Grassmann-odd three-point functions of conserved supercurrents in 3D $\mathcal{N} = 1$ SCFT

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(Received 16 May 2023; accepted 25 July 2023; published 7 August 2023)

We consider the analytic construction of three-point functions of conserved higher-spin supercurrents in three-dimensional $\mathcal{N} = 1$ superconformal field theory which are Grassmann-odd in superspace. In particular, these include the three-point functions of the supercurrent and flavor currents, which contain the three-point functions of the energy-momentum tensor and conserved vector currents at the component level. We present an analytic proof for arbitrary superspins that these correlators do not possess a parity-violating contribution. We also prove that the parity-even contribution is unique, and exists (under an assumption that is well supported by the computational approach of arXiv:2302.00593) for arbitrary superspins. The construction of the parity-even sector is shown to reduce to solving a system of linear homogeneous equations with a tridiagonal matrix of corank one, which we solve explicitly for arbitrary superspins.

DOI: 10.1103/PhysRevD.108.046001

I. INTRODUCTION

In conformal field theory (CFT), the general structure of correlation functions is highly constrained by conformal symmetry. In particular, the three-point functions of conserved currents such as the energy-momentum tensor, flavor currents and more generally higher-spin currents, are fixed up to finitely many independent structures [1–12]. For conformal field theories in three dimensions (3D), it has been proven that the three-point functions of conserved higher-spin currents are constrained up to only three independent structures [13–17]. Two of the structures are parity-even (corresponding to free theories), and one is parity-odd (or parity violating), which has been shown to correspond to theories of a Chern-Simons gauge field interacting with parity-violating matter (see e.g. [18–28]). In superconformal field theory (SCFT), three-point functions are further constrained.[1] For example, in 3D $\mathcal{N} = 1$ superconformal field theory it was shown in [38] that there is an apparent tension between supersymmetry and the existence of parity-violating structures in three-point functions. In contrast with the nonsupersymmetric case, it was shown that parity-odd structures are not found in the three-point functions of the energy-momentum tensor and conserved vector currents, which were studied using a manifestly supersymmetric approach in [35–38,45].

The general structure of three-point functions of higher-spin supercurrents in 3D $\mathcal{N} = 1$ SCFT was elucidated in [45]. Conformal higher-spin supercurrents of superspin-s (integer or half-integer) are defined as totally symmetric spin-tensor superfields, $\mathbf{J}_s \equiv \mathbf{J}_{\alpha_1 \ldots \alpha_{2s}}(z) = \mathbf{J}_{(\alpha_1 \ldots \alpha_{2s})}(z)$, and satisfy the conservation equation

$$D^{\alpha_1} \mathbf{J}_{\alpha_1 \alpha_2 \ldots \alpha_{2s}}(z) = 0, \tag{1.1}$$

where D^α is the spinor covariant derivative in 3D $\mathcal{N} = 1$ Minkowski superspace. The most important examples of conserved supercurrents in superconformal field theory are the flavor current and supercurrent multiplets, corresponding to the cases $s = \frac{1}{2}$ and $s = \frac{3}{2}$ respectively (for a review of the properties of flavor current and supercurrent multiplets in three dimensions, see [35,46] and the references therein). The flavor current multiplet contains a conserved vector current, while the supercurrent multiplet contains the energy-momentum tensor and the supersymmetry current. For higher-spin supercurrents, it was shown by explicit calculations up to a high computational bound ($s_i \leq 20$) that the general structure of the three-point function $\langle \mathbf{J}_{s_1}(z_1) \mathbf{J}'_{s_2}(z_2) \mathbf{J}''_{s_3}(z_3) \rangle$ is fixed up to the following form [45]:

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[1] The study of correlation functions in superconformal theories has been carried out in diverse dimensions using the group-theoretic approach developed in the following publications [29–44].

$$\langle \mathbf{J}_{s_1} \mathbf{J}'_{s_2} \mathbf{J}''_{s_3} \rangle = a \langle \mathbf{J}_{s_1} \mathbf{J}'_{s_2} \mathbf{J}''_{s_3} \rangle_E + b \langle \mathbf{J}_{s_1} \mathbf{J}'_{s_2} \mathbf{J}''_{s_3} \rangle_O, \qquad (1.2)$$

where $\langle \mathbf{J}_{s_1} \mathbf{J}'_{s_2} \mathbf{J}''_{s_3} \rangle_E$ is a parity-even solution, and $\langle \mathbf{J}_{s_1} \mathbf{J}'_{s_2} \mathbf{J}''_{s_3} \rangle_O$ is a parity-odd solution. For the three-point functions which are Grassmann-even (bosonic) in superspace, the existence of the parity-odd solution is subject to the following superspin triangle inequalities:

$$s_1 \leq s_2 + s_3, \qquad s_2 \leq s_1 + s_3, \qquad s_3 \leq s_1 + s_2. \qquad (1.3)$$

When the triangle inequalities are simultaneously satisfied there is one even solution and one odd solution, however, if any of the above inequalities are not satisfied then the odd solution is incompatible with the superfield conservation equations. On the other hand, for the Grassmann-odd (fermionic) three-point functions it was shown that the parity-odd solution appears to vanish in general. Despite being limited by computational power to consider superspins $s_i \leq 20$, the pattern was clear and we proposed in [45] that these results hold in general.

The aim of this paper is to study the Grassmann-odd three-point functions analytically for arbitrary superspins. We use a different approach to [45], based on a method of irreducible decomposition of tensors. Quite remarkably, we find that these three-point functions can be constructed explicitly for arbitrary superspins. For the parity-violating sector we give a completely analytic proof that it vanishes for arbitrary superspins. For the parity-even sector we found that its construction is reduced to solving a homogeneous system of linear equations with a tridiagonal matrix of corank one, which proves that the parity-even sector is fixed up to a single structure in general. We also found the solution to this system for arbitrary superspins, thus obtaining the explicit form of the parity-even contribution. Our analysis uses one simplifying assumption which is, however, well supported by our computational approach [45]. It was noticed in [45] that if the third superspin satisfies the triangle inequality $s_3 \leq s_1 + s_2$ it is not necessary to impose the supercurrent conservation condition at the third point because it is automatically satisfied and does not give any further restrictions. In this paper we assume that this property continues to hold for arbitrary superspins. However, we should stress that our proof that the parity-odd sector vanishes does not rely on this assumption. It is also inconsequential for our analysis of the parity-even sector, as after imposing the conservation conditions for the first two supercurrents we prove that it is already fixed up to an overall coefficient. Since on general grounds we should expect at least one parity-even solution,[2] it follows that the conservation condition for the third supercurrent is, indeed, unnecessary.

The results of this paper are organized as follows. In Sec. II we provide a brief review of the general structure of the three-point functions of conserved currents in 3D $\mathcal{N} = 1$ SCFT. In Sec. III we study Grassmann-odd three-point functions which consist of three conserved supercurrents of arbitrary half-integer superspins. We show that the construction of both the parity-even and parity-odd sector is governed by a homogeneous system of linear equations with tridiagonal matrix. By computing the determinants of the tridiagonal matrices for the parity-even and parity-odd sectors, in the former case we prove that the matrix has corank one, and hence the parity-even solution is unique for arbitrary superspins. In the latter case, we prove that the matrix is nondegenerate meaning that the parity-odd solution vanishes in general. In Sec. IV we perform a similar analysis for Grassmann-odd three-point functions consisting of one fermionic and two bosonic supercurrents. Appendix is dedicated to our 3D conventions and notation.

II. SUPERCONFORMAL BUILDING BLOCKS AND CORRELATION FUNCTIONS

In this section we will review the essentials of the group-theoretic formalism used to compute three-point correlation functions of primary superfields in 3D $\mathcal{N} = 1$ superconformal field theories. For a more detailed review of the formalism and our conventions, the reader may consult [17,32,35,45].

Given two superspace points z_1 and z_2, we define the two-point functions

$$\boldsymbol{x}_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta_1^{(\alpha}\theta_2^{\beta)} - \mathrm{i}\theta_{12}^{\alpha}\theta_{12}^{\beta}, \qquad \theta_{12}^{\alpha} = \theta_1^{\alpha} - \theta_2^{\alpha}. \qquad (2.1)$$

The two-point function $\boldsymbol{x}_{12}^{\alpha\beta}$ can be split into symmetric and antisymmetric parts as follows:

$$\boldsymbol{x}_{12}^{\alpha\beta} = x_{12}^{\alpha\beta} + \frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\theta_{12}^2, \quad \theta_{12}^2 = \theta_{12}^{\alpha}\theta_{12\alpha}, \quad \boldsymbol{x}_{12}^2 = -\frac{1}{2}\boldsymbol{x}_{12}^{\alpha\beta}\boldsymbol{x}_{12\alpha\beta}, \qquad (2.2)$$

where the symmetric component

$$x_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta_1^{(\alpha}\theta_2^{\beta)}, \qquad (2.3)$$

is the standard bosonic two-point superspace interval. It is useful to introduce the normalized two-point functions, denoted by $\hat{\boldsymbol{x}}_{12}$,

$$\hat{\boldsymbol{x}}_{12\alpha\beta} = \frac{\boldsymbol{x}_{12\alpha\beta}}{(\boldsymbol{x}_{12}^2)^{1/2}}, \qquad \hat{\boldsymbol{x}}_{12}^{\alpha\sigma}\hat{\boldsymbol{x}}_{21\sigma\beta} = \delta_{\beta}^{\alpha}. \qquad (2.4)$$

From here we can now construct an operator analogous to the conformal inversion tensor acting on the space of

[2] In some cases the parity-even solution and, hence, the entire three-point function, vanishes. However, this occurs only when the three-point function is required to be invariant under permutations of superspace points. These cases were analyzed systematically in [45].

symmetric traceless spin-tensors of arbitrary rank. Given a normalized two-point function $\hat{\boldsymbol{x}}$, we define the operator

$$\mathcal{I}_{\alpha(k)\beta(k)}(\boldsymbol{x}) = \hat{\boldsymbol{x}}_{(\alpha_1(\beta_1}\ldots\hat{\boldsymbol{x}}_{\alpha_k)\beta_k)}. \tag{2.5}$$

This object is essential to the construction of correlation functions of primary operators of arbitrary superspins [45].

Now given three superspace points, z_1, z_2, z_3, we define the three-point building blocks, $\mathcal{Z}_k = (\boldsymbol{X}_{ij}, \Theta_{ij})$ as follows:

$$\begin{aligned}\boldsymbol{X}_{ij\alpha\beta} &= -(\boldsymbol{x}_{ik}^{-1})_{\alpha\gamma}\boldsymbol{x}_{ij}^{\gamma\delta}(\boldsymbol{x}_{kj}^{-1})_{\delta\beta},\\ \Theta_{ij\alpha} &= (\boldsymbol{x}_{ik}^{-1})_{\alpha\beta}\theta_{ki}^{\beta} - (\boldsymbol{x}_{jk}^{-1})_{\alpha\beta}\theta_{kj}^{\beta},\end{aligned} \tag{2.6}$$

where the labels (i, j, k) are a cyclic permutation of (1, 2, 3). They satisfy many properties similar to those of the two-point building blocks [for simplicity we consider $(\boldsymbol{X}_{12}, \Theta_{12})$]

$$\boldsymbol{X}_{12}^{\alpha\sigma}\boldsymbol{X}_{21\sigma\beta} = \boldsymbol{X}_{12}^2\delta_\beta^\alpha, \qquad \boldsymbol{X}_{12}^2 = -\frac{1}{2}\boldsymbol{X}_{12}^{\alpha\beta}\boldsymbol{X}_{12\alpha\beta}, \tag{2.7a}$$

$$\boldsymbol{X}_{12}^2 = \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{13}^2\boldsymbol{x}_{23}^2}, \qquad \Theta_{12}^2 = \Theta_{12}^\alpha\Theta_{12\alpha}, \tag{2.7b}$$

and may be decomposed into symmetric and antisymmetric parts similar to (2.2) as follows:

$$\boldsymbol{X}_{12\alpha\beta} = X_{12\alpha\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\Theta_{12}^2, \qquad X_{12\alpha\beta} = X_{12\beta\alpha}. \tag{2.8}$$

The symmetric spin-tensor, $X_{12\alpha\beta}$, can be equivalently represented by the three-vector $X_{12m} = -\frac{1}{2}(\gamma_m)^{\alpha\beta}X_{12\alpha\beta}$. One may also identify the superconformal invariant

$$\boldsymbol{J} = \frac{\Theta_{12}^2}{\sqrt{\boldsymbol{X}_{12}^2}} = \frac{\Theta_{23}^2}{\sqrt{\boldsymbol{X}_{23}^2}} = \frac{\Theta_{31}^2}{\sqrt{\boldsymbol{X}_{31}^2}}. \tag{2.9}$$

