Is the diagonal case a general picture for loop quantum cosmology?

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The correct implementation of the loop quantum gravity to the early homogeneous Universe has been the subject of a long debate in the literature because the $SU(2)$ symmetry cannot be properly retained. The role of this symmetry is expressed by the Gauss constraint. Here, a nonvanishing Gauss constraint is found. However, we show that using suitable variables, it can be recast into three Abelian constraints, justifying the absence of such a symmetry in loop quantum cosmology.

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I. INTRODUCTION

The most promising proposal to quantize the gravitational field is, until now, the so-called loop quantum gravity [1–3]. This claim is based on the idea that such a proposal, starting from a classical formulation of General Relativity, which is (on shell) equivalent to the Einstein-Hilbert formulation [4–8], arrives, via the introduction of the $SU(2)$ symmetry, to describe geometrical operators, like areas and volumes of space, as associated to discrete spectrum [9]. As a consequence, the implementation of loop quantum gravity to the cosmological setting led to a big bounce for the primordial Universe [10–12], due to an anomaly of the classical limit.

The reliability of the so-called loop quantum cosmology has been debated over the years [13–15], because the symmetry restriction induced by the homogeneity constraint prevents the preservation of the $SU(2)$ symmetry in the classical and quantum formulation. Actually, the its-self implementation of the dynamics for homogeneous models is a step forward in the general formulation, for which a reliable implementation of the regularized scalar constraint [16,17] is not viable [18].

An interesting attempt to restore a gauge $SU(2)$ symmetry in cosmology, along with the associated Gauss constraint, has been formulated in Refs. [19–22]. There, a kinematical Hilbert space has been constructed by emulating the basic formulation in loop quantum gravity. The idea is that the homogeneity of the space still allows for a local time-dependent Lorentz rotation of the triad vectors, so restoring a nonidentically vanishing Gauss constraint as for the original formulation of the Ashtekar School [23].

The present analysis starts from the same theoretical setup of a local time-dependent gauge transformation of the triad, but, investigating in detail the relation of the Ashtekar-Barbero-Immirzi connection and conjugate momentum to the standard Arnowitt-Deser-Misner (ADM)-Hamiltonian variables, it arrives at a rather different conclusion: the resulting picture is closer to the formulation of the Ashtekar School than to real "spin-network" construction [1]. When we express the $SU(2)$ gauge connection in terms of the metric variables (three scale factors, three Euler angles and, eventually three gauge angles), a local expansion of the involved functions outlines a linear dependence of the Gauss constraint from the three momenta variables, associated to the gauge angles (i.e., those responsible for the local Lorentz rotation). This result suggests pursuing, *ab initio* a Holst formulation [6], by expressing the $SU(2)$ connection in terms of the metric variables. This calculus strategy provides the net and relevant issue of a linear relationship between the three Gauss constraint components and the three null momenta of the gauge angles: the Gauss constraint validity is ensured by the simultaneous vanishing behavior of the three momenta and vice versa.

Particularly, we demonstrate that the Gauss constraints can be suitably restated into three Abelian constraints, simply stating the gauge nature of the three angles which rotates the dreibein. The explicit expression of the matrix linking the two sets of constraints is provided here.

Finally, the most important consequence of the present study is that the physical kinematical states of the theory cannot depend on the three gauge angles (simply because in a canonical formulation they are annihilated by the three null momenta) so that the quantization of the model reduces to the analysis provided in Ref. [24] on the nondiagonal Bianchi models. In other words, the present study allows validation of the original idea that the space of the almost-periodic functions is the suitable approach to implement a canonical loop quantum gravity in cosmology. Even if we start with all the nine nonzero triad components, three of

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them are actually gauge angles, leading to a Gauss constraint that is reducible to three Abelian vanishing momenta. The quantization coincides with that one of a nondiagonal Bianchi Universe, which in Ref. [24] was associated with a diagonal representation of the fluxes, in agreement with the analysis in Refs. [25–27].

II. ROTATIONS AS GAUGE TRANSFORMATIONS

We recall the classical description of Ashtekar variables in a homogeneous Universe. In a homogeneous model, the space-time is a manifold $\mathcal{M} \cong \mathbb{R} \times \Sigma$, where Σ is a three-dimensional Riemannian homogeneous space. We require that the isometry group S of Σ acts transitively and freely [22].

