Proof of the Rényi quantum null energy condition for free fermions

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The quantum null energy condition (QNEC) is usually stated as a bound on the expectation value of null components of the stress-energy tensor at a point in terms of second null shape variations of the entanglement entropy at the same point. It can be recast as the statement that the sign of the second null shape variation of the relative entropy of any state with respect to the vacuum is positive. Using instead a Rényi generalization of relative entropy, called sandwiched Rényi divergence (SRD), leads to what is termed the Rényi QNEC: The second null shape variation of SRD of any state with respect to the vacuum is positive. In this work, we prove the Rényi QNEC for free and super-renormalizable fermionic quantum field theories in spacetime dimensions greater than 2 using null quantization, for the case where the Rényi parameter $n > 1$. We end with comments on multiple possible generalizations.

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I. INTRODUCTION

The holographic principle [1,2] (see also [3]) and its explicit example, the AdS/CFT correspondence [4–6], have revolutionized our understanding of (quantum) gravity over the past three decades. An entry of the AdS/CFT "dictionary," the Hubeny-Rangamani-Ryu-Takayanagi formula [7,8], provides a translation between the entanglement structure of quantum field theories and geometric properties of extremal surfaces in asymptotically AdS spacetimes. This has led to major advances in our understanding of the AdS/CFT correspondence, on both sides of the story.

An interesting development in recent years has been the quantum focusing conjecture [9], a generalization of the classical focusing theorem. The quantum focusing conjecture reduces in a particular limit to what has been termed the quantum null energy condition (QNEC), which states that, along null congruences with vanishing expansion and shear, one has

$$\langle T_{kk} \rangle \geq \frac{\hbar}{2\pi} S'', \tag{1}$$

where $T_{kk} = T_{ab}k^a k^b$, T_{ab} is the stress-energy tensor of a quantum field theory (QFT) regarded as a quantum operator, k^a is the tangent to the null congruence, and S'' denotes the second variation of the entanglement entropy as the entangling region is deformed along k^a at one point. A more precise statement can be found in Ref. [10]. We note that the QNEC does not involve the bulk gravitational constant G_N and, as such, is a statement purely about QFTs.

As discussed in Ref. [9], the variation S'' of the entanglement entropy appearing in QNEC is of the form

$$S'' = \lim_{y' \to y} \frac{\delta^2 S}{\delta\lambda(y)\delta\lambda(y')}, \tag{2}$$

with λ the affine parameter along the geodesics of the null plane. Entanglement entropy of subregions is known to be universally UV divergent in QFTs (see, e.g., [11]), and one would like to have a better formulation of the QNEC. This is in fact easily achieved by rewriting Eq. (1) in terms of the relative entropy [12]. For density matrices associated with a region R, we have

$$\begin{aligned} S_{\mathrm{rel}}(\rho_R|\sigma_R) &= \mathrm{tr}\rho_R \log\rho_R - \mathrm{tr}\rho_R \log\sigma_R \\ &= \Delta\langle K_R^\sigma \rangle - (\Delta S)_R, \end{aligned} \tag{3}$$

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where $K^\sigma = -\log \sigma$ is the modular Hamiltonian associated to a density matrix σ.[1] For arbitrary cuts of a null plane, one has the result that [13]

$$K_R^\sigma = 2\pi \int_{V(y)}^{\infty} (v - V(y)) T_{vv}(y) dv d^{d-2}y, \qquad (4)$$

where v is the null direction along the null surface, y are the transverse directions, and $V(y)$ is an arbitrary curve on the null plane describing the choice of the cut. Choosing σ_R to be the vacuum state, one can define a vacuum subtracted modular Hamiltonian:

$$\Delta\langle K\rangle = 2\pi \int_{V(y)}^{\infty} (v - V(y)) \langle T_{vv}(y)\rangle dv d^{d-2}y. \qquad (5)$$

Taking two variations along v and using the fact that the entanglement entropy for the vacuum state is stationary for null cuts, it is straightforward to see that Eq. (1) is equivalent to

$$\lim_{y'\to y} \frac{\delta^2 S_{\rm rel}(\rho_R|\sigma_R)}{\delta\lambda(y)\delta\lambda(y')} \geq 0. \qquad (6)$$

Positivity of the off-diagonal components ($\lambda \neq \lambda'$) of the second variation of $S_{\rm rel}$ follows from strong subadditivity [10]. However, we focus on only the "diagonal" terms of the variation, i.e., the limit $y' \to y$, throughout the work.

Evidence for validity of QNEC has been accumulating over the years. Reference [10] gave the first proof of QNEC for free and superrenormalizable bosons based on replica trick calculations. Reference [14] later generalized this proof to the case of free fermions. Reference [15] proved QNEC for CFTs with holographic duals and their relevant deformations. Reference [16] proved the QNEC for general states of a CFT on Minkowski space using properties of modular Hamiltonians under shape deformations and causality. Finally, REf. [17] provided a rigorous proof of QNEC for cases where the entangling region is a null cut in a general Poincaré invariant QFT, using Tomita-Takesaki theory and the theory of half-sided modular inclusions.

Relative entropy is a measure of the distinguishability of a state ρ given a state σ. It is defined to be always positive, $S_{\rm rel}(\rho|\sigma) \geq 0$, and is monotonic under completely positive trace-preserving maps, $S_{\rm rel}(\Phi\rho|\Phi\sigma) \leq S_{\rm rel}(\rho|\sigma)$ [11,18].[2] This monotonicity property is often referred to in the literature as the data processing inequality (DPI). For states that are close to each other, DPI can be thought of as a constraint on the sign of the first derivative of $S_{\rm rel}$. QNEC is a constraint on the sign of the second derivative of $S_{\rm rel}$ in local quantum physics. Given the purely information-theoretic nature of QNEC (6) and the amount of supporting evidence, it is natural to ask if generalizations of the relative entropy that are positive and monotonic also satisfy such constraints on their second derivatives.

One such measure is the sandwiched Rényi divergence [20,21], a Rényi generalization of the relative entropy. Reference [22] showed that some sandwiched Rényi divergences (SRDs) can be written as correlation functions in quantum field theory. Based on the path integral expression for SRD in Ref. [22], Ref. [23] put constraints on correlation functions in QFT based on some known properties of the divergence. It was further conjectured there that the second null variation of the SRD should also be positive, thus providing a Rényi generalization of QNEC, which we will refer to as Rényi QNEC. A few examples where the conjecture holds were also demonstrated. Very recently, Ref. [24] gave a proof of the Rényi QNEC for free and superrenormalizable bosons in spacetime dimensions $D > 2$.

In this work, we provide a proof of Rényi QNEC for free and superrenormalizable fermions in $D > 2$ spacetime dimensions. Our proof closely follows the one presented in Ref. [24]. We use the formalism of null quantization to write an arbitrary state as an expansion around the vacuum and reduce the Rényi QNEC for arbitrary states to a statement regarding a state perturbatively close to the vacuum. One can then evaluate the relevant SRD variations using expansions known in the literature and show that the conjecture indeed is true in some cases. We note that the conjecture is not true for some values of the Rényi parameter, as also found in Ref. [24]. We do not investigate these.

We now present an outline of the rest of the paper. Section II begins with a review of the definition and properties of sandwiched Rényi divergence followed by a discussion of the Rényi QNEC conjecture. Section III provides details of the setup where we perform our computations. We briefly review the formalism of null quantization. Rényi QNEC is then reformulated as a perturbative statement, simplifying the calculations significantly. In Sec. IV, we proceed to prove Rényi QNEC, doing it in two ways, once for integer values of the Rényi parameter and then for general values. The two methods provide complementary insight. Finally, we conclude with a discussion of possible generalizations and applications of Rényi QNEC. Two appendixes contain details of the fermionic theory that we consider and the calculation of correlation functions needed for completing the proof.

II. SANDWICHED RÉNYI DIVERGENCE AND RÉNYI QNEC

We begin this section by defining sandwiched Rényi divergence for finite-dimensional quantum systems, in terms of density matrices. We then provide the basic

[1] Note that, here and in Eq. (5) below, Δx denotes a change in the quantity x, as opposed to Sec. II, where Δ is used to denote the modular Hamiltonian.

[2] It is, in fact, monotonic under positive maps [19].

definitions of modular theory, to be able to define SRD in QFTs in general. As discussed in Refs. [22,23], sandwiched Rényi divergence for integer $n > 1$ can be expressed in terms of one-sheeted Euclidean $2n$-point correlation functions. One can then evaluate the SRD explicitly and check the statement of Eq. (14), at least in free field theories. We provide a brief introduction to these ideas to finish this section.

A. Preliminaries

A quantum version of Rényi relative entropy, termed sandwiched Rényi divergence, was proposed for type I von Neumann algebras in Refs. [20,21]; see also Sec. 3.3 in Ref. [25]. States in these systems can be described in terms of density matrices. For two states described by density matrices ρ, σ, sandwiched Rényi relative α entropy[3] of ρ with respect to σ is defined to be

$$S_\alpha(\rho|\sigma) = \frac{1}{\alpha-1}\log \mathrm{tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right], \tag{7}$$

for $\alpha > 0$, $\alpha \neq 1$, if the support of ρ is contained in the support of σ; otherwise, $S_\alpha(\rho|\sigma) = \infty$. For $\alpha \in [1/2, \infty)$, $S_\alpha(\rho|\sigma)$ was shown to satisfy the DPI [26,27]. The limit $\alpha \to 1$ corresponds to Umegaki's relative entropy defined in Eq. (3).

Our interest is in the Rényi relative entropy for QFTs, which correspond to type III von Neumann algebras [28–31]. There is no notion of a density matrix for general von Neumann algebras, and the above definitions of relative entropy and Rényi relative entropy are not useful. It becomes essential to use the language of algebraic QFT, in particular, the theory of Tomita and Takesaki [32,33]. See Ref. [11] for an accessible introduction to these ideas. We briefly review the basics needed to define SRD in QFT. The reader is also referred to Ref. [34] for a quick introduction to, and to Ref. [25] for a detailed treatment of, the algebraic approach in the finite-dimensional context.