Analogous to the two-point functions, it is also useful to introduce the following normalized three-point building blocks, denoted by $\hat{\boldsymbol{X}}_{12}$, $\hat{\Theta}_{12}$:

$$\hat{\boldsymbol{X}}_{12\alpha\beta} = \frac{\boldsymbol{X}_{12\alpha\beta}}{(\boldsymbol{X}_{12}^2)^{1/2}}, \qquad \hat{\Theta}_{12}^\alpha = \frac{\Theta_{12}^\alpha}{(\boldsymbol{X}_{12}^2)^{1/4}}, \tag{2.10}$$

such that

$$\hat{\boldsymbol{X}}_{12}^{\alpha\sigma}\hat{\boldsymbol{X}}_{21\sigma\beta} = \delta_\beta^\alpha, \qquad \boldsymbol{J} = \hat{\Theta}_{12}^2. \tag{2.11}$$

Now given an arbitrary three-point building block, $\boldsymbol{X}$, we construct the following higher-spin inversion operator:

$$\mathcal{I}_{\alpha(k)\beta(k)}(\boldsymbol{X}) = \hat{\boldsymbol{X}}_{(\alpha_1(\beta_1}\ldots\hat{\boldsymbol{X}}_{\alpha_k)\beta_k)}. \tag{2.12}$$

This operators possess properties similar to the two-point higher-spin inversion operators (2.5). Let us now introduce the following analogs of the covariant spinor derivative and supercharge operators involving the three-point building blocks, where $(X, \Theta) = (X_{12}, \Theta_{12})$:

$$\begin{aligned}\mathcal{D}_\alpha &= \frac{\partial}{\partial\Theta^\alpha} + \mathrm{i}(\gamma^m)_{\alpha\beta}\Theta^\beta\frac{\partial}{\partial X^m},\\ \mathcal{Q}_\alpha &= \mathrm{i}\frac{\partial}{\partial\Theta^\alpha} + (\gamma^m)_{\alpha\beta}\Theta^\beta\frac{\partial}{\partial X^m},\end{aligned} \tag{2.13}$$

which obey the commutation relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = 2\mathrm{i}(\gamma^m)_{\alpha\beta}\frac{\partial}{\partial X^m}. \tag{2.14}$$

Now given a function $f(\boldsymbol{X}_{12}, \Theta_{12})$, there are the following differential identities which are essential for imposing differential constraints on three-point correlation functions of primary superfields:

$$D_{(1)\gamma}f(\boldsymbol{X}_{12}, \Theta_{12}) = (\boldsymbol{x}_{13}^{-1})_{\alpha\gamma}\mathcal{D}^\alpha f(\boldsymbol{X}_{12}, \Theta_{12}), \tag{2.15a}$$

$$D_{(2)\gamma}f(\boldsymbol{X}_{12}, \Theta_{12}) = \mathrm{i}(\boldsymbol{x}_{23}^{-1})_{\alpha\gamma}\mathcal{Q}^\alpha f(\boldsymbol{X}_{12}, \Theta_{12}). \tag{2.15b}$$

Here by $D_{(1)\gamma}$ and $D_{(2)\gamma}$ we denote the ordinary superspace covariant derivatives acting on the superspace points $z_1 = (x_1, \theta_1)$ and $z_2 = (x_2, \theta_2)$, respectively.

Now consider a primary tensor superfield $\Phi_{\mathcal{A}}(z)$ of dimension Δ transforming in an irreducible representation of the Lorentz group. The two-point correlation function $\langle\Phi_{\mathcal{A}}(z_1)\Phi^{\mathcal{B}}(z_2)\rangle$ is constrained by superconformal symmetry to the following form:

$$\langle\Phi_{\mathcal{A}}(z_1)\Phi^{\mathcal{B}}(z_2)\rangle = c\frac{\mathcal{I}_{\mathcal{A}}{}^{\mathcal{B}}(\boldsymbol{x}_{12})}{(\boldsymbol{x}_{12}^2)^\Delta}, \tag{2.16}$$

where $\mathcal{I}$ is an appropriate representation of the inversion tensor and c is a constant real parameter. The denominator of the two-point function is determined by the conformal dimension of $\Phi_{\mathcal{A}}$, which guarantees that the correlation function transforms with the appropriate weight under scale transformations.

For three-point functions, let Φ, Ψ, Π be primary superfields with scale dimensions Δ_1, Δ_2, and Δ_3 respectively. The three-point function is constructed using the general ansatz [31,32]

$$\langle\Phi_{\mathcal{A}_1}(z_1)\Psi_{\mathcal{A}_2}(z_2)\Pi_{\mathcal{A}_3}(z_3)\rangle = \frac{\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{A}_1'}(\boldsymbol{x}_{13})\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}_2'}(\boldsymbol{x}_{23})}{(\boldsymbol{x}_{13}^2)^{\Delta_1}(\boldsymbol{x}_{23}^2)^{\Delta_2}}\mathcal{H}_{\mathcal{A}_1'\mathcal{A}_2'\mathcal{A}_3}(\boldsymbol{X}_{12}, \Theta_{12}), \tag{2.17}$$

where the tensor $\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}$ encodes all information about the correlation function, and is related to the leading singular operator product expansion (OPE) coefficient [11].

In this work we are primarily interested in the structure of three-point correlation functions of conserved (higher-spin) supercurrents. In 3D $\mathcal{N}=1$ theories, a conserved (higher-spin) supercurrent of superspin-s (integer or half-integer), is defined as a totally symmetric spin-tensor of rank $2s$, $\mathbf{J}_{\alpha_1\ldots\alpha_{2s}}(z)=\mathbf{J}_{(\alpha_1\ldots\alpha_{2s})}(z)=\mathbf{J}_{\alpha(2s)}(z)$, satisfying a conservation equation of the form

$$D^{\alpha_1}\mathbf{J}_{\alpha_1\alpha_2\ldots\alpha_{2s}}(z)=0. \tag{2.18}$$

Conserved currents are primary superfields, and the dimension $\Delta_{\mathbf{J}}$ of $\mathbf{J}$ is fixed by the conservation condition (2.18) to $\Delta_{\mathbf{J}}=s+1$. At the component level, a higher-spin supercurrent of superspin-s contains conserved conformal currents of spin-s and spin-$(s+\frac{1}{2})$ respectively. Indeed, for conserved supercurrents of superspin s, the dimension Δ of the two-point function (2.16) is fixed by conservation to $\Delta=s+1$. If we now consider the three-point function of the conserved primary superfields $\mathbf{J}_{\alpha(I)}$, $\mathbf{J}'_{\beta(J)}$, $\mathbf{J}''_{\gamma(K)}$, where $I=2s_1$, $J=2s_2$, $K=2s_3$, then the general ansatz is

$$\begin{aligned}&\langle\mathbf{J}_{\alpha(I)}(z_1)\mathbf{J}'_{\beta(J)}(z_2)\mathbf{J}''_{\gamma(K)}(z_3)\rangle\\&=\frac{\mathcal{I}_{\alpha(I)}{}^{\alpha'(I)}(\boldsymbol{x}_{13})\mathcal{I}_{\beta(J)}{}^{\beta'(J)}(\boldsymbol{x}_{23})}{(\boldsymbol{x}_{13}^2)^{\Delta_1}(\boldsymbol{x}_{23}^2)^{\Delta_2}}\mathcal{H}_{\alpha'(I)\beta'(J)\gamma(K)}(\boldsymbol{X}_{12},\Theta_{12}),\end{aligned} \tag{2.19}$$

where $\Delta_i=s_i+1$. Below we summarize the constraints on $\mathcal{H}$.

(i) Homogeneity:

$$\begin{aligned}&\mathcal{H}_{\alpha(I)\beta(J)\gamma(K)}(\lambda^2\boldsymbol{X},\lambda\Theta)\\&=(\lambda^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{H}_{\alpha(I)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta),\end{aligned} \tag{2.20}$$

It is often convenient to introduce $\hat{\mathcal{H}}_{\alpha(I)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta)$, such that

$$\mathcal{H}_{\alpha(I)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta)=\boldsymbol{X}^{\Delta_3-\Delta_3-\Delta_1}\hat{\mathcal{H}}_{\alpha(I)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta), \tag{2.21}$$

where $\hat{\mathcal{H}}_{\alpha(I)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta)$ is homogeneous degree 0 in $(\boldsymbol{X},\Theta)$, i.e.

$$\hat{\mathcal{H}}_{\alpha(I)\beta(J)\gamma(K)}(\lambda^2\boldsymbol{X},\lambda\Theta)=\hat{\mathcal{H}}_{\alpha(I)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta). \tag{2.22}$$

(ii) Differential constraints:

After application of the identities (2.15a), (2.15b) we obtain the following constraints:

$$\text{Conservation at } z_1:\ \mathcal{D}^{\alpha}\mathcal{H}_{\alpha\alpha(I-1)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta)=0, \tag{2.23a}$$

$$\text{Conservation at } z_2:\ \mathcal{Q}^{\beta}\mathcal{H}_{\alpha(I)\beta\beta(J-1)\gamma(K)}(\boldsymbol{X},\Theta)=0, \tag{2.23b}$$

$$\text{Conservation at } z_3:\ \mathcal{Q}^{\gamma}\tilde{\mathcal{H}}_{\alpha(I)\beta(J)\gamma\gamma(K-1)}(\boldsymbol{X},\Theta)=0, \tag{2.23c}$$

where

$$\begin{aligned}&\tilde{\mathcal{H}}^{(\pm)}_{\alpha(I)\beta(J)\gamma(K)}(\boldsymbol{X},\Theta)\\&=(\boldsymbol{X}^2)^{\Delta_1-\Delta_3}\mathcal{I}_{\beta(J)}{}^{\beta'(J)}(\boldsymbol{X})\mathcal{H}^{I(\pm)}_{\alpha(I)\beta'(J)\gamma(K)}(\boldsymbol{X},\Theta).\end{aligned} \tag{2.24}$$

(iii) Point-switch symmetries:

If the fields $\mathbf{J}$ and $\mathbf{J}'$ coincide, then we obtain the following point-switch identity

$$\mathcal{H}_{\alpha(I)\beta(I)\gamma(K)}(\boldsymbol{X},\Theta)=(-1)^{\epsilon(\mathbf{J})}\mathcal{H}_{\beta(I)\alpha(I)\gamma(K)}(-\boldsymbol{X}^{\mathrm{T}},-\Theta), \tag{2.25}$$

where $\epsilon(\mathbf{J})$ is the Grassmann parity of $\mathbf{J}$. Likewise, if the fields $\mathbf{J}$ and $\mathbf{J}''$ coincide, then we obtain the constraint

$$\tilde{\mathcal{H}}_{\alpha(I)\beta(J)\gamma(I)}(\boldsymbol{X},\Theta)=(-1)^{\epsilon(\mathbf{J})}\mathcal{H}_{\gamma(I)\beta(J)\alpha(I)}(-\boldsymbol{X}^{\mathrm{T}},-\Theta). \tag{2.26}$$

In the next sections, we will demonstrate that conservation on z_1 and z_2 is sufficient to constrain the structure of the three-point function to a unique parity-even solution, while the parity-odd solution must vanish.

III. GRASSMANN-ODD THREE-POINT FUNCTIONS $\langle\mathbf{J}_F\mathbf{J}'_F\mathbf{J}''_F\rangle$

There are only two possibilities for Grassmann-odd three-point functions in superspace (up to permutations of the fields), they are:

$$\langle\mathbf{J}_F\mathbf{J}'_F\mathbf{J}''_F\rangle,\qquad\langle\mathbf{J}_F\mathbf{J}'_B\mathbf{J}''_B\rangle, \tag{3.1}$$

where "B" represents a Grassmann-even (bosonic) field, and "F" represents a Grassmann-odd (fermionic) field. Each of these correlation functions require separate analysis, however, they have a similar underlying structure.