On Σ exists a basis of left-invariant one-forms ω^I (i.e. $F^*\omega^I = \omega^I,\ \forall\ F \in S$) such that

$$d\omega^I + \frac{1}{2} f^I_{JK}\omega^J \wedge \omega^K = 0. \tag{1}$$

The dual vector fields ξ_I [defined by $\omega^I(\xi_J) = \delta^I_J$] are the generators of the Lie algebra $\mathfrak{s}$ of S

$$[\xi_I, \xi_J] = f^K_{IJ}\xi_K, \tag{2}$$

thus, f^K_{IJ} are the structure constants.

The induced Riemannian metric h on Σ is left invariant due to the homogeneous hypothesis, hence, it can be written in terms of ω^I

$$h = \eta_{IJ}\omega^I \otimes \omega^J, \tag{3}$$

where η_{IJ} is a symmetric matrix constant on Σ.

A homogeneous connection A on Σ is determined by a linear map $\phi\colon\ \mathfrak{s} \to \mathfrak{su}(2)$ and it is written as $A = \phi \circ \theta_{\mathrm{MC}}$, where $\theta_{\mathrm{MC}} = \xi_I \otimes \omega^I$ is the Maurer-Cartan form [19].

Using a coordinate system (t, x^i) adapted to the space-time decomposition, the components of the left-invariant one-forms ω^I_i and the dual vector fields ξ^i_I depend only on x^i, while the other quantities that are constant on Σ are functions on t. Thus, the Ashtekar variables read [25]

$$\begin{aligned} A^a_i(t,x) &= \phi^a_I(t)\omega^I_i(x), \\ E^i_a(t,x) &= |\det(\omega^J_j(x))| p^I_a(t)\xi^i_I(x). \end{aligned} \tag{4}$$

We can also characterize the space-time metric g via its component:

$$g_{00} = -N^2 + N^i N^j h_{ij}, \qquad g_{0i} = N^j h_{ij}, \qquad g_{ij} = h_{ij},$$

where N and N^i are the lapse function and the shift vector, respectively, and h is the induced Riemannian metric $h_{ij} = \eta_{IJ}(t)\omega^I_i(x)\omega^J_j(x)$. In a homogeneous model, the lapse function is a function of time only $N = N(t)$, while the shift vector can be factorized as $N^i = N^I(t)\xi^i_I(x)$.

Now, we are interested in the gauge freedom of the Ashtekar variables. The gauge transformation for the densitized triads is known $p^I_a \mapsto p^I_b O^b_a$, with $O \in SO(3)$ [22].

Due to the homogeneity hypothesis, p^I_a only depends on time and this property must hold also after the gauge transformation. Hence, although O can be arbitrary and does not contribute in any physical sense, it must depend on time only too.

Moreover, the gauge transformation can be seen as a rotation of the dreibein $e^i_a \mapsto O^b_a e^i_b$. This interpretation allows us to find the associated gauge transformation of the connection variables ϕ^a_I.

Consider the usual expression of the Ashtekar connection $A^a_i = \Gamma^a_i + \gamma K^a_i$ where γ is the Barbero-Immirzi parameter. We can treat the two terms separately. The second term $K^a_i = K_{ij}e^{aj}$ contains the external curvature K_{ij} which is a geometrical quantity and is not affected by gauge transformations, while e^{ja} is a dreibein vector, so it rotates under a gauge transformation. It is easy to check that the rotation matrix is the inverse of the transformation matrix that acts on e^i_a because $\delta^i_j = e^i_a e^a_j$ must be invariant. Then, $K_{ij}e^{ja} \mapsto K_{ij}(O^{-1})^a_b e^{jb}$.

