 \mathrm{diag}(3,1,1,1). \tag{B4}$$

The energy-momentum tensor for the AdS soliton is computed in Ref. [52]. One can obtain it by the double Wick rotation from the SAdS result:

$$T_{\mu\nu} = \frac{\pi^2}{8} \frac{N_c^2}{l^4} \mathrm{diag}(-1,1,1,-3). \tag{B5}$$

It remains traceless. The nonvanishing energy density is interpreted as the Casimir energy. The directions (t,x,y) have the Lorentz invariance, so $T_{ij}(i = t,x,y)$ has the Lorentz invariance and is proportional to η_{ij}. The Lorentz invariance is broken in the z-direction, so one obtains an anisotropic pressure.

3. Mass gap

For the SAdS black hole, one imposes the incoming-wave boundary condition at the horizon. Because a perturbation is absorbed by the black hole, one obtains quasinormal modes; namely poles are located in the complex ω-plane. The AdS soliton does not have a horizon, and the geometry smoothly ends at $u = 1$, so one obtains normal modes.

For simplicity, consider the massless scalar field $0 = \nabla^2\phi$, and consider the perturbation of the form $e^{ip_i x^i}$. The field has the asymptotic behavior

$$\phi \sim A + Bu^2 \qquad (u \to 0), \tag{B6}$$

and Green's function is given by

$$G^R \propto \frac{B}{A}. \tag{B7}$$

So, a pole corresponds to $A = 0$. Then, it is enough to solve the perturbation equation under the boundary condition $A = 0$.

Near the tip of the cigar $u = 1$, the field equation becomes

$$0 \sim \phi'' + \frac{1}{u-1}\phi' \qquad (u \to 1), \tag{B8}$$

when $q_z = 0$, so the solution is

$$\phi \sim C_1 + C_2 \ln(1-u). \tag{B9}$$

We impose $C_2 = 0$ from the regularity condition at $u = 1$.

One can solve the perturbation equation by a power series expansion around the tip of the cigar $u = 1$:

$$\phi = \sum_{n=0}^{\infty} a_n (u-1)^{n+\lambda}. \tag{B10}$$

Substituting this into the field equation, one gets the indicial equation at the lowest order: $\lambda^2 = 0$. The boundary condition $A = 0$ corresponds to

$$\phi|_{u=0} = \sum_{n=0}^{N} (-)^n a_n = 0. \tag{B11}$$

One truncates the series after a large number of terms $n = N$. One can check the accuracy as one goes to higher series. The problem has a nontrivial solution only for particular values of $p^2 = -m_3^2$ which give the mass spectrum. The first few states are $m_3^2 \approx 11.59, 34.53, \ldots$.

APPENDIX C: GAUGE-INVARIANT VARIABLES

We follow Ref. [16] with slight changes in the conventions. First, decompose the background spacetime into a two-dimensional space x^a and a p-dimensional spacetime y^i:

$$ds_{p+2}^2 = g_{ab}(x) dx^a dx^b + e^{2\varphi} g_{ij}(y) dy^i dy^j. \tag{C1}$$

Here, the decomposition is chosen so that the y^i-part remains maximally symmetric:

(1) For the SAdS_5, the metric is Eq. (2.1), so $x^a = (t, u)$, $y^i = (x, y, z)$, and $g_{ij} = \delta_{ij}$.

(2) For the AdS soliton, the metric is Eq. (3.2), so $x^a = (z, u)$, $y^i = (t, x, y)$, and $g_{ij} = \eta_{ij}$.

Below, we focus on the AdS soliton case: $e^{2\varphi} = 1/u$, and

$$g_{ab} = \begin{pmatrix} g_{zz} & 0 \\ 0 & g_{uu} \end{pmatrix}. \tag{C2}$$

We decompose perturbations under the transformation of y^i. The scalar (vector) mode transforms as a scalar (vector) under the transformation. We consider the perturbation of the form

$$\phi(u) e^{ip_i x^i + iq_z z}, \tag{C3}$$

where $p_i = (-\omega, q_x, 0)$.[2]

1. Maxwell perturbations

The Maxwell perturbations consist of $A_M = (A_a, A_i)$. A_i can be decomposed as

$$A_i = \partial_i A_L + A_{Ti}, \qquad \partial^i A_{Ti} = 0, \tag{C4}$$

or

$$A_i = ip_i A_L + A_{Ti}, \qquad p^i A_{Ti} = 0. \tag{C5}$$

Thus, for $p = 3$,

(1) The scalar mode consists of three perturbations $A_a(A_z, A_u)$ and A_L.