A. Method of irreducible decomposition

First let us analyse the case $\langle\mathbf{J}_F\mathbf{J}'_F\mathbf{J}''_F\rangle$; we consider three Grassmann-odd currents: $\mathbf{J}_{\alpha(2A+1)}$, $\mathbf{J}'_{\alpha(2B+1)}$, $\mathbf{J}''_{\gamma(2C+1)}$, where A, B, C are positive integers. Therefore, the superfields $\mathbf{J}$, $\mathbf{J}'$, $\mathbf{J}''$ are of superspin $s_1=A+\frac{1}{2}$, $s_2=B+\frac{1}{2}$, $s_3=C+\frac{1}{2}$ respectively. Using the formalism above, all information about the correlation function

$$\langle\mathbf{J}_{\alpha(2A+1)}(z_1)\mathbf{J}'_{\beta(2B+1)}(z_2)\mathbf{J}''_{\gamma(2C+1)}(z_3)\rangle, \tag{3.2}$$

is encoded in a homogeneous tensor field $\mathcal{H}_{\alpha(2A+1)\beta(2B+1)\gamma(2C+1)}(\boldsymbol{X},\Theta)$, which is a function of a single superspace variable $\mathcal{Z}=(\boldsymbol{X},\Theta)$ and satisfies the scaling property

$$\mathcal{H}_{\alpha(2A+1)\beta(2B+1)\gamma(2C+1)}(\lambda^2\boldsymbol{X},\lambda\Theta) = (\lambda^2)^{C-A-B-\frac{3}{2}}\mathcal{H}_{\alpha(2A+1)\beta(2B+1)\gamma(2C+1)}(\boldsymbol{X},\Theta). \tag{3.3}$$

To simplify the problem, for each set of totally symmetric spinor indices (the α's, β's and γ's respectively), we convert pairs of them into vector indices as follows:

$$\begin{aligned}\mathcal{H}_{\alpha(2A+1)\beta(2B+1)\gamma(2C+1)}(\boldsymbol{X},\Theta) &\equiv \mathcal{H}_{\alpha\alpha(2A),\beta\beta(2B),\gamma\gamma(2C)}(\boldsymbol{X},\Theta)\\ &= (\gamma^{m_1})_{\alpha_1\alpha_2}\cdots(\gamma^{m_A})_{\alpha_{2A-1}\alpha_{2A}}\\ &\quad\times(\gamma^{n_1})_{\beta_1\beta_2}\cdots(\gamma^{n_B})_{\beta_{2B-1}\beta_{2B}}\\ &\quad\times(\gamma^{k_1})_{\gamma_1\gamma_2}\cdots(\gamma^{k_C})_{\gamma_{2C-1}\gamma_{2C}}\\ &\quad\times\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A,n_1\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta).\end{aligned} \tag{3.4}$$

The equality above holds only if and only if $\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A n_1\ldots n_B k_1\ldots k_C}(\boldsymbol{X},\Theta)$ is totally symmetric

$$\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A n_1\ldots n_B k_1\ldots k_C}(\boldsymbol{X},\Theta) = \mathcal{H}_{\alpha\beta\gamma,(m_1\ldots m_A)(n_1\ldots n_B)(k_1\ldots k_C)}(\boldsymbol{X},\Theta), \tag{3.5}$$

and traceless in each group of vector indices, i.e. $\forall\, i, j$

$$\eta^{m_i m_j}\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_i m_j\ldots m_A,n_1\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta)=0, \tag{3.6a}$$

$$\eta^{n_i n_j}\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A,n_1\ldots n_i n_j\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta)=0, \tag{3.6b}$$

$$\eta^{k_i k_j}\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A,n_1\ldots n_B,k_1\ldots k_i k_j\ldots k_C}(\boldsymbol{X},\Theta)=0. \tag{3.6c}$$

It is also required that $\mathcal{H}$ is subject to the γ-trace constraints

$$(\gamma^{m_1})_\sigma{}^\alpha\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A,n_1\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta)=0, \tag{3.7a}$$

$$(\gamma^{n_1})_\sigma{}^\beta\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A,n_1\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta)=0, \tag{3.7b}$$

$$(\gamma^{k_1})_\sigma{}^\gamma\mathcal{H}_{\alpha\beta\gamma,m_1\ldots m_A,n_1\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta)=0. \tag{3.7c}$$

Now since $\mathcal{H}$ is Grassmann-odd, it must be linear in Θ, and, using the property (2.8), we decompose $\mathcal{H}$ as follows (raising all indices for convenience):

$$\mathcal{H}^{\alpha\beta\gamma,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=\sum_{i=1}^{4}\mathcal{H}_i^{\alpha\beta\gamma,m(A)n(B)k(C)}(X,\Theta), \tag{3.8a}$$

$$\mathcal{H}_1^{\alpha\beta\gamma,m(A)n(B)k(C)}(X,\Theta)=\varepsilon^{\alpha\beta}\Theta^\gamma A^{m(A)n(B)k(C)}(X), \tag{3.8b}$$

$$\mathcal{H}_2^{\alpha\beta\gamma,m(A)n(B)k(C)}(X,\Theta)=\varepsilon^{\alpha\beta}(\gamma_r)^\gamma{}_\delta\Theta^\delta B^{r,m(A)n(B)k(C)}(X), \tag{3.8c}$$

$$\mathcal{H}_3^{\alpha\beta\gamma,m(A)n(B)k(C)}(X,\Theta)=(\gamma_p)^{\alpha\beta}\Theta^\gamma C^{p,m(A)n(B)k(C)}(X), \tag{3.8d}$$

$$\mathcal{H}_4^{\alpha\beta\gamma,m(A)n(B)k(C)}(X,\Theta)=(\gamma_p)^{\alpha\beta}(\gamma_r)^\gamma{}_\delta\Theta^\delta D^{pr,m(A)n(B)k(C)}(X). \tag{3.8e}$$

Here it is convenient to view the contributions $\mathcal{H}_i$ as functions of the symmetric tensor X (which is equivalent to a three-dimensional vector) rather than of $\boldsymbol{X}$. In fact, since $\mathcal{H}$ is linear in Θ and $\Theta^3=0$ we have $\mathcal{H}(\boldsymbol{X},\Theta)=\mathcal{H}(X,\Theta)$. The tensors A, B, C, D are constrained by conservation equations and any algebraic symmetry properties which $\mathcal{H}$ possesses. In particular, the conservation equations (2.23a), (2.23b), are now equivalent to the following constraints on $\mathcal{H}$ with vector indices

$$\mathcal{D}_\alpha\mathcal{H}^{\alpha\beta\gamma,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0, \tag{3.9a}$$

$$\mathcal{Q}_\beta\mathcal{H}^{\alpha\beta\gamma,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0. \tag{3.9b}$$

We also need consider the constraint for conservation at the third point (2.23c), however, this is technically challenging to impose using this analytic approach and we will not do it here. Instead we will comment on it at the end of Secs. III C and III D. Since $\mathcal{H}$ is linear in Θ, the conservation conditions (3.9a) split up into constraints $O(\Theta^0)$

$$\frac{\partial}{\partial\Theta^\alpha}\mathcal{H}^{\alpha\beta\gamma,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0, \tag{3.10a}$$

$$\frac{\partial}{\partial\Theta^\beta}\mathcal{H}^{\alpha\beta\gamma,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0, \tag{3.10b}$$

and $O(\Theta^2)$

$$(\gamma^m)_{\alpha\delta}\Theta^\delta\frac{\partial}{\partial X^m}\mathcal{H}^{\alpha\beta\gamma,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0, \tag{3.11a}$$

$$(\gamma^m)_{\beta\delta}\Theta^\delta\frac{\partial}{\partial X^m}\mathcal{H}^{\alpha\beta\gamma,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0. \tag{3.11b}$$

Using the irreducible decomposition (3.8), Eq. (3.10a) results in the algebraic relations

$$A^{m(A)n(B)k(C)}+\eta_{pr}D^{pr,m(A)n(B)k(C)}=0, \tag{3.12a}$$

$$B^{q,m(A)n(B)k(C)}+C^{q,m(A)n(B)k(C)}-\epsilon^q{}_{pr}D^{pr,m(A)n(B)k(C)}=0, \tag{3.12b}$$

while (3.10b) gives

$$-A^{m(A)n(B)k(C)} + \eta_{pr}D^{pr,m(A)n(B)k(C)} = 0, \tag{3.13a}$$

$$-B^{q,m(A)n(B)k(C)} + C^{q,m(A)n(B)k(C)} - \epsilon^q{}_{pr}D^{pr,m(A)n(B)k(C)} = 0. \tag{3.13b}$$

Hence, equations (3.12a), (3.13a) together result in $A = B = 0$, while C and D satisfy:

$$\eta_{pr}D^{pr,m(A)n(B)k(C)} = 0, \tag{3.14a}$$

$$C^{q,m(A)n(B)k(C)} - \epsilon^q{}_{pr}D^{pr,m(A)n(B)k(C)} = 0. \tag{3.14b}$$

Next we consider the relations arising from the conservation equations at $O(\Theta^2)$. Using the decomposition (3.8), from a straightforward computation we obtain

$$\partial_t(\epsilon^t{}_{pr}D^{pr,m(A)n(B)k(C)} + C^{t,m(A)n(B)k(C)}) = 0, \tag{3.15a}$$

$$\partial_t(-\epsilon^{tq}{}_p C^{p,m(A)n(B)k(C)} + D^{qt,m(A)n(B)k(C)} + D^{tq,m(A)n(B)k(C)}) = 0. \tag{3.15b}$$

After substituting the algebraic relations (3.14) into (3.15a), we obtain

$$\partial_p C^{p,m(A)n(B)k(C)} = 0, \qquad \partial_p D^{pr,m(A)n(B)k(C)} = 0. \tag{3.16}$$

We now must impose the γ-trace conditions, starting with (3.7a) and (3.7b). Making use of the decomposition (3.8), Eq. (3.7a) results in the algebraic constraints

$$\eta_{mp}C^{p,mm(A-1)n(B)k(C)} = 0, \qquad \epsilon_{qmp}C^{p,mm(A-1)n(B)k(C)} = 0, \tag{3.17a}$$

$$\eta_{mp}D^{pr,mm(A-1)n(B)k(C)} = 0, \qquad \epsilon_{qmp}D^{pr,mm(A-1)n(B)k(C)} = 0, \tag{3.17b}$$

while from (3.7b) we find

$$\eta_{np}C^{p,m(A)nn(B-1)k(C)} = 0, \qquad \epsilon_{qnp}C^{p,m(A)nn(B-1)k(C)} = 0, \tag{3.18a}$$

$$\eta_{np}D^{pr,m(A)nn(B-1)k(C)} = 0, \qquad \epsilon_{qnp}D^{pr,m(A)nn(B-1)k(C)} = 0. \tag{3.18b}$$

Altogether these relations imply that both C and D are symmetric and traceless in the indices $p, m_1, \ldots, m_A, n_1, \ldots, n_B$, i.e.,

$$C^{p,m(A)n(B)k(C)} \equiv C^{(pm(A)n(B))k(C)}, \qquad D^{pr,m(A)n(B)k(C)} \equiv D^{(pm(A)n(B)),rk(C)}. \tag{3.19}$$

Next, from the γ_k-trace constraint (3.7c), we obtain the algebraic relations

$$\eta_{kr}D^{pr,m(A)n(B)kk(C-1)} = 0, \tag{3.20a}$$

$$C^{p,m(A)n(B)kk(C-1)} + \epsilon^k{}_{rs}D^{pr,m(A)n(B)sk(C-1)} = 0. \tag{3.20b}$$

To make use of these relations, we decompose D into symmetric and antisymmetric parts on the indices $r, k_1, \ldots, k_C$ as follows:

$$D^{(pm(A)n(B)),r(k_1\ldots k_C)} = D_S^{(pm(A)n(B)),(rk_1\ldots k_C)} + \sum_{i=1}^{C}\epsilon^{rk_i}{}_t D_A^{(pm(A)n(B)),(tk_1\ldots\hat{k}_i\ldots k_C)}, \tag{3.21}$$

where the notation $\hat{k}_i$ denotes removal of the index k_i from D_A. After substituting this decomposition into (3.20), we obtain:

$$\eta_{kr}D_S^{(pm(A)n(B)),(rkk(C-1))} = 0, \tag{3.22a}$$

$$D_A^{(pm(A)n(B)),(rk(C-1))} = \frac{1}{C+1}C^{(pm(A)n(B)),(rk(C-1))}. \tag{3.22b}$$

We see that the tensor D_A is fully determined in terms of C. To continue we now substitute (3.21) into (3.14) to obtain equations relating C and D_S; we obtain:

$$\eta_{pr} D_S^{(pm(A)n(B)),(rk(C))} + \frac{1}{C+1}\sum_{i=1}^{C} \epsilon^{k_i}{}_{pr} C^{(pm(A)n(B)),(rk_1\ldots\hat{k}_i\ldots k_C)} = 0, \tag{3.23a}$$

$$\begin{aligned}&\epsilon^p{}_{qr} D_S^{(qm(A)n(B)),(rk(C))} - C^{(pm(A)n(B)),(rk(C))}\\ &\quad + \frac{1}{C+1}\sum_{i=1}^{C}\{\eta^{pk_i}\eta_{qr} C^{(qm(A)n(B)),(rk_1\ldots\hat{k}_i\ldots k(C))} - C^{(k_i m(A)n(B)),(pk_1\ldots\hat{k}_i\ldots k(C))}\} = 0.\end{aligned} \tag{3.23b}$$

Further, the conservation equations (3.16) are now equivalent to

$$\partial_p C^{(pm(A)n(B)),k(C)} = 0, \qquad \partial_p D_S^{(pm(A)n(B)),(rk(C))} = 0. \tag{3.24}$$

Hence, finding the solution for the three-point function $\langle \mathbf{J}_F \mathbf{J}'_F \mathbf{J}''_F \rangle$ is now equivalent to finding two transverse tensors C and D_S, which are related by the algebraic constraints (3.23). It may be checked for $A = B = C = 1$ that the constraints reproduce those of the supercurrent three-point function found in [35].