Moreover, also the spin part transforms as $\Gamma^a_i \mapsto (O^{-1})^a_b\Gamma^b_i$. Thus, under a gauge transformation, a matrix rotation appears:

$$A^a_i = \phi^a_I\omega^I_i \mapsto (O^{-1})^a_b A^b_i = (O^{-1})^a_b\phi^b_I\omega^I_i. \tag{5}$$

Therefore, on the phase space (ϕ^a_I, p^J_b) the gauge transformation acts as

$$p^I_a \mapsto p^I_b O^b_a, \qquad \phi^a_I \mapsto (O^t)^a_b\phi^b_I. \tag{6}$$

We can check that such a transformation leaves the Gauss constraint weakly vanishing. In fact, the transformation of the Gauss constraint $G_a = \epsilon_{ab}{}^c\phi^b_I p^I_c$ reads

$$G_a \mapsto \epsilon_{ab}{}^c(O^t)^b_d O^e_c\phi^d_I p^I_e = \epsilon_{bd}{}^e O^b_a\phi^d_I p^I_e = O^b_a G_b \approx 0. \tag{7}$$

Now, we look for a description in metric variables like the ones in Ref. [24]. The new phase space of the metric variables, composed of the three scale factors a, b, c and the three Euler angles of the physics rotation θ, ψ, φ, needs to include variables of the gauge freedom.

Since $O \in SO(3)$, it can be written in terms of Euler angles

$$O = \exp(\alpha j_3)\exp(\beta j_2)\exp(\gamma j_3), \tag{8}$$

where j_i are the real matrix generators of $SO(3)$. Then, the three gauge variables are these three Euler angles (α, β, γ), they are seen as a chart on $SO(3)$, $\alpha, \gamma \in (0, 2\pi)$, $\beta \in (0, \pi)$.

Hence, the new configuration coordinates are $\{a, b, c, \theta, \psi, \varphi, \alpha, \beta, \gamma\}$.

In order to construct a theory in which the Gauss constraint does not vanish identically and in which the role of the cosmological quantities is made explicit, the assumption of phase space with configuration variables $\{a, b, c, \theta, \psi, \varphi, \alpha, \beta, \gamma\}$ seems to be a reasonable solution. The idea is to impose a canonical transformation between the two phase spaces such that the conjugate momenta to the gauge variables are included in the expression of the Ashtekar variables. These momenta will play a role in the Gauss constraint and they can be later removed from the theory to recover the expressions in Ref. [24].

III. EXAMINATION OF THE BIANCHI I MODEL

For the nondiagonal Bianchi I model, we have a simple expression of Ashtekar variables in terms of metric variables which allows us to do some computations. We want to use the connection and fluxes expression in Eqs. (39) and (42) from Ref. [24] properly gauge rotated

$$\phi_I^a = \frac{\gamma}{2Na_b}\Lambda_b^J \dot{\eta}_{JI}(O^t)_b^a, \tag{9}$$

$$p_a^I = a_b a_c \Lambda_d^I O_a^d \quad \text{with} \quad \epsilon_{abc} = 1, \tag{10}$$

where Λ is the physical rotation, and a_1, a_2, a_3 are the scale factors.

A direct computation shows that, despite the gauge freedom, the Gauss constraint identically vanishes (as well as in Ref. [28]). Thus, the gauge momenta play a fundamental role in a nonvanishing Gauss constraint description. Now, we want to analyze this aspect.

A. The Lie condition approach

Consider the two phase spaces: the metric one $(a, b, c, \theta, \psi, \varphi, \alpha, \beta, \gamma, p_a, p_b, p_c, p_\theta, p_\psi, p_\varphi, p_\alpha, p_\beta, p_\gamma)$, and the connection-fluxes one (ϕ_I^a, p_a^I). We want to impose a canonical transformation between them starting from the expression of p_a^I in Eq. (10) to find a formulation of the connection which includes a dependence on the gauge momenta. We expect that the gauge momenta can give a nontrivial contribution to the Gauss constraint.

For simplicity, we switch coordinates and momenta of (ϕ_I^a, p_a^I) and we implement a canonical transformation via the Lie condition with a null total differential

$$\phi_I^a dp_a^I - p_i dq_i = \left(\phi_I^a \frac{\partial p_a^I}{\partial q_i} - p_i\right) dq_i = 0. \tag{11}$$

