Slow rotation black hole perturbation theory

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(Received 8 June 2023; accepted 27 July 2023; published 30 August 2023)

In this paper, we present a detailed analysis of first-order perturbations of the Kerr metric in the slow-rotation limit. We perform the calculation by perturbing the Schwarzschild metric plus up to second-order corrections in the spin in the Regge-Wheeler gauge. The apparent coupling between different angular momentum axial-led and polar-led modes can be removed by suitably combining the perturbation equations and projecting them onto spin-weighted spherical harmonics. In this way, we derive the corrections to the Regge-Wheeler and the Zerilli equations up to second order in the spin. We show that the two potentials remain isospectral as in the nonrotating limit. However, it is easy to demonstrate it only for a precise choice of the tortoise coordinate. The isospectrality with a slowly rotating Teukolsky equation is also verified. We discuss the main implication of this result for the problem of vacuum metric reconstruction, providing the transformation rule between slow-spinning Teukolsky variables and metric perturbations. The existence of this relation leaves us with the conjecture that a resummation of the expansion in the spin is possible, leading to two decoupled differential equations for perturbations of the Kerr metric.

DOI: 10.1103/PhysRevD.108.044079

I. INTRODUCTION

Black hole perturbation theory (BHPT) is the branch of gravitational physics that studies the response of black holes (BHs) to small generic fluctuations of the spacetime. It was initially developed in the 1957 breakthrough work by Regge and Wheeler [1], who for the first time obtained the first-order equations for a perturbation with axial parity on top of a Schwarzschild BH. In 1970, Zerilli then obtained a similar equation for even parity perturbations [2]. A few years later, Teukolsky managed to derive an equation that governs linear perturbations on top of a Kerr BH, by implementing a different formalism based on the perturbations of the curvature [3]. These three milestones are still the basic equations for the analysis of small perturbations of rotating and nonrotating BHs in general relativity (GR). Notable examples of its applications are the study of the quasinormal modes of BHs [4,5] and the waveform generation of black hole binaries with very large mass ratios [6–9].

The Regge-Wheeler and the Zerilli equations both have the form of a Schrödinger equation, but their effective potentials have different analytical expressions. Nevertheless, Chandrasekhar found that one can transform the Regge-Wheeler equation into the Zerilli equation and back [10,11]. The existence of this transformation confirms the *isospectrality* of the two potentials, which means that their spectrum of quasinormal modes is completely equivalent [12]. In addition, Chandrasekhar found that the Regge-Wheeler and the Zerilli equations can be related, with a slightly more complex transformation, to the nonrotating limit of the Teukolsky equation [11], also known as the Bardeen-Press equation [13]. This latter result by Chandrasekhar confirms a somewhat expected property: the spectrum of oscillations of a BH does not depend on the perturbation scheme used to calculate it. A direct perturbation of the spacetime metric must lead to an equivalent result as if one performs a different perturbation scheme, like for the derivation of the Teukolsky equation.

The transformations derived by Chandrasekhar not only prove the isospectrality between the three different equations but also provide the transformation rules to move from one to another. While this result in the nonrotating case is rather immediate, having an exact formula that links the solution of the Teukolsky equation to the small perturbations of the Kerr metric is a much more involved problem commonly known as "metric reconstruction." Such reconstruction appears to be necessary in all those problems of BHPT which involve "perturbations of perturbations": second-order perturbations of Kerr BHs in self-force computations [14] or in the case of modified Teukolsky equations in alternative theories of gravity [15–17].

The main issue in the rotating cases is that an equivalent of the Regge-Wheeler and the Zerilli equations cannot be found, and the perturbations of a rotating BH can only be understood via the Teukolsky equation. The procedure to

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obtain it uses a decomposition of the spacetime for which it is not straightforward to trace it back to the actual perturbations of the metric. Nevertheless, there are techniques that aim precisely to obtain the metric perturbations starting from the Teukolsky variables, and they differ among themselves upon the gauge choice [14]. In the radiation gauge, one possibility is to integrate over the Hertz potentials [18] (see also [19] for an overview of the technique and possible implementations). It is worth noting that recent progress showed an alternative formulation based on the calculations of Chandrasekhar [11] which avoids the use of Hertz potentials [20]. Another alternative is to perform the reconstruction in the Lorentz gauge [21].

In this paper, we analyze the problem of metric reconstruction in the Regge-Wheeler gauge, extending the results of [22] for a vacuum spacetime up to second order in the spin. Indeed, it has been known since the early works of Kojima on the perturbation of slowly rotating neutron stars [23,24] that first-order corrections in the spin maintain the same structure of the perturbation equations as in the nonrotating case. The effect of the spin on BH linear perturbations is just to modify the Regge-Wheeler and Zerilli potential [25–27]. The slow-spin expansion is particularly relevant in those cases where the perturbative approach to the full rotating problem is not possible, like in alternative theories of gravity where fully rotating solutions are not known analytically [28–32] or when exotic matter fields coupled to gravity do not lead to an evident separation of the variables in the perturbation equations [33,34].

The outline of the paper is the following. In Sec. II we present the slow-spinning Kerr metric and the perturbation scheme, followed by a revision of the method of Kojima to perform the separation of the equations. The method is extended up to second order in the spin, as it was done in [32], and we show how to manipulate the equations in order to find a correction in the spin for the Regge-Wheeler and the Zerilli potentials. In Sec. III we discuss how the property of isospectrality remains satisfied at second order in the spin, as well as with respect to the slow-spinning Teukolsky equation as shown in Sec. IV. We discuss the main results of the paper in Sec. V, where we summarize the steps necessary to perform the metric reconstruction, provide a formula that generates the corrections due to the spin to the effective potential of the Regge-Wheeler and the Zerilli equations, and conjecture the existence of two fully rotating versions of those equations. Finally, in Sec. VI, we argue how the results of this paper could address some currently open problems and which future directions can be investigated.

Throughout the paper we use a mostly minus signature $(+,-,-,-)$. This choice was made in order to conform to the notation of [11]. Moreover, we use units such that $c = G_{\rm N} = 1$.

II. SLOWLY ROTATING REGGE-WHEELER AND ZERILLI EQUATIONS

A. Perturbation scheme

With $g^{\rm K}_{ab}$, we denote the Kerr metric in Boyer-Lindquist coordinates $x^a = (t, r, \theta, \varphi)$, whose line element $\mathrm{d}s^2_{\rm K} = g^{\rm K}_{ab}\mathrm{d}x^a\mathrm{d}x^b$ is

$$\mathrm{d}s^2_{\rm K} = \left(1 - \frac{r}{\Sigma}\right)\mathrm{d}t^2 + \frac{2ar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\mathrm{d}\theta^2 - \sin^2\theta\left(r^2 + a^2 + \frac{a^2 r\sin^2\theta}{\Sigma}\right)\mathrm{d}\varphi^2, \quad (1)$$

where $\Delta = r^2 - r + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. Without loss of generality we choose units such as $M = 1/2$, with M the mass parameter of the Kerr metric.

Since we are interested in the slow-rotation BHPT, we need to perform a double perturbation scheme: the usual linear perturbation for which we introduce a formal bookkeeping parameter ε, as well as an expansion in the spin parameter a, which for the purposes of this paper we truncate at second order. In this way, we formally consider the following ansatz for the metric:

$$g_{ab} = g^{\rm SRK}_{ab} + \varepsilon h_{ab}, \quad (2)$$

with $g^{\rm SRK}_{ab}$ being the expansion of the metric in Eq. (1) up to second order in a,

$$g^{\rm SRK}_{tt} = f_0 + a^2\frac{\cos^2\theta}{r^3}, \qquad g^{\rm SRK}_{t\varphi} = \frac{2a\sin^2\theta}{r}, \quad (3)$$

$$g^{\rm SRK}_{rr} = -\frac{1}{f_0} + a^2\frac{1 - f_0\cos^2\theta}{r^2 f_0^2}, \quad (4)$$

$$g^{\rm SRK}_{\theta\theta} = -r^2 - a^2\cos^2\theta, \quad (5)$$

$$g^{\rm SRK}_{\varphi\varphi} = -\sin^2\theta\left[r^2 + a^2\frac{r + \sin^2\theta}{r}\right], \quad (6)$$

with $f_0 = 1 - 1/r$. If we treat $g^{\rm SRK}_{ab}$ as a correction to the Schwarzschild metric, we can express the perturbation metric h_{ab} in the Regge-Wheeler gauge, which decomposes in a sum of axial $h^{(-)}_{ab}$ and polar $h^{(+)}_{ab}$ contributions. Their explicit expressions, respectively, read as

$$h^{(-)}_{ab}\mathrm{d}x^a\mathrm{d}x^b = 2[h_0^{\ell m}(r)\mathrm{d}t + h_1^{\ell m}(r)\mathrm{d}r] \times [S^{\ell m}_\theta(\theta,\varphi)\mathrm{d}\theta + S^{\ell m}_\varphi(\theta,\varphi)\mathrm{d}\varphi]\mathrm{e}^{\rho t}, \quad (7)$$

$$h_{ab}^{(+)}\mathrm{d}x^a\mathrm{d}x^b = \left[f_0 H_0^{\ell m}(r)\mathrm{d}t^2 + 2H_1^{\ell m}(r)\mathrm{d}t\mathrm{d}r + \frac{H_2^{\ell m}(r)}{f_0}\mathrm{d}r^2 + K^{\ell m}(r)\mathrm{d}\Omega^2\right]Y^{\ell m}(\theta,\varphi)\mathrm{e}^{\rho t}, \tag{8}$$

where we performed a Fourier decomposition in modes of frequency $\omega = -\mathrm{i}\rho$, the functions $Y^{\ell m}(\theta,\varphi)$ are scalar spherical harmonics, and

$$(S_\theta^{\ell m}, S_\varphi^{\ell m}) = \left(-\frac{Y^{\ell m}_{,\varphi}}{\sin\theta}, \sin\theta Y^{\ell m}_{,\theta}\right), \tag{9}$$

where the comma followed by a coordinate stands for the partial derivative with respect to that variable. Note how the functions $h_0^{\ell m}$, $h_1^{\ell m}$, $H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$, $K^{\ell m}$ are free radial functions, and we will see how to relate them to their nonrotating counterpart. We can plug the linearized ansatz (2) into the Einstein equations and solve them with a double-perturbation scheme in ε and a. In total there are ten equations which we schematically dub

$$\delta\mathscr{E}_{ab} \equiv R_{ab}^{(1)} = 0, \tag{10}$$

where we defined the linear perturbation of the Ricci tensor as $R_{ab} = R_{ab}^{(0)} + \varepsilon R_{ab}^{(1)}$. In principle, the presence of the spin introduces an angular dependency in θ, which makes the equations nonmanifestly separable. In the next section, we show a scheme, which is valid up to second order in a, but it is, in principle, extendable to arbitrary order, allowing us to separate the equations and obtain two master radial equations.