Let us now briefly comment on the analysis of three-point functions involving flavor currents, i.e. when A, B, $C = 0$. In these cases one can simply ignore the relevant tracelessness and γ-trace conditions (3.6), (3.7) respectively and omit the appropriate groups of tensor indices. The analysis of the conservation equations proves to be essentially the same and we will not elaborate further on these cases.

B. Conservation equations

Let us now summarize the constraint analysis in the previous subsection in a way that makes the symmetries more apparent. For the $\langle \mathbf{J}_F \mathbf{J}'_F \mathbf{J}''_F \rangle$ correlators we have two tensors; C of rank $N + C + 1$, and D of rank $N + C + 2$, where $N = A + B$, which possess the following symmetries:

$$C^{(r_1\ldots r_{N+1})(k_1\ldots k_C)}, \qquad D_S^{(r_1\ldots r_{N+1})(k_1\ldots k_{C+1})}. \tag{3.25}$$

The tensors C and D_S are totally symmetric and traceless in the groups of indices r and k respectively. They also satisfy the conservation conditions

$$\partial_{r_1} C^{(r_1\ldots r_{N+1})(k_1\ldots k_C)} = 0, \qquad \partial_{r_1} D_S^{(r_1\ldots r_{N+1})(k_1\ldots k_{C+1})} = 0, \tag{3.26}$$

and the algebraic relations

$$\eta_{r_1k_1} D_S^{(r_1\ldots r_{N+1})(k_1\ldots k_{C+1})} + \frac{1}{C+1}\sum_{i=2}^{C+1} \epsilon^{k_i}{}_{r_1k_1} C^{(r_1\ldots r_{N+1})(k_1k_2\ldots\hat{k}_i\ldots k_{C+1})} = 0, \tag{3.27a}$$

$$\begin{aligned}&\epsilon^p{}_{r_1k_1} D_S^{(r_1\ldots r_{N+1})(k_1\ldots k_{C+1})} - C^{(pr_2\ldots r_{N+1})(k_2\ldots k_{C+1})}\\ &\quad + \frac{1}{C+1}\sum_{i=2}^{C+1}\{\eta^{pk_i}\eta_{r_1k_1} C^{(r_1\ldots r_{N+1})(k_1k_2\ldots\hat{k}_i\ldots k_{C+1})} - C^{(k_ir_2\ldots r_{N+1})(pk_2\ldots\hat{k}_i\ldots k_{C+1})}\} = 0.\end{aligned} \tag{3.27b}$$

The "full" tensor D, present in the decomposition (3.8), is constructed from C and D_S as follows:

$$D^{q,(r_1\ldots r_{N+1})(k_1\ldots k_C)} = D_S^{(r_1\ldots r_{N+1})(qk_1\ldots k_C)} + \frac{1}{C+1}\sum_{i=1}^{C} \epsilon^{qk_i}{}_t C^{(r_1\ldots r_{N+1})(tk_1\ldots\hat{k}_i\ldots k_C)}. \tag{3.28}$$

As we will see later, the algebraic relations (3.27) are sufficient to determine D completely in terms of C. However, before we prove this it is prudent to analyze the conservation equation (3.26) for C.

Since we have identified the algebraic symmetries of C, it is convenient to convert C back into spinor notation and contract it with commuting auxiliary spinors as follows:

$$C(X; u(2N+2), v(2C)) = C_{\alpha(2N+2)\beta(2C)}(X) u^{\alpha_1}\ldots u^{\alpha_{2N+2}} v^{\beta_1}\ldots v^{\beta_{2C}}. \tag{3.29}$$

Since the auxiliary spinors are commuting they satisfy

$$\varepsilon_{\alpha\beta}u^{\alpha}u^{\beta} = 0, \qquad \varepsilon_{\alpha\beta}v^{\alpha}v^{\beta} = 0. \tag{3.30}$$

We now introduce a basis of monomials out of which C can be constructed. Adapting the results of [17,45], we use

$$P_3 = \varepsilon_{\alpha\beta}u^{\alpha}v^{\beta}, \qquad Q_3 = \hat{X}_{\alpha\beta}u^{\alpha}v^{\beta}, \qquad Z_1 = \hat{X}_{\alpha\beta}u^{\alpha}u^{\beta}, \qquad Z_2 = \hat{X}_{\alpha\beta}v^{\alpha}v^{\beta}. \tag{3.31}$$

A general ansatz for $C(X;u,v)$ which is homogeneous degree $2(N+1)$ in u, $2C$ in v and $C-N-2$ in X is of the following form:

$$C(X;u,v) = X^{C-N-2}\sum_{a,b}\alpha(a,b)P_3^a Q_3^{2(C-b)-a}Z_1^{b+N-C+1}Z_2^b. \tag{3.32}$$

However, there is linear dependence between the monomials (3.31) of the form

$$Z_1 Z_2 = Q_3^2 - P_3^2, \tag{3.33}$$

which allows for elimination of Z_2. Hence, the ansatz becomes:

$$C(X;u,v) = X^{C-N-2}\sum_{k=0}^{2C}\alpha_k P_3^k Q_3^{2C-k}Z_1^{N-C+1}. \tag{3.34}$$

This expansion is valid for $C \le N+1$, which is always fulfilled for field configurations where the third superspin triangle inequality, $s_3 \le s_1 + s_2$, is satisfied. Note that if $s_3 > s_1 + s_2$ it then follows that $s_1 \le s_2 + s_3$ and $s_2 \le s_1 + s_3$. That is, if one of the triangle inequalities is violated the remaining two are necessarily satisfied. It implies that one can always arrange the fields in the three-point function to obtain a configuration for which $s_3 \le s_1 + s_2$, or equivalently, $C \le N < N+1$. We will assume that we have performed such an arrangement and use Eq. (3.34).

Requiring that the three-point function is conserved at z_1 and z_2 is now tantamount to imposing

$$\frac{\partial}{\partial u^{\alpha}}\frac{\partial}{\partial u^{\beta}}\frac{\partial}{\partial X_{\alpha\beta}}C(X;u,v) = 0. \tag{3.35}$$

By acting with this operator on the ansatz (3.34), we obtain

$$X^{C-N-3}\sum_{k=0}^{2C}\alpha_k P_3^{k-2}Q_3^{2C-2-k}Z_1^{N-C}\{\sigma_1(k)P_3^2Q_3^2 + \sigma_2(k)Q_3^4 + \sigma_3(k)P_3^4\} = 0, \tag{3.36}$$

where

$$\sigma_1(k) = -2k^3 + 4Ck^2 + 2k(1 + C + 2N(2+N)) - 2C(3 + 4N(2+N)), \tag{3.37a}$$

$$\sigma_2(k) = -k(k-1)(2N-k+3), \tag{3.37b}$$

$$\sigma_3(k) = (2C-k-1)(2C-k)(2N+k+3). \tag{3.37c}$$

The sum above may now be split up into three contributions so that the coefficients may be easily read off

$$\sum_{k=0}^{2C}\alpha_k\sigma_1(k)P_3^kQ_3^{2C-k} + \sum_{k=0}^{2C-2}\alpha_{k+2}\sigma_2(k+2)P_3^kQ_3^{2C-k} + \sum_{k=2}^{2C}\alpha_{k-2}\sigma_3(k-2)P_3^kQ_3^{2C-k} = 0. \tag{3.38}$$

Hence, we obtain the following linear system:

$$\alpha_k\sigma_1(k) + \alpha_{k+2}\sigma_2(k+2) + \alpha_{k-2}\sigma_3(k-2) = 0, \qquad 2 \le k \le 2C-2, \tag{3.39a}$$

$$\alpha_0\sigma_1(0) + \alpha_2\sigma_2(2) = 0, \qquad \alpha_{2C}\sigma_1(2C) + \alpha_{2C-2}\sigma_3(2C-2) = 0, \tag{3.39b}$$

$$\alpha_1\sigma_1(1) + \alpha_3\sigma_2(3) = 0, \qquad \alpha_{2C-1}\sigma_1(2C-1) + \alpha_{2C-3}\sigma_3(2C-3) = 0. \tag{3.39c}$$

It must be noted that the equations above for the variables α_k split into independent $C+1$ and C dimensional systems of linear homogeneous equations corresponding to the parity-even and parity-odd sectors respectively. The terms for which k is even are denoted parity-even, while the terms for which k is odd are denoted parity-odd, so that

$$C(X;u,v) = C_E(X;u,v) + C_O(X;u,v), \tag{3.40}$$

where

$$C_E(X;u,v) = X^{C-N-2} \sum_{k=0}^{C} \alpha_{2k} P_3^{2k} Q_3^{2(C-k)} Z_1^{N-C+1}, \tag{3.41a}$$

$$C_O(X;u,v) = X^{C-N-2} \sum_{k=1}^{C} \alpha_{2k-1} P_3^{2k-1} Q_3^{2(C-k)+1} Z_1^{N-C+1}. \tag{3.41b}$$

Indeed, this convention is consistent with that of [17,45]. Hence, in the linear homogeneous system (3.39), we define the parity-even coefficients, a_k, b_k, c_k, as

$$a_k = \sigma_1(2k-2), \qquad 1 \le k \le C+1, \tag{3.42a}$$

$$b_k = \sigma_2(2k), \qquad 1 \le k \le C, \tag{3.42b}$$

$$c_k = \sigma_3(2k-2), \qquad 1 \le k \le C, \tag{3.42c}$$

and the parity-odd coefficients, $\tilde{a}_k$, $\tilde{b}_k$, $\tilde{c}_k$ as

$$\tilde{a}_k = \sigma_1(2k-1), \qquad 1 \le k \le C, \tag{3.43a}$$

$$\tilde{b}_k = \sigma_2(2k+1), \qquad 1 \le k \le C-1, \tag{3.43b}$$

$$\tilde{c}_k = \sigma_3(2k-1), \qquad 1 \le k \le C-1. \tag{3.43c}$$

With the above definitions, the linear homogeneous equations (3.39) split into two independent systems which can be written in the form $\mathbf{M}\vec{\alpha} = \mathbf{0}$. More explicitly

$$\textbf{Even:} \qquad \mathbf{M}_E = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & c_2 & a_3 & b_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & c_{C-1} & a_C & b_C \\ 0 & 0 & \dots & 0 & c_C & a_{C+1} \end{bmatrix}, \qquad \vec{\alpha}_E = \begin{bmatrix} \alpha_0 \\ \alpha_2 \\ \alpha_4 \\ \vdots \\ \alpha_{2C-2} \\ \alpha_{2C} \end{bmatrix}, \tag{3.44a}$$

$$\textbf{Odd:} \qquad \mathbf{M}_O = \begin{bmatrix} \tilde{a}_1 & \tilde{b}_1 & 0 & 0 & \dots & 0 \\ \tilde{c}_1 & \tilde{a}_2 & \tilde{b}_2 & 0 & \dots & 0 \\ 0 & \tilde{c}_2 & \tilde{a}_3 & \tilde{b}_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \tilde{c}_{C-2} & \tilde{a}_{C-1} & \tilde{b}_{C-1} \\ 0 & 0 & \dots & 0 & \tilde{c}_{C-1} & \tilde{a}_C \end{bmatrix}, \qquad \vec{\alpha}_O = \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_5 \\ \vdots \\ \alpha_{2C-3} \\ \alpha_{2C-1} \end{bmatrix}, \tag{3.44b}$$

where $\mathbf{M}_E$, $\mathbf{M}_O$ are square matrices of dimension $C+1$, C respectively. The system of Eq. (3.44) is associated with the solution $C_E(X;u,v)$, while the system (3.44b) is associated with the solution $C_O(X;u,v)$. The question now is whether there exists explicit solutions to these linear homogeneous systems for arbitrary A, B, C.