Now, we want to find ϕ_I^a. It is useful to introduce the matrix M defined as $M = \frac{\partial p_a^I}{\partial q_i}$, seen as a 9×9 matrix, such that the term $\phi_I^a dp_a^I$ can be written as a product "matrix per vector"

$$\phi_I^a \frac{\partial p_a^I}{\partial q_i} = \begin{pmatrix} \frac{\partial p_1^1}{\partial a} & \frac{\partial p_1^2}{\partial a} & \cdots & \frac{\partial p_3^3}{\partial a} \\ \frac{\partial p_1^1}{\partial b} & \frac{\partial p_1^2}{\partial b} & \cdots & \frac{\partial p_3^3}{\partial b} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial p_1^1}{\partial \gamma} & \frac{\partial p_1^2}{\partial \gamma} & \cdots & \frac{\partial p_3^3}{\partial \gamma} \end{pmatrix} \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \\ \vdots \\ \phi_3^3 \end{pmatrix}. \tag{12}$$

In such a way, the Lie condition can be written as $\vec{p} = M\vec{\phi}$, where $\vec{p} = (p_a, p_b, p_c, p_\theta, p_\psi, p_\varphi, p_\alpha, p_\beta, p_\gamma)$ is the vector of the momenta, while $\vec{\phi}$ is the vector of the ϕ_I^a, ordered as in Eq. (12).

Using the expression of p_a^I in terms of metric variables, M can be written explicitly, but the expression is too long to write it here. However, the computation of the inverse is too difficult, and also computing program such as Wolfram *Mathematica* does not give a solution. At least, the determinant does not vanish identically and the matrix is invertible in an open subset of the configuration space.

The complexity of the matrix is due to the abundant presence of trigonometric functions. Consequently, to do an approximate analysis, we need to fix the values of the angles such that the matrix is invertible. These values are $\pi/2$ for all the angles. At this point, the matrix M is invertible, and ϕ_I^a can be written in terms of the gauge momenta. From which, a nonidentically vanishing Gauss constraint is provided

$$G = (p_\beta, p_\alpha, p_\gamma). \tag{13}$$

We can check that, imposing the constraint (i.e. putting the gauge momenta to zero), we recover the same expression of ϕ_I^a as in Eq. (9).

Since the nonvanishing condition of the determinant is an open condition, there exists an open neighborhood of $\pi/2$ in which the matrix is invertible. So, now we can do an expansion of M at the first order with respect to a small parameter ε, and, for ε small enough, we obtain an invertible matrix. We need a small parameter for each angle: the small parameter that affects the θ angle is defined as $\varepsilon_\theta = \theta - \pi/2$, and similarly for all the other angles. From this, we compute M at the first order. The inverse of the matrix can be found as well as the connection. In the first-order approximation, the Gauss constraint has a simple expression

$$G \simeq (p_\beta + \varepsilon_\gamma p_\alpha, p_\alpha + \varepsilon_\beta p_\gamma - \varepsilon_\gamma p_\beta, p_\gamma). \tag{14}$$

It trivial to check that, for $\varepsilon \to 0$, one finds the quantities computed at $\pi/2$.

A nonvanishing Gauss constraint is found, this means that a formulation in terms of the gauge variables is the right way forward. However, this result is not satisfying. The expression for the Gauss constraint is an approximated form around an arbitrary point in which the limit does not bring back to any well-known theory. Moreover, no explanation is given about the identically vanishing Gauss constraint that emerges in the previous formulations. Hence, a deeper analysis is required.

B. The Lagrangian approach

We want to investigate what happens to the Hamiltonian formulation starting from the Hols action [3,6]

$$\mathcal{S}_H = \frac{c^3}{8\pi G\gamma}\int dt\left(p_a^I \dot{\phi}_I^a + \lambda^a G_a - N^I \mathcal{V}_I - \frac{N}{2}\gamma\mathcal{S}\right), \quad (15)$$

and considering the connection and the dreibein with respect to the metric variables (9), (10). Here, λ^a are Lagrange multipliers and

$$G_a = \epsilon_{ab}{}^c \phi_I^b p_c^I, \qquad \mathcal{V}_I = G_b \phi_I^b,$$
$$\mathcal{S} = -\frac{1}{\gamma^2|\det(p_c^K)|}(p_a^I \phi_I^a p_b^J \phi_J^b - p_a^I \phi_J^a p_b^J \phi_I^b),$$

are the Gauss, diffeomorphism and scalar constraints, respectively.