B. Decoupling the equations

Here, we apply the scheme for the decoupling of the equations in the slow-spin expansion as in [23], and we extend it up to the second order in the spin, as it was done in [32]. The ten equations (10) can be divided into three different groups, according to their functional form. From now on, to avoid cluttering of indices we omit the superscript index m since different values are always decoupled in the equations. The first group schematically reads as

$$\delta\mathscr{E}_{(i)} \equiv (A_{0,\ell}^{(i)} + A_{1,\ell}^{(i)}\cos\theta + A_{2,\ell}^{(i)}\cos\theta^2)Y^\ell + (B_{1,\ell}^{(i)} + B_{2,\ell}^{(i)}\cos\theta)\sin\theta Y^\ell_{,\theta} = 0, \tag{11}$$

where the index i runs from 0 to 3 and corresponds to $\delta\mathscr{E}_{tt} = 0$, $\delta\mathscr{E}_{tr} = 0$, $\delta\mathscr{E}_{rr} = 0$, and $\delta\mathscr{E}_{\theta\theta} + \delta\mathscr{E}_{\varphi\varphi}/\sin^2\theta = 0$, respectively. The second group reads

$$\delta\mathscr{E}_{(j\theta)} \equiv \left(\sum_{n=0}^{2}\alpha_{n,\ell}^{(j)}\cos^n\theta + \tilde{\alpha}_{2,\ell}^{(j)}\sin^2\theta\right)Y^\ell_{,\theta} - \left(\sum_{n=0}^{2}\beta_{n,\ell}^{(j)}\cos^n\theta + \tilde{\beta}_{2,\ell}^{(j)}\sin^2\theta\right)\frac{Y^\ell_{,\varphi}}{\sin\theta} + (\eta_{1,\ell}^{(j)} + \eta_{2,\ell}^{(j)}\cos\theta)\sin\theta Y^\ell + \chi_{1,\ell}^{(j)}\sin\theta W^\ell + (\xi_{1,\ell}^{(j)} + \xi_{2,\ell}^{(j)}\cos\theta)X^\ell = 0, \tag{12a}$$

$$\delta\mathscr{E}_{(j\varphi)} \equiv \left(\sum_{n=0}^{2}\beta_{n,\ell}^{(j)}\cos^n\theta - \tilde{\beta}_{2,\ell}^{(j)}\sin^2\theta\right)Y^\ell_{,\theta} + \left(\sum_{n=0}^{2}\alpha_{n,\ell}^{(j)}\cos^n\theta - \tilde{\alpha}_{2,\ell}^{(j)}\sin^2\theta\right)\frac{Y^\ell_{,\varphi}}{\sin\theta} + (\zeta_{1,\ell}^{(j)} + \zeta_{2,\ell}^{(j)}\cos\theta)\sin\theta Y^\ell + \chi_{1,\ell}^{(j)}X^\ell - (\xi_{1,\ell}^{(j)} + \xi_{2,\ell}^{(j)}\cos\theta)\sin\theta W^\ell = 0, \tag{12b}$$

where $j = 0, 1$ corresponds to $\delta\mathscr{E}_{t\theta} = 0$ and $\delta\mathscr{E}_{r\theta} = 0$ for the first equation and $\delta\mathscr{E}_{t\varphi} = 0$ and $\delta\mathscr{E}_{r\varphi} = 0$ for the second equation. The symbols X^ℓ and W^ℓ are related to the spin-2 spherical harmonics and are defined as

$$X^\ell = 2Y^\ell_{,\theta\varphi} - 2\cot\theta Y^\ell_{,\varphi}, \tag{13}$$

$$W^\ell = Y^\ell_{,\theta\theta} - \cot\theta Y^\ell_{,\theta} - \frac{Y^\ell_{,\varphi\varphi}}{\sin^2\theta}. \tag{14}$$

Finally, the third group reads

$$\delta\mathscr{E}_{(\theta\varphi)} \equiv (f_{1,\ell} + f_{2,\ell}\cos\theta)\sin\theta Y^\ell_{,\theta} + (g_{1,\ell} + g_{2,\ell}\cos\theta)Y^\ell_{,\varphi} + h_{2,\ell}\sin^2\theta Y^\ell + (j_{0,\ell} + j_{2,\ell}\cos^2\theta)\frac{X^\ell}{\sin\theta} + (k_{0,\ell} + k_{2,\ell}\cos^2\theta)W^\ell = 0, \tag{15a}$$

$$\delta\mathscr{E}_{(-)} \equiv (g_{1,\ell} + g_{2,\ell}\cos\theta)\sin\theta Y^\ell_{,\theta} - (f_{1,\ell} + f_{2,\ell}\cos\theta)Y^\ell_{,\varphi} + \tilde{h}_{2,\ell}\sin^2\theta Y^\ell - (k_{0,\ell} + k_{2,\ell}\cos^2\theta)\frac{X^\ell}{\sin\theta} + (j_{0,\ell} + j_{2,\ell}\cos^2\theta)W^\ell = 0, \tag{15b}$$

with $\delta\mathscr{E}_{(-)} = \delta\mathscr{E}_{\theta\theta} - \delta\mathscr{E}_{\varphi\varphi}/\sin^2\theta = 0$. With this schematic representation of the equations, the functions $A_{n,\ell}^{(i)}$, $B_{n,\ell}^{(i)}$, $\alpha_{n,\ell}^{(j)}$, $\beta_{n,\ell}^{(j)}$, $\tilde{\alpha}_{n,\ell}^{(j)}$, $\tilde{\beta}_{n,\ell}^{(j)}$, $\eta_{n,\ell}^{(j)}$, $\xi_{n,\ell}^{(j)}$, $\chi_{n,\ell}^{(j)}$, $f_{n,\ell}$, $g_{n,\ell}$, $h_{n,\ell}$, $\tilde{h}_{n,\ell}$, $k_{n,\ell}$, $j_{n,\ell}$ are purely radial functions, which contain combinations of the metric perturbation functions and their radial derivative. We label each function with the index n such that it contains at least $\mathcal{O}(a)^n$ terms. Their explicit expression can be found in Appendix F. To separate radial and angular components from the equations, we make use of the completeness relation of spherical harmonics,

$$\int \mathrm{d}\Omega Y^{\ell} Y^{*\ell'} = \delta^{\ell\ell'}, \tag{16}$$

where $*$ denotes complex conjugation, by the fact that spherical harmonics satisfy the equation

$$Y^{\ell}_{,\theta\theta} + \cot\theta Y^{\ell}_{,\theta} + \frac{Y^{\ell}_{,\varphi\varphi}}{\sin^2\theta} = -\ell(\ell+1)Y^{\ell}, \tag{17}$$

as well as the following relations among combinations of spherical harmonics and trigonometric functions:

$$\cos\theta Y^{\ell} = \mathcal{Q}_{\ell+1} Y^{\ell+1} + \mathcal{Q}_{\ell} Y^{\ell-1}, \tag{18}$$

$$\sin\theta Y^{\ell}_{,\theta} = \ell \mathcal{Q}_{\ell+1} Y^{\ell+1} - (\ell+1)\mathcal{Q}_{\ell} Y^{\ell-1}, \tag{19}$$

where we defined $\mathcal{Q}_{\ell} = \sqrt{(\ell^2 - m^2)/(4\ell^2 - 1)}$. One can repeatedly apply the formulas (18) and (19) to find similar expressions for $\cos^n\theta Y^{\ell}$ and $\cos^n\theta \sin\theta Y^{\ell}_{,\theta}$. The useful ones that we used for our calculation are shown in Appendix A. The equations of the three groups can be separated by taking suitable linear combinations of the equations and integrating them over the 2-sphere. The decoupled equations are obtained in the following schematic form:

$$\mathcal{E}^{\mathrm{I}}_{(i+)} \equiv \int \mathrm{d}\Omega \delta\mathscr{E}_{(i)} Y^{*\ell'} = \sum_{n=0}^{2} \left[\mathcal{C}_n A^{(i)}_{n,\ell} + \mathcal{S}_n B^{(i)}_{n,\ell} \right], \tag{20}$$

$$\begin{aligned}\mathcal{E}^{\mathrm{II}}_{(j+)} \equiv \int \mathrm{d}\Omega \left(\delta\mathscr{E}_{(j\theta)} Y^{*\ell'}_{,\theta} + \frac{\delta\mathscr{E}_{(j\varphi)} Y^{*\ell'}_{,\varphi}}{\sin\theta} \right) &= \sum_{n=0}^{2} \left[\mathcal{A}_n \alpha^{(j)}_{n,\ell} + \mathcal{B}_n \beta^{(j)}_{n,\ell} \right] + \tilde{\mathcal{A}}_2 \tilde{\alpha}^{(j)}_{2,\ell} + \tilde{\mathcal{B}}_2 \tilde{\beta}^{(j)}_{2,\ell} \\ &+ \sum_{n=1}^{2} \left[\bar{\mathcal{S}}_n \eta^{(j)}_{n,\ell} + \mathrm{i} m \mathcal{C}_{n-1} \zeta^{(j)}_{n,\ell} + \mathcal{X}_{n-1} \xi^{(j)}_{n,\ell} + \bar{\mathcal{X}}_{n-1} \chi^{(j)}_{n,\ell} \right],\end{aligned} \tag{21}$$

$$\begin{aligned}\mathcal{E}^{\mathrm{II}}_{(j-)} \equiv \int \mathrm{d}\Omega \left(\delta\mathscr{E}_{(j\varphi)} Y^{*\ell'}_{,\theta} - \frac{\delta\mathscr{E}_{(j\theta)} Y^{*\ell'}_{,\varphi}}{\sin\theta} \right) &= \sum_{n=0}^{2} \left[-\mathcal{A}_n \beta^{(j)}_{n,\ell} + \mathcal{B}_n \alpha^{(j)}_{n,\ell} \right] + \tilde{\mathcal{A}}_2 \tilde{\beta}^{(j)}_{2,\ell} - \tilde{\mathcal{B}}_2 \tilde{\alpha}^{(j)}_{2,\ell} \\ &+ \sum_{n=1}^{2} \left[\bar{\mathcal{S}}_n \zeta^{(j)}_{n,\ell} - \mathrm{i} m \mathcal{C}_{n-1} \eta^{(j)}_{n,\ell} + \mathcal{X}_{n-1} \chi^{(j)}_{n,\ell} - \bar{\mathcal{X}}_{n-1} \xi^{(j)}_{n,\ell} \right],\end{aligned} \tag{22}$$

$$\mathcal{E}^{\mathrm{III}}_{(+)} \equiv \int \mathrm{d}\Omega \left(\delta\mathscr{E}_{(-)} W^{*\ell'} + \frac{\delta\mathscr{E}_{(\theta\varphi)} X^{*\ell'}}{\sin\theta} \right) = \sum_{n=0}^{1} \left[\mathcal{F}_n f_{n+1,\ell} + \mathcal{G}_n g_{n+1,\ell} + \mathcal{J}_{2n} j_{2n,\ell} + \mathcal{K}_{2n} k_{2n,\ell} \right] + \bar{\mathcal{H}} h_{2,\ell} + \mathcal{H} \bar{h}_{2,\ell}, \tag{23}$$

$$\mathcal{E}^{\mathrm{III}}_{(-)} \equiv \int \mathrm{d}\Omega \left(\delta\mathscr{E}_{(\theta\varphi)} W^{*\ell'} - \frac{\delta\mathscr{E}_{(-)} X^{*\ell'}}{\sin\theta} \right) = \sum_{n=0}^{1} \left[\mathcal{G}_n f_{n+1,\ell} - \mathcal{F}_n g_{n+1,\ell} + \mathcal{J}_{2n} k_{2n,\ell} - \mathcal{K}_{2n} j_{2n,\ell} \right] - \bar{\mathcal{H}} \bar{h}_{2,\ell} + \mathcal{H} h_{2,\ell}. \tag{24}$$

We label the equations with a Roman number that shows from which group they have been obtained, and with a $(+)/(-)$, we indicate the equations whose limit $a \to 0$ contains only polar and axial quantities. We refer to the first and second groups of equations as polar and axial-led, respectively. The operators $\mathcal{C}_n$, $\mathcal{S}_n$, $\mathcal{A}_n$, $\mathcal{B}_n$, $\tilde{\mathcal{A}}_2$, $\tilde{\mathcal{B}}_2$, $\bar{\mathcal{S}}_n$, $\mathcal{X}_n$, $\bar{\mathcal{X}}_n$, $\mathcal{F}_n$, $\mathcal{G}_n$, $\mathcal{J}_{2n}$, $\mathcal{K}_{2n}$, $\mathcal{H}$, and $\bar{\mathcal{H}}$ are integrals which mix modes with different angular momenta ℓ, and they are explicitly provided in Appendix A.