The matrices of the form (3.44) are referred to as tridiagonal matrices, before we continue with the analysis let us comment on some of their features. The sufficient conditions under which a tridiagonal matrix is invertible for arbitrary sequences a_k, b_k, c_k has been discussed in e.g. [47,48]. Now consider the determinant of a tridiagonal matrix

$$\Delta_k = \begin{vmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 \\ c_1 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & c_2 & a_3 & b_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & c_{k-2} & a_{k-1} & b_{k-1} \\ 0 & 0 & \dots & 0 & c_{k-1} & a_k \end{vmatrix}. \tag{3.45}$$

One of its most important properties is that it satisfies the recurrence relation

$$\Delta_k = a_k \Delta_{k-1} - b_{k-1} c_{k-1} \Delta_{k-2}, \quad \Delta_0 = 1, \quad \Delta_{-1} = 0, \quad (3.46)$$

which may be seen by performing a Laplace expansion on the last row. The sequence Δ_k is often called the "generalized continuant" with respect to the sequences a_k, b_k, c_k. There are methods to compute this determinant in closed form for simple cases, for example if the tridiagonal matrix under consideration is "Toeplitz" (i.e. if the sequences are constant $a_k = a$, $b_k = b$, $c_k = c$). For Toeplitz tridiagonal matrices one obtains

$$\Delta_k = \frac{1}{\sqrt{a^2 - 4bc}} \times \left\{ \left(\frac{a + \sqrt{a^2 - 4bc}}{2} \right)^{k+1} - \left(\frac{a - \sqrt{a^2 - 4bc}}{2} \right)^{k+1} \right\}, \quad (3.47)$$

for $a^2 - 4bc \neq 0$, while for $a^2 - 4bc = 0$ we obtain $\Delta_k = (k+1)(\frac{a}{2})^k$. For general sequences a_k, b_k, c_k there is no straightforward approach to compute Δ_k and it must computed recursively using (3.46). Another important feature of tridiagonal matrices is that in general their nullity (corank) is either 0 or 1, which implies that any system of linear homogeneous equations with a tridiagonal matrix has at most one nontrivial solution.

In the next subsections, we study the continuants of $\mathbf{M}_E$, $\mathbf{M}_O$ for the homogeneous systems (3.44a), (3.44b), and obtain their explicit form for arbitrary A, B, C. This determines whether $\mathbf{M}_E$, $\mathbf{M}_O$ are invertible, and the number of solutions for the homogeneous systems (3.44a), (3.44b).

C. Parity-odd case

First we will analyse the system of equations for the parity-odd sector. Note that if $\det[\mathbf{M}_O] \neq 0$ then the system of equations $\mathbf{M}_O \vec{\alpha}_O = \mathbf{0}$ admits only the trivial solution. Let us denote $\mathbf{M}_O \coloneqq \mathbf{M}_O^{(C)}$ to indicate that the dimension of the tridiagonal matrix in (3.44b) is $C \times C$. We also introduce the $k \times k$ continuant $\tilde{\Delta}_k^{(C)}$, where $1 \leq k \leq C$. It satisfies the continuant equation (3.46) with

$$\tilde{a}_k = 4C(-1 + k(4k-3) - 2N(N+2)) - 4(2k-1)(2k(k-1) - N(N+2)), \quad 1 \leq k \leq C, \quad (3.48a)$$

$$\tilde{b}_k = -4k(1+2k)(N-k+1), \quad 1 \leq k \leq C-1, \quad (3.48b)$$

$$\tilde{c}_k = 4(1+2C-2k)(C-k)(1+k+N), \quad 1 \leq k \leq C-1. \quad (3.48c)$$

We also have $\tilde{\Delta}_C^{(C)} = \det[\mathbf{M}_O^{(C)}]$. Below we present some examples of the matrix $\mathbf{M}_O$ and the determinant $\tilde{\Delta}_C^{(C)}$ for arbitrary $N = A + B$ and fixed C:

$$\mathbf{M}_O^{(1)} = [-4N(N+2)], \quad (3.49a)$$

$$\mathbf{M}_O^{(2)} = \begin{bmatrix} -12N(N+2) & -12N \\ 12(N+2) & 24 - 4N(N+2) \end{bmatrix}, \quad (3.49b)$$

$$\mathbf{M}_O^{(3)} = \begin{bmatrix} -20N(N+2) & -12N & 0 \\ 40(N+2) & 60 - 12N(N+2) & -40(N-1) \\ 0 & 12(N+3) & 72 - 4N(N+2) \end{bmatrix}, \quad (3.49c)$$

$$\mathbf{M}_O^{(4)} = \begin{bmatrix} -28N(N+2) & -12N & 0 & 0 \\ 84(N+2) & 96 - 20N(N+2) & -40(N-1) & 0 \\ 0 & 40(N+3) & 176 - 12N(N+2) & -84(N-2) \\ 0 & 0 & 12(N+4) & 144 - 4N(N+2) \end{bmatrix}. \quad (3.49d)$$

The corresponding determinants are

$$\tilde{\Delta}_1^{(1)} = -4N(N+2), \quad (3.50a)$$

$$\tilde{\Delta}_2^{(2)} = 48(N-1)N(N+2)(N+3), \quad (3.50b)$$

$$\tilde{\Delta}_3^{(3)} = -960(N-2)(N-1)N(N+2)(N+3)(N+4), \quad (3.50c)$$

$$\tilde{\Delta}_4^{(4)} = 26880(N-3)(N-2)(N-1)N(N+2)(N+3)(N+4)(N+5). \tag{3.50d}$$

Indeed the matrices above are invertible as $\det[\mathbf{M}_O] \neq 0$, therefore we have the solution $\vec{\alpha}_O = \mathbf{0}$. The determinant can be efficiently computed using the recursion formula (3.46), and the pattern appears holds for large integers N, C. Using *Mathematica* we explicitly computed $\tilde{\Delta}_C^{(C)}$ for arbitrary N, up to $C = 500$. In all cases we found it is nontrivial.

However, the continuant $\tilde{\Delta}_k^{(C)}$ can also be obtained explicitly for all $1 \le k \le C$, and arbitrary C. First, one can show that $\tilde{\Delta}_k^{(C)}$ satisfies the following recurrence relation:

$$\tilde{\Delta}_k^{(C)} = \tilde{\gamma}_{k-1}^{(C)} \tilde{\Delta}_{k-1}^{(C)}, \qquad \tilde{\Delta}_1^{(C)} = -4(2C-1)N(2+N), \qquad 1 \le k \le C, \tag{3.51}$$

where $\tilde{\gamma}_k^{(C)}$ is given by

$$\tilde{\gamma}_k^{(C)} = -4(N-k)(2+k+N)(-1+2C-2k). \tag{3.52}$$

To see this, consider the combination $\tilde{\Delta}_{k+2}^{(C)} - \tilde{a}_{k+2}\tilde{\Delta}_{k+1}^{(C)} + \tilde{b}_{k+1}\tilde{c}_{k+1}\tilde{\Delta}_k^{(C)}$. Using Eqs. (3.64), (3.52), this combination becomes $(\tilde{\gamma}_{k+1}\tilde{\gamma}_k - \tilde{a}_{k+2}\tilde{\gamma}_k + \tilde{b}_{k+1}\tilde{c}_{k+1})\tilde{\Delta}_k^{(C)}$. However it may be shown using Eqs. (3.48) and (3.52) that

$$\tilde{\gamma}_{k+1}\tilde{\gamma}_k - \tilde{a}_{k+2}\tilde{\gamma}_k + \tilde{b}_{k+1}\tilde{c}_{k+1} = 0, \tag{3.53}$$

for arbitrary k, N, C, which implies that the recurrence relation (3.46) is indeed satisfied. We find the following general solution for (3.51)[3]:

$$\tilde{\Delta}_k^{(C)} = 2^{3k} \frac{\Gamma(\frac{1}{2} - C + k)}{\Gamma(\frac{1}{2} - C)} \frac{(N+k+1)!}{(N+1)(N-k)!}, \qquad 1 \le k \le C. \tag{3.54}$$

One can check (using e.g. *Mathematica*) that it solves the continuant equation (3.46). Recalling that $\det[\mathbf{M}_O^{(C)}] = \tilde{\Delta}_C^{(C)}$, it then follows that

$$\det[\mathbf{M}_O^{(C)}] = \frac{2^{3C}\sqrt{\pi}}{\Gamma(\frac{1}{2} - C)} \frac{(N+C+1)!}{(N+1)(N-C)!}, \qquad C \le N, \tag{3.55}$$

which is always nontrivial. This implies that $\mathbf{M}_O$ is of full rank and, hence, the system of equations (3.44b) admits only the trivial solution $\vec{\alpha}_O = \mathbf{0}$. Therefore, recalling that C_O is the parity-odd solution corresponding to the system of equations (3.44b), from the analysis above we have shown that $C_O(X; u, v) = 0$ for arbitrary A, B, C.

Now we will show that the tensor D_S associated with C_O, which are related by the algebraic relations (3.27), also vanishes. Since we have shown that $C_O = 0$ in general, Eq. (3.27) implies

$$\eta_{r_1 k_1} D_S^{(r_1 \ldots r_{N+1})(k_1 \ldots k_{C+1})} = 0, \tag{3.56a}$$

$$\epsilon^p{}_{r_1 k_1} D_S^{(r_1 \ldots r_{N+1})(k_1 \ldots k_{C+1})} = 0. \tag{3.56b}$$

Hence, D_S is totally symmetric and traceless in all tensor indices

$$D_S^{(r_1 \ldots r_{N+1})(k_1 \ldots k_{C+1})} \equiv D_S^{(r_1 \ldots r_{N+1} k_1 \ldots k_{C+1})}. \tag{3.57}$$

We can now construct a solution for D_S using auxiliary spinors as follows:

$$D_S(X; u(2N+2C+4)) = D_{S\alpha(2N+2C+4)}(X) u^{\alpha_1} \ldots u^{\alpha_{2N+2C+4}}. \tag{3.58}$$

Since D_S is transverse, the polynomial $D_S(X; u)$ must satisfy the conservation equation

$$\frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial u^\beta} \frac{\partial}{\partial X_{\alpha\beta}} D_S(X; u) = 0. \tag{3.59}$$

The only possible structure (up to a constant coefficient) for $D_S(X; u)$ is of the form

$$D_S(X; u(2N+2C+4)) = X^{C-N-2} Z_1^{N+C+2}. \tag{3.60}$$

[3]Recall that we have arranged the operators in the three-point function so that $s_3 \le s_1 + s_2$ which implies $C \le N$.

Explicit computation of (3.59) gives

$$\frac{\partial}{\partial u^{\alpha}}\frac{\partial}{\partial u^{\beta}}\frac{\partial}{\partial X_{\alpha\beta}}D_S(X;u) = -4(N+C+2)^2(1+2C)X^{C-N-3}Z_1^{N+C+1}, \quad (3.61)$$

which is always nonzero. Therefore $D_S = 0$ for the parity-odd sector. Hence, for Grassmann-odd three-point functions of the form $\langle \mathbf{J}_F \mathbf{J}'_F \mathbf{J}''_F \rangle$, there is no parity-odd solution for arbitrary superspins.

Let us point out that we did not need to use the constraint arising from conservation on the third point. Imposing the conservation equations at the first two points was sufficient to prove vanishing of the parity-odd contribution. Let us, however, indicate that our consideration is sensitive to the fact that the operator inserted at the third point is a conserved supercurrent. Indeed, we used the fact that its dimension is $s_3 + 1$, saturating the unitarity bound, which implies that this operator is a conserved supercurrent.