For an open set R in D-dimensional Minkowski space, M_D, we consider the local algebra $\mathcal{A}_R$ of operators supported in R. Denote the vacuum state by Ω. States formed by acting with a finite number of operators on the vacuum comprise the vacuum sector $\mathcal{H}_0$ of the Hilbert space. A state $\Psi \in \mathcal{H}_0$ is called cyclic for $\mathcal{A}_R$ if the states $A|\Psi\rangle, A \in \mathcal{A}_R$, are dense in $\mathcal{H}_0$. The state Ψ is called separating for $\mathcal{A}_R$ if, for $A \in \mathcal{A}_R$, $A|\Psi\rangle = 0 \Rightarrow A = 0$. The Reeh-Schlieder theorem implies that the vacuum is a cyclic separating vector for the algebra associated to any subregion. Including what are known as weak limits of sequences of operators gives a closed algebra of operators, and we denote the corresponding Hilbert space by $\mathcal{H}$.

[3] Alternatively called sandwiched Rényi divergence and Rényi relative entropy.

Let $\Psi \in \mathcal{H}$ be a cyclic separating state for $\mathcal{A}_R$. The Tomita operator (for the state Ψ) is the antilinear[4] operator defined by $S_\Psi A|\Psi\rangle = A^\dagger|\Psi\rangle \;\; \forall \;\; A \in \mathcal{A}_R$.[5] S_Ψ is invertible and has a unique polar decomposition as

$$S_\Psi = J_\Psi \Delta_\Psi^{1/2}, \tag{8}$$

where J_Ψ is an antiunitary called the modular conjugation operator and $\Delta_\Psi^{1/2}$ is Hermitian and positive definite; Δ_Ψ is called the modular operator.

Now, let $\Phi \in \mathcal{H}'$ be another state, with $\mathcal{H}'$ not necessarily the same as $\mathcal{H} \ni \Psi$. Let $\mathcal{A}_R$ be an algebra that acts on both $\mathcal{H}$ and $\mathcal{H}'$. The relative Tomita operator is defined by

$$S_{\Psi|\Phi}\colon \mathcal{H} \to \mathcal{H}', \qquad S_{\Psi|\Phi}A|\Psi\rangle = A^\dagger|\Phi\rangle, \tag{9}$$

where $\langle\Psi|\Psi\rangle = \langle\Phi|\Phi\rangle = 1$, and one again needs to take a closure. The relative Tomita operator satisfies $S_{\Omega|\Phi}S_{\Phi|\Omega} = 1$ and $S^\dagger_{\Omega|\Phi}S^\dagger_{\Phi|\Omega} = 1$. The relative modular operator is defined to be

$$\Delta_{\Psi|\Phi} = S^\dagger_{\Psi|\Phi}S_{\Psi|\Phi}, \tag{10}$$

which is unbounded and positive semidefinite. It is positive definite iff $S_{\Psi|\Phi}$ is invertible. The polar decomposition of the relative Tomita operator takes the form $S_{\Psi|\Phi} = J_{\Psi|\Phi}\Delta^{1/2}_{\Psi|\Phi}$. We note that the modular operator acts as, e.g.,

$$\langle B\Psi|\Delta_{\Psi|\Phi}|A\Psi\rangle = \langle A^\dagger\Phi|B^\dagger\Phi\rangle, \tag{11}$$

which can be alternatively written as $B\Delta_{\Psi|\Phi}A = AB$. The inverse of the relative modular operator satisfies $\Delta^{-1}_{\Psi|\Phi} = S_{\Phi|\Psi}S^\dagger_{\Phi|\Psi}$. The relative modular operator comprises the core component of Tomita-Takesaki theory.

The relative modular operator can be used to define relative entropy in QFT [35] as

$$S_{\mathrm{rel}}(\Psi|\Phi) = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle. \tag{12}$$

This reduces to Eq. (3) for von Neumann algebras of type I. For regions $\tilde{R} \subset R$, so that $\mathcal{A}_{\tilde{R}} \subset \mathcal{A}_R$, the relative modular operators associated with given regions satisfy the

[4] For an antilinear operator W, its adjoint for states λ, χ is defined by

$$\langle\lambda|W\chi\rangle = \overline{\langle W^\dagger\lambda|\chi\rangle} = \langle\chi|W^\dagger\lambda\rangle.$$

If W is antiunitary, then $\langle W\lambda|W\chi\rangle = \langle\chi|\lambda\rangle$.

[5] One has to take a closure for the Tomita operator to be completely well defined. We consider only the closed operator throughout.

monotonicity property: $\Delta^{R}_{\Psi|\Phi} \leq \Delta^{\tilde{R}}_{\Psi|\Phi}$.[6] This can be used to prove the monotonicity of $S_{\rm rel}(\Psi|\Phi)$ under restrictions of subsets: $S_{\rm rel}(\Psi|\Phi)(R) \geq S_{\rm rel}(\Psi|\Phi)(\tilde{R})$.

B. SRD for QFT and Rényi QNEC

Sandwiched Rényi divergence between states Φ and Ψ is defined by the relations [23,36,37]

$$S^R_p(\Phi|\Psi) = \frac{p}{p-1}\sup_{|\chi\rangle\in\mathcal{H}} \log\langle\Phi|(\Delta_{\chi|\Psi})^{\frac{1}{p}-1}|\Phi\rangle, \qquad p\in(1,\infty)$$
$$S^R_p(\Phi|\Psi) = \frac{p}{p-1}\inf_{|\chi\rangle\in\mathcal{H}} \log\langle\Phi|(\Delta_{\chi|\Psi})^{\frac{1}{p}-1}|\Phi\rangle, \qquad p\in\left[\frac{1}{2},1\right). \tag{13}$$

This definition is a generalization of the p-norm of a matrix to unbounded operators.[7] For $p > 1$, SRD is defined to be infinite if the vector Φ is not in the intersection of the domains of $\Delta^{-1}_{\chi|\Psi}$ for all $|\chi\rangle \in \mathcal{H}$. For $p < 1$, SRD is finite if Ψ is cyclic separating [24].

SRD is non-negative and satisfies the data processing inequality; i.e., it monotonically decreases for algebras of smaller subregions [23,37–39]. This makes it an interesting quantity to study from an information theoretic perspective. SRD is also monotonic in p, $S^R_p(\Phi|\Psi) > S^R_q(\Phi|\Psi)$ for $p > q \in [\frac{1}{2}, 1) \cup (1,\infty)$. In the limit $p \to 1$, $S^R_p(\Phi|\Psi) \to S_{\rm rel}(\Phi|\Psi)$.

Based on the similarities between the relative entropy and the sandwiched Rényi divergence, Ref. [23] conjectured that its second variation should also be positive as it is for relative entropy:

$$\lim_{y'\to y}\frac{\delta^2 S^R_p(\Phi|\Psi)}{\delta\lambda(y)\delta\lambda(y')} \geq 0, \tag{14}$$

where the variation being considered is the same as in Eq. (6). We note that we are still concerned with only the "diagonal" component of the variation. This was termed the Rényi QNEC conjecture in Ref. [24], where a proof for the case of free (and superrenormalizable) bosons was provided. The goal of this paper is to extend their proof to the case of free (and superrenormalizable) fermions.

C. Correlation functions from path integrals

Let us write D-dimensional Minkowski space as M_D, with metric $ds^2 = -dt^2 + dx^2 + d\vec{y}\cdot d\vec{y}$. Let Σ be the initial value surface $t = 0$, which we divide into two half-spaces, $x > 0$ and $x < 0$, denoted $V_{R,L}$. Their respective domains of dependence will be denoted $U_{R,L}$ with the corresponding local algebras being $\mathcal{A}_{R,L}$. Let Ω be the vacuum of a QFT on M_D. States in this QFT can be prepared by inserting appropriately smeared operators in the path integral over the lower half of Euclidean time, $\tau \leq 0$. We will be interested in only the set of states that have finite SRD with respect to the vacuum.

The modular operator on the vacuum state, Δ_Ω, leaves the vacuum invariant. However, positive powers $\Delta^\alpha_\Omega, \alpha\in(0,1/2]$, take operators in $\mathcal{A}_R$ and rotate them to the location $\theta = -2\pi\alpha$ in the path integral over $\tau \leq 0$ [11]. Here, (r,θ) are polar coordinates on the (τ, x) plane, with $z = x + i\tau = re^{i\theta}$. Consider then an operator $\mathcal{O}_R \in \mathcal{A}_R$, and construct the excited states

$$|\Phi\rangle = \Delta^{\theta/2\pi}_\Omega \mathcal{O}_R(r,0)|\Omega\rangle = \mathcal{O}(r,\theta)|\Omega\rangle, \tag{15}$$

with $\theta\in[0,\pi]$, where we have used $\Delta^{\theta/2\pi}_\Omega \mathcal{O}_R(r,0)\Delta^{-\theta/2\pi}_\Omega = \mathcal{O}(r,\theta)$. One can bound the SRD $S^R_p(\Phi|\Omega)$ for $1 \leq p \in \mathbb{R}$, by choosing $\chi = \Omega$ in the supremum in Eq. (13):

$$S^R_p(\Phi|\Omega) \geq \frac{p}{p-1}\log\langle\Phi|(\Delta_{\Omega|\Omega})^{\frac{1}{p}-1}|\Phi\rangle$$
$$= \frac{p}{p-1}\log\|(\Delta_\Omega)^{\frac{\theta}{2\pi}+\frac{1}{2p}-\frac{1}{2}}\mathcal{O}_R(r,0)|\Omega\rangle\|^2.$$

States of the form $\Delta^\alpha_\Omega\mathcal{O}_R|\Omega\rangle$ generically have infinite norm for $\alpha > 1/2$, which implies that $S^R_p(\Phi|\Omega)$ diverges for $\theta < \pi - \pi/p$ and for $\theta > 2\pi - \pi/p$. On the other hand, as discussed in Refs. [22,23], sandwiched Rényi divergence for integer $p = n > 1$ can be expressed in terms of one-sheeted Euclidean $2n$-point correlation functions, which are known to be finite. An overview of their construction follows.