The general structure of all the radial equations obtained with this procedure is

$$\mathcal{E}_{\ell} = \mathcal{P}^{\ell} + a\bar{\mathcal{P}}^{\ell\pm1} + a^2 \mathcal{P}^{\ell\pm2}, \tag{25}$$

where $\mathcal{P}$ refers to a combination of the functions and their derivatives for a given parity, whereas $\bar{\mathcal{P}}$ are combinations of functions and their derivatives of opposite parity. The ℓ label signals that functions of a chosen parity of angular momentum ℓ couple at least at first order in the spin with functions of opposite parity and angular momentum $\ell \pm 1$ and at least at second order in the spin with functions of the same parity and angular momentum $\ell \pm 2$. The spin factor outside each component signals the minimum order at which the modification enters.

Note that thanks to the linearity of the equations one can combine different equations and their radial derivatives in such a way that the structure denoted in Eq. (25) does not change. If we have any two equations $\mathcal{E}_1^\ell = 0$ and $\mathcal{E}_2^\ell = 0$ with the same parity in the nonspinning limit, as well as an equation $\bar{\mathcal{E}}_3^\ell = 0$ with different parity in the nonspinning limit, the structure (25) up to second order with the spin is preserved if

(i) one takes a linear combination of $\mathcal{E}_1^\ell$, $\mathcal{E}_2^\ell$ as well as their radial derivatives;

(ii) one takes a linear combination of $\mathcal{E}_1^\ell$, $a\bar{\mathcal{E}}_3^{\ell\pm1}$ as well as their radial derivatives; and

(iii) one takes a linear combination of $\mathcal{E}_1^\ell$, $a^2\mathcal{E}_2^{\ell\pm2}$ as well as their radial derivatives.

We will use these three transformations extensively to drastically simplify the ten equations.

Finally, let us notice that the terms $\mathcal{P}_\ell$, $\bar{\mathcal{P}}_{\ell\pm1}$, and $\mathcal{P}_{\ell\pm2}$ appearing in Eq. (25) have the following structure:

$$\mathcal{P}^\ell = [A_0^\ell + amA_1^\ell + a^2(A_2^\ell + m^2\bar{A}_2^\ell + \tilde{A}_2^\ell \mathcal{Q}_{\ell+1}^2 + \tilde{A}_2^{-\ell-1}\mathcal{Q}_\ell^2)]f_\ell, \tag{26}$$

$$\bar{\mathcal{P}}^{\ell\pm1} = \mathcal{Q}_{\ell+1}(B_1^\ell + amB_2^\ell)\bar{f}_{\ell+1} + \mathcal{Q}_\ell(B_1^{-\ell-1} + amB_2^{-\ell-1})\bar{f}_{\ell-1}, \tag{27}$$

$$\mathcal{P}^{\ell\pm2} = \mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}C_2^\ell f_{\ell+2} + \mathcal{Q}_{\ell-1}\mathcal{Q}_\ell C_2^{-\ell-1} f_{\ell-2}, \tag{28}$$

where with the symbol f_ℓ we refer to any of the perturbation functions h_0^ℓ, h_1^ℓ, H_0^ℓ, H_1^ℓ, H_2^ℓ, K^ℓ (and their derivatives) of a given parity, whereas $\bar{f}$ stands for the same functions but opposite parity, and A_0^ℓ, A_1^ℓ, A_2^ℓ, $\bar{A}_2^\ell$, $\tilde{A}_2^\ell$, B_1^ℓ, B_2^ℓ, and C_2^ℓ are functions of r and ℓ only. In this way, we completely determined how the index m enters in the equations, and it is clear that different values of m never couple to each other. In the next section we show how to redefine the perturbation variables such that they satisfy differential equations where the coupling between different ℓ is also removed.

C. Spin corrections to the Regge-Wheeler and Zerilli equations

Let us start by denoting the polar-led equations as $Z_1 = \mathcal{E}^{\rm I}_{(0+)}$, $Z_2 = \mathcal{E}^{\rm I}_{(1+)}$, $Z_3 = \mathcal{E}^{\rm I}_{(2+)}$, $Z_4 = \mathcal{E}^{\rm I}_{(3+)}$, $Z_5 = \mathcal{E}^{\rm II}_{(0+)}$, $Z_6 = \mathcal{E}^{\rm II}_{(1+)}$, $Z_7 = \mathcal{E}^{\rm III}_{(+)}$ and the axial-led equations as $Q_1 = \mathcal{E}^{\rm II}_{(0-)}$, $Q_2 = \mathcal{E}^{\rm II}_{(1-)}$, $Q_3 = \mathcal{E}^{\rm III}_{(-)}$. We now show that these ten equations are not independent, and we explain how they can be recast into two independent equations which generalize the Regge-Wheeler and the Zerilli equations up to second order in the spin.

Let us revise the derivation of the two equations in the limit $a = 0$. From $Q_2 = 0$, we can find an expression for $\partial_r h_0^\ell$, while from $Q_3 = 0$ we obtain an expression for $\partial_r h_1^\ell$. It is straightforward to check that $Q_1 = 0$ is automatically satisfied, by taking combinations of Q_2, Q_3 and their derivatives. At this point, one can define

$$h_1^\ell(r) = \frac{r}{f_0}\Phi_{(-)}^\ell(r), \qquad h_0^\ell(r) = \frac{f_0}{\rho}\partial_r[r\Phi_{(-)}^\ell(r)]. \tag{29}$$

By inserting these expressions into, e.g., the equation $Q_2 = 0$, one finds that the function $\Phi_{(-)}^\ell$ satisfies the so-called Regge-Wheeler equation [1]

$$\frac{d^2\Phi_{(-)}^\ell}{dr_{*,0}^2} - (\rho^2 + V_{(-),0}^\ell)\Phi_{(-)}^\ell = 0, \tag{30}$$

where $dr_{*,0} = dr/f_0$ is the Schwarzschild tortoise coordinate, and the Regge-Wheeler potential reads

$$V_{(-),0}^\ell = f_0\left[\frac{\ell(\ell+1)}{r^2} - \frac{3}{r^3}\right]. \tag{31}$$

On the polar side, we can solve $Z_7 = 0$ to find an algebraic expression for H_2^ℓ; then the combined solution to $Z_5 = Z_2 = Z_6 = 0$ leads to an expression for $\partial_r H_1^\ell$, $\partial_r K^\ell$, $\partial_r H_0^\ell$, respectively. The combination of the former results inserted into $Z_4 = 0$ leads to an algebraic expression for H_0^ℓ. Again, with some algebraic manipulation one can show that $Z_1 = Z_3 = 0$ is automatically satisfied. The definition of the Zerilli function is made through

$$K^\ell = \left[\frac{\lambda-2}{2r} - \frac{3f_0}{r(3+\lambda r)}\right]\Phi_{(+)}^\ell + f_0\partial_r\Phi_{(+)}^\ell, \tag{32}$$

$$H_1^\ell = \left[\frac{2r-3}{2rf_0} - \frac{3}{3+\lambda r}\right]\rho\Phi_{(+)}^\ell + \rho r\partial_r\Phi_{(+)}^\ell, \tag{33}$$

where $\lambda = \ell(\ell+1) - 2$. By inserting these expressions into, e.g., $Z_5 = 0$, one obtains the so-called Zerilli equation [2]

$$\frac{d^2\Phi_{(+)}^\ell}{dr_{*,0}^2} - (\rho^2 + V_{(+),0}^\ell)\Phi_{(+)}^\ell = 0, \tag{34}$$

with the Zerilli potential being

$$V^{\ell}_{(+),0} = f_0\left[\frac{\ell(\ell+1)}{r^2} - \frac{3}{r^3}\frac{r^2\lambda(\lambda+4)+6r-3}{(3+r\lambda)^2}\right]. \quad (35)$$

Let us now turn back to the full problem. One can formally perform the same calculations to obtain second order in a expressions for $\partial_r h_0^\ell$ and $\partial_r h_1^\ell$ in the axial-led sector and for $\partial_r H_1^\ell$, $\partial_r K^\ell$, $\partial_r H_0^\ell$, H_2^ℓ, and H_0^ℓ in the polar-led sector. The structure of these expressions is the same as Eq. (25) because it is obtained by repeated use of the three combination rules enumerated in the previous section. Thus, for each given equation of a given parity, one can use the expressions found for the functions of opposite parity and $\ell \pm 1$ up to first order in the spin, and for the functions of the same parity and $\ell \pm 2$ up to zeroth order in the spin. The overall outcome is very similar to the spinless case, as here $Q_1 = Z_1 = Z_3 = 0$ are also automatically satisfied up to $\mathcal{O}(a^2)$.

In order to obtain a generalization of the RW and the Zerilli equations, we propose a modified redefinition of Eqs. (29), (32), and (33) that takes into account couplings with functions of different angular momenta, which are introduced at each order in the spin. Given that all the equations can be written just in terms of h_0^ℓ, h_1^ℓ, K^ℓ, and H_1^ℓ, we guess the following ansatz for the redefinition of variables, based on the transformation rules that maintain the structure of Eq. (25) unchanged,[1]

$$\begin{aligned} h_1^\ell = {}& \frac{r}{f_0}\Phi^\ell_{(-)} + \frac{\mathrm{i}am}{\rho}c_1^\ell\Phi^\ell_{(-)} + a^2 d_1^\ell \Phi^\ell_{(-)} \\ &+ a\Big[\mathcal{Q}_\ell(s_1^{-\ell-1}H_1^{\ell-1} + t_1^{-\ell-1}K^{\ell-1}) \\ &+ \mathcal{Q}_{\ell+1}(s_1^\ell H_1^{\ell+1} + t_1^\ell K^{\ell+1})\Big] \\ &+ a^2\Big[\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell(u_1^{-\ell-1}h_1^{\ell-2} + v_1^{-\ell-1}h_0^{\ell-2}) \\ &+ \mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}(u_1^\ell h_1^{\ell+2} + v_1^\ell h_0^{\ell+2})\Big], \end{aligned} \quad (36)$$

$$\begin{aligned} h_0^\ell = {}& \frac{f_0}{\rho}\partial_r(r\Phi^\ell_{(-)}) + \frac{\mathrm{i}am}{\rho}c_0^\ell\Phi^\ell_{(-)} + a^2 d_0^\ell \Phi^\ell_{(-)} \\ &+ a\Big[\mathcal{Q}_\ell(s_0^{-\ell-1}H_1^{\ell-1} + t_0^{-\ell-1}K^{\ell-1}) \\ &+ \mathcal{Q}_{\ell+1}(s_0^\ell H_1^{\ell+1} + t_0^\ell K^{\ell+1})\Big] \\ &+ a^2\Big[\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell(u_0^{-\ell-1}h_1^{\ell-2} + v_0^{-\ell-1}h_0^{\ell-2}) \\ &+ \mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}(u_0^\ell h_1^{\ell+2} + v_0^\ell h_0^{\ell+2})\Big], \end{aligned} \quad (37)$$

for the axial-led variables, and

$$\begin{aligned} K^\ell = {}& \left[\frac{\lambda-2}{2r} - \frac{3f_0}{r(3+\lambda r)}\right]\Phi^\ell_{(+)} + f_0\partial_r\Phi^\ell_{(+)} + \frac{\mathrm{i}am}{\rho}c_K^\ell\Phi^\ell_{(+)} + a^2 d_K^\ell\Phi^\ell_{(+)} + a\Big[\mathcal{Q}_\ell(s_K^{-\ell-1}h_1^{\ell-1} + t_K^{-\ell-1}h_0^{\ell-1}) \\ &+ \mathcal{Q}_{\ell+1}(s_K^\ell h_1^{\ell+1} + t_K^\ell h_0^{\ell+1})\Big] + a^2\Big[\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell(u_K^{-\ell-1}K^{\ell-2} + v_K^{-\ell-1}H_1^{\ell-2}) + \mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}(u_K^\ell K^{\ell+2} + v_K^\ell H_1^{\ell+2})\Big], \end{aligned} \quad (38)$$

$$\begin{aligned} H_1^\ell = {}& \left[\frac{2r-3}{2rf_0} - \frac{3}{3+\lambda r}\right]\rho\Phi^\ell_{(+)} + \rho r\partial_r\Phi^\ell_{(+)} + \frac{\mathrm{i}am}{\rho}c_H^\ell\Phi^\ell_{(+)} + a^2 d_H^\ell\Phi^\ell_{(+)} + a\Big[\mathcal{Q}_\ell(s_H^{-\ell-1}h_1^{\ell-1} + t_H^{-\ell-1}h_0^{\ell-1}) \\ &+ \mathcal{Q}_{\ell+1}(s_H^\ell h_1^{\ell+1} + t_H^\ell h_0^{\ell+1})\Big] + a^2\Big[\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell(u_H^{-\ell-1}K^{\ell-2} + v_H^{-\ell-1}H_1^{\ell-2}) + \mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}(u_H^\ell K^{\ell+2} + v_H^\ell H_1^{\ell+2})\Big], \end{aligned} \quad (39)$$

for the polar-led ones. The coefficients s_i^ℓ and t_i^ℓ can be further split as

$$s_i^\ell = s_{0,i}^\ell + a s_{1,i}^\ell \qquad t_i^\ell = t_{0,i}^\ell + a t_{1,i}^\ell. \quad (40)$$