D. Parity-even case

For the parity even case, we follow the same approach. Let us denote $\mathbf{M}_E \coloneqq \mathbf{M}_E^{(C)}$ (recall that the dimension of the matrix $\mathbf{M}_E^{(C)}$ in (3.44) is now $(C+1)\times(C+1)$) and consider the continuant $\Delta_k^{(C)}$, $1 \le k \le C+1$, with $\Delta_{C+1}^{(C)} = \det[\mathbf{M}_E^{(C)}]$. The continuant satisfies Eq. (3.46), where

$$\begin{aligned} a_k &= 2C(3 + 2k(4k-7) - 4N(N+2)) \\ &\quad - 4(k-1)(3 + 4k(k-2) - 2N(N+2)), \qquad 1 \le k \le C+1, \end{aligned} \quad (3.62a)$$

$$b_k = -2k(2k-1)(3+2N-2k), \qquad 1 \le k \le C, \quad (3.62b)$$

$$c_k = 2(1+2C-2k)(1+C-k)(1+2k+2N), \qquad 1 \le k \le C. \quad (3.62c)$$

Below are some examples of the matrix $\mathbf{M}_E$ and the determinant $\Delta_{C+1}^{(C)}$ for fixed C:

$$\mathbf{M}_E^{(N,1)} = \begin{bmatrix} -2(2N+1)(2N+3) & -2(2N+1) \\ 2(2N+3) & 2 \end{bmatrix}, \quad (3.63a)$$

$$\mathbf{M}_E^{(N,2)} = \begin{bmatrix} -4(2N+1)(2N+3) & -2(2N+1) & 0 \\ 12(2N+3) & -8(N^2+2N-2) & -12(2N-1) \\ 0 & 2(2N+5) & 12 \end{bmatrix}, \quad (3.63b)$$

$$\mathbf{M}_E^{(N,3)} = \begin{bmatrix} -6(2N+1)(2N+3) & -2(2N+1) & 0 & 0 \\ 30(2N+3) & -2(8N^2+16N-15) & -12(2N-1) & 0 \\ 0 & 12(2N+5) & -2(4N^2+8N-39) & -30(2N-3) \\ 0 & 0 & 2(2N+7) & 30 \end{bmatrix}, \quad (3.63c)$$

$$\mathbf{M}_E^{(N,4)} = \begin{bmatrix} -8(2N+1)(2N+3) & -2(2N+1) & 0 & 0 & 0 \\ 56(2N+3) & -4(6N^2+12N-11) & -12(2N-1) & 0 & 0 \\ 0 & 30(2N+5) & -16(N^2+2N-9) & -30(2N-3) & 0 \\ 0 & 0 & 12(2N+7) & -4(2N^2+4N-45) & -56(2N-5) \\ 0 & 0 & 0 & 2(2N+9) & 56 \end{bmatrix}. \quad (3.63d)$$

Contrary to the parity-odd case, all of the matrices above are singular, i.e. $\Delta_2^{(1)} = \Delta_3^{(2)} = \Delta_4^{(3)} = \Delta_5^{(4)} = 0$. For large integers we use the recursion formula (3.46) to analyse the general structure of $\Delta_{C+1}^{(C)}$. Analogous to the parity-odd case, we computed it for arbitrary N up to $C = 500$ and found in all cases $\Delta_{C+1}^{(C)} = 0$. It then follows that the matrix $\mathbf{M}_E^{(C)}$ is of corank one and the solution, $\vec{\alpha}_E$, to $\mathbf{M}_E^{(N,C)}\vec{\alpha}_E = \mathbf{0}$, is fixed up to a single overall constant in all cases, and therefore C_E is unique.

For the parity-even case it also turns out to be possible to solve for the continuants $\Delta_k^{(C)}$ for $1 \leq k \leq C+1$. One can show that $\Delta_k^{(C)}$ satisfies the following recurrence relation:

$$\Delta_k^{(C)} = \gamma_{k-1}^{(C)} \Delta_{k-1}^{(C)}, \qquad \Delta_1^{(C)} = -2C(1+2N)(3+2N), \quad 1 \leq k \leq C+1, \tag{3.64}$$

where $\gamma_k^{(C)}$ is given by

$$\gamma_k^{(C)} = -2(C-k)(1+2N-2k)(3+2N+2k). \tag{3.65}$$

To show this, consider again the combination $\Delta_{k+2}^{(C)} - a_{k+2}\Delta_{k+1}^{(C)} + b_{k+1}c_{k+1}\Delta_k^{(C)}$. Using Eqs. (3.64), (3.65), we obtain $(\gamma_{k+1}\gamma_k - a_{k+2}\gamma_k + b_{k+1}c_{k+1})\Delta_k^{(C)}$. It is then simple to show using Eqs. (3.62) and (3.65) that this combination vanishes for arbitrary k, N, C, which implies that the recurrence relation (3.46) is indeed satisfied. Analogous to the parity-odd case, it is possible to find an explicit solution for the recurrence relation (3.64), and we find the following general formula for the continuant:

$$\Delta_k^{(C)} = (-1)^{N+1} \frac{2^{3k} C!}{\pi (C-k)!} \Gamma\left(\frac{1}{2} - N + k\right) \Gamma\left(\frac{3}{2} + N + k\right), \quad 1 \leq k \leq C. \tag{3.66}$$

For $k = C+1$, we have $\Delta_{C+1}^{(C)} = \gamma_C^{(C)} \Delta_C^{(C)}$. However, we note that $\gamma_C^{(C)} = 0$, which implies $\Delta_{C+1}^{(C)} = \det[\mathbf{M}_E^{(C)}] = 0$, for arbitrary N, C. Hence, we have shown that the matrix $\mathbf{M}_E$ is always of corank one and the system (3.44a) has a unique nontrivial solution.[4] This, in turn, implies that the tensor $C_E(X; u, v)$ is unique up to an overall coefficient. The explicit solution to the system $\mathbf{M}\vec{\alpha} = \mathbf{0}$ and, thus, for the tensor C_E can also be found for arbitrary N, C. Indeed, by analysing the nullspace of $\mathbf{M}_E$ we obtain the following solution for the coefficients α_{2k} of (3.41a):

$$\alpha_{2k} = \frac{(-1)^k 2^{2k} \Gamma(C+1) \Gamma(k + N + \frac{3}{2})}{\Gamma(2k+1)\Gamma(C-k+1)\Gamma(N+\frac{3}{2})} \alpha_0, \qquad 1 \leq k \leq C. \tag{3.67}$$

This is also a solution to the parity-even sector of the recurrence relations (3.39), which can be explicitly checked. Hence, we have obtained a unique solution (up to an overall coefficient) for C_E in explicit form for arbitrary superspins.

Now recall that the three-point functions under consideration are determined not just by the tensor $C^{(r_1 \ldots r_{N+1})(k_1 \ldots k_C)}$ but also by the tensor $D_S^{(r_1 \ldots r_{N+1})(k_1 \ldots k_{C+1})}$ which is transverse [see Eq. (3.24)] and related to $C^{(r_1 \ldots r_{N+1})(k_1 \ldots k_C)}$ by the algebraic relation (3.23). Remarkably, it is possible to solve for D_S in terms of C using (3.23). To show it let us begin by constructing irreducible decompositions for C and D. Since we know that C is parity-even, it cannot contain ϵ, and we use the decompositions

$$\begin{aligned} C^{(r_1 \ldots r_{N+1})(k_1 \ldots k_C)} &= C_1^{(r_1 \ldots r_{N+1} k_1 \ldots k_C)} + \sum_{i=1}^{N+1} \sum_{j=1}^{C} \eta^{r_i k_j} C_2^{(r_1 \ldots \hat{r}_i \ldots r_{N+1} k_1 \ldots \hat{k}_j \ldots k_C)} \\ &\quad + \sum_{j>i=1}^{N+1} \eta^{r_i r_j} C_3^{(r_1 \ldots \hat{r}_i \hat{r}_j \ldots r_{N+1} k_1 \ldots k_C)} + \sum_{j>i=1}^{C} \eta^{k_i k_j} C_4^{(r_1 \ldots r_{N+1} k_1 \ldots \hat{k}_i \hat{k}_j \ldots k_C)}, \end{aligned} \tag{3.68}$$

where C_2, C_3, C_4 are the irreducible components of rank $N+C-1$ (C_4 exists only for $C>1$). Requiring that the above ansatz is traceless in the appropriate groups of indices fixes C_3 and C_4 in terms of C_2 as follows (indices suppressed):

$$C_3 = -\frac{2C}{2N+1} C_2, \qquad C_4 = -\frac{2(N+1)}{2C-1} C_2. \tag{3.69}$$

Hence, C is determined completely in terms of the totally symmetric and traceless tensors C_1 and C_2. Now let us construct an irreducible decomposition of D_S. Due to the algebraic relation (3.27), we know that D_S must be linear in ϵ. The only way to construct D_S such that it contains ϵ is by using the following decomposition:

$$D_S^{(r_1 \ldots r_{N+1})(k_1 \ldots k_{C+1})} = \sum_{i=1}^{N+1} \sum_{j=1}^{C+1} \epsilon^{q r_i k_j} T^{q, (r_1 \ldots \hat{r}_i \ldots r_{N+1} k_1 \ldots \hat{k}_j \ldots k_{C+1})}, \tag{3.70}$$

[4] Alternatively, note that (3.66) implies that the largest nontrivial minor of $\mathbf{M}_E$ is of dimension $C \times C$. Therefore, $\mathrm{Rank}(\mathbf{M}_E) = C$, which implies $\mathrm{Nullity}(\mathbf{M}_E) = 1$.

where the tensor T (of rank $N+C+1$) is decomposed as follows:

$$T^{q,(r_1\ldots r_N)(k_1\ldots k_C)} = T_1^{(r_1\ldots r_N k_1\ldots k_C q)} + \sum_{i=1}^{N}\sum_{j=1}^{C}\eta^{r_i k_j} T_2^{(r_1\ldots\hat{r}_i\ldots r_N k_1\ldots\hat{k}_j\ldots k_C q)} + \sum_{j>i=1}^{N}\eta^{r_i r_j} T_3^{(r_1\ldots\hat{r}_i\hat{r}_j\ldots r_N k_1\ldots k_C q)} + \sum_{j>i=1}^{C}\eta^{k_i k_j} T_4^{(r_1\ldots r_N k_1\ldots\hat{k}_i\hat{k}_j\ldots k_C q)}. \tag{3.71}$$

Here T_1 is the irreducible component of rank $N+C+1$ and T_2, T_3, T_4 are the irreducible components of rank $N+C-1$ (where T_4 exists only for $C>1$). It should be noted that one could also consider contributions to T proportionate to η^{qr_i}, η^{qk_j}, but such contributions will cancel when substituted into (3.70) and, hence, they do not contribute to the irreducible decomposition of D_S. Requiring that D is traceless on the appropriate groups of indices fixes T_3 and T_4 in terms of T_2 as follows (indices suppressed):

$$T_3 = -\frac{2C}{2N+1}T_2, \qquad T_4 = -\frac{2N}{2C-1}T_2. \tag{3.72}$$

Hence, D_S is described completely in terms of the totally symmetric and traceless tensors T_1 and T_2. If we now consider the algebraic relations (3.27) and substitute in the above decompositions, after some tedious calculation one obtains:

$$\begin{aligned} 0 &= \eta_{r_1 k_1} D_S^{(r_1\ldots r_{N+1})(k_1\ldots k_{C+1})} + \frac{1}{C+1}\sum_{i=2}^{C+1}\epsilon^{k_i}{}_{r_1 k_1} C^{(r_1\ldots r_{N+1})(k_1 k_2\ldots\hat{k}_i\ldots k_{C+1})} \\ &= \sum_{i=2}^{N+1}\sum_{j=2}^{C+1}\epsilon^{q r_i k_j}\left\{\tau(N,C) T_2^{(\hat{r}_1\ldots\hat{r}_i\ldots r_{N+1}\hat{k}_1\ldots\hat{k}_j\ldots k_{C+1}q)} + \frac{1}{C+1}\left(1+\frac{2C}{2N+1}\right) C_2^{(\hat{r}_1\ldots\hat{r}_i\ldots r_{N+1}\hat{k}_1\ldots\hat{k}_j\ldots k_{C+1}q)}\right\}, \end{aligned} \tag{3.73}$$

where the constant $\tau(N,C)$ is defined as follows:

$$\tau(N,C) = \frac{6C^2+2N^2+3C(1+4N)-5N-3}{(2C-1)(1+2N)}. \tag{3.74}$$

Requiring that the above combination vanishes gives $T_2 = \xi_2 C_2$, where

$$\xi_2 = -\frac{(2C-1)(1+2C+2N)}{(1+C)(6C^2+2N^2+3C(1+4N)-5N-3)}. \tag{3.75}$$

One can proceed in a similar way with the second algebraic relation in (3.27). After some calculation we found that it relates T_1 and C_1 as $T_1 = \xi_1 C_1$, where

$$\xi_1 = -\frac{2C+1}{(C+1)(N+C+2)}. \tag{3.76}$$

A similar calculation was performed in [35], where it was shown that for $A=B=C=1$, $\xi_1 = -\frac{3}{10}$, $\xi_2 = -\frac{1}{8}$, which is in full agreement with the expressions above.

Finally, we need to show that D_S found this way is transverse, i.e.

$$\partial_p D_S^{(pr_1\ldots r_N)(k_1\ldots k_{C+1})} = 0. \tag{3.77}$$

For this let us define the tensor

$$E^{(r_1\ldots r_N)(k_1\ldots k_{C+1})} = \partial_p D_S^{(pr_1\ldots r_N)(k_1\ldots k_{C+1})}. \tag{3.78}$$

Our aim is to show that $E^{(r_1\ldots r_N)(k_1\ldots k_{C+1})} = 0$. To find $E^{(r_1\ldots r_N)(k_1\ldots k_{C+1})}$ we contract D_S in Eqs. (3.70)–(3.72) with the derivative. However, from the algebraic relations (3.23) and the fact that C is transverse it follows that $E^{(r_1\ldots r_N)(k_1\ldots k_{C+1})}$ is totally symmetric and traceless:

$$E^{(r_1\ldots r_N)(k_1\ldots k_{C+1})} \equiv E^{(r_1\ldots r_N k_1\ldots k_{C+1})}. \tag{3.79}$$

Using Eqs. (3.70)–(3.72) we find that the totally symmetric and traceless contribution is given by

$$\begin{aligned} E^{(r_2\ldots r_{N+1}k_1\ldots k_{C+1})} = &-\sum_{i=2}^{N+1}\epsilon_{pq}{}^{r_i}\partial^p T_1^{(qr_2\ldots\hat{r}_i\ldots r_{N+1}k_1\ldots k_{C+1})} \\ &-\sum_{j=1}^{C+1}\epsilon_{pq}{}^{k_j}\partial^p T_1^{(qr_2\ldots r_{N+1}k_1\ldots\hat{k}_j\ldots k_{C+1})}. \end{aligned} \tag{3.80}$$

Since the tensor T_1 is proportional to C_1, it is constructed out of the vector X^m and the Minkowski metric η^{mn}. It is not difficult to see that for a tensor T_1 constructed out of X^m and η^{mn}, the combination (3.80) must vanish. Hence $E=0$ and D_S is transverse.