We recall that ϕ_I^a is computed from the usual expression of the connection $A_i^a = \Gamma_i^a + \gamma K_i^a$, while p_a^I has the geometrical meaning as the homogeneous part of the dreibein vectors. We want the Holst action to be explicit in terms of metric variables, the calculation of the single terms provides that the Gauss constraint vanishes $G_a = 0$, and so the diffeomorphism constraint $\mathcal{V}_i = 0$, while the scalar constraint has the same expression as in Eq. (46) from Ref. [24].

As we expect, the gauge freedom does not appear in the Lagrangian, and it is invariant under gauge transformation. Hence, the momenta can be computed and the gauge momenta are null (i.e. $p_\alpha = 0$, $p_\beta = 0$, $p_\gamma = 0$), while the others are the same presented in Ref. [24]. We can now perform the Legendre transformation with Lagrangian multipliers λ_i to find the Hamiltonian

$$H = \frac{c^3}{8\pi G}\left(\lambda_1 p_\alpha + \lambda_2 p_\beta + \lambda_3 p_\gamma - \frac{N}{2}\mathcal{S}\right). \quad (16)$$

Thus, the theory of the nondiagonal Bianchi I model in metric variables is a constrained Hamiltonian theory with phase space $(a, b, c, \theta, \psi, \varphi, \alpha, \beta, \gamma, p_a, p_b, p_c, p_\theta, p_\psi, p_\varphi, p_\alpha, p_\beta, p_\gamma)$ with four constraints

$$p_\alpha \approx 0, \qquad p_\beta \approx 0, \qquad p_\gamma \approx 0, \qquad \mathcal{S} \approx 0 \quad (17)$$

and with a Hamiltonian which is a linear combination of such constraints.

Notice that the same theory written in terms of connection and dreibein (ϕ_I^a, p_b^I) has four constraints given by $G_a \approx 0$ and $\mathcal{S} \approx 0$.

The scalar constraint is the same in both formulations in the sense that it is possible to switch from one to the other using transformation (9). This property does not hold for the Gauss constraint, which also in gauge variables vanishes. However, it is replaced by the three constraints on the pure gauge momenta.

We interpret this result as follows: the Gauss constraint after the canonical transformation becomes the gauge momenta constraint. In such a way, the dependence on the gauge momenta we introduce in the Ashtekar variables vanishes on the constraints' hypersurface, recovering the usual description.

IV. EQUIVALENCE BETWEEN GAUSS CONSTRAINT AND PURE GAUGE MOMENTA

In this section, we want to find an explicit expression for the Gauss constraint. Previously, we showed that there exists a relation between the Gauss constraint and the momenta constraint, which we interpreted as

$$G_a = 0 \Leftrightarrow p_g = 0 \quad (18)$$

where $g \in \{\alpha, \beta, \gamma\}$.

Such a relation is satisfied if the Gauss constraint is linearly dependent on pure gauge momenta p_g only. For simplicity, it will be our ansatz. Thus, we enunciate the following conjecture:

Conjecture 1. The Gauss constraint depends on the gauge momenta via a 3×3 matrix L_{ag}:

$$G_a = L_{ag} p_g, \quad (19)$$

with a is $SU(2)$ internal index and $g \in \{\alpha, \beta, \gamma\}$.

Using this ansatz, we can explicitly compute the coefficients of the linear combination without using M, nor an explicit expression of ϕ_I^a. Let p_a^I as in Eq. (10) and the gauge momenta p_g given by the transformation that satisfies the Lie condition (11).

With this assumption, the Gauss constraint reads

$$G_a = \epsilon_{ab}{}^c \phi_I^b p_c^I = \phi_I^b (p^{\text{ph}})_d^I \epsilon_{ab}{}^c O_c^d$$
$$= L_{ag} p_g = L_{ag} \phi_I^b \frac{\partial p_b^I}{\partial q_g} = \phi_I^b (p^{\text{ph}})_d^I L_{ag} \frac{\partial O_b^d}{\partial q_g},$$

where $(p^{\text{ph}})_d^I$ is the physical part of the dreibein (i.e. the one not gauge rotated). From this, we can derive the following equation:

$$\epsilon_{ab}{}^c = L_{ag} (O^t)_d^c \frac{\partial O_b^d}{\partial q_g}. \quad (20)$$