One has to choose the coefficients c_i^ℓ, d_i^ℓ, s_i^ℓ, t_i^ℓ, u_i^ℓ, and v_i^ℓ (with $i = 0, 1, K, H$) such that if one inverts Eqs. (36)–(39) to find a consistent expression for $\Phi^\ell_{(\pm)}$ and $\partial_r\Phi^\ell_{(\pm)}$. We can equate the second expression to the radial derivative of the first one, and, provided that the functions h_0, h_1, K, and H_1 are linearly independent, we can set to zero all the coefficients that multiply them. This requirement uniquely fixes the coefficients c_i^ℓ and d_i^ℓ and relates the coefficients s_i^ℓ to the t_i^ℓ and the u_i^ℓ to the v_i^ℓ. This freedom of reparametrization will be exploited in the next steps to fully diagonalize the equations for the functions $\Phi^\ell_{(\pm)}$.

By inserting the definitions (36)–(39) into the equations $Q_2 = 0$ and $Z_5 = 0$, one finds the following generalized Regge-Wheeler and Zerilli equations:

[1] In principle, one should be able to express the functions h_1, h_0, K, and H_1 only in terms of the functions $\Phi_{(\pm)}$ with angular momenta $\ell \pm 0$, 1, 2. However, since the transformations are invertible at each order in a, and they all have the form of Eq. (25), we find that the explicit expression of the coefficients is more compact if expressed in terms of the original metric functions (see Appendix B for linear coefficients and Appendix G for quadratic ones).

$$\frac{d\Phi^{\ell}_{(\pm)}}{dr_*^2} - (\rho^2 + V^{\ell}_{(\pm)})\Phi^{\ell}_{(\pm)} = a(A^{\ell}_{(\pm)}\Phi^{\ell+1}_{(\mp)} + A^{-\ell-1}_{(\pm)}\Phi^{\ell-1}_{(\mp)}) + a^2(B^{\ell}_{(\pm)}\Phi^{\ell+2}_{(\pm)} + B^{-\ell-1}_{(\pm)}\Phi^{\ell-2}_{(\pm)}), \tag{41}$$

where the tortoise coordinate is $dr_* = dr/f_T$, with

$$f_T = f_0\left(1 + \frac{iamf_1}{\rho} + a^2 f_2\right). \tag{42}$$

The functions f_1 and f_2 are left unspecified for now, as this can always be done by simultaneously rescaling the equation and $\Phi_{(\pm)}$. We aim to set $A^{\ell}_{(\pm)} = 0$ and $B^{\ell}_{(\pm)} = 0$. This result can be achieved by completely fixing the yet unspecified functions t^{ℓ}_i and v^{ℓ}_i. The full list of coefficients up to first order in the spin is shown in Appendix B. The steps taken up until now finally lead to the two completely decoupled equations

$$\frac{d\Phi^{\ell}_{(\pm)}}{dr_*^2} - (\rho^2 + V^{\ell}_{(\pm)})\Phi^{\ell}_{(\pm)} = 0. \tag{43}$$

The potentials are corrected as

$$V^{\ell}_{(\pm)} = V^{\ell}_{(\pm),0} + \frac{iam}{\rho}V^{\ell}_{(\pm),1} + a^2 V^{\ell}_{(\pm),2}. \tag{44}$$

If we do not specify the tortoise coordinate defined in Eq. (42), the first- and second-order potentials have the following form:

$$V^{\ell}_{(\pm),1} = \bar{V}^{\ell}_{(\pm),1} - \frac{1}{2}\frac{d^2 f_1}{dr_{*,0}^2} + 2(\rho^2 + V^{\ell}_{(\pm),0})f_1 \tag{45}$$

$$V^{\ell}_{(\pm),2} = \bar{V}^{\ell}_{(\pm),2} - \frac{1}{2}\frac{d^2 f_2}{dr_{*,0}^2} + 2(\rho^2 + V^{\ell}_{(\pm),0})f_2 - \frac{m^2}{\rho^2}\left[\frac{3}{4}\left(\frac{df_1}{dr_{*,0}}\right)^2 + 2\bar{V}^{\ell}_{(\pm),1}f_1\right], \tag{46}$$

where the corrections at first order in the spin are

$$\bar{V}^{\ell}_{(-),1} = \frac{6f_0(7-6r)}{\ell(\ell+1)r^6} + \frac{2\rho^2}{r^3}, \tag{47}$$

$$\bar{V}^{\ell}_{(+),1} = \frac{f_0}{(\lambda+2)r^7(3+r\lambda)^4}[6\lambda^4(\lambda+4)r^6 + \lambda^3(-7\lambda^2 - 4\lambda + 96)r^5 - 6\lambda^2(5\lambda^2 + 8\lambda - 8)r^4 - 3\lambda(29\lambda^2 - 84\lambda + 90)r^3 - 18(21\lambda^2 - 64\lambda + 18)r^2 - 162(6\lambda - 7)r - 864] + \frac{2\rho^2}{r^3} + 4f_0\frac{(\lambda-2)\lambda^2 r^3 + 9\lambda^2 r^2 + 15\lambda r + 9}{(\lambda+2)r^3(3+r\lambda)^3}\rho^2, \tag{48}$$

while second-order corrections are given in Appendix C. We check that if $f_1 = 0$ the first-order corrections to the two potentials are consistent with those already calculated in [27]. With this calculation, we show that even though rotation introduces coupling between different parities, it is nevertheless possible to diagonalize the system into two separate, axial-led and polar-led, modes. One can conjecture whether these modes are related to each other as in the nonrotating limit and whether this behavior persists at any order in the spin. While an answer to the second question is beyond the scope of this work, in the next section we show how the modes $\Phi_{(+)}$ and $\Phi_{(-)}$ are actually connected.

III. ISOSPECTRALITY OF POLAR-LED AND AXIAL-LED EQUATIONS

The Regge-Wheeler and the Zerilli potentials are known to be isospectral. This means that, even though the functional form of the potentials in the two equations is drastically different, their spectra of QNMs identically coincide. The motivation for this result was shown for the first time by Chandrasekhar [10], who realized that the two potentials can be generated by a superpotential W^{ℓ}_0,

$$V^{\ell}_{(\pm),0} = \beta_0^2 W^{\ell 2}_0 + \beta_0\frac{dW^{\ell}_0}{dr_{*,0}} + \kappa_0 W^{\ell}_0, \tag{49}$$

with $\beta_0 = \pm 3$, and the choice of the sign is either the Regge-Wheeler or the Zerilli potential, $\kappa_0 = \lambda(\lambda+2)$ and

$$W^{\ell}_0 = \frac{f_0}{r(3+\lambda r)}. \tag{50}$$

This simple-looking relation can be explained in the context of Darboux transformation, whose applications to BHPT have been extensively discussed in [35]. For the nonspinning case, the Darboux transformation is of the form [11]

$$\Phi^{\ell}_{(\pm)} = \frac{\mathrm{d}\Phi^{\ell}_{(\mp)}}{\mathrm{d}r_{*,0}} + \left(\frac{\kappa_0}{2\beta_0} + \beta_0 W^{\ell}_0\right)\Phi^{\ell}_{(\mp)}, \qquad (51)$$

where a positive (negative) sign of β_0 determines whether the transformation is from polar (axial) to axial (polar) functions.

We now analyze whether this structure still exists for the slow-spinning potentials found in the previous section. We seek a superpotential W^{ℓ} that can generate the potentials $V^{\ell}_{(\pm)}$ through a generalized version of Eq. (49),

$$V^{\ell}_{(\pm)} = \beta_0^2 W^{\ell 2} + \beta_0 \frac{\mathrm{d}W^{\ell}}{\mathrm{d}r_*} + \kappa_0 W^{\ell} + \kappa_0 \kappa, \qquad (52)$$

where κ is a constant, the sign of β_0 determines whether to retrieve the axial-led or polar-led potentials, and we truncate the calculation at second order in the spin,

$$W^{\ell} = W^{\ell}_0 + \frac{\mathrm{i}am}{\rho} W^{\ell}_1 + a^2 W^{\ell}_2, \qquad (53)$$

$$\kappa = \frac{\mathrm{i}am}{\rho}\kappa_1 + a^2\kappa_2. \qquad (54)$$

From Ref. [35], we know that the function W^{ℓ}, if it exists, must satisfy the following constraints:

$$\frac{\mathrm{d}W^{\ell}}{\mathrm{d}r_*} = -\frac{1}{2}\left(V^{\ell}_{(-)} - V^{\ell}_{(+)}\right), \qquad (55)$$

$$W^{\ell} = \frac{1}{2\left(V^{\ell}_{(-)} - V^{\ell}_{(+)}\right)} \frac{\mathrm{d}}{\mathrm{d}r_*}\left(V^{\ell}_{(-)} + V^{\ell}_{(+)}\right). \qquad (56)$$

We solve Eq. (55) at first order in the spin separately for each power of ρ. Hence, we find it convenient to perform the further splitting

$$W^{\ell}_1 = \tilde{W}^{\ell}_1 + \bar{W}^{\ell}_1 \rho^2, \qquad (57)$$

$$\kappa_1 = \tilde{\kappa}_1 + \bar{\kappa}_1 \rho^2. \qquad (58)$$

For the terms proportional to ρ, we obtain

$$\bar{W}^{\ell}_1 = \bar{k}_1 - \frac{3 + 6r\lambda + r^2\lambda(\lambda - 2)}{3r^2(\lambda + 2)(3 + r\lambda)^2}, \qquad (59)$$

where $\bar{k}_1$ is an integration constant. By checking the self-consistency of the result with Eq. (56), we find that it is satisfied only for a specific choice of the tortoise coordinate (42), which until now was left unspecified. Explicitly, one has that the first-order function of the tortoise coordinate must assume the form

$$f_1 = \frac{\kappa_0 \bar{\kappa}_1}{2} - \frac{1}{r^3} + \left(\frac{\kappa_0}{2} + \beta_0^2 W^{\ell}_0\right)\bar{W}^{\ell}_1 - \frac{3}{2}\frac{\mathrm{d}\bar{W}^{\ell}_1}{\mathrm{d}r_{*,0}}. \qquad (60)$$

In order to have consistency with Eqs. (55) and (56) also for the terms proportional to ρ^{-1}, we find

$$\frac{\tilde{W}^{\ell}_1}{W^{\ell}_0} = \frac{r^2(3r-4)\lambda(\lambda+10) + 12r - 36}{6r^4(\lambda+2)(3+r\lambda)} + 2f_1, \qquad (61)$$

$$\tilde{\kappa}_1 = 0, \qquad \bar{\kappa}_1 = -\bar{k}_1 = \frac{2}{9}. \qquad (62)$$

The calculation at second order in the spin proceeds analogously. Since it is longer, but essentially similar to that at first order, we show it in Appendix D. Indeed, we find that in order to find a generator of the potentials, we need to fully specify the tortoise function f_2. Differently from the first-order case though, the consistency of the Darboux transformation does not fully specify all the integration constants.