The vacuum density matrix of the right half-space, ω, is given by a path integral over the whole τ plane, with cuts at $\tau = 0^\pm, x > 0$. Let ϕ be the density matrix with operator insertions of $\Phi^\dagger, \Phi$ at $\pm(\pi-\theta)$. For $\theta \leq \pi/n$, with integer $n > 1$, the operator $\omega^{\frac{1}{2n}-\frac{1}{2}}\phi\omega^{\frac{1}{2n}-\frac{1}{2}}$ has a path integral representation as a wedge of opening angle $2\pi/n$ with two operator insertions. Sewing together n such wedges evaluates ${\rm tr}[(\omega^{\frac{1}{2n}-\frac{1}{2}}\phi\omega^{\frac{1}{2n}-\frac{1}{2}})^n]$ as a $2n$-point correlation function of $\Phi, \Phi^\dagger$, with operators inserted at $z^\pm_k = re^{i(\frac{2\pi k}{n}\pm\theta)}$ for $k = 0,\ldots,n-1$,

$${\rm tr}[(\omega^{\frac{1}{2n}-\frac{1}{2}}\phi\omega^{\frac{1}{2n}-\frac{1}{2}})^n] = \left\langle\prod_{k=0}^{n-1}\Phi^\dagger(z^+_k)\Phi(z^-_k)\right\rangle, \tag{16}$$

where the other directions $\vec{y}$ are suppressed. Then, for integer $n > 1$, we can evaluate SRD for this configuration of fields with respect to the vacuum as

[6] This means that the operator $\Delta^{\tilde{R}}_{\Psi|\Phi} - \Delta^{R}_{\Psi|\Phi}$ is a positive operator on $\tilde{R}$.

[7] For $p \geq 1$, the (Schatten) p-norm of a matrix A is defined to be $|A|_p = [{\rm Tr}(\sqrt{A^\dagger A}^p)]^{1/p}$.

$$S_n^R(\Phi|\Omega) = \frac{1}{n-1}\log\left(\frac{\langle\prod_{k=0}^{n-1}\Phi^\dagger(z_k^+)\Phi(z_k^-)\rangle}{\langle\Phi^\dagger(z_0^+)\Phi(z_0^-)\rangle^n}\right). \quad (17)$$

This relation between SRD and correlation functions was used in Ref. [23] to put constraints on correlation functions in QFT from known properties of SRD. It will be very useful for us, too.

As a final point, note that, although we defined SRD for QFTs using the abstract modular theory of Tomita and Takesaki, we have here resorted to using density matrices and will continue to do so in what follows. This implicitly assumes that one is working with regularized entropy and energy-momentum tensor using an appropriate renormalization scheme. See Ref. [40] for a detailed discussion. It has recently been shown that this assumption holds in various situations [41–44].

III. SETTING UP

In this section, we introduce the field theory that we work with and the null quantization scheme that allows us to prove the Rényi QNEC. We also give the precise statement that we will prove and reformulate it in a much more convenient form as a calculation regarding a state perturbatively close to the vacuum.

A. Null quantization and the state

For our proof, we will need to assume that the QFT describing the matter has a valid null-hypersurface initial-value formulation; i.e., a field algebra $\mathcal{A}(N)$ can be defined on a stationary null-surface N without making reference to anything outside N. Our treatment of null quantization closely follows Ref. [10]; see Refs. [40,45] for more details. The details of our setup are summarized in Fig. 1. We spell them out in detail below.

Let γ be a spacelike codimension-2 surface that splits a Cauchy surface Σ into two sides. The proof we present here applies when γ is a section of a general stationary null surface N in $D > 2$. Specifying the state of the QFT on the Cauchy surface Σ is unitarily equivalent to specifying the state on the null plane N and parts of past and future null infinities.

Now discretize N along the transverse directions into small regions of transverse area $\mathcal{A}$. These regions fully extended along the null directions are called pencils. See Fig. 1. We use $\mathcal{A}$ as an expansion parameter and take the limit $\mathcal{A} \to 0$ at the end. Degrees of freedom on different horizon generators are independent systems [40], and so the Hilbert space on N factorizes into a product of Hilbert spaces on the pencils. On each pencil, there exists a $1+1$-dimensional free chiral fermionic conformal field theory. Details of the CFT on the pencils are provided in Appendix A.

We want to consider the second shape variation along N of SRD at some point q on γ. This point is contained in one specific pencil, which we denote by P. Decompose the Hilbert space of the system as $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_A$, where $\mathcal{H}_P$ is the Hilbert space of our specific pencil P and $\mathcal{H}_A \equiv \mathcal{H}_{\mathrm{auxiliary}}$ contains all the remaining degrees of freedom. Consider a density matrix on $\mathcal{H}$, which we deform to obtain a one-parameter family of density matrices $\rho(\lambda)$ by tracing out the part of the pencil P in the past of affine parameter λ.

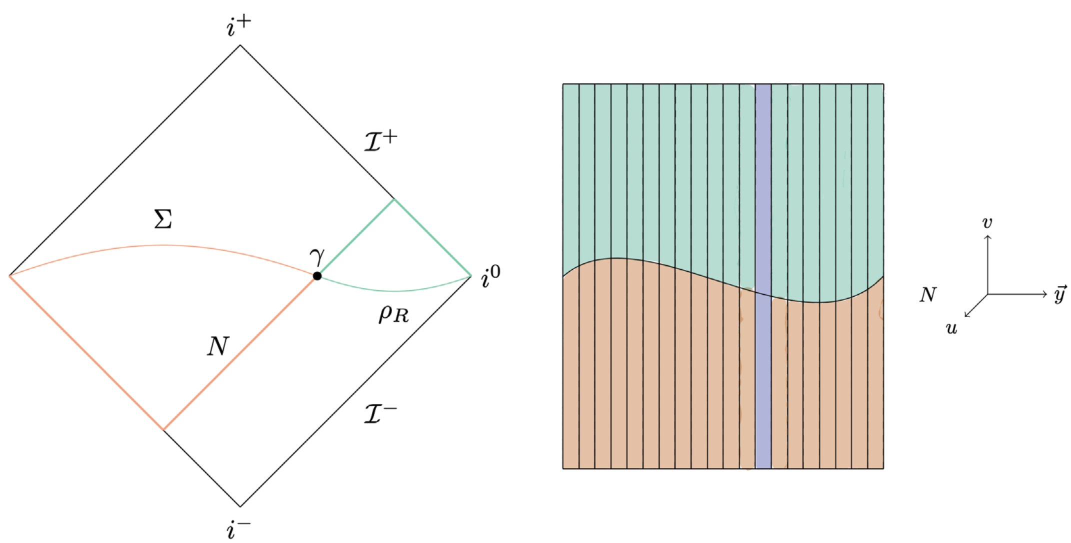

FIG. 1. An illustration of the setup that we work with. Σ is a Cauchy surface in Minkowski space, which is split in half by the codimension-2 surface γ. The reduced state ρ_R on the green half of the Cauchy surface is equivalent to the state on (green) parts of the null hypersurface N and null infinity $\mathcal{I}^+$. The right figure is a front view of N. The vertical segments bounded by solid lines are the pencils on N, with the blue pencil being our specific pencil P; the rest are all part of the auxiliary system. The parameter λ is $-\infty$ at the bottom of N in the right figure, and $\lambda \to \infty$ at the top. The curve that divides the green and orange regions on N is γ. The point p used above is the intersection of P with γ. The green part of Σ extends behind the plane of the page.

For small $\mathcal{A}$, the state of the pencil is near the vacuum, and the global state can be written as

$$\rho(\lambda) = \sigma_P(\lambda) \otimes \rho_A^{(0)} + \mathcal{A}^{1/2}\rho^{(1)}(\lambda) + \cdots. \tag{18}$$

This can be seen as follows. Since the probability to have n particles on a pencil should scale extensively with the area, it is proportional to $\mathcal{A}^n$ for each pencil. Then, terms of the form $|n\rangle\langle m|$ in the pencil Fock basis scale as $\mathcal{A}^{(n+m)/2}$. Thus, the leading contribution to the perturbation, $\rho^{(1)}(\lambda)$, is of the order of $\mathcal{A}^{1/2}$, with terms of the form $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$. Explicitly, the state on N must take the form

$$\rho = |0\rangle\langle 0| \otimes \rho_A^{(0)} + \mathcal{A}^{1/2}\sum_{ij}(|0\rangle\langle\psi_{ij}^1| + |\psi_{ij}^2\rangle\langle 0|) \otimes |i\rangle\langle j| + \mathcal{A}^{1/2}\sum_{ij}(|0\rangle\langle\psi_{ij}^2| + |\psi_{ij}^1\rangle\langle 0|) \otimes |j\rangle\langle i| + \cdots,$$

where $|\psi_{ij}\rangle$ are single-particle states on the pencil CFT, which can be constructed by inserting single operators in the fermionic path integral, and $\{|i\rangle, |j\rangle\}$ together form an arbitrary basis of $\mathcal{H}_A$ such that $\{|i\rangle\langle j|, |j\rangle\langle i|\}$ form a Grassmann-odd basis of operators on $\mathcal{H}_A$. The basis elements need to be Grassmann-odd, since the pencil CFT has one fermion excited in the leading contribution to the perturbation. We also note for later use that physical observables can be only Grassmann-even. We can define the needed Grassmann-odd basis, e.g., by taking $|i\rangle(|j\rangle)$ to be states with an even (odd) number of creation operators acting on the vacuum. We stick to this convention throughout the paper. A resolution of the identity operator for the auxiliary system in this basis is

$$\mathbf{1}_a = \sum_i |i\rangle\langle i| + \sum_j |j\rangle\langle j| \equiv \sum_\mu |\mu\rangle\langle\mu|, \tag{19}$$

where the sum over i (respectively, j) runs over states with even (respectively, odd) numbers of creation operators. We use the convention that indices μ, ν can correspond to either even (i) or odd (j) states. Note that the auxiliary system $\rho_A^{(0)}$ is not necessarily in the vacuum state. Entanglement between the pencil and the auxiliary system is contained in $\rho^{(1)}(\lambda)$ and the further subleading terms.