This equation is enough to fully characterize the matrix L_{ag}, in fact, the rhs has the same skew-symmetric property as the Levi-Civita symbol. Therefore, we obtain nine linear independent equations. The equations' system has nine equations in nine variables and the associated determinant is $\sin^3\beta$, so, it is nondegenerate. Hence, it exists one and only one solution. The solution L_{ag} can be found easily and it reads

$$L_{ag} = \begin{pmatrix} -\csc\beta\cos\gamma & \sin\gamma & \cot\beta\cos\gamma \\ \csc\beta\sin\gamma & \cos\gamma & -\cot\beta\sin\gamma \\ 0 & 0 & 1 \end{pmatrix}. \quad (21)$$

Finally, the Gauss constraint can be written explicitly in terms of gauge momenta

$$G_a = \begin{pmatrix} -\csc\beta\cos\gamma p_\alpha + \sin\gamma p_\beta + \cot\beta\cos\gamma p_\gamma \\ \csc\beta\sin\gamma p_\alpha + \cos\gamma p_\beta - \cot\beta\sin\gamma p_\gamma \\ p_\gamma \end{pmatrix}. \quad (22)$$

The conjecture enables us to find the explicit dependence of the Gauss constraint on the gauge momenta. The matrix of coefficients is invertible since its determinant is $\det(L_{ag}) = -\csc\beta$, then the equivalence condition (18) holds.

This explains the vanishing Gauss constraint in our initial approach, and in general, in the similar approaches of loop quantum cosmology. In fact, the connection $A_i^a = \Gamma_i^a + \gamma K_i^a$ is reduced and, when it is described in terms of metric variables, it results in a function defined on the constraints' hypersurface, as well as the dreibein, and so, the linear dependence (19) implies that the Gauss constraint computed from such a connection must vanish.

A. Relation with $\mathfrak{su}(2)$

The Gauss constraint is deeply linked with the gauge group $SU(2)$ and, in particular, with its Lie algebra $\mathfrak{su}(2)$. We want to explore this link in view of the previous results.

The expression in (22) has no evident $SU(2)$ symmetry. However, via a change of coordinates, a contribution of $\mathfrak{su}(2)$ appears in the expression.

Consider the matrix O of the gauge transformation parametrized by three angles α_1, α_2, α_3 such that $O = \exp(\alpha_i j_i) = \exp[\alpha(n_i j_i)]$, where j_i are the generators of $SU(2)$, $n_i = \alpha_i/\alpha$ and $\alpha = \sqrt{(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2}$. Imposing Eq. (20) we find a new set of equations, they admit only one solution, and the matrix L_{ag} can be found. An expansion for small α of L_{ag} give us

$$L_{ag} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix} + \mathcal{O}(\alpha^2). \quad (23)$$

It is interesting to notice that the first-order term is a skew-symmetric 3×3 matrix, so it is an element of the Lie algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$.

There is another and more profound link between the Gauss constraint and the $SU(2)$ group that will be examined in the next section.

V. GAUSS CONSTRAINT AS THE GENERATOR OF GAUGE TRANSFORMATIONS

It is well known that the Gauss constraint G_a is the generator of the gauge transformations on the phase space (ϕ_I^a, p_b^J) [1,2,19,24]. This feature should hold in the new variables. Therefore, we want to compute the canonical Poisson brackets with respect to $\{\alpha, \beta, \gamma, p_\alpha, p_\beta, p_\gamma\}$ of the Gauss constraint in (22). We obtain

$$\{G_a, G_b\} = -\epsilon_{abc} G_c. \quad (24)$$

The sign is not relevant. We expect that this formulation comes out from the canonical transformation in (11) in which the connection and dreibein are switched, then a sign in the Poisson bracket appears.

Hence, the Gauss constraint respects the $\mathfrak{su}(2)$-Lie algebra and generates the $SU(2)$ gauge transformations. Furthermore, it is linear in gauge momenta, so the hypersurface defined by $G_a = 0$ is also described by $p_\alpha = p_\beta = p_\gamma = 0$. Thus, the Gauss constraint is equivalent to three constraints on the momenta. Consequently, the generators of the gauge transformation can be decomposed into three generators which commute each other

$$\{p_\alpha, p_\beta\} = \{p_\alpha, p_\gamma\} = \{p_\beta, p_\gamma\} = 0. \quad (25)$$

This decomposition is particularly useful in simplifying the implementation of the Gauss constraint in a quantum theory.