(2) One can use the Lorenz gauge condition to eliminate a component of A_{Ti}, e.g., A_{Tt}. Then, the vector mode consists of two perturbations A_{Tx} and $A_{Ty} = A_y$. They satisfy the identical equation.

In total, there are five perturbations.

The scalar mode has three perturbations, but one is redundant due to the gauge symmetry. The gauge transformation $\delta A_M = -\partial_M \lambda$ becomes

$$\delta A_a = -\partial_a \lambda, \tag{C6a}$$

$$\delta A_i = ip_i \delta A_L + \delta A_{Ti} = -ip_i \lambda \tag{C6b}$$

so that

$$\delta A_{Ti} = 0, \qquad \delta A_L = -\lambda. \tag{C7}$$

The gauge-invariant variables are obtained by eliminating the gauge parameter λ. The variables A_{Ti} are gauge invariant by themselves. From Eq. (C7), λ is expressed by A_L as $\lambda = -\delta A_L$. Substituting this into Eq. (C6a) gives

$$\delta(A_a - \partial_a A_L) = 0, \tag{C8}$$

so the gauge-invariant scalar perturbations are given by

$$\mathfrak{A}_a := A_a - \partial_a A_L. \tag{C9}$$

2. Gravitational perturbations

The gravitational perturbations consist of $h_{MN} = (h_{ab}, h_{ai}, h_{ij})$. Again, perturbations are decomposed as scalar, vector, and tensor modes. h_{ab} gives three scalar perturbations. Just as the Maxwell perturbations, h_{ai} is decomposed as

$$h_{ai} = \partial_i h_a + h^{(1)}_{ai}, \qquad \partial^i h^{(1)}_{ai} = 0. \tag{C10}$$

h_a gives two scalar perturbations, and $h^{(1)}_{ai}$ gives $2(p-1)$ vector perturbations. The superscript "(1)" refers to the number of index i ("spin"). Similarly, h_{ij} is decomposed as

$$h_{ij} =: h_L \eta_{ij} + \mathcal{P}_{ij} h_T + 2\partial_{(i} h^{(1)}_{Tj)} + h^{(2)}_{Tij}, \tag{C11}$$

where

$$\partial^i h^{(1)}_{Ti} = 0, \qquad \partial^j h^{(2)}_{Tij} = 0, \qquad h^{(2)i}_{Ti} = 0, \tag{C12}$$

and $\mathcal{P}_{ij}$ is the projection operator given by

$$\mathcal{P}_{ij} := \partial_i \partial_j - \frac{1}{p} \eta_{ij} \partial_k^2. \tag{C13}$$

The first term of h_{ij} (h_L) is the trace part which is a scalar perturbation. The rest is the traceless part which is decomposed as a scalar h_T, vector $h^{(1)}_{Tj}$, and tensor perturbations $h^{(2)}_{Tij}$. Thus,

(1) The scalar mode consists of seven perturbations (h_{ab}, h_a, h_L, h_T).

(2) The vector mode consists of perturbations ($h^{(1)}_{ai}$, $h^{(1)}_{Ti}$). The former has $2(p-1)$ components, and the latter has $(p-1)$ components, so there are $3(p-1)$ components.

(3) The tensor mode consists of $h^{(2)}_{Tij}$ which has $(p+1)(p-2)/2$ components.

In total, there are $(p+2)(p+3)/2$ components.

Again consider the gauge transformation $\delta x^M = -\xi^M$. The infinitesimal transformation ξ^i is decomposed as

$$\xi_i =: \partial_i \xi_L + \xi_{Ti}, \qquad \partial^i \xi_{Ti} = 0. \tag{C14}$$

[2] For the quantities defined in the p-dimensional subspacetime such as p^i, the index i is raised and lowered with η_{ij}, i.e., $p^i = \eta^{ij} p_j$.