Since the theory on each pencil is chiral, we can replace translations (and derivatives) along λ by translations (and derivatives) along the spatial direction. Also, the single-particle states $|\psi_{ij}\rangle$ of the Fock space can be constructed by a Euclidean path integral over the lower half plane, $\tau < 0$, with insertions of a single field.

The affine parameter $\lambda = V_P$, V_P some constant, describes the location of the entangling surface on the pencil or, equivalently, the location $x = \lambda$ on the $\tau = 0$ slice of the Euclidean path integral. Using null translation invariance of the free theory, we choose to fix the entangling surface to be at $x = 0$ by moving the operator insertions simultaneously. The reduced density matrix $\rho(\lambda)$ then corresponds to tracing out the region $x < 0$. Deformations of the entangling region now amount to changing the location of the operator insertions, allowing us to evaluate shape derivatives with relative ease. Furthermore, the density matrix $\sigma_P^{(0)}$ is now independent of λ. In fact,

$$\sigma_P = e^{-2\pi K_P}, \tag{20}$$

where K_P is the modular Hamiltonian on the pencil [46,47].

We can now write an arbitrary state as

$$\begin{aligned}\rho(\lambda) = {}& \sigma_P \otimes \rho_A^{(0)} \\ &+ \mathcal{A}^{1/2}\sum_{ij}\left(\sigma_P\int dr d\theta f_{ij}(r,\theta)\psi(re^{i\theta}-\lambda)\right)\otimes|i\rangle\langle j| \\ &+ \mathcal{A}^{1/2}\sum_{ij}\left(\sigma_P\int dr d\theta f_{ji}(r,\theta)\psi(re^{i\theta}-\lambda)\right)\otimes|j\rangle\langle i| \\ &+ \mathcal{O}(\mathcal{A}),\end{aligned}$$

which we write as

$$\begin{aligned}\rho(\lambda) = {}& \sigma_P \otimes \rho_A^{(0)} \\ &+ \mathcal{A}^{1/2}\sum_{\mu\nu}\left(\sigma_P\int dr d\theta f_{\mu\nu}(r,\theta)\psi(re^{i\theta}-\lambda)\right)\otimes|\mu\rangle\langle\nu| \\ &+ \mathcal{O}(\mathcal{A}).\end{aligned}$$

Defining

$$\rho^{(1)}(\lambda) = \sum_{\mu\nu}\left(\sigma_P\int dr d\theta f_{\mu\nu}(r,\theta)\psi(re^{i\theta}-\lambda)\right)\otimes|\mu\rangle\langle\nu|,$$

we succinctly write

$$\rho(\lambda) = \sigma_P \otimes \rho_A^{(0)} + \mathcal{A}^{1/2}\rho^{(1)}(\lambda) + \mathcal{O}(\mathcal{A}), \tag{21}$$

with the factor of $\mathcal{A}^{1/2}$ explicit. To ensure that the state is Hermitian, we need to impose

$$f_{\mu\nu}(r,\theta) = -i f_{\nu\mu}^*(r, 2\pi - \theta), \tag{22}$$

where we emphasize that μ, ν indices can be either i or j.

It is necessary to require that $f_{\mu\nu}(r,\theta)$ vanishes at $\theta = 0, \pi$ so that the state is normalizable. To ensure that SRD is finite for the states we study, it is further necessary to restrict the support of the functions $f_{\mu\nu}$ to a wedge of size $2\pi/n$ centered at $\theta = \pi$, as discussed in Sec. II C. We will refer to this requirement as the wedge condition.

B. Statement of Rényi QNEC

Recall that, for density matrices, SRD of ρ with respect to σ is given by

$$S_n^R(\rho|\sigma) = \frac{1}{n-1}\log \hat{Z}_n(\rho|\sigma), \tag{23}$$

where

$$\hat{Z}_n(\rho|\sigma) = \mathrm{Tr}[(\sigma^{\frac{1-n}{2n}}\rho\sigma^{\frac{1-n}{2n}})^n]. \tag{24}$$

For the Rényi QNEC, ρ is the density matrix $\rho(\lambda)$ introduced above, defined for the subregion $v > V(\vec{y})$, along the null surface N, and σ is the vacuum density matrix. The deformation relevant for Rényi QNEC is then simply translation along the affine parameter λ on the pencil P near $\lambda = 0$, and Rényi QNEC is the statement that, for $1 < n \in \mathbb{R}$,

$$\lim_{\mathcal{A}\to 0}\frac{1}{\mathcal{A}}\frac{d^2}{d\lambda^2}S_n(\rho(\lambda)|\sigma)\bigg|_{\lambda=0} \geq 0. \tag{25}$$

If we expand the state $\rho(\lambda)$ as

$$\rho(\lambda) = \sigma_P\otimes\rho_A^{(0)} + \mathcal{A}^{1/2}\rho^{(1)}(\lambda) + \mathcal{A}\rho^{(2)}(\lambda) + \mathcal{O}(\mathcal{A}^{3/2}), \tag{26}$$

$\hat{Z}_n$ can be expanded as

$$\hat{Z}_n(\rho(\lambda)|\sigma) = \hat{Z}_n^{(0)} + \mathcal{A}^{1/2}\hat{Z}_n^{(1)}(\lambda) + \mathcal{A}(\hat{Z}_n^{(2)}(\lambda) + \hat{Z}_n^{(1,1)}(\lambda)) + \mathcal{O}(\mathcal{A}^{3/2}),$$

where $\hat{Z}_n^{(i)}$ contains contributions from the ith term of the $\rho(\lambda)$ expansion and $\hat{Z}_n^{(1,1)}$ contains two powers of $\rho^{(1)}$. Using this in Eq. (25), one can recast the Rényi QNEC statement as [24]

$$\frac{d^2}{d\lambda^2}Z_n^{(1,1)}(\lambda)\bigg|_{\lambda=0} \geq 0, \tag{27}$$

where

$$Z_n^{(1,1)}(\lambda) = \frac{1}{n-1}\hat{Z}_n^{(1,1)}(\lambda). \tag{28}$$

C. Reformulation as a perturbative calculation

As it stands, we need to perform a calculation of SRD between an arbitrary state and the vacuum. As demonstrated in Ref. [24], however, we can improve the situation and restate our problem as a calculation of SRD between two perturbatively close states.

Let us first define

$$\tilde{\rho}^{(0)} = \sigma_P\otimes\tilde{\rho}_A^{(0)}, \qquad \tilde{\rho}_A^{(0)} = \left(\sigma_a^{\frac{1-n}{2n}}\rho_A^{(0)}\sigma_a^{\frac{1-n}{2n}}\right)^n, \tag{29}$$

and choose a basis $\{|i\rangle, |j\rangle\}$ such that $\tilde{\rho}_A^{(0)}$ acts diagonally:

$$\tilde{\rho}_A^{(0)}|\mu\rangle = e^{-2\pi K_\mu}|\mu\rangle. \tag{30}$$

We also define

$$E_{\mu\nu} \equiv e^{\theta(K_\mu - K_\nu)}|\mu\rangle\langle\nu|. \tag{31}$$

Then, inserting a complete set of states, we can write

$$\sigma^{\frac{1-n}{2n}}\rho^{(1)}(\lambda)\sigma^{\frac{1-n}{2n}} = \sum_{\mu_1\mu\nu\mu_2}\sigma_P^{\frac{1-n}{2n}+1}\left(\int dr d\theta f_{\mu\nu}(r,\theta)\psi(re^{i\theta}-\lambda)\right)\sigma_P^{\frac{1-n}{2n}} \otimes |\mu_1\rangle\langle\mu_1|\sigma_a^{\frac{1-n}{2n}}|\mu\rangle\langle\nu|\sigma_a^{\frac{1-n}{2n}}|\mu_2\rangle\langle\mu_2|.$$

Note that, since σ_a is a Grassmann-even operator, elements such as $\langle i|\sigma_a^m|j\rangle$ will vanish.

Now define

$$\tilde{f}_{\mu\nu}(r,\theta) = \sum_{\mu_1\nu_1}f_{\mu_1\nu_1}(r,\theta)\langle\mu|\sigma_a^{\frac{1-n}{2n}}|\mu_1\rangle\langle\nu_1|\sigma_a^{\frac{1-n}{2n}}|\nu\rangle,$$

which satisfies the same reality conditions as $f_{\mu\nu}$. This allows us to write

$$\sigma^{\frac{1-n}{2n}}\rho^{(1)}(\lambda)\sigma^{\frac{1-n}{2n}} = \sum_{\mu\nu}\sigma_P^{\frac{1-n}{2n}}\left(\sigma_P\int dr d\theta\tilde{f}_{\mu\nu}(r,\theta)\psi(re^{i\theta}-\lambda)\right)\sigma_P^{\frac{1-n}{2n}}\otimes|\mu\rangle\langle\nu|.$$

Note that any function that satisfies the same reality and support conditions as $f_{\mu\nu}(r,\theta)$ is equally valid as a support function for the field. We use this freedom to write [24]

$$\sigma^{\frac{1-n}{2n}}\rho^{(1)}(\lambda)\sigma^{\frac{1-n}{2n}} = \sum_{\mu\nu}\sigma_P^{\frac{1-n}{2n}}\left(\sigma_P\int dr d\theta\tilde{f}_{\mu\nu}(r,\theta)\psi(re^{i\theta}-\lambda)\right)\sigma_P^{\frac{1-n}{2n}} \otimes ((\tilde{\rho}_A^{(0)})^{\frac{1-n}{2n}}\tilde{\rho}_A^{(0)}E_{\mu\nu}(\theta)(\tilde{\rho}_A^{(0)})^{\frac{1-n}{2n}}).$$

Defining

$$\tilde{\rho}^{(1)}(\lambda) = \sum_{\mu\nu}\tilde{\rho}^{(0)}\int dr d\theta\tilde{f}_{\mu\nu}(r,\theta)(\psi(re^{i\theta}-\lambda)\otimes E_{\mu\nu}(\theta)), \tag{32}$$

we have the result that

$$Z_n(\lambda) = \frac{1}{n-1}\mathrm{Tr}[(\sigma^{\frac{1-n}{2n}}\rho\sigma^{\frac{1-n}{2n}})^n]$$
$$= \frac{1}{n-1}\mathrm{Tr}[((\tilde{\rho}^{(0)})^{1/n} + \mathcal{A}^{1/2}(\tilde{\rho}^{(0)})^{\frac{1-n}{2n}}\tilde{\rho}^{(1)}(\lambda)(\tilde{\rho}^{(0)})^{\frac{1-n}{2n}})^n].$$

This is equivalent to

$$Z_n(\lambda) = Z_n(\tilde{\rho}^{(0)} + \mathcal{A}^{1/2}\tilde{\rho}^{(1)}(\lambda)|\tilde{\rho}^{(0)}), \tag{33}$$

which is just the SRD between two perturbatively close states. This quantity has been studied previously in Refs. [24,48] and is much simpler than the original statement (27) involving a state arbitrarily far from the vacuum.