With this calculation, we show how one can transform the Regge-Wheeler equation to the Zerilli equation and vice versa up to second order in the spin, upon a suitable choice of the tortoise coordinate. We conclude by explicitly showing the Darboux transformation between polar-led and axial-led quantities,

$$\Phi^{\ell}_{(\pm)} = \frac{\mathrm{d}\Phi^{\ell}_{(\mp)}}{\mathrm{d}r_*} + \left(\frac{\kappa_0}{2\beta_0} + \beta_0 W^{\ell}\right)\Phi^{\ell}_{(\mp)}. \qquad (63)$$

The existence of this transformation implies that the spectrum of the two potentials is exactly equivalent since it does not modify the boundary conditions, as it happens in the nonrotating case [35]. The only difference is that the Darboux relation is manifest only for one specific choice of the tortoise coordinate. We check that this functional form of the tortoise coordinate cannot be related with any previous choices already existing in the literature. We assume that one can find a more general transformation that links the two equations if the tortoise coordinate is left unspecified. In any case, we want to stress that there is nothing special about truncating the expansion at second order in the spin, and we expect this trend to continue even at higher orders. We now turn our focus to the relation between the Regge-Wheeler and the Zerilli equations and the Teukolsky equation in the slow rotation limit.

IV. ISOSPECTRALITY WITH SLOW-SPINNING TEUKOLSKY EQUATION

A. Teukolsky equation

BHPT for Kerr BHs is most commonly studied within the Teukolsky formalism [3]. This approach works by projecting the spacetime onto the null Newman-Penrose tetrad and by analyzing the geometrical relations between some components of the projected Weyl tensor, which is the trace-free Riemann tensor. All the information of the projected Weyl tensor is contained in five complex scalars. By considering the perturbation of these Weyl scalars, only two of them cannot be set to zero with an infinitesimal gauge transformation. We commonly refer to them as $\Psi^{(0)}$ and $\Psi^{(4)}$, which can be treated as linear perturbation quantities since their background vanishes for the Kerr solution. The Teukolsky equations are linear second-order differential equations for $\Psi^{(0,4)}$, which can be considered as perturbations of internal spin $s = \pm 2$, but they are also valid for $s = 0$ (scalar perturbations), $s = \pm 1/2$ (spinor perturbations), and $s = \pm 1$ (electromagnetic perturbations). Their entire derivation can be found in detail in [11]. Here, we report the general form of the Teukolsky master equation,

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\partial_t^2\psi^{(s)} + \frac{2ar}{\Delta}\partial_t\partial_\phi\psi^{(s)} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\partial_\phi^2\psi^{(s)} - \Delta^{-s}\partial_r(\Delta^{s+1}\partial_r\psi^{(s)}) - \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi^{(s)}) + (s^2\cot\theta^2 - s)\psi^{(s)} - 2s\left[\frac{a(2r-1)}{2\Delta} + \mathrm{i}\frac{\cos\theta}{\sin^2\theta}\right]\partial_\phi\psi^{(s)} - 2s\left[\frac{(r^2-a^2)}{2\Delta} - r - \mathrm{i}a\cos\theta\right]\partial_t\psi^{(s)} = 0. \tag{64}$$

For the analysis pursued in this paper, we are only interested in tensor perturbations ($s = \pm 2$), for which the perturbation function $\psi^{(s)}$ can be linked to first-order perturbations of the Weyl scalars as

$$\psi^{(2)} = \Psi^{(0)}, \qquad \psi^{(-2)} = (r - \mathrm{i}a\cos\theta)^4\Psi^{(4)}. \tag{65}$$

In the Teukolsky equation one can decouple the radial and the angular part by choosing the following mode decomposition:

$$\psi^{(s)} = R^{(s)}_{\ell m}(r)S^{(s)}_{\ell m}(\theta)\mathrm{e}^{\rho t + \mathrm{i}m\varphi}. \tag{66}$$

For the angular part one gets

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta S^{(s)}_{\ell m}) + \left[2\mathrm{i}sa\rho\cos\theta - a^2\rho^2\cos^2\theta - \frac{(m+s\cos\theta)^2}{\sin^2\theta} + s + \lambda^{(s)}_{\ell m}\right]S^{(s)}_{\ell m} = 0, \tag{67}$$

where the functions $S^{(s)}_{\ell m}$ are known as spin-weighted spheroidal harmonics; $\lambda^{(s)}_{\ell m}$ is the separation constant, and it depends on the frequency ρ. We find it more convenient to work with the reduced constant $\tilde{\lambda}^{(s)}_{\ell m} = \lambda^{(s)}_{\ell m} + (s + |s|)$. The constants $\tilde{\lambda}^{(s)}_{\ell m}$ are only known numerically, but we report here their value up to second order in the spin $\tilde{\lambda}^{(s)}_{\ell m} \simeq \lambda + \mathrm{i}am\rho\lambda^{(s)}_1 + a^2\rho^2\lambda^{(s)}_2$, where [36]

$$\lambda^{(s)}_1 = -\frac{2s^2}{\lambda+2}, \qquad \lambda^{(s)}_2 = \frac{1 + \mathcal{P}^{(s)}_\ell - \mathcal{P}^{(s)}_{\ell+1}}{\lambda+2}, \tag{68}$$

where we define

$$\mathcal{P}^{(s)}_\ell = 2\mathcal{Q}^2_\ell\frac{(\ell^2 - s^2)^2}{\ell^3}. \tag{69}$$

Finally, the radial part of the Teukolsky equation must satisfy the following equation:

$$\Delta^{-s}\partial_r(\Delta^{s+1}\partial_r)R^{(s)}_{\ell m} + \left[\frac{K^2 - \mathrm{i}s(2r-1)K}{\Delta} - a^2\omega^2 + 2ma\omega + 4\mathrm{i}s\omega r - \lambda^{(s)}_{\ell m}\right]R^{(s)}_{\ell m} = 0, \tag{70}$$

with $K = (r^2 + a^2)\omega - am$.

B. Isospectrality relation

In order to show the isospectrality between metric and Weyl scalar perturbations up to second order in the spin, we borrow the notation to write the equations introduced by Chandrasekhar in [11]. First of all, we introduce the operators

$$\Lambda_\pm = \frac{\mathrm{d}}{\mathrm{d}r_*} \pm \rho. \tag{71}$$

Then, after redefining the radial functions as

$$U^{(s)}(r) = \frac{\Delta^{(s\pm 2)/2}}{r^3}R^{(s)}(r), \tag{72}$$

one can compactly write the Teukolsky equation for the perturbation variable U,[2]

$$\Lambda_+\Lambda_-U + P\Lambda_-U - QU = 0, \tag{73}$$

where

$$P = \frac{\mathrm{d}}{\mathrm{d}r_*}\log\left(\frac{f_P^2}{f_T^2}\right), \tag{74}$$

Note that f_T is the tortoise function defined in Eq. (42), and f_P and Q are functions that must be specified such that they

[2]The two $\pm s$ polarizations defined in this way satisfy complex conjugate equations [11]. Moreover, the separation constants $\lambda^{(s)}_i$ are identical for both choices of the spin. From now on, we focus on $s = 2$ only and drop the index s everywhere.

match the Teukolsky equation. We find that a convenient definition is

$$f_P = \frac{r^2}{2}\left[1 + \frac{f_T}{f_0} + a^2\left(\frac{m^2}{\rho^2}\frac{f_1^2}{4} - \frac{1}{r^2 f_0}\right)\right]. \tag{75}$$

With this choice, the effective potential schematically takes the form

$$Q = Q_0 + \frac{\mathrm{i}am}{\rho}Q_1 + a^2 Q_2, \tag{76}$$

where we labeled with Q_0 the effective potential of the Bardeen-Press equation, which, as we know from the relations obtained by Chandrasekhar [11], it can be related with the superpotential W_0 defined in Eq. (50) as

$$Q_0 = \frac{f_0^2}{r^4 W_0}. \tag{77}$$

We furthermore split first- and second-order corrections by powers of m and ρ as

$$Q_1 = \sum_{i=0}^{2} Q_{1,i}\rho^i, \tag{78}$$

$$Q_2 = \sum_{i=0}^{2} Q_{2,i}\rho^i + \frac{m^2}{\rho^2}\sum_{i=1}^{2}\bar{Q}_{2,i}\rho^i, \tag{79}$$

with $Q_{i,j}$ and $\bar{Q}_{i,j}$ functions of r and ℓ. The choice of f_P given in Eq. (75) is such that the first-order expressions for Q_1 take the rather simple forms

$$Q_{1,0} = 2Q_0 f_1, \tag{80}$$

$$Q_{1,1} = \frac{\mathrm{d}f_1}{\mathrm{d}r_{*,0}} - 2\frac{3-2r}{r^2}f_1 + 2\frac{1-2r}{r^4}, \tag{81}$$

$$Q_{1,2} = \frac{2}{r^3} + 2f_1 + f_0\frac{\lambda_1}{r^2}, \tag{82}$$

whereas the expressions of Q_2 are

$$Q_{2,0} = \frac{3 - r(\lambda+6)}{r^5} + 2Q_0 f_2, \tag{83}$$

$$Q_{2,1} = \frac{\mathrm{d}f_2}{\mathrm{d}r_{*,0}} - 2\frac{3-2r}{r^2}f_2 + \frac{(11-6r)r-2}{f_0 r^5}, \tag{84}$$

$$Q_{2,2} = -\frac{r^2+1}{r^4 f_0} + 2f_2 + f_0\frac{\lambda_2}{r^2}, \tag{85}$$

$$\bar{Q}_{2,1} = \left[Q_{1,1} - \frac{3}{r^2}\left(2\frac{1-2r}{r^2} + (2r-3)f_1\right)\right]f_1, \tag{86}$$

$$\bar{Q}_{2,2} = -2(Q_{1,2} - 2f_1)f_1. \tag{87}$$

In the limit $a \to 0$, the Teukolsky equation reduces to the Bardeen-Press equation. In [11], Chandrasekhar found an elegant way, which he labeled transformation theory, to transform it to the Regge-Wheeler or the Zerilli equation, which, in the notation used in this section, appears as

$$\Lambda_+\Lambda_-\Phi_{(\pm)} - V_{(\pm)}\Phi_{(\pm)} = 0, \tag{88}$$

and back. The transformation found by Chandrasekhar is more general than the Darboux transformation used in Sec. III, and it falls within the class of generalized Darboux transformations [35]. It is remarkable that the transformation theory naturally embeds the isospectrality relation between the Regge-Wheeler and the Zerilli equation, as we made manifest by explicitly showing the relation between the Bardeen-Press potential Q_0 and the superpotential W_0 in Eq. (77). We now want to see if the slowly rotating Regge-Wheeler and Zerilli equations can be linked to the slowly rotating expansion of the Teukolsky equation in a similar manner.