A. Quantum Gauss constraint

In Ref. [24] is shown that the quantization of the nondiagonal Bianchi I model can be provided in diagonal fluxes and angles variables. It is reasonable that a similar quantization can be provided for the other nondiagonal models, given a loop quantization of homogeneous Universes. However, to complete the description in the loop framework, we need to include the gauge transformations and a nonvanishing Gauss constraint.

Supposing that we have a quantization like in the nondiagonal Bianchi I case, it is enough to add the gauge variables to the phase space of the diagonal fluxes and angles. These gauge variables will be the Euler angles of the gauge rotation and they will be quantized independently (as the physical angles [24]) via the Schrödinger picture. Thus, the wave functions are $\Psi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3, \alpha, \beta, \gamma)$, where p_1, p_2, p_3 are the diagonal fluxes and θ_1, θ_2, θ_3 are the physical angles.

Moreover, the Hamiltonian (such as the Lagrangian) is independent of the gauge variables, hence the wave function factorizes $\Psi = \phi(\alpha, \beta, \gamma)\Phi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3)$.

On these functions, the action on the Gauss constraint is essentially a first-order derivative, so the imposition of the weak constraint $\hat{G}_a\Psi = 0$ is equivalent to

$$-i\hbar\frac{\partial\Psi}{\partial\alpha} = 0, \qquad -i\hbar\frac{\partial\Psi}{\partial\beta} = 0, \qquad -i\hbar\frac{\partial\Psi}{\partial\gamma} = 0. \quad (26)$$

The solution of this set of equations is trivial: $\phi(\alpha, \beta, \gamma) =$ const. Thus, the Gauss constraint in this Hilbert space imposes the independence of the wave function on the gauge angles. Therefore, the kinematical Hilbert space for the nondiagonal Bianchi I model presented in Ref. [24] remains the same also including the gauge transformations.

VI. CONCLUDING REMARK

The analysis above deepens the idea proposed in Ref. [22], that a nonvanishing Gauss constraint can be restored also in the minisuperspace of a Bianchi model, as soon as the most general for the Ashtekar-Barbero-Immirzi connection is considered.

Actually, we interpret this general formulation in terms of the ADM metric variables. The introduction of gauge variables is responsible for restoring the $SU(2)$ symmetry and ensuring that the corresponding connection has to verify a Gauss constraint. However, the main result we obtained is that the components of such a Gauss constraint are linearly dependent on the three momenta corresponding to the gauge angles. Thus, in terms of metric variables, the $SU(2)$ symmetry reduces to the vanishing behavior of these three momenta, i.e. it is, *de facto* reduced to a set of Abelian constraints.

We also clarified how the noncommutative character of the Gauss constraint components is restored via the transformation linking the two representations, associated with the $SU(2)$ generators.

This issue has a deep impact on the Dirac quantization of the model since the three momenta operators, associated with the gauge angles, must annihilate the state function, which is therefore independent of such angles. Hence, our quantization of the model is equivalent to a nondiagonal quantum Bianchi cosmology, as discussed in Ref. [24], especially concerning the kinematical Hilbert space structure. Since in Ref. [24], the quantum picture is associated with a diagonal set of flux variables, plus the three Euler angles expected to be canonically quantized, the present analysis allows us to claim that the quantization of the Bianchi I model, discussed in Ref. [26], see also Ref. [14] for a critical revision, is actually a rather general formulation, and the only one available in a minisuperspace dynamics. In other words, the scale factors associated in a Bianchi cosmology to independent space directions are the most relevant subjects of a loop quantum cosmology quantization procedure and are characterized by an almost-periodic functions representation.

The reason why the minisuperspace $SU(2)$ symmetry can be decomposed on an Abelian symmetry of the phase space kinematics is reliably due to the fact that for the space Ashtekar-Barbero-Immirzi connection, a local Lorentz rotation depending on time only retains a global character. Thus, a genuine $SU(2)$-formulation in the sense of loop quantum gravity is still forbidden.

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