IV. PROOF

We prove Rényi QNEC in two ways. We first deal with the case where the Rényi index $1 < n \in \mathbb{Z}^+$. This proof is much shorter and neater, following just from reflection positivity, the Euclidean analog of unitarity. Then we proceed to calculate the second variation of SRD for arbitrary n and calculate correlation functions of the pencil and the auxiliary systems to show that the relevant shape variation is, in fact, positive.

A. For integer $n > 1$

For $n \in \mathbb{Z}^+$, it is easy to see from Eq. (33) that

$$Z_n^{(1,1)}(\lambda) = \frac{n}{2(n-1)} \times \sum_{k=1}^{n-1}\mathrm{Tr}((\tilde{\rho}^{(0)})^{-1+\frac{k}{n}}\tilde{\rho}^{(1)}(\lambda)(\tilde{\rho}^{(0)})^{-\frac{k}{n}}\tilde{\rho}^{(1)}(\lambda)). \tag{34}$$

Denoting λ derivatives by dots, we are interested in the second derivative

$$\ddot{Z}_n^{(1,1)}(\lambda) = \frac{n}{2(n-1)}\sum_{k=1}^{n-1}[\mathrm{Tr}((\tilde{\rho}^{(0)})^{-1+\frac{k}{n}}\ddot{\tilde{\rho}}^{(1)}(\lambda)(\tilde{\rho}^{(0)})^{-\frac{k}{n}}\tilde{\rho}^{(1)}(\lambda))$$
$$+ 2\mathrm{Tr}((\tilde{\rho}^{(0)})^{-1+\frac{k}{n}}\dot{\tilde{\rho}}^{(1)}(\lambda)(\tilde{\rho}^{(0)})^{-\frac{k}{n}}\dot{\tilde{\rho}}^{(1)}(\lambda))$$
$$+ \mathrm{Tr}((\tilde{\rho}^{(0)})^{-1+\frac{k}{n}}\tilde{\rho}^{(1)}(\lambda)(\tilde{\rho}^{(0)})^{-\frac{k}{n}}\ddot{\tilde{\rho}}^{(1)}(\lambda))].$$

Using the explicit form of $\tilde{\rho}^{(1)}$ given in Eq. (32) and setting $\lambda = 0$, this becomes

$$\ddot{Z}_n^{(1,1)} = \frac{n}{2(n-1)}\sum_{k=1}^{n-1}\int d\alpha \times [\mathrm{Tr}(\tilde{\rho}^{(0)}(\tilde{\rho}^{(0)})^{-\frac{k}{n}}\mathcal{O}_{\mu_2\nu_2}(r_2,\theta_2)(\tilde{\rho}^{(0)})^{\frac{k}{n}}\ddot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1))$$
$$+ 2\mathrm{Tr}(\tilde{\rho}^{(0)}(\tilde{\rho}^{(0)})^{-\frac{k}{n}}\dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)(\tilde{\rho}^{(0)})^{\frac{k}{n}}\dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1))$$
$$+ \mathrm{Tr}(\tilde{\rho}^{(0)}(\tilde{\rho}^{(0)})^{-\frac{k}{n}}\ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)(\tilde{\rho}^{(0)})^{\frac{k}{n}}\mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1))],$$

where we have defined

$$\int d\alpha = \sum_{\mu_1,\nu_1\mu_2\nu_2}\prod_{i=1}^{2}\int dr_i\int_{\pi-\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} d\theta_i\tilde{f}_{\mu_i\nu_i}(r_i,\theta_i) \tag{35}$$

and

$$\mathcal{O}_{\mu\nu} = \psi(re^{i\theta})\otimes E_{\mu\nu}(\theta),$$
$$\dot{\mathcal{O}}_{\mu\nu} = \partial\psi(re^{i\theta})\otimes E_{\mu\nu}(\theta),$$
$$\ddot{\mathcal{O}}_{\mu\nu} = \partial^2\psi(re^{i\theta})\otimes E_{\mu\nu}(\theta). \tag{36}$$

Since the conformal weight of $\partial^m\psi$ is $(\frac{1}{2}+m, 0)$, it transforms under conjugation by σ_P as

$$\sigma_P^{-\frac{k}{2n}}\partial^m\psi(re^{i\theta})\sigma_P^{\frac{k}{2n}} = e^{i\pi\frac{k}{n}(m+\frac{1}{2})}\partial^m\psi(re^{i\theta+i\pi\frac{k}{n}}). \tag{37}$$

The other factor $E_{\mu\nu}(\theta)$ transforms under conjugation by $\tilde{\rho}_A^{(0)}$ as

$$(\tilde{\rho}_A^{(0)})^{-\frac{k}{2n}}E_{\mu\nu}(\theta)(\tilde{\rho}_A^{(0)})^{\frac{k}{2n}} = E_{\mu\nu}(\theta + \pi k/n). \tag{38}$$

Combining these, we can conjugate the operators $\mathcal{O}_{\mu\nu}$ and their derivatives to get

$$\ddot{Z}_n^{(1,1)} = \frac{n}{2(n-1)}\sum_{k=1}^{n-1}\int d\alpha\left[e^{-i\frac{5\pi k}{2n}}\mathrm{Tr}\left(\tilde{\rho}^{(0)}\mathcal{O}_{\mu_2\nu_2}\left(r_2,\theta_2+\frac{\pi k}{n}\right)\ddot{\mathcal{O}}_{\mu_1\nu_1}\left(r_1,\theta_1-\frac{\pi k}{n}\right)\right)\right.$$
$$+ 2\mathrm{Tr}\left(\tilde{\rho}^{(0)}\dot{\mathcal{O}}_{\mu_2\nu_2}\left(r_2,\theta_2+\frac{\pi k}{n}\right)\dot{\mathcal{O}}_{\mu_1\nu_1}\left(r_1,\theta_1-\frac{\pi k}{n}\right)\right)$$
$$\left.+ e^{i\frac{5\pi k}{2n}}\mathrm{Tr}\left(\tilde{\rho}^{(0)}\ddot{\mathcal{O}}_{\mu_2\nu_2}\left(r_2,\theta_2+\frac{\pi k}{n}\right)\mathcal{O}_{\mu_1\nu_1}\left(r_1,\theta_1-\frac{\pi k}{n}\right)\right)\right].$$

Because of the wedge condition, the operator insertions are angle ordered and the terms above can be written as correlation functions:

$$\ddot{Z}_n^{(1,1)} = \frac{n}{2(n-1)} \sum_{k=1}^{n-1} \int d\alpha \times \Big[e^{-i\frac{5\pi k}{2n}} \Big\langle \mathcal{O}_{\mu_2\nu_2}\Big(r_2, \theta_2 + \frac{\pi k}{n}\Big) \ddot{\mathcal{O}}_{\mu_1\nu_1}\Big(r_1, \theta_1 - \frac{\pi k}{n}\Big) \Big\rangle + 2\Big\langle \dot{\mathcal{O}}_{\mu_2\nu_2}\Big(r_2, \theta_2 + \frac{\pi k}{n}\Big) \dot{\mathcal{O}}_{\mu_1\nu_1}\Big(r_1, \theta_1 - \frac{\pi k}{n}\Big) \Big\rangle + e^{i\frac{5\pi k}{2n}} \Big\langle \ddot{\mathcal{O}}_{\mu_2\nu_2}\Big(r_2, \theta_2 + \frac{\pi k}{n}\Big) \mathcal{O}_{\mu_1\nu_1}\Big(r_1, \theta_1 - \frac{\pi k}{n}\Big) \Big\rangle \Big].$$

Now we note that, since

$$\langle \partial_z^2 \psi(z)\psi(w)\rangle_p = -\langle \partial_z\psi(z)\partial_w\psi(w)\rangle_p = \langle \psi(z)\partial_w^2\psi(w)\rangle_p,$$

the correlators appearing in $\ddot{Z}_n^{(1,1)}$ are, in fact, all proportional to each other. This allows us to write

$$\ddot{Z}_n^{(1,1)} = \frac{2n}{n-1} \sum_{k=1}^{n-1} \int d\alpha \sin^2\Big(\frac{5\pi k}{4n}\Big) \times \langle \dot{\mathcal{O}}_{\mu_2\nu_2}(r_2, \theta_2 + \pi k/n) \dot{\mathcal{O}}_{\mu_1\nu_1}(r_1, \theta_1 - \pi k/n)\rangle.$$