Let us summarize the results of this subsection. We have shown that the parity-even contribution is fixed up to an overall coefficient. Moreover, it can be explicitly found for arbitrary superspins by solving a linear homogeneous system of equations with the tridiagonal matrix (3.44a). Once the solution for the parity-even coefficients α_{2k} is found, the tensor C is obtained using eq. (3.41a) and the tensor D_S is obtained from C as discussed above. Note that our analysis for the parity even solution is incomplete since we have not imposed the conservation condition at the third point. This is technically difficult to impose using the approach outlined in the present paper, and from this viewpoint the computational approach developed in [45] is far more useful. However, it was verified in [45] up to $s_i = 20$ that if $s_3 \leq s_1 + s_2$ (that is, the third triangle inequality is satisfied) then the third conservation equation is automatically satisfied and does not result in any new restrictions on the three-point function. The analysis in this subsection assumes that this property continues to hold for arbitrary superspins.

E. Point-switch symmetries

For the $\langle \mathbf{J}_F \mathbf{J}'_F \mathbf{J}''_F \rangle$ three-point functions we can also examine the case where $\mathbf{J} = \mathbf{J}'$ for arbitrary superspins. We want to determine the conditions under which the parity-even solution satisfies the point-switch symmetry. If we consider the condition (2.25) and the irreducible decomposition (3.8), we obtain the following conditions on C and D:

$$C_E^{(r_1 \ldots r_{N+1})(k_1 \ldots k_C)}(X) - C_E^{(r_1 \ldots r_{N+1})(k_1 \ldots k_C)}(-X) = 0, \tag{3.81a}$$

$$D_E^{(r_1 \ldots r_{N+1})(k_1 \ldots k_{C+1})}(X) - D_E^{(r_1 \ldots r_{N+1})(k_1 \ldots k_{C+1})}(-X) = 0. \tag{3.81b}$$

Let us consider (3.81a) first. Using auxiliary spinors, this condition may be written as

$$C_E(X; u, v) - C_E(-X; u, v) = 0. \tag{3.82}$$

However, recall that $C_E(X; u, v)$ is of the form

$$C_E(X; u, v) = X^{C-N-2} \sum_{k=0}^{C} \alpha_{2k} P_3^{2k} Q_3^{2(C-k)} Z_1^{N-C+1}. \tag{3.83}$$

For $\mathbf{J} = \mathbf{J}'$, we have $N = A + B = 2A$, and from (3.82) we obtain

$$\sum_{k=0}^{C} (1 + (-1)^C) \alpha_{2k} P_3^{2k} Q_3^{2(C-k)} Z_1^{N-C+1} = 0. \tag{3.84}$$

Hence, the parity-even solution $C_E(X; u, v)$ satisfies the point-switch only for C odd, i.e. for $s_3 = 2k + \frac{3}{2}$, $k \in \mathbb{Z}_{\geq 0}$, which is consistent with the results of [45].

Now assume that C_E satisfies the point-switch symmetry (3.81a). We want to show that D_E, which is fully determined by C_E, satisfies (3.81b). Since (3.81a) is satisfied, from (3.68) we must have $C_1(X) = C_1(-X)$, $C_2(X) = C_2(-X)$ (indices suppressed). Now consider (3.28) and the irreducible decomposition for D_S given by (3.70). Since $T_1 \propto C_1$, $T_2 \propto C_2$, we have $T_1(X) = T_1(-X)$, $T_2(X) = T_2(-X)$. It is then easy to see by substituting (3.28), (3.70) into (3.81b) that D_E also satisfies the point-switch symmetry. Concerning point-switch symmetries that involve $\mathbf{J}''$, these are difficult to check using the approach outlined in this paper, as one must compute $\tilde{\mathcal{H}}$ using (2.24) and check (2.26). However, the results of [45] cover these cases in more detail and so we will not discuss them here.

IV. GRASSMANN-ODD THREE-POINT FUNCTIONS $\langle \mathbf{J}_F \mathbf{J}'_B \mathbf{J}''_B \rangle$

Let us now consider the case $\langle \mathbf{J}_F \mathbf{J}'_B \mathbf{J}''_B \rangle$, which proves to be considerably simpler. Let us begin with making important comments on the arrangement of the operators in this three-point function. First, we arrange them in such a way that the operator at the third position is bosonic. Second, we arrange them in such a way that the third superspin satisfies the triangle inequality, that is $s_3 \leq s_1 + s_2$. As was mentioned in the previous section if one of the triangle inequalities is violated the remaining two are necessarily satisfied. This means that we can always place a bosonic operator with the superspin satisfying the triangle inequality at the third position.

We consider one Grassmann-odd current, $\mathbf{J}_{\alpha(2A+1)}$, of spin $s_1 = A + \frac{1}{2}$, and two Grassmann-even currents $\mathbf{J}'_{\alpha(2B)}$, $\mathbf{J}''_{\gamma(2C)}$, of spins $s_2 = B$, $s_3 = C$ respectively, where A, B, C are positive integers. All information about the correlation function

$$\langle \mathbf{J}_{\alpha(2A+1)}(z_1) \mathbf{J}'_{\beta(2B)}(z_2) \mathbf{J}''_{\gamma(2C)}(z_3) \rangle, \tag{4.1}$$

is now encoded in a homogeneous tensor field $\mathcal{H}_{\alpha(2A+1)\beta(2B+1)\gamma(2C+1)}(\boldsymbol{X}, \Theta)$, which satisfies the scaling property

$$\begin{aligned} &\mathcal{H}_{\alpha(2A+1)\beta(2B)\gamma(2C)}(\lambda^2 \boldsymbol{X}, \lambda\Theta) \\ &\quad = (\lambda^2)^{C-A-B-\frac{3}{2}} \mathcal{H}_{\alpha(2A+1)\beta(2B)\gamma(2C)}(\boldsymbol{X}, \Theta). \end{aligned} \tag{4.2}$$

Analogous to the previous case, for each set of totally symmetric spinor indices (the α's, β's and γ's respectively), we convert pairs of them into vector indices as follows:

$$\begin{aligned}\mathcal{H}_{\alpha(2A+1)\beta(2B)\gamma(2C)}(\boldsymbol{X},\Theta) &\equiv \mathcal{H}_{\alpha\alpha(2A),\beta(2B),\gamma(2C)}(\boldsymbol{X},\Theta)\\ &= (\gamma^{m_1})_{\alpha_1\alpha_2}\cdots(\gamma^{m_A})_{\alpha_{2A-1}\alpha_{2A}}\\ &\quad\times(\gamma^{n_1})_{\beta_1\beta_2}\cdots(\gamma^{n_B})_{\beta_{2B-1}\beta_{2B}}\\ &\quad\times(\gamma^{k_1})_{\gamma_1\gamma_2}\cdots(\gamma^{k_C})_{\gamma_{2C-1}\gamma_{2C}}\\ &\quad\times\mathcal{H}_{\alpha,m_1\ldots m_A,n_1\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta).\end{aligned} \tag{4.3}$$

Again, the equality above holds only if and only if $\mathcal{H}_{\alpha,m_1\ldots m_A n_1\ldots n_B k_1\ldots k_C}(\boldsymbol{X},\Theta)$ is totally symmetric and traceless in each group of vector indices. It is also required that $\mathcal{H}$ is subject to the γ-trace constraint

$$(\gamma^{m_1})_\sigma{}^\alpha\mathcal{H}_{\alpha,m_1\ldots m_A,n_1\ldots n_B,k_1\ldots k_C}(\boldsymbol{X},\Theta)=0. \tag{4.4}$$

Indeed, since $\mathcal{H}$ is Grassmann-odd it is linear in Θ, and we decompose $\mathcal{H}$ it follows (again, raising all indices for convenience):

$$\mathcal{H}^{\alpha,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=\sum_{i=1}^{2}\mathcal{H}_i^{\alpha,m(A)n(B)k(C)}(X,\Theta), \tag{4.5a}$$

$$\mathcal{H}_1^{\alpha,m(A)n(B)k(C)}(X,\Theta)=\Theta^\alpha A^{m(A)n(B)k(C)}(X), \tag{4.5b}$$

$$\mathcal{H}_2^{\alpha,m(A)n(B)k(C)}(X,\Theta)=(\gamma_p)^\alpha{}_\delta\Theta^\delta B^{p,m(A)n(B)k(C)}(X). \tag{4.5c}$$

Hence, in this case there are only two contributions to consider. The conservation equations (2.23a), (2.23b) are now equivalent to the following constraints on $\mathcal{H}$ with vector indices:

$$\mathcal{D}_\alpha\mathcal{H}^{\alpha,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0, \tag{4.6a}$$

$$(\gamma_n)_\sigma{}^\beta\mathcal{Q}_\beta\mathcal{H}^{\alpha,m(A)nn(B-1)k(C)}(\boldsymbol{X},\Theta)=0. \tag{4.6b}$$

They split up into constraints $O(\Theta^0)$

$$\frac{\partial}{\partial\Theta^\alpha}\mathcal{H}^{\alpha,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0, \tag{4.7a}$$

$$(\gamma_n)_\sigma{}^\beta\frac{\partial}{\partial\Theta^\beta}\mathcal{H}^{\alpha,m(A)nn(B-1)k(C)}(\boldsymbol{X},\Theta)=0, \tag{4.7b}$$

and $O(\Theta^2)$

$$(\gamma^m)_{\alpha\delta}\Theta^\delta\frac{\partial}{\partial X^m}\mathcal{H}^{\alpha,m(A)n(B)k(C)}(\boldsymbol{X},\Theta)=0, \tag{4.8a}$$

$$(\gamma_n)_\sigma{}^\beta(\gamma^m)_{\beta\delta}\Theta^\delta\frac{\partial}{\partial X^m}\mathcal{H}^{\alpha,m(A)nn(B-1)k(C)}(\boldsymbol{X},\Theta)=0. \tag{4.8b}$$

Using the irreducible decomposition (4.5), Eq. (4.7a) immediately results in $A=0$, while (4.7b) gives

$$\eta_{pn}B^{p,m(A)nn(B-1)k(C)}=0,\qquad \epsilon_{qpn}B^{p,m(A)nn(B-1)k(C)}=0. \tag{4.9}$$

Next, after imposing the γ-trace condition (4.4), we find that B must satisfy

$$\eta_{pm}B^{p,mm(A-1)n(B)k(C)}=0,\quad \epsilon_{qpm}B^{p,mm(A-1)n(B)k(C)}=0. \tag{4.10}$$

Altogether (4.9) and (4.10) imply that B is symmetric and traceless in the indices p, $m_1,\ldots,m_A$, $n_1,\ldots,n_B$, i.e.

$$B^{p,m(A)n(B)k(C)}\equiv B^{(pm(A)n(B)),k(C)}. \tag{4.11}$$

If we now consider the equations arising from conservation at $O(\Theta^2)$, a simple computation shows that B must satisfy

$$\partial_p B^{(pm(A)n(B)),k(C)}=0. \tag{4.12}$$

Therefore we need to construct a single transverse tensor B of rank $A+B+C+1$. We see that the tensor B has absolutely same properties as the tensor C from the previous section. Hence, the analysis becomes exactly the same as for C in the $\langle\mathbf{J}_F\mathbf{J}'_F\mathbf{J}''_F\rangle$ case and we will not repeat it. Recalling the results from the previous section, we find that $\langle\mathbf{J}_F\mathbf{J}'_B\mathbf{J}''_B\rangle$ has vanishing parity-odd contribution for all values of the superspins and a unique parity-even structure.[5] The explicit form of the parity-even solution can be found from the tridiagonal system of linear equations just like in the previous section.

ACKNOWLEDGMENTS

The authors are grateful to Sergei Kuzenko for valuable discussions. The work of E. I. B. is supported in part by the Australian Research Council, Projects DP200101944 and DP230101629. The work of B. S. is supported by the *Bruce and Betty Green Postgraduate Research Scholarship* under the Australian Government Research Training Program.