Let us revise how we obtain the transformation proposed by Chandrasekhar, slightly generalizing it. The first step is to assume that field U transforms as[3]

$$U = \mathscr{f}\Lambda_+^2\Phi + (T - 2\rho\mathscr{f})\Lambda_+\Phi, \tag{89}$$

where $\mathscr{f}$ and T are the functions that we want to find. Alternatively, by using the fact that Φ satisfies Eq. (88), one can rewrite the transformation as

$$U = \mathscr{f}V\Phi + T\Lambda_+\Phi. \tag{90}$$

The application of the operator Λ_- to both sides of the equation yields

$$\Lambda_- U = -\frac{f_T^2}{f_P^2}\beta\Phi + R\Lambda_+\Phi, \tag{91}$$

where we introduced the functions β and R defined from the two relations

$$\mathscr{R}_1 \equiv \frac{f_T^2}{f_P^2}\beta + \frac{\mathrm{d}}{\mathrm{d}r_*}(\mathscr{f}V) + (T - 2\rho\mathscr{f})V = 0, \tag{92}$$

$$\mathscr{R}_2 \equiv R - \mathscr{f}V - \frac{\mathrm{d}T}{\mathrm{d}r_*} = 0. \tag{93}$$

[3]From now on, we drop the label $\pm$ from the Regge-Wheeler and Zerilli functions, as each transformation must be intended for the two fields separately but with the same functional form.

Now, we require that the function Y satisfies Eq. (73), which leads to two additional relations for the free functions that we defined as

$$\mathscr{R}_3 \equiv RV - \frac{f_T^2}{f_P^2}\frac{\mathrm{d}\beta}{\mathrm{d}r_*} - \ell QV = 0, \tag{94}$$

$$\mathscr{R}_4 \equiv \frac{\mathrm{d}R}{\mathrm{d}r_*} + RP - (QT - 2\rho R) - \frac{f_T^2}{f_P^2}\beta = 0. \tag{95}$$

Finally, it is worth noticing that one can combine Eqs. (92)–(95) into one single integral relation

$$\mathscr{R}_5 \equiv \frac{f_P^2}{f_T^2} RV\ell + \beta T = K = \text{const}. \tag{96}$$

This last relation assures that, if a transformation is found, the inverse transformation also exists,

$$K\Phi = \frac{f_P^2}{f_T^2}(RU - T\Lambda\_U), \tag{97}$$

$$K\Lambda_+\Phi = \beta U + \frac{f_P^2}{f_T^2} V\ell\Lambda\_U. \tag{98}$$

Now, we assume that the unknown functions ℓ, R, T, and β as well as the constant K can be expanded in powers of a, m, and ρ in the same form of Q given in Eqs. (76), (78), and (79). In the limit $a \to 0$, they must reproduce the same value found by Chandrasekhar for the generalized Darboux transformation between the Bardeen-Press and the Regge-Wheeler and Zerilli equations,

$$\ell\ell_0 = 1, \qquad \beta_0 = \pm 3, \qquad R_0 = Q_0, \tag{99}$$

$$K_0 = \kappa_0, \qquad \tilde{T}_0 = \frac{1}{\beta_0}\left(\kappa_0 - \frac{V_0}{W_0}\right), \tag{100}$$

and $T_0 = \tilde{T}_0 + 2\rho$. For the next orders in the spin we solve the equations $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_4$, and $\mathscr{R}_5$, assuming that they vanish independently for each power in ρ and m. We were able to find a solution to the problem, as schematically shown below.

At first order in the spin the functions to be specified are $\ell\ell_{1,i}$, $\beta_{1,i}$, $T_{1,i}$, $\tilde{R}_{1,i} = R_{1,i} - Q_{1,i}$, and $K_{1,i}$ with $i = [0, 1, 2]$. We also find it convenient to rewrite $V_1 = V_{1,0} + V_{1,2}\rho^2$. We find terms proportional to ρ^3 only in $\mathscr{R}_1$, $\mathscr{R}_4$, and $\mathscr{R}_5$. By making them vanish, we find that

$$\ell\ell_{1,2} = \beta_{1,2} = \tilde{R}_{1,2} = 0. \tag{101}$$

We can manipulate the terms proportional to ρ^2 into three algebraic relations and one first-order differential equation. Their solution reads

$$2\ell\ell_{1,1} = T_{1,2} + \tilde{T}_0\frac{V_{1,2}}{V_0} + \frac{1}{V_0}\frac{\mathrm{d}V_{1,2}}{\mathrm{d}r_{*,0}}, \tag{102}$$

$$2\beta_{1,1} = K_{1,2} - \frac{1}{W_0}\left(\frac{Q_{1,2}V_0}{Q_0} + V_{1,2}\right) - \beta_0 T_{1,2}, \tag{103}$$

$$2\tilde{R}_{1,1} = Q_0 T_{1,2} + \left(\tilde{T}_0 + 2\frac{3-2r}{r^2}\right)Q_{1,2} - \frac{\mathrm{d}Q_{1,2}}{\mathrm{d}r_{*,0}}, \tag{104}$$

$$T_{1,2} = -\frac{\lambda_1}{r} - (3+\beta_0)(\bar{W}_1 - \bar{k}_1) + t_2, \tag{105}$$

where t_2 is an integration constant. Analogously, the terms proportional to ρ can be written as three algebraic relations and one first-order differential equation, whose solution is

$$2\ell\ell_{1,0} = T_{1,1} - \tilde{T}_0\ell\ell_{1,1} + \frac{W_0Q_0}{V_0}(\beta_{1,1} - \beta_0\ell\ell_{1,1}) + \frac{\mathrm{d}\ell\ell_{1,1}}{\mathrm{d}r_{*,0}}, \tag{106}$$

$$2\beta_{1,0} = K_{1,1} + \frac{V_0}{W_0}\left(\frac{\beta_{1,1}}{\beta_0} - \ell\ell_{1,1} - \frac{R_{1,1}}{Q_0}\right) - \beta_0 T_{1,1} - \frac{\kappa_0\beta_{1,1}}{\beta_0}, \tag{107}$$

$$2\tilde{R}_{1,0} = Q_0(T_{1,1} + \beta_{1,1}W_0) + \tilde{T}_0 Q_{1,1} + 2\frac{3-2r}{r^2}R_{1,1} - \frac{\mathrm{d}R_{1,1}}{\mathrm{d}r_{*,0}}, \tag{108}$$

$$2T_{1,1} = T_{1,2}\tilde{T}_0 - V_{1,2} - Q_{1,2} - 2f_1 + 2f_0\frac{\lambda_1}{r^2} + \frac{4(r+1)}{r^3} + \kappa_0\bar{k}_1 + t_1, \tag{109}$$

where t_1 is another integration constant. Finally, from the last group of equations we obtain

$$T_{1,0} = \frac{\mathrm{d}f_1}{\mathrm{d}r_{*,0}} + \frac{1}{Q_0}\frac{\mathrm{d}R_{1,0}}{\mathrm{d}r_{0,*}} - \beta_{1,0}W_0 - 2\frac{3-2r}{r^2}\frac{R_{1,0}}{Q_0} + (\tilde{T}_0 + 2\beta_0 W_0)f_1, \tag{110}$$

$$K_{1,2} = 2\beta_0 t_2 - 4\lambda, \qquad K_{1,1} = 2\beta_0 t_1 + \kappa_0 t_2, \tag{111}$$

$$K_{1,0} = \kappa_0 t_1. \tag{112}$$

We check that these relations can be found only for the same choice of the tortoise coordinate f_1 as given in Eq. (60). This shows that the deep intimacy between metric perturbations and the Teukolsky formalism is maintained at first order in the spin. We argue that a different choice for the tortoise coordinate can still admit a generalized Darboux transformation between the Regge-Wheeler, the Zerilli, and the Teukolsky equations at first order in the spin but not with a compact form polynomial in ρ. It is worth noting that the integration constants t_1 and t_2 are left completely unspecified from the calculation.

The calculation at second order is complicated because it has more free functions, but it can be performed in the same fashion: treating each term with a different power in ρ and in m as independent and setting it to zero. In this way, we are able to find a consistent solution, provided that the tortoise coordinate is the same as that selected by the isospectrality analysis of the Regge-Wheeler and Zerilli potentials described in Sec. III. The whole, generalized, Darboux transformation is presented in Appendix E.

V. DISCUSSION

A. Metric reconstruction

Throughout the calculations performed in this paper, we managed to find a set of analytic transformations that bring slowly rotating Regge-Wheeler and Zerilli equations to the spin-2, slow spin limit of the Teukolsky equations and vice versa. The second direction is of particular interest for various fields of gravitational physics because it allows one to obtain the perturbations of the metric h_{ab} once the gauge invariant Weyl scalars Ψ^0 and Ψ^4 are known. We summarize the main steps of this transformation, with direct links to the necessary equations provided in the paper.

(i) The Weyl scalars Ψ^0 and Ψ^4 must be decomposed according to Eqs. (65) and (66). The radial functions $R^{(\pm 2)}_{\ell m}(r)$ can then be transformed into the variables $U^{(\pm 2)}_{\ell m}(r)$ as specified in Eq. (72).

(ii) The relation that tells us how to transform the Teukolsky variable to either the Regge-Wheeler or the Zerilli variables is given in Eq. (97). This relation depends on the quantities R, T, and K, which have been provided up to second order in the spin. It is worth noting that these quantities depend on the parameter $\beta_0 = \pm 3$, for which the sign choice tells us how $\Phi^{\ell m}_{(\pm)}(r)$ are given in terms of $U^{(2)}_{\ell m}(r)$ and its radial derivative (up to an integration constant). The transformations for $U^{(-2)}_{\ell m}$ can be found analogously by following the calculations provided in this paper.

(iii) Once one knows $\Phi^{\ell m}_{(\pm)}(r)$, the functions $h^{\ell m}_0(r)$, $h^{\ell m}_1(r)$, $H^{\ell m}_1(r)$, and $K^{\ell m}(r)$ can be constructed from Eqs. (36)–(39). Indeed, by repeated use of these equations, and by keeping terms up to second order in a, one can write

$$\begin{aligned} h^{\ell}_{i,(\pm)} &= c^{\ell}_{i,(\pm)}\Phi^{\ell}_{(\pm)} + d^{\ell}_{i,(\pm)}\partial_r\Phi^{\ell}_{(\pm)} + a[\mathcal{Q}_\ell(s^{-\ell-1}_{i,(\pm)}\Phi^{\ell-1}_{(\mp)} \\ &\quad + t^{-\ell-1}_{i,(\pm)}\partial_r\Phi^{\ell-1}_{(\mp)}) + \mathcal{Q}_{\ell+1}(s^{\ell}_{i,(\pm)}\Phi^{\ell+1}_{(\mp)} \\ &\quad + t^{\ell+1}_{i,(\pm)}\partial_r\Phi^{\ell+1}_{(\mp)})] + a^2[\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell(u^{-\ell-1}_{i,(\pm)}\Phi^{\ell-2}_{(\pm)} \\ &\quad + v^{-\ell-1}_{i,(\pm)}\partial_r\Phi^{\ell-2}_{(\pm)}) + \mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}(u^{\ell}_{i,(\pm)}\Phi^{\ell+2}_{(\pm)} \\ &\quad + v^{\ell+1}_{i,(\pm)}\partial_r\Phi^{\ell+2}_{(\pm)})], \end{aligned} \tag{113}$$

where $h^{\ell}_{0,(-)} = h^{\ell}_0$, $h^{\ell}_{1,(-)} = h^{\ell}_1$, $h^{\ell}_{K,(+)} = K^{\ell}$, and $h^{\ell}_{1,(+)} = H^{\ell}_1$, and the coefficients $c^{\ell}_{i,(\pm)}$, $d^{\ell}_{i,(\pm)}$, $s^{\ell}_{i,(\pm)}$, $t^{\ell}_{i,(\pm)}$, $u^{\ell}_{i,(\pm)}$, and $v^{\ell}_{i,(\pm)}$ can be related to the coefficients c^{ℓ}_i, d^{ℓ}_i, s^{ℓ}_i, t^{ℓ}_i, u^{ℓ}_i, and v^{ℓ}_i introduced in Sec. II C. Finally, the remaining functions $H^{\ell}_0(r)$ and $H^{\ell}_2(r)$ are determined by the constraints set by the polar-led equations discussed in Sec. II C.