Defining the (Grassmann-even) operators

$$\Psi_k = \sum_{\mu\nu} \int dr \int_{\pi-\pi/n}^{\pi+\pi/n} d\theta \tilde{f}_{\mu\nu}(r,\theta) \dot{\mathcal{O}}_{\mu\nu}(r, \theta - \pi k/n),$$

$$\bar{\Psi}_k = \sum_{\mu\nu} \int dr \int_{\pi-\pi/n}^{\pi+\pi/n} d\theta \tilde{f}_{\mu\nu}(r,\theta) \dot{\mathcal{O}}_{\mu\nu}(r, \theta + \pi k/n),$$

we see that

$$\ddot{Z}_n^{(1,1)} = \frac{2n}{n-1} \sum_{k=1}^{n-1} \sin^2\Big(\frac{5\pi k}{4n}\Big) \langle \bar{\Psi}_k \Psi_k\rangle. \qquad (39)$$

Under the change of variables $\theta \to 2\pi - \theta$, we have $f_{\mu\nu}(r, 2\pi - \theta) = -i f^*_{\nu\mu}(r,\theta)$ and

$$\dot{\mathcal{O}}_{\mu\nu}(r, 2\pi - \theta) = i(\tilde{\rho}^{(0)})^{-1} \dot{\mathcal{O}}^\dagger_{\nu\mu}(r,\theta) \tilde{\rho}^{(0)}. \qquad (40)$$

Using both of these, we get

$$\bar{\Psi}_k = (\tilde{\rho}^{(0)})^{-1} \Psi_k^\dagger \tilde{\rho}^{(0)}. \qquad (41)$$

The correlators in $\ddot{Z}_n^{(1,1)}$ now become

$$\langle \bar{\Psi}_k \Psi_k\rangle = \langle (\tilde{\rho}^{(0)})^{-1} \Psi_k^\dagger \tilde{\rho}^{(0)} \Psi_k\rangle = \langle \Psi_k \Psi_k^\dagger\rangle > 0, \qquad (42)$$

where the last inequality is just the statement of reflection positivity.[8] Thus, all terms in Eq. (39) are individually positive, and we have proved Rényi QNEC for integer $n > 1$.

B. Second variation of SRD for arbitrary *n*

Let $|\kappa\rangle$ denote a basis in which $\tilde{\rho}^{(0)}$ is diagonal, i.e.,

$$\tilde{\rho}^{(0)}|\kappa\rangle = e^{-2\pi\kappa}|\kappa\rangle, \qquad (43)$$

where κ can be negative since $\tilde{\rho}^{(0)}$ is not a normalized density matrix. Recall from Eq. (34) that

$$Z_n^{(1,1)}(\lambda) = \frac{n}{2(n-1)} \times \sum_{k=1}^{n-1} \mathrm{Tr}\big((\tilde{\rho}^{(0)})^{-1+\frac{k}{n}} \tilde{\rho}^{(1)}(\lambda) (\tilde{\rho}^{(0)})^{-\frac{k}{n}} \tilde{\rho}^{(1)}(\lambda)\big).$$

One can show that, in this basis, we have the following result for arbitrary n [24]:

$$Z_n^{(1,1)}(\lambda) = \frac{1}{2} \int d\kappa \int d\kappa' e^{2\pi\kappa'} F_n(\kappa - \kappa') |\langle \kappa|\tilde{\rho}^{(1)}(\lambda)|\kappa'\rangle|^2,$$

where

$$F_n(x) = -\frac{n}{n-1} \frac{e^{2\pi(\frac{n-1}{n})x} - 1}{e^{-2\pi x/n} - 1}. \qquad (44)$$

Taking two derivatives with respect to λ gives us

$$\ddot{Z}_n^{(1,1)}(\lambda) = \int d\kappa d\kappa' \, e^{2\pi\kappa'} F_n(\kappa - \kappa') \times \big(\langle \kappa|\tilde{\rho}^{(1)}(\lambda)|\kappa'\rangle \langle \kappa'|\ddot{\tilde{\rho}}^{(1)}(\lambda)|\kappa\rangle + \langle \kappa|\dot{\tilde{\rho}}^{(1)}(\lambda)|\kappa'\rangle \langle \kappa'|\dot{\tilde{\rho}}^{(1)}(\lambda)|\kappa\rangle\big), \qquad (45)$$

where dots again denote λ derivatives. Using the definition of $\tilde{\rho}^{(1)}$ in Eq. (32) and setting $\lambda = 0$,

$$\ddot{Z}_n^{(1,1)}|_{\lambda=0} = \int d\alpha \int d\kappa d\kappa' e^{-2\pi\kappa} F_n(\kappa - \kappa') \times \big(\langle \kappa|\mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1)|\kappa'\rangle \langle \kappa'|\ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)|\kappa\rangle + \langle \kappa|\dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1)|\kappa'\rangle \langle \kappa'|\dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)|\kappa\rangle\big),$$

where $\int d\alpha$ is defined in Eq. (35) and $\mathcal{O}_{\mu\nu}$ in Eq. (36).

[8] See, e.g., Ref. [49] for an overview of reflection positivity.

Making the angular ordering explicit, this is

$$\ddot{Z}_n^{(1,1)} = \int_{\theta_1>\theta_2} d\alpha \int d\kappa d\kappa' e^{-2\pi\kappa} F_n(\kappa-\kappa')(\langle\kappa|\mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1)|\kappa'\rangle\langle\kappa'|\ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)|\kappa\rangle + \langle\kappa|\dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1)|\kappa'\rangle\langle\kappa'|\dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)|\kappa\rangle) + \int_{\theta_2>\theta_1} d\alpha \int d\kappa d\kappa' e^{-2\pi\kappa'} F_n(\kappa'-\kappa) \times (\langle\kappa|\mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1)|\kappa'\rangle\langle\kappa'|\ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)|\kappa\rangle + \langle\kappa|\dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1)|\kappa'\rangle\langle\kappa'|\dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)|\kappa\rangle),$$

where we have used the fact that $F_n(x) = e^{2\pi x}F_n(-x)$. Decomposing $F_n(x)$ into Fourier modes as

$$F_n(x) = \int_{-\infty}^{\infty} ds e^{isx}\mathcal{F}_n(s), \tag{46}$$

one obtains

$$\ddot{Z}_n^{(1,1)} = \int_{\theta_1>\theta_2} d\alpha \int_{-\infty}^{\infty} ds\mathcal{F}_n(s)\{\mathrm{Tr}[(\tilde{\rho}^{(0)})^{1-\frac{is}{2\pi}}\mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1)(\tilde{\rho}^{(0)})^{\frac{is}{2\pi}}\ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)] + \mathrm{Tr}[(\tilde{\rho}^{(0)})^{1-\frac{is}{2\pi}}\dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1)(\tilde{\rho}^{(0)})^{\frac{is}{2\pi}}\dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)]\} + \int_{\theta_2>\theta_1} d\alpha \int_{-\infty}^{\infty} ds\mathcal{F}_n(-s)\{\mathrm{Tr}[(\tilde{\rho}^{(0)})^{1+\frac{is}{2\pi}}\ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)(\tilde{\rho}^{(0)})^{-\frac{is}{2\pi}}\mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1)] + \mathrm{Tr}[(\tilde{\rho}^{(0)})^{1+\frac{is}{2\pi}}\dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)(\tilde{\rho}^{(0)})^{-\frac{is}{2\pi}}\dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1)]\}.$$

Since correlation functions are defined to be implicitly ordered in angular time, we can now write

$$\ddot{Z}_n^{(1,1)} = \int d\alpha \int_{-\infty}^{\infty} ds\mathcal{F}_n(\mathrm{sgn}(\theta_{12})s)\mathcal{G}(s), \tag{47}$$

where we have defined $\theta_{ij} = \theta_i - \theta_j$ and

$$\mathcal{G}(s) = \langle(\tilde{\rho}^{(0)})^{-\frac{is}{2\pi}}\mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1)(\tilde{\rho}^{(0)})^{\frac{is}{2\pi}}\ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)\rangle + \langle(\tilde{\rho}^{(0)})^{-\frac{is}{2\pi}}\dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1)(\tilde{\rho}^{(0)})^{\frac{is}{2\pi}}\dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)\rangle.$$

Using the Fourier transform of $\mathcal{G}(s)$, defined by

$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds e^{-is\omega}\mathcal{G}(s), \tag{48}$$

we obtain

$$\ddot{Z}_n^{(1,1)} = \int d\alpha \int_{-\infty}^{\infty} d\omega F_n(\mathrm{sgn}(\theta_{12})\omega)G(\omega). \tag{49}$$

We write this instead as

$$\ddot{Z}_n^{(1,1)} = \int d\alpha \int_{-\infty}^{\infty} d\omega \tilde{F}_n(\omega)e^{\mathrm{sgn}(\theta_{12})\pi\omega}G(\omega), \tag{50}$$

with

$$\tilde{F}_n(\omega) = e^{-\pi\omega}F_n(\omega) = \frac{n}{n-1}\frac{\sinh\pi\omega\frac{n-1}{n}}{\sinh\pi\omega/n}. \tag{51}$$

The correlation function $\mathcal{G}(s)$ and its Fourier transform $G(\omega)$ are calculated explicitly in Appendix B. The result we obtain is

$$G(\omega) = -\frac{i}{8}\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1}e^{-\pi(K_{\mu_1}+K_{\mu_2})}e^{-\mathrm{sgn}(\theta_{12})\pi\omega} \times (r_1e^{i\theta_1})^{-\frac{3}{2}}(r_2e^{i\theta_2})^{-\frac{3}{2}}\left(\frac{r_1}{r_2}\right)^{iK_{\mu_1\mu_2}} \times \left[Q(z-i)\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)^{1-i\omega} + Q(z)\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)^{-i\omega}\right],$$

where $z = K_{\mu_1\mu_2} - \omega$ and

$$Q(x) = \frac{4x^2+1}{\cosh\pi x}. \tag{52}$$

We finally have that

$$\ddot{Z}_n^{1,1} = -\frac{i}{8}\int d\tilde{\alpha}e^{-\pi(K_\mu+K_\nu)}(r_1e^{i\theta_1})^{-\frac{3}{2}}(r_2e^{i\theta_2})^{-\frac{3}{2}} \times \left(\frac{r_1}{r_2}\right)^{iK_{\mu\nu}}\int_{-\infty}^{\infty} d\omega\tilde{F}_n(\omega)\left[Q(z)\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)^{-i\omega} + Q(z-i)\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)^{1-i\omega}\right],$$

where

$$\int d\tilde{\alpha} = \sum_{\mu\nu} \int dr_1 dr_2 \int d\theta_1 d\theta_2 \tilde{f}_{\mu\nu}(r_1,\theta_1)\tilde{f}_{\nu\mu}(r_2,\theta_2).$$