APPENDIX: 3D CONVENTIONS AND NOTATION

For the Minkowski metric we use the "mostly plus" convention: $\eta_{mn}=\mathrm{diag}(-1,1,1)$. Spinor indices are then raised and lowered with the $\mathrm{SL}(2,\mathbb{R})$ invariant antisymmetric ε-tensor

[5]Here we also have assumed that with our arrangement of the operators the conservation condition at the third point is automatically satisfied for all superspins. It is verified in our computational approach in [45] up to $s_i=20$.

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta, \tag{A1a}$$

$$\phi_\alpha = \varepsilon_{\alpha\beta}\phi^\beta, \qquad \phi^\alpha = \varepsilon^{\alpha\beta}\phi_\beta. \tag{A1b}$$

The γ-matrices are chosen to be real, and are expressed in terms of the Pauli matrices, σ, as follows:

$$(\gamma_0)_\alpha{}^\beta = -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma_1)_\alpha{}^\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A2a}$$

$$(\gamma_2)_\alpha{}^\beta = -\sigma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A2b}$$

$$(\gamma_m)_{\alpha\beta} = \varepsilon_{\beta\delta}(\gamma_m)_\alpha{}^\delta, \qquad (\gamma_m)^{\alpha\beta} = \varepsilon^{\alpha\delta}(\gamma_m)_\delta{}^\beta. \tag{A2c}$$

The γ-matrices are traceless and symmetric

$$(\gamma_m)^\alpha{}_\alpha = 0, \qquad (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha}, \tag{A3}$$

and also satisfy the Clifford algebra

$$\gamma_m\gamma_n + \gamma_n\gamma_m = 2\eta_{mn}. \tag{A4}$$

For products of γ-matrices we make use of the identities

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\beta = \eta_{mn}\delta_\alpha{}^\beta + \epsilon_{mnp}(\gamma^p)_\alpha{}^\beta, \tag{A5a}$$

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\sigma(\gamma_p)_\sigma{}^\beta = \eta_{mn}(\gamma_p)_\alpha{}^\beta - \eta_{mp}(\gamma_n)_\alpha{}^\beta + \eta_{np}(\gamma_m)_\alpha{}^\beta + \epsilon_{mnp}\delta_\alpha{}^\beta, \tag{A5b}$$

where we have introduced the 3D Levi-Civita tensor ϵ, with $\epsilon^{012} = -\epsilon_{012} = 1$. We also have the orthogonality and completeness relations for the γ-matrices

$$(\gamma^m)_{\alpha\beta}(\gamma_m)^{\rho\sigma} = -\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\alpha{}^\sigma\delta_\beta{}^\rho, (\gamma_m)_{\alpha\beta}(\gamma_n)^{\alpha\beta} = -2\eta_{mn}. \tag{A6}$$

The γ-matrices are used to swap from vector indices to spinor indices. For example, given some three-vector x_m, it may equivalently be expressed in terms of a symmetric second-rank spinor $x_{\alpha\beta}$ as follows:

$$x_{\alpha\beta} = (\gamma^m)_{\alpha\beta}x_m, \qquad x_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}x_{\alpha\beta}, \tag{A7a}$$

$$\det(x_{\alpha\beta}) = \frac{1}{2}x^{\alpha\beta}x_{\alpha\beta} = -x^m x_m = -x^2. \tag{A7b}$$

The same conventions are also adopted for the spacetime partial derivatives ∂_m

$$\partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial_m, \qquad \partial_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}\partial_{\alpha\beta}, \tag{A8a}$$

$$\partial_m x^n = \delta_m^n, \qquad \partial_{\alpha\beta}x^{\rho\sigma} = -\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\alpha{}^\sigma\delta_\beta{}^\rho, \tag{A8b}$$

$$\xi^m\partial_m = -\frac{1}{2}\xi^{\alpha\beta}\partial_{\alpha\beta}. \tag{A9}$$

We also define the supersymmetry generators Q_α

$$Q_\alpha = \mathrm{i}\frac{\partial}{\partial\theta^\alpha} + (\gamma^m)_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^m}, \tag{A10}$$

and the covariant spinor derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\gamma^m)_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^m}, \tag{A11}$$

which anticommute with the supersymmetry generators, $\{Q_\alpha, D_\beta\} = 0$, and obey the standard anticommutation relations

$$\{D_\alpha, D_\beta\} = 2\mathrm{i}(\gamma^m)_{\alpha\beta}\partial_m. \tag{A12}$$

[1] A. M. Polyakov, Conformal symmetry of critical fluctuations, JETP Lett. **12**, 381 (1970).
[2] E. J. Schreier, Conformal symmetry and three-point functions, Phys. Rev. D **3**, 980 (1971).
[3] A. A. Migdal, On hadronic interactions at small distances, Phys. Lett. **37B**, 98 (1971).
[4] A. A. Migdal, Conformal invariance and bootstrap, Phys. Lett. **37B**, 386 (1971).
[5] S. Ferrara, A. F. Grillo, and R. Gatto, Manifestly conformal-covariant expansion on the light cone, Phys. Rev. D **5**, 3102 (1972).
[6] S. Ferrara, A. F. Grillo, and R. Gatto, Tensor representations of conformal algebra and conformally covariant operator product expansion, Ann. Phys. (N.Y.) **76**, 161 (1973).
[7] K. Koller, The significance of conformal inversion in quantum field theory, Commun. Math. Phys. **40**, 15 (1974).
[8] G. Mack, Convergence of operator product expansions on the vacuum in conformal invariant quantum field theory, Commun. Math. Phys. **53**, 155 (1977).
[9] E. S. Fradkin and M. Y. Palchik, Recent developments in conformal invariant quantum field theory, Phys. Rep. **44**, 249 (1978).

[10] Y. S. Stanev, Stress—energy tensor and U(1) current operator product expansions in conformal QFT, Bulgarian Journal of Physics **15**, 93 (1988).

[11] H. Osborn and A. C. Petkou, Implications of conformal invariance in field theories for general dimensions, Ann. Phys. (N.Y.) **231**, 311 (1994).

[12] J. Erdmenger and H. Osborn, Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions, Nucl. Phys. **B483**, 431 (1997).

[13] S. Giombi, S. Prakash, and X. Yin, A note on CFT correlators in three dimensions, J. High Energy Phys. 07 (2013) 105.

[14] J. Maldacena and A. Zhiboedov, Constraining conformal field theories with a higher spin symmetry, J. Phys. A **46**, 214011 (2013).

[15] A. Zhiboedov, A note on three-point functions of conserved currents, arXiv:1206.6370.

[16] S. Giombi, V. Gurucharan, V. Kirilin, S. Prakash, and E. Skvortsov, On the higher-spin spectrum in large N Chern-Simons vector models, J. High Energy Phys. 01 (2017) 058.

[17] E. I. Buchbinder and B. J. Stone, Three-point functions of conserved currents in 3D CFT: General formalism for arbitrary spins, Phys. Rev. D **107**, 046007 (2023).

[18] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia, and X. Yin, Chern-Simons theory with vector fermion matter, Eur. Phys. J. C **72**, 2112 (2012).

[19] O. Aharony, G. Gur-Ari, and R. Yacoby, $d = 3$ bosonic vector models coupled to Chern-Simons gauge theories, J. High Energy Phys. 03 (2012) 037.

[20] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, Comments on Chern-Simons contact terms in three dimensions, J. High Energy Phys. 09 (2012) 091.

[21] G. Gur-Ari and R. Yacoby, Correlators of large N fermionic Chern-Simons vector models, J. High Energy Phys. 02 (2013) 150.

[22] O. Aharony, G. Gur-Ari, and R. Yacoby, Correlation functions of large N Chern-Simons-matter theories and bosonization in three dimensions, J. High Energy Phys. 12 (2012) 028.

[23] S. Jain, S. P. Trivedi, S. R. Wadia, and S. Yokoyama, Supersymmetric Chern-Simons theories with vector matter, J. High Energy Phys. 10 (2012) 194.

[24] S. D. Chowdhury, J. R. David, and S. Prakash, Constraints on parity violating conformal field theories in $d = 3$, J. High Energy Phys. 11 (2017) 171.

[25] E. Skvortsov, Light-front bootstrap for Chern-Simons matter theories, J. High Energy Phys. 06 (2019) 058.

[26] K. Inbasekar, S. Jain, V. Malvimat, A. Mehta, P. Nayak, and T. Sharma, Correlation functions in $\mathcal{N} = 2$ Supersymmetric vector matter Chern-Simons theory, J. High Energy Phys. 04 (2020) 207.

[27] S. Jain and R. R. John, Relation between parity-even and parity-odd CFT correlation functions in three dimensions, J. High Energy Phys. 12 (2021) 067.

[28] S. Prakash, Spectrum of a Gross-Neveu Yukawa model with flavor disorder in three dimensions, Phys. Rev. D **107**, 066025 (2023).

[29] H. Osborn, $N = 1$ superconformal symmetry in four-dimensional quantum field theory, Ann. Phys. (N.Y.) **272**, 243 (1999).

[30] J.-H. Park, Superconformal symmetry in six-dimensions and its reduction to four-dimensions, Nucl. Phys. **B539**, 599 (1999).

[31] J. H. Park, Superconformal symmetry and correlation functions, Nucl. Phys. **B559**, 455 (1999).

[32] J.-H. Park, Superconformal symmetry in three-dimensions, J. Math. Phys. (N.Y.) **41**, 7129 (2000).

[33] S. M. Kuzenko and S. Theisen, Correlation functions of conserved currents in $N = 2$ superconformal theory, Classical Quantum Gravity **17**, 665 (2000).

[34] A. A. Nizami, T. Sharma, and V. Umesh, Superspace formulation and correlation functions of 3d superconformal field theories, J. High Energy Phys. 07 (2014) 022.

[35] E. I. Buchbinder, S. M. Kuzenko, and I. B. Samsonov, Superconformal field theory in three dimensions: Correlation functions of conserved currents, J. High Energy Phys. 06 (2015) 138.

[36] E. I. Buchbinder, S. M. Kuzenko, and I. B. Samsonov, Implications of $\mathcal{N} = 4$ superconformal symmetry in three spacetime dimensions, J. High Energy Phys. 08 (2015) 125.

[37] S. M. Kuzenko and I. B. Samsonov, Implications of $\mathcal{N} = 5$, 6 superconformal symmetry in three spacetime dimensions, J. High Energy Phys. 08 (2016) 084.

[38] E. I. Buchbinder and B. J. Stone, Mixed three-point functions of conserved currents in three-dimensional superconformal field theory, Phys. Rev. D **103**, 086023 (2021).

[39] E. I. Buchbinder, J. Hutomo, and S. M. Kuzenko, Correlation functions of spinor current multiplets in $\mathcal{N} = 1$ superconformal theory, J. High Energy Phys. 07 (2021) 165.

[40] E. I. Buchbinder, J. Hutomo, and S. M. Kuzenko, Three-point functions of higher-spin spinor current multiplets in $\mathcal{N} = 1$ superconformal theory, J. High Energy Phys. 10 (2021) 058.

[41] E. I. Buchbinder and B. J. Stone, Three-point functions of a superspin-2 current multiplet in 3D, $N = 1$ superconformal theory, Phys. Rev. D **104**, 106004 (2021).

[42] A. Jain and A. A. Nizami, Superconformal invariants and spinning correlators in 3d $\mathcal{N} = 2$ SCFTs, Eur. Phys. J. C **82**, 1065 (2022).

[43] E. I. Buchbinder and B. J. Stone, Three-point functions of a fermionic higher-spin current in 4D conformal field theory, Phys. Rev. D **105**, 125004 (2022).

[44] E. I. Buchbinder, J. Hutomo, and G. Tartaglino-Mazzucchelli, Three-point functions of higher-spin supercurrents in 4D $N = 1$ superconformal field theory, Fortschr. Phys. **70**, 2200133 (2022).

[45] E. I. Buchbinder and B. J. Stone, Three-point functions of conserved supercurrents in 3D $N = 1$ SCFT: General formalism for arbitrary superspins, Phys. Rev. D **107**, 106001 (2023).

[46] Y. Korovin, S. M. Kuzenko, and S. Theisen, The conformal supercurrents in diverse dimensions and conserved superconformal currents, J. High Energy Phys. 05 (2016) 134.

[47] L. Brugnano and D. Trigiante, Tridiagonal matrices: Invertibility and conditioning, Linear Algebra Appl. **166**, 131 (1992).

[48] Y. Huang and W. F. McColl, Analytical inversion of general tridiagonal matrices, J. Phys. A **30**, 7919 (1997).