All of these steps completely determine how to construct, in vacuum, the first-order perturbation metric h_{ab} in the Regge-Wheeler gauge, up to second order in the spin starting from our knowledge of the Weyl scalars.

B. Generator of the slowly rotating Regge-Wheeler and Zerilli effective potentials

Another very remarkable result of this paper is that we could find the generalization of the Chandrasekhar superpotential up to second order in the spin. This allows one to have a handy and compact formula to generate the first order to the Regge-Wheeler and Zerilli equations (47) and (48),

$$\begin{aligned} V^{\ell}_{(\pm),1} &= (\kappa_0 + 2\beta_0^2 W^{\ell}_0)W^{\ell}_1 + \kappa_0\kappa_1 \\ &\quad + \beta_0 f_0\left[\frac{\mathrm{d}}{\mathrm{d}r}W^{\ell}_1 + f_1\frac{\mathrm{d}W^{\ell}_0}{\mathrm{d}r}\right], \end{aligned} \tag{114}$$

and second-order corrections (see Appendix C)

$$\begin{aligned} V^{\ell}_{(\pm),2} &= (\kappa_0 + 2\beta_0^2 W^{\ell}_0)W^{\ell}_2 + \kappa_0\kappa_2 + \beta_0 f_0\left[\frac{\mathrm{d}}{\mathrm{d}r}W^{\ell}_2 + f_2\frac{\mathrm{d}W^{\ell}_0}{\mathrm{d}r}\right] \\ &\quad - \beta_0\frac{m^2}{\rho^2}\left[\beta_0 W^{\ell 2}_1 + f_1 f_0\frac{\mathrm{d}W^{\ell}_1}{\mathrm{d}r}\right]. \end{aligned} \tag{115}$$

The explicit expressions of W^{ℓ}_0, W^{ℓ}_1, κ_0, κ_1, and f_1 are given in Sec. III, whereas expressions for W^{ℓ}_2, κ_2, and f_2 can be found in Appendix D.

We stress that the function W^{ℓ} and the selection of the tortoise coordinate f_T are naturally consistent with the procedure that links the slowly rotating Teukolsky potential with the Regge-Wheeler and the Zerilli potentials. This procedure also leaves some integration constants unspecified. We argue that this result can be interpreted as the freedom of scaling the functions $U^{(\pm 2)}_{\ell m}(r)$ through the Teukolsky-Starobinsky identities [37,38].

C. Kerr metric perturbation conjecture

The deep link between metric perturbations and curvature perturbations revealed by Chandrasekhar's transformation theory solidly holds up to second order in the spin. For this reason we conjecture the existence of a couple of yet to be discovered equations that generalize the Regge-Wheeler and the Zerilli equations for any value in the spin. These equations would describe the perturbations of a Kerr metric, rather than the perturbations of its curvature as in the

Teukolsky equations. Past investigations failed in finding these equations, but we hope that the results of this paper can fuel new research in this direction. The possible discovery of such equations would give insight into the isospectrality of modes of definite parity for fully spinning metric (see, e.g., the Appendix of [39] for a discussion on reconstructing parity definite metric perturbations on a Kerr background).

We remark that we compared the slowly rotating Regge-Wheeler and Zerilli potentials with those obtained by Chandrasekhar and Detweiler in [40]. The Chandrasekhar-Detweiler potentials are obtained by performing a generalized Darboux transformation to the Teukolsky equation that brings it into a Schrödinger-like form. Among these four potentials, two reduce to the Regge-Wheeler potential in the nonrotating limit, while the other two reduce to the Zerilli potential. These potentials have been useful to compute Kerr quasinormal modes with a Wentzel-Kramers-Brillouin method [41]. Moreover, it was shown numerically that the lowest-order quasinormal modes of one of the Chandrasekhar-Detweiler potentials, which reduces to the Regge-Wheeler equation, agree with those computed from the Teukolsky equation until second order in the spin [42]. Unfortunately, the shape of the first-order correction in the spin to the Chandrasekhar-Detweiler potentials is always different from the corrections found in this paper in Eqs. (47) and (48). We also studied whether transforming the equations into a different gauge or using a different tortoise coordinate could bring the potentials into the same form, but it was not possible. We argued that the two classes of potentials must then be linked by a generalized Darboux transformation, even if we were not able to prove it.

VI. CONCLUSIONS

In this paper we provided a complete formalism to study vacuum BHPT in the regime of slow rotation up to second order in the spin. We described how to perturb a slowly spinning Kerr metric in the Regge-Wheeler gauge in the frequency domain, and the prescription to decouple the radial and the angular contribution to the equations. In this way we obtained seven polar-led equations and three axial-led equations, where each mode of angular momentum ℓ couples, starting from the first order in the spin, to modes of different parity and angular momentum $\ell \pm 1$ and, starting from second order in the spin, to modes of the same parity and $\ell \pm 2$. We showed that a suitable redefinition of the variables allows one to completely decouple modes of different angular momentum and parity, leading to two diagonalized second-order differential equations that generalize the Regge-Wheeler and the Zerilli equations up to second order in the spin.

We then proved that these two potentials are not independent from each other, as a transformation that brings one to the other and vice versa was found. The existence of this transformation ensures the isospectrality of the two potentials, as well as the existence of a function that generates them. This generating function, also known as a superpotential, can be understood by comparing the metric perturbation equations to the Teukolsky equations. Indeed, we found that a third transformation that links Teukolsky, Regge-Wheeler, and Zerilli equations still exists at second order in the spin. We discussed how the existence of this transformation naturally embeds a procedure of metric reconstruction.

The approach taken in the paper does not seem to single out a reason why these results should not hold at any higher order in the spin expansion. The only impediment is the increasing difficulty in the calculations, especially for the computation of the integrals that appear in the process of decoupling the equations of motion, such as those listed in Appendix A. For this reason it would be revolutionary to find a prescription to study gravitational perturbations of a Kerr metric for any spin. Moreover, we expect that the covariant and gauge-independent formalism developed in [43,44] should also hold at any order in the spin.

The results of this paper are very relevant for different fields of application. The difficulty of studying quasinormal modes of rotating solutions in alternative theories of gravity requires some sort of simplification of the problem. One possibility often encountered is to evaluate them perturbatively in the spin, as it was done in [28–32]. It would be interesting to see if, at least in the small coupling limit, one would still get a slowly rotating Regge-Wheeler equation and a Zerilli equation plus a correction due to the modifications of GR. By writing the problem in this form, one can use more accurate methods for the computation of the quasinormal modes, rather than the direct integration usually employed in these cases, e.g., the continued fraction method [45].

In the context of quasinormal modes computation beyond GR, there was recent progress in developing a generalized Teukolsky equation for any metric that is modified from the Kerr metric by a small parameter [15–17]. Such a derivation requires the knowledge of metric perturbations, which appear as a “source” term for the perturbations of the Weyl scalars. Even though the procedure outlined in this paper is only perturbative in the spin, we stress that all the terms that require the metric reconstruction are multiplied by the small perturbative parameter of the theory. In an analogous case for scalar field perturbations on top of a non-Kerr BH, it was shown that treating this term in a small-spin expansion does not strongly affect the computation of the quasinormal modes [46].

Finally, it would be interesting to generalize the metric reconstruction procedure to the case where a point particle source is present in the spacetime, along the lines of [22]. This prescription would be useful in the context of self-force calculations, as it was noted that second-order contributions are necessary to compute accurate waveforms of extreme mass ratio inspirals [9].

ACKNOWLEDGMENTS

I am indebted to Stas Babak, Enrico Barausse, Emanuele Berti, Éric Chassande-Mottin, Kostas Glampedakis, Leonardo Gualtieri, Luca Santoni, Sebastian Völkel, and Niels Warburton for useful discussions, fruitful comments on the interpretation of the results, and valuable feedback on the manuscript.

APPENDIX A: USEFUL INTEGRALS

According to the author's knowledge, the integrals appearing in the decoupling of the equations in Sec. II B are not explicitly computed in the existing literature. For this reason we report them here for the interested reader. All the expressions are obtained using simplifications due to the completeness relation of spherical harmonics (16), as well as the spherical harmonics relation (17). By recursively applying the mixing of the spherical harmonics and their derivatives with the trigonometric functions (18) and (19), we obtain the following expression:

$$\mathcal{C}_1 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos\theta Y^\ell Y^{*\ell'} = f_{\ell-1}\mathcal{Q}_\ell + f_{\ell+1}\mathcal{Q}_{\ell+1}, \tag{A1}$$

$$\mathcal{C}_2 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos^2\theta Y^\ell Y^{*\ell'} = f_{\ell-2}\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell + f_\ell(\mathcal{Q}_\ell^2 + \mathcal{Q}_{\ell+1}^2) + f_{\ell+2}\mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}, \tag{A2}$$

$$\mathcal{S}_1 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin\theta Y^\ell_{,\theta} Y^{*\ell'} = (\ell-1) f_{\ell-1}\mathcal{Q}_\ell - (\ell+2) f_{\ell+1}\mathcal{Q}_{\ell+1}, \tag{A3}$$

$$\mathcal{S}_2 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin\theta\cos\theta Y^\ell_{,\theta} Y^{*\ell'} = (\ell-2) f_{\ell-2}\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell + f_\ell(\ell\mathcal{Q}_{\ell+1}^2 - (\ell+1)\mathcal{Q}_\ell^2) - (\ell+3) f_{\ell+2}\mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}, \tag{A4}$$

$$\bar{\mathcal{S}}_1 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin\theta Y^\ell Y^{*\ell'}_{,\theta} = -(\ell+1) f_{\ell-1}\mathcal{Q}_\ell + \ell f_{\ell+1}\mathcal{Q}_{\ell+1}, \tag{A5}$$

$$\bar{\mathcal{S}}_2 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin\theta\cos\theta Y^\ell Y^{*\ell'}_{,\theta} = -(\ell+1) f_{\ell-2}\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell + f_\ell(\ell\mathcal{Q}_{\ell+1}^2 - (\ell+1)\mathcal{Q}_\ell^2) + \ell f_{\ell+2}\mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}, \tag{A6}$$

$$\mathcal{S}\bar{\mathcal{S}} f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin^2\theta Y^\ell_{,\theta} Y^{*\ell'}_{,\theta} = -(\ell-2)(\ell+1) f_{\ell-2}\mathcal{Q}_{\ell-1}\mathcal{Q}_\ell + f_\ell(\ell^2\mathcal{Q}_{\ell+1}^2 + (\ell+1)^2\mathcal{Q}_\ell^2) - \ell(\ell+3) f_{\ell+2}\mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}. \tag{A7}$$