The scalar part has three components, ξ_a and ξ_L, and the vector part has $(p-1)$ components, ξ_{Ti}.

a. Tensor mode

The tensor mode is gauge invariant by itself:

$$\delta h^{(2)}_{Tij} = 0. \tag{C15}$$

The combination $(u h^{(2)}_{Tij})$ satisfies the field equation for the minimally coupled massless scalar field, so we do not discuss this mode further.

b. Vector mode

Under the gauge transformation $\delta x^M = -\xi^M$, the vector mode transforms as

$$\delta h^{(1)}_{ai} = \partial_a \xi_{Ti} - 2\xi_{Ti}(\partial_a \varphi), \tag{C16a}$$

$$\delta h^{(1)}_{Ti} = \xi_{Ti}. \tag{C16b}$$

To obtain gauge-invariant variables, we again express gauge parameters ξ_{Ti} by perturbations. Equation (C16b) expresses ξ_{Ti} by $\delta h^{(1)}_{Ti}$. Substituting Eq. (C16b) into Eq. (C16a) gives

$$\delta(h^{(1)}_{ai} - \partial_a h^{(1)}_{Ti} + 2h^{(1)}_{Ti}\partial_a\varphi) = 0, \tag{C17}$$

so the gauge-invariant vector perturbations are

$$\mathfrak{h}_{ai} := h^{(1)}_{ai} - e^{2\varphi}\partial_a(e^{-2\varphi}h^{(1)}_{Ti}). \tag{C18}$$

Eliminating ξ_{Ti} gives $2(p-1)$ gauge-invariant perturbations.

c. Scalar mode

1. The gauge-invariant variables

Under the gauge transformation, the scalar mode transforms as

$$\delta h_{ab} = 2\,{}^{(2)}\nabla_{(a}\xi_{b)}, \tag{C19a}$$

$$\delta h_a = \xi_a + \partial_a \xi_L - 2\xi_L(\partial_a\varphi), \tag{C19b}$$

$$\delta h_L = \xi^{a\,(2)}\nabla_a e^{2\varphi} + \frac{2}{p}\partial_k^2 \xi_L, \tag{C19c}$$

$$\delta h_T = 2\xi_L, \tag{C19d}$$

where ${}^{(2)}\nabla_a$ is the covariant derivative with respect to g_{ab}.

Equation (C19d) expresses ξ_L by δh_T. Substituting Eq. (C19d) into Eq. (C19b), ξ_a is expressed by δh_a and δh_T:

$$\xi_a = -\delta\eta_a, \tag{C20a}$$

$$\eta_a := \frac{1}{2}\partial_a h_T - h_T\partial_a\varphi - h_a. \tag{C20b}$$

Substituting ξ_a into Eq. (C19a), one obtains

$$\delta(h_{ab} + 2\,{}^{(2)}\nabla_{(a}\eta_{b)}) = 0, \tag{C21}$$

so one gets the gauge-invariant perturbations

$$\mathfrak{h}_{ab} := h_{ab} + 2\,{}^{(2)}\nabla_{(a}\eta_{b)}. \tag{C22}$$

Similarly, Eq. (C19c) becomes

$$\delta\left(h_L + \eta^{a\,(2)}\nabla_a e^{2\varphi} - \frac{1}{p}\partial_i^2 h_T\right) = 0, \tag{C23}$$

which gives

$$\mathfrak{h}_L := h_L + \eta^{a\,(2)}\nabla_a e^{2\varphi} - \frac{1}{p}\partial_i^2 h_T. \tag{C24}$$

The scalar mode has seven perturbations, but four gauge-invariant perturbations remain after one eliminates ξ_L and ξ_a.

Writing these formulas in components for $p=3$, one gets

$$\mathfrak{h}_{zz} = h_{zz} + 2iq_z\eta_z + \frac{g'_{zz}}{g_{uu}}\eta_u, \tag{C25a}$$

$$\mathfrak{h}_{zu} = h_{zu} + iq_z\eta_u - \frac{g'_{zz}}{g_{zz}}\eta_z + \eta'_z, \tag{C25b}$$

$$\mathfrak{h}_{uu} = h_{uu} - \frac{g'_{uu}}{g_{uu}}\eta_u + 2\eta'_u, \tag{C25c}$$

$$\mathfrak{h}_L = h_L + \frac{p_i^2}{3}h_T - \frac{1}{u^2 g_{uu}}\eta_u. \tag{C25d}$$