C. Proving Rényi QNEC for $n > 1$

Note that $Q(K_{\mu\nu} - \omega)$ has no poles in the strip $-1/2 \leq \mathrm{Im}(\omega) \leq 1/2$ and that the poles of $\tilde{F}_n(\omega)$ are at $\omega = inp, p \in \mathbb{Z}$. Then, for $n > 1$, we can make a contour deformation $\omega \to \omega \pm i/2$ without crossing any poles. This leads to the equalities

$$\int_{-\infty}^{\infty} d\omega \tilde{F}_n(\omega) Q(z) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega} = \int_{-\infty}^{\infty} d\omega \tilde{F}_n(\omega - i/2) Q(z + i/2) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega - 1/2},$$

$$\int_{-\infty}^{\infty} d\omega \tilde{F}_n(\omega) Q(z - i) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{1-i\omega} = \int_{-\infty}^{\infty} d\omega \tilde{F}_n(\omega - i/2) Q(z - i/2) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega + 1/2}.$$

We note that this deformation makes use of the fact that the source functions $\tilde{f}_{\mu\nu}$ are nonvanishing only inside $|\theta - \pi| < \pi/n$, in which range the integrand vanishes as $\mathrm{Re}(\omega) \to \pm\infty$. Using these relations, we can write

$$\ddot{Z}_n^{1,1} = -\frac{i}{8} \int d\tilde{\alpha} e^{-\pi(K_\mu + K_\nu)} (r_1 e^{i\theta_1})^{-\frac{3}{2}} (r_2 e^{i\theta_2})^{-\frac{3}{2}} \left(\frac{r_1}{r_2}\right)^{iK_{\mu\nu}} \times \int_{-\infty}^{\infty} d\omega \tilde{F}_n(\omega - i/2) \left[Q(z + i/2) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega - 1/2} + Q(z - i/2) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega + 1/2} \right].$$

Under $\mu \leftrightarrow \nu$, $(r_1, \theta_1) \leftrightarrow (r_2, \theta_2)$, $\omega \to -\omega$, we get

$$\ddot{Z}_n^{1,1} = -\frac{i}{8} \int d\tilde{\alpha} e^{-\pi(K_\mu + K_\nu)} (r_1 e^{i\theta_1})^{-\frac{3}{2}} (r_2 e^{i\theta_2})^{-\frac{3}{2}} \left(\frac{r_1}{r_2}\right)^{iK_{\mu\nu}} \times \int_{-\infty}^{\infty} d\omega \tilde{F}_n(\omega + i/2) \left[Q(z - i/2) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega + 1/2} + Q(z + i/2) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega - 1/2} \right].$$

This gives us

$$\ddot{Z}_n^{1,1} = -\frac{i}{16} \int d\tilde{\alpha} e^{-\pi(K_\mu + K_\nu)} (r_1 e^{i\theta_1})^{-\frac{3}{2}} (r_2 e^{i\theta_2})^{-\frac{3}{2}} \left(\frac{r_1}{r_2}\right)^{iK_{\mu\nu}} \int_{-\infty}^{\infty} d\omega [\tilde{F}_n(\omega + i/2) + \tilde{F}_n(\omega - i/2)] \times \left[Q\left(z + \frac{i}{2}\right) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega - \frac{1}{2}} + Q\left(z - \frac{i}{2}\right) \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^{-i\omega + \frac{1}{2}} \right].$$

We want to show that this expression is positive definite. It can be written as

$$-\frac{i}{16} \sum_{\mu\nu} e^{-\pi(K_\mu + K_\nu)} \int_{-\infty}^{\infty} d\omega \left[\tilde{F}_n\left(\omega + \frac{i}{2}\right) + \tilde{F}_n\left(\omega - \frac{i}{2}\right) \right] \times \left[Q\left(K_{\mu\nu} - \omega + \frac{i}{2}\right) \int dr_1 dr_2 \int d\theta_1 d\theta_2 \tilde{f}_{\mu\nu}(r_1, \theta_1) r_1^{iK_{\mu\nu} - i\omega - 2} e^{-(2i - \omega)\theta_1} \tilde{f}_{\nu\mu}(r_2, \theta_2) r_2^{-iK_{\mu\nu} + i\omega - 1} e^{-(i + \omega)\theta_2} + Q\left(K_{\mu\nu} - \omega - \frac{i}{2}\right) \int dr_1 dr_2 \int d\theta_1 d\theta_2 \tilde{f}_{\mu\nu}(r_1, \theta_1) r_1^{iK_{\mu\nu} - i\omega - 1} e^{-(i - \omega)\theta_1} \tilde{f}_{\nu\mu}(r_2, \theta_2) r_2^{-iK_{\mu\nu} + i\omega - 2} e^{-(2i + \omega)\theta_2} \right].$$

One can show that, for $n \geq 1$,

$$\tilde{F}_n(\omega + i/2) + \tilde{F}_n(\omega - i/2) = -\frac{2n}{n-1} \frac{\cosh \pi\omega \sin \frac{\pi}{n}}{\cos \frac{\pi}{n} - \cosh \frac{2\pi\omega}{n}} \leq 0.$$

We now make the substitution $\theta_2 \to 2\pi - \theta_2$ and use $\tilde{f}_{\mu\nu}(r, \theta) = -i\tilde{f}^*_{\nu\mu}(r, 2\pi - \theta)$. The term within the square brackets above then becomes

$$-ie^{-2\pi\omega} \left[Q(K_{\mu\nu} - \omega + i/2) \int dr_1 dr_2 \int d\theta_1 d\theta_2 \tilde{f}_{\mu\nu}(r_1, \theta_1) r_1^{iK_{\mu\nu} - i\omega - 2} e^{-(2i - \omega)\theta_1} \tilde{f}^*_{\mu\nu}(r_2, \theta_2) r_2^{-iK_{\mu\nu} + i\omega - 1} e^{(i + \omega)\theta_2} + Q(K_{\mu\nu} - \omega - i/2) \int dr_1 dr_2 \int d\theta_1 d\theta_2 \tilde{f}_{\mu\nu}(r_1, \theta_1) r_1^{iK_{\mu\nu} - i\omega - 1} e^{-(i - \omega)\theta_1} \tilde{f}^*_{\mu\nu}(r_2, \theta_2) r_2^{-iK_{\mu\nu} + i\omega - 2} e^{(2i + \omega)\theta_2} \right].$$

If we now again perform the substitutions $\omega \to \omega \pm i/2$ in the first and second terms, respectively, we get

$$\ddot{Z}_n^{1,1} = \frac{n}{4(n-1)} \sum_{\mu\nu} e^{-\pi(K_\mu + K_\nu)} \int_{-\infty}^{\infty} d\omega \times e^{-2\pi\omega} \sin^2\left(\frac{\pi}{n}\right) \frac{\sinh \pi\omega \coth \frac{\pi\omega}{n}}{\cosh \frac{2\pi\omega}{n} - \cos \frac{2\pi}{n}} \times Q(K_{\mu\nu} - \omega) \int dr_1 d\theta_1 \tilde{f}_{\mu\nu}(r_1 \theta_1) r_1^{iz-3/2} e^{(-\frac{3i}{2}+\omega)\theta_1} \times \int dr_2 d\theta_2 \tilde{f}^*_{\mu\nu}(r_2, \theta_2) r_2^{-iz-3/2} e^{(\frac{3i}{2}+\omega)\theta_2}.$$

The last two integrals are complex conjugates of each other, and the other terms are all positive for $n \geq 1$. Thus,

$$\ddot{Z}_n^{1,1} \geq 0, \tag{53}$$

proving the Rényi QNEC for free fermions for arbitrary $n \geq 1$. We note that the limit $n \to 1^+$ correctly reproduces the answer in Ref. [14].

V. DISCUSSION

Sandwiched Rényi divergence is a new measure of distance between states in Hilbert spaces, which has the desirable properties of being positive and satisfying the data processing inequality. Previous studies of SRD in the context of QFT and holography include Refs. [48,50–53]. Motivated by the purely information theoretic formulation of QNEC, a Rényi QNEC was conjectured [23] and was soon proven for free bosons [24]. In this work, we have generalized the arguments of Ref. [24] to the case of free fermions and showed that the Rényi QNEC indeed holds for $n \geq 1$.

Other related distance measures between quantum states are the Petz divergence [23,54,55], the α-z-Rényi relative entropy [56], optimized quantum f divergences [57,58], and the refined Rényi divergence defined in Ref. [59]. A visual summary of the interrelations between various entropy measures is found in Ref. [60]. Another measure is the recently defined multistate quantum f divergence [61]. A Rényi mutual information in QFT was defined very recently in Ref. [62]. It is natural to expect that some of these measures also satisfy a QNEC-like constraint on second null shape derivatives. This was already pointed out for some divergences in Ref. [24]. It will be very interesting to see if the techniques used in this work can be used to check such conjectural inequalities, and we hope to report on this front in a future work. An important technical direction to explore is proving Rényi QNEC beyond the free regime using methods of algebraic QFT as in Ref. [17].

Inverting the question, it is as important to find and understand examples where such general inequalities fail to hold. The first example to demonstrate the violation of QNEC to the author's best knowledge is Ref. [63], which attributed the violation to IR effects. In Ref. [64], the authors studied the evolution of QNEC after a quench in $\mathrm{AdS}_3/\mathrm{CFT}_2$. Reference [65] found very interestingly that QNEC can, in fact, be violated in these situations and that nonviolation of QNEC places bounds on the thermodynamics of the system postquench. This was further studied in Ref. [66] in the context of inhomogeneous quenches. Performing similar calculations for Rényi QNEC in tractable setups should also lead to nontrivial constraints on the dynamics postquench.

It seems important to the author to understand the holographic dual of the statement of Rényi QNEC, since it might lead to a generalization of the quantum focusing conjecture. This requires first elucidating the holographic dual of the sandwiched Rényi divergence. The holographic dual of the Rényi entropy was proposed in Ref. [67] and further studied in Refs. [68–72], among others. In particular, Ref. [59] studied the holographic dual to the refined Rényi relative entropy. We hope that the techniques of these works might be further developed and progress made toward a holographic statement and proof for Rényi QNEC.

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APPENDIX A: FREE FERMION FIELD

A Majorana fermion in two-dimensional Minkowski space is described by the action

$$S = k \int d^2x (-i) \bar{\chi} \gamma^\mu \partial_\mu \chi, \tag{A1}$$

where $\chi^T = (\chi_1\ \chi_2)$, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\bar{\chi} = \chi^\dagger \gamma^0$, and k is some normalization factor. We choose

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A2}$$

so that the Majorana condition on χ implies that both χ_1 and χ_2 are real. After rotating to Euclidean time, $t \to -i\tau$, writing $S_E = -iS$, and defining $z = x - i\tau, \bar{z} = x + i\tau$, we get

$$S_E = k \int d\tau dx (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}), \tag{A3}$$

where we have defined $\psi = \sqrt{-i}\chi_1, \bar{\psi} = \sqrt{i}\chi_2$. We focus only on the left-moving chiral field ψ throughout the work, which has the property that

$$\psi(r)^\dagger = i\psi(r). \tag{A4}$$

Since ψ is a conformal primary with weight $(h, \bar{h}) = (\frac{1}{2}, 0)$, we also have

$$\psi(re^{i\theta}) = e^{i\theta/2} e^{\theta K_P} \psi(r) e^{-\theta K_P}, \tag{A5}$$

where K_P is the modular Hamiltonian generating θ rotations for the pencil system.

We choose the normalization factor k to be such that the two-point correlation function is given by

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}. \tag{A6}$$

APPENDIX B: CALCULATING CORRELATION FUNCTIONS

We want to evaluate

$$\mathcal{G}(s) = \langle (\tilde{\rho}^{(0)})^{-\frac{is}{2\pi}} \mathcal{O}_{\mu_1\nu_1}(r_1,\theta_1)(\tilde{\rho}^{(0)})^{\frac{is}{2\pi}} \ddot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)\rangle + \langle (\tilde{\rho}^{(0)})^{-\frac{is}{2\pi}} \dot{\mathcal{O}}_{\mu_1\nu_1}(r_1,\theta_1)(\tilde{\rho}^{(0)})^{\frac{is}{2\pi}} \dot{\mathcal{O}}_{\mu_2\nu_2}(r_2,\theta_2)\rangle.$$

Recalling Eq. (36) and that $\tilde{\rho}^{(0)} = \sigma_P \otimes \tilde{\rho}_A^{(0)}$, this correlation function factorizes into separate correlation functions for the pencil and the auxiliary system:

$$\mathcal{G}(s) = \mathcal{G}_P(s) \cdot \mathcal{G}_A(s), \tag{B1}$$

where

$$\mathcal{G}_P(s) = \langle \sigma_P^{-\frac{is}{2\pi}} \psi(r_1 e^{i\theta_1}) \sigma_P^{\frac{is}{2\pi}} \partial^2 \psi(r_2 e^{i\theta_2})\rangle_P + \langle \sigma_P^{-\frac{is}{2\pi}} \partial\psi(r_1 e^{i\theta_1}) \sigma_P^{\frac{is}{2\pi}} \partial \psi(r_2 e^{i\theta_2})\rangle_P$$

and

$$\mathcal{G}_A(s) = \langle (\tilde{\rho}_A^{(0)})^{-\frac{is}{2\pi}} E_{\mu_1\nu_1}(\theta_1)(\tilde{\rho}_A^{(0)})^{\frac{is}{2\pi}} E_{\mu_2\nu_2}(\theta_2)\rangle_A.$$

The auxiliary system correlation function can be calculated straightforwardly. For $\theta_1 > \theta_2$, we get

$$\begin{aligned}\mathcal{G}_A(s) &= e^{-2\pi K_{\mu_1}} e^{is(K_{\mu_1}-K_{\mu_2})} e^{\theta_1(K_{\mu_1}-K_{\nu_1})} e^{\theta_2(K_{\mu_2}-K_{\nu_2})} \\ &\quad \times \mathrm{Tr}_a[|\mu_1\rangle\langle\nu_1| \cdot |\mu_2\rangle\langle\nu_2|] \\ &= e^{-2\pi K_{\mu_1}} e^{(is+\theta_{12})K_{\mu_1\mu_2}} \delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1},\end{aligned}$$

where, as above, $\theta_{ij} = \theta_i - \theta_j$ and also $K_{\mu_i\mu_j} = K_{\mu_i} - K_{\mu_j}$. Similarly, for $\theta_2 > \theta_1$, we have

$$\mathcal{G}_A(s) = -e^{-2\pi K_{\mu_2}} e^{(is+\theta_{12})K_{\mu_1\mu_2}} \delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1},$$

where the minus sign appears due to angle ordering the $E_{\mu\nu}$, which have fermionic statistics. We can combine the above two equations to write, for all θ_1 and θ_2,

$$\mathcal{G}_A(s) = \mathrm{sgn}(\theta_{12}) e^{-\pi(K_{\mu_1}+K_{\mu_2})} e^{-\mathrm{sgn}(\theta_{12})\pi K_{\mu_1\mu_2}} \times e^{(is+\theta_{12})K_{\mu_1\mu_2}} \delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1}. \tag{B2}$$

Let us now look at $\mathcal{G}_P(s)$. Using

$$\sigma_P^{-\alpha} \partial^m \psi(re^{i\theta}) \sigma_P^{\alpha} = e^{i2\pi\alpha(m+\frac{1}{2})} \partial^m \psi(re^{i(\theta+2\pi\alpha)}),$$

and the fermion two-point function (A6), we get

$$\mathcal{G}_P(s) = \frac{2}{(r_1 e^{i\theta_1 - s} - r_2 e^{i\theta_2})^3} (e^{-s/2} - e^{-3s/2}). \tag{B3}$$

Multiplying $\mathcal{G}_A(s)$ and $\mathcal{G}_P(s)$,

$$\mathcal{G}(s) = -\mathrm{sgn}(\theta_{12}) 2 e^{-\pi(K_{\mu_1}+K_{\mu_2})} e^{-\mathrm{sgn}(\theta_{12})\pi K_{\mu_1\mu_2}} \times e^{\theta_{12}K_{\mu_1\mu_2}} \delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1} \frac{(e^{-s/2} - e^{-3s/2}) e^{isK_{\mu_1\mu_2}}}{(r_2 e^{i\theta_2} - r_1 e^{i\theta_1 - s})^3},$$

where we have collected the s-dependent terms.

We now want to take a Fourier transform to calculate $G(\omega)$:

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-is\omega} \mathcal{G}(s).$$

Note that $\mathcal{G}(s) \to 0$ as $\mathrm{Re}(s) \to \pm\infty$. Also note that

$$\mathcal{G}(s + 2\pi i) = -e^{-2\pi K_{\mu_1\mu_2}} \mathcal{G}(s).$$

These allow us to write

$$G(\omega) = \frac{1}{2\pi} \frac{1}{1 + e^{2\pi(\omega - K_{\mu_1\mu_2})}} \oint_C ds e^{-is\omega} \mathcal{G}(s), \tag{B4}$$

where C is the closed contour

$$C: (-\infty, \infty) \cup (\infty, \infty + 2\pi i) \cup (\infty + 2\pi i, -\infty + 2\pi i) \cup (-\infty + 2\pi i, -\infty).$$

This contour integral can now be evaluated using the residue theorem, noting the fact that $\mathcal{G}(s)$ has only one pole inside C, given by

$$s = s_* = \log\frac{r_1}{r_2} + i(\theta_{12} + \pi(1 - \mathrm{sgn}(\theta_{12}))), \qquad \text{(B5)}$$

where the extra terms take care of the branches when $\theta_{12} < 0$. For $z \in \mathbb{C}$, one can calculate that

$$\mathrm{Res}\left[\frac{e^{-3s/2}e^{isz}}{(1 - re^{i\theta - s})^3}, s = s_*\right] = -\frac{1}{8}\mathrm{sgn}(\theta)e^{-\pi z(1-\mathrm{sgn}(\theta))}(re^{i\theta})^{-\frac{3}{2}+iz}(1 + 4z^2),$$

$$\mathrm{Res}\left[\frac{e^{-s/2}e^{isz}}{(1 - re^{i\theta - s})^3}, s = s_*\right] = -\frac{1}{8}\mathrm{sgn}(\theta)e^{-\pi z(1-\mathrm{sgn}(\theta))}(re^{i\theta})^{-\frac{1}{2}+iz}(-3 - 8iz + 4z^2).$$

Putting everything together and writing $z = K_{\mu_1\mu_2} - \omega$,

$$\begin{aligned} G(\omega) &= \frac{ie^{\pi(K_{\mu_1\mu_2} - \omega)}}{2\cosh\pi(K_{\mu_1\mu_2} - \omega)}\mathrm{Res}[e^{-is\omega}\mathcal{G}(s), s = s_*] \\ &= -\frac{i}{8}\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1}e^{-\pi(K_{\mu_1}+K_{\mu_2})}e^{-\mathrm{sgn}(\theta_{12})\pi\omega} \\ &\quad\times (r_1e^{i\theta_1})^{-\frac{3}{2}}(r_2e^{i\theta_2})^{-\frac{3}{2}}\left(\frac{r_1}{r_2}\right)^{iK_{\mu_1\mu_2}} \\ &\quad\times\left[Q(z-i)\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)^{1-i\omega} + Q(z)\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)^{-i\omega}\right], \end{aligned}$$

where we have defined

$$Q(x) = \frac{4x^2 + 1}{\cosh\pi x}. \qquad \text{(B6)}$$

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