From Eq. (C20b), η_a becomes

$$\eta_z = \frac{1}{2}iq_z h_T - h_z, \tag{C26a}$$

$$\eta_u = \frac{1}{2u}(uh_T)' - h_u. \tag{C26b}$$

2. The field equations

The scalar mode has four variables $\mathfrak{h}_{zz}$, $\mathfrak{h}_{zu}$, $\mathfrak{h}_{uu}$, $\mathfrak{h}_L$. The linearized Einstein equation reduces to

$$0 = \mathfrak{h}_{zz} + f\mathfrak{h}_L + 4uf^2\mathfrak{h}_{uu}, \tag{C27a}$$

$$0 = \frac{2\mathfrak{p}^2 + 3u}{2f}\mathfrak{h}_{zz} + \{3\mathfrak{q}_z^2 + \mathfrak{p}^2(2-f)\}\frac{\mathfrak{h}_{zu}}{i\mathfrak{q}_z} + \frac{6\mathfrak{q}_z^2 + 4\mathfrak{p}^2 f - 3u(2-f)}{2f}\mathfrak{h}_L, \tag{C27b}$$

$$0 = \left(u\frac{d}{du} + \frac{1}{f}\right)\mathfrak{h}_{zz} - 2(1-f)\mathfrak{h}_L + 2\frac{3\mathfrak{q}_z^2 + 2\mathfrak{p}^2 f}{3} \times \frac{u\mathfrak{h}_{zu}}{i\mathfrak{q}_z} + 2uf(2-f)\mathfrak{h}_{uu}, \tag{C27c}$$

$$0 = \left(u\frac{d}{du} + \frac{1}{f}\right)\mathfrak{h}_L - \frac{2\mathfrak{p}^2}{3}\frac{u\mathfrak{h}_{zu}}{i\mathfrak{q}_z} + 2uf\mathfrak{h}_{uu}. \tag{C27d}$$

Here, we arranged field equations. Namely, these equations are not just the bare Einstein equation components. The bare equations involve first and second derivatives in u. Combining the equations appropriately, one can eliminate all second derivatives. The resulting equations involve various first derivatives. Combining equations further eliminates some first derivatives, and one obtains equations where the first derivative appears at most once in each equation. Finally, we use the constraint equations for cosmetic purposes.

There are two constraint equations without u-derivatives and two differential equations with one u-derivative. The constraint equations (C27a) and (C27b) allow us to choose two independent variables. Both obey first-order differential equations, so one gets a second-order differential equation for a single variable.

APPENDIX D: THE CHOICE OF MASTER FIELD

As mentioned in Sec. IV B, the choice of a master field is not unique. One needs to find which variable is most suitable, or one needs to take all variables into account [16]. This is sometimes problematic for the pole-skipping analysis. One cannot find some pole-skipping points if one chooses a different variable. In this appendix, we show this explicitly for the Maxwell scalar mode and for the gravitational vector mode.

For the Maxwell scalar mode, there are two gauge-invariant variables $\mathfrak{A}_z$, $\mathfrak{A}_u$, and we choose $\mathfrak{A}_z$. If one chooses $\mathfrak{A}_u$ as the master variable, the master equation is given by

$$0 = Z_2'' + \left(\frac{f'}{f} + \frac{1}{u}\right)Z_2' - \frac{\mathfrak{q}_z^2 + \mathfrak{p}^2 f}{uf^2}Z_2, \tag{D1}$$

where $Z_2 = f\mathfrak{A}_u$. In this case, one obtains

$$b_1 = -\frac{2\mathfrak{p}^2 + \mathfrak{q}_z(4 + 3\mathfrak{q}_z)}{4(\mathfrak{q}_z + 1)}, \tag{D2}$$

so the first pole-skipping point is given by $(\mathfrak{q}_z, \mathfrak{p}^2) = (-1, 1/2)$. Namely, one cannot see the hydrodynamic pole-skipping $(\mathfrak{q}_z, \mathfrak{p}^2) = (0, 0)$.

Similarly, for the gravitational vector mode, there are two gauge-invariant variables $\mathfrak{h}_{zy}$, $\mathfrak{h}_{uy}$, and we choose $\mathfrak{h}_{zy}$. If one chooses $\mathfrak{h}_{uy}$ as the master variable, the master equation is given by

$$0 = Z_2'' + \left(\frac{f'}{f} + \frac{2}{u}\right)Z_2' - \frac{\mathfrak{q}_z^2 + \mathfrak{p}^2 f}{uf^2}Z_2, \tag{D3}$$

where $Z_2 = f\mathfrak{h}_{uy}$. In this case, one obtains

$$b_1 = -\frac{-4 + 2\mathfrak{p}^2 + 2\mathfrak{q}_z + 3\mathfrak{q}_z^2}{4(\mathfrak{q}_z + 1)}, \tag{D4}$$

so the first pole-skipping point is given by $(\mathfrak{q}_z, \mathfrak{p}^2) = (-1, 3/2)$, and again one cannot see the hydrodynamic pole skipping.

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