With the knowledge of these integrals we can compute the following operators, where we defined $\mathcal{S}_0 f_\ell \equiv 0$ and $\mathcal{C}_0 f_\ell \equiv f_\ell$. For the class of equations belonging to the second group, one would need to use

$$\mathcal{A}_n f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos^n\theta \left(Y^\ell_{,\theta} Y^{*\ell'}_{,\theta} + \frac{Y^\ell_{,\varphi} Y^{*\ell'}_{,\varphi}}{\sin^2\theta} \right) = [\mathcal{C}_n(\lambda+2) + n\mathcal{S}_n] f_\ell, \tag{A8}$$

$$\mathcal{B}_n f_\ell \equiv f_{\ell'} \int \frac{\mathrm{d}\Omega}{\sin\theta} \cos^n\theta (Y^\ell_{,\varphi} Y^{*\ell'}_{,\theta} - Y^\ell_{,\theta} Y^{*\ell'}_{,\varphi}) = \mathrm{i} mn \mathcal{C}_{n-1} f_\ell, \tag{A9}$$

$$\mathcal{X}_n f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos^n\theta (X^\ell Y^{*\ell'}_{,\theta} - W^\ell Y^{*\ell'}_{,\varphi}) = \mathrm{i} m[\mathcal{C}_n(\lambda - 2n) + 2n\mathcal{S}_n] f_\ell, \tag{A10}$$

$$\bar{\mathcal{X}}_n f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos^n\theta \left(\sin\theta W^\ell Y^{*\ell'}_{,\theta} + \frac{X^\ell Y^{*\ell'}_{,\varphi}}{\sin\theta} \right) = 2\left[nm^2 \mathcal{C}_{n-1} - (n+1)\mathcal{S}_{n+1} - \left(\frac{\bar{\mathcal{S}}_{n+1}}{2} + \mathcal{C}_{n+1} \right) \frac{\kappa_0}{\lambda} \right] f_\ell, \tag{A11}$$

$$\tilde{\mathcal{A}}_2 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega (\sin^2\theta Y^\ell_{,\theta} Y^{*\ell'}_{,\theta} - Y^\ell_{,\varphi} Y^{*\ell'}_{,\varphi}) = (\mathcal{S}\bar{\mathcal{S}} - m^2) f_\ell, \tag{A12}$$

$$\tilde{\mathcal{B}}_2 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin\theta (Y^\ell_{,\varphi} Y^{*\ell'}_{,\theta} + Y^\ell_{,\theta} Y^{*\ell'}_{,\varphi}) = \mathrm{i} m(\bar{\mathcal{S}}_1 - \mathcal{S}_1) f_\ell, \tag{A13}$$

while for the third group one has

$$\mathcal{F}_n f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos^n\theta (Y^\ell_{,\theta} X^{*\ell'} - Y^\ell_{,\varphi} W^{*\ell'}) = \mathrm{i}m[(\lambda+2n+4)\mathcal{C}_n - 2\mathcal{C}_n(\lambda+2) - 2n\mathcal{S}_n] f_\ell, \tag{A14}$$

$$\mathcal{G}_n f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos^n\theta \left(\sin\theta Y^\ell_{,\theta} W^{*\ell'} + \frac{Y^\ell_{,\varphi} X^{*\ell'}}{\sin\theta} \right) = [2m^2 n\mathcal{C}_{n-1} - (\lambda+2n+4)\mathcal{S}_{n+1} - 2\mathcal{C}_{n+1}(\lambda+2)] f_\ell, \tag{A15}$$

$$\mathcal{H} f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin^2\theta Y^\ell W^{*\ell'} = [(\lambda+2)(\mathcal{C}_2 - 1) - 2\bar{\mathcal{S}}_2 + 2m^2] f_\ell, \tag{A16}$$

$$\bar{\mathcal{H}} f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \sin^2\theta Y^\ell X^{*\ell'} = -2\mathrm{i}m(\bar{\mathcal{S}}_1 - \mathcal{C}_1) f_\ell, \tag{A17}$$

$$\mathcal{J}_0 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \left(W_\ell W^{*\ell'} + \frac{X^\ell X^{*\ell'}}{\sin^2\theta} \right) = \kappa_0 f_\ell, \tag{A18}$$

$$\mathcal{J}_2 f_\ell \equiv f_{\ell'} \int \mathrm{d}\Omega \cos^2\theta \left(W_\ell W^{*\ell'} + \frac{X^\ell X^{*\ell'}}{\sin^2\theta} \right) = [8m^2 + ((\lambda-2)\mathcal{C}_2 - 2\bar{\mathcal{S}}_2 - 2)(\lambda+2) - 2(\lambda+6)\mathcal{S}_2] f_\ell, \tag{A19}$$

$$\mathcal{K}_0 f_\ell \equiv f_{\ell'} \int \frac{\mathrm{d}\Omega}{\sin\theta} (W_\ell X^{*\ell'} - X^\ell W^{*\ell'}) = 0, \tag{A20}$$

$$\mathcal{K}_2 f_\ell \equiv f_{\ell'} \int \frac{\mathrm{d}\Omega}{\sin\theta} \cos^2\theta (W_\ell X^{*\ell'} - X^\ell W^{*\ell'}) = -2\mathrm{i}m[(\bar{\mathcal{S}}_1 + 3\mathcal{C}_1)(\lambda+2) + (\lambda+6)(\mathcal{S}_1 - \mathcal{C}_1)] f_\ell, \tag{A21}$$

where we note that $\lambda = \ell^2 + \ell - 2$ and $\kappa_0 = \lambda(\lambda+2)$.

APPENDIX B: DECOUPLING COEFFICIENTS

Let us present the explicit form of the coefficients that appear in Eqs. (36)–(39) and that allow a complete decoupling of the slow-spinning Regge-Wheeler and Zerilli equations. We report here only the coefficients proportional to a, while those proportional to a^2 are reported in Appendix G,

$$c^\ell_0 = -\frac{6f_0}{\ell(\ell+1)r^3}, \qquad c^\ell_1 = \frac{1}{r^2 f_0}, \qquad c^\ell_K = \frac{f_0}{\rho r} c^\ell_H + \frac{\rho^2}{r^2(\lambda+2)} - \frac{24 + 3r(\lambda-6) + 8r^2\lambda + r^3\lambda^2(\lambda+6)}{4r^6(\lambda+2)(3+\lambda r)}, \tag{B1}$$

$$\begin{aligned} c^\ell_H = {} & \frac{21 + 18r\lambda + r^2\lambda(3\lambda-2)}{rf_0(\lambda+2)(3+r\lambda)^2}\rho^3 + \frac{12r-11}{4r^4 f_0}\rho \\ & + \frac{8\lambda^3 r^5 - 2\lambda^2(19\lambda+20)r^4 + \lambda(29\lambda^2 - 76\lambda - 312)r^3 + 12(9\lambda^2 + 22\lambda - 36)r^2 + 27(\lambda+22)r - 180}{2(\lambda+2)r^5 f_0(3+\lambda r)^3}\rho, \end{aligned} \tag{B2}$$

$$s^\ell_{0,1} = -\frac{1}{\ell+1}, \qquad t^\ell_{0,1} = \frac{2\rho r}{f_0(\ell+1)^2} \qquad s^\ell_{0,0} = \frac{f_0}{(\ell^2 r + 3\ell r + 3)} \left[\frac{2(\ell+3)r^2\rho^2}{(\ell+1)^2} - \frac{(\ell+2)(2r\ell^2 + 6r\ell - \ell + 3)}{2r(\ell+1)} \right], \tag{B3}$$

$$t^\ell_{0,0} = \frac{\rho}{(\ell^2 r + 3\ell r + 3)} \left[\frac{-4r^2\ell(\ell^2 + 5\ell + 6) + 2r(\ell^3 + 5\ell^2 + 2\ell - 6) + 3(\ell+1)}{2r(\ell+1)^2} - \frac{2(\ell+3)r^3\rho^2}{(\ell+1)^2} \right], \tag{B4}$$

$$s^\ell_{0,K} = \frac{2(\ell+3)f_0}{(\ell+1)\rho r} \left[\frac{\ell(\ell r + 2r - 1)}{r^3} + \frac{2\rho^2}{\ell+1} \right] \qquad t^\ell_{0,K} = \frac{2(\ell+3)(\ell+2)}{(\ell+1)r^2}, \tag{B5}$$

$$s^\ell_{0,H} = \frac{\ell(\ell+3)[4(\ell+2)r - 3(\ell+3)] - 4}{(\ell+1)^2 r^3} + \frac{4(\ell+3)\rho^2}{(\ell+1)^2} \qquad t^\ell_{0,H} = \frac{2(\ell+3)(2\ell r + 4r - 3)}{(\ell+1)^2 f_0 r^2}\rho. \tag{B6}$$

APPENDIX C: EFFECTIVE POTENTIAL

In Sec. II we found that the perturbations of a slowly rotating Kerr BH can be recast into the same form of the Regge-Wheeler and Zerilli equations and that the potentials receive a correction in the spin of the form given in Eq. (44). Here, we report the explicit form of $\bar{V}^{\ell}_{(\pm),2}$ defined in Eq. (46). For the axial sector we have

$$\begin{aligned}\bar{V}^{\ell}_{(-),2} = &-\frac{m^2}{r^4} + m^2 f_0\left[-\frac{24(7-6r)}{(\lambda+2)^3 r^6} - \frac{12(47-40r)}{(\lambda+2)^2 r^6} + \frac{2(6r^2-250r-315)}{(\lambda+2)r^6} + \frac{420(6-5r)}{(5+4\lambda)r^6}\right]\\ &- m^2 f_0\left[\frac{2(\lambda-10)}{(\lambda+2)(5+4\lambda)r^2}\rho^2 + 3\frac{6r^2(4\lambda-19)-26r(\lambda-13)-231}{(\lambda+2)^2 r^{10}\rho^2}\right],\end{aligned} \tag{C1}$$

while for the polar sector we find

$$\begin{aligned}\bar{V}^{\ell}_{(+),2} = &-\frac{m^2}{r^4} + \frac{m^2 f_0}{(3+r\lambda)^5}\left[\frac{2(4\lambda-1)\lambda^4 r}{\lambda+2} - \frac{2(34\lambda^5+135\lambda^4-513\lambda^3-1753\lambda^2-312\lambda+60)\lambda^3}{(\lambda+2)^3(5+4\lambda)}\right.\\
&-\frac{(70\lambda^5+810\lambda^4+5599\lambda^3-398\lambda^2-26676\lambda+1704)\lambda^3}{(\lambda+2)^3(5+4\lambda)r}\\
&-\frac{(929\lambda^5+5310\lambda^4+40121\lambda^3+33516\lambda^2-128980\lambda+16272)\lambda^2}{(\lambda+2)^3(5+4\lambda)r^2}\\
&-\frac{3(2993\lambda^5+8760\lambda^4+93012\lambda^3+146824\lambda^2-243912\lambda+19728)\lambda}{2(\lambda+2)^3(5+4\lambda)r^3} - \frac{1944(8\lambda^2-27\lambda-134)}{(\lambda+2)^3(5+4\lambda)r^7}\\
&-\frac{9(6403\lambda^3-20642\lambda^2-54548\lambda+135720)}{2(\lambda+2)^3(5+4\lambda)r^6} - \frac{9(5375\lambda^4-11512\lambda^3+22328\lambda^2+229320\lambda-65664)}{2(\lambda+2)^3(5+4\lambda)r^5}\\
&\left.-\frac{3(8503\lambda^5+1374\lambda^4+160020\lambda^3+455216\lambda^2-349416\lambda+11664)}{2(\lambda+2)^3(5+4\lambda)r^4}\right]\\
&+\frac{2m^2 f_0\rho^2}{(\lambda+2)(5+4\lambda)(3+r\lambda)^4}\left[\frac{3(3\lambda^4+26\lambda^3+214\lambda^2+528\lambda-24)\lambda}{(\lambda+2)^2} - (\lambda^2-6\lambda+2)\lambda^3 r^2\right.\\
&+\frac{18(17\lambda^2-64\lambda-304)}{(\lambda+2)^2 r^3} + \frac{6(71\lambda^3-136\lambda^2-754\lambda+468)}{(\lambda+2)^2 r^2} + \frac{12(19\lambda^3+16\lambda^2+4\lambda+312)\lambda}{(\lambda+2)^2 r}\\
&\left.-\frac{4(2\lambda^4-7\lambda^3-57\lambda^2-76\lambda+12)\lambda^2 r}{(\lambda+2)^2}\right]\\
&+\frac{m^2 f_0}{(\lambda+2)^2(3+r\lambda)^4\rho^2}\left[-\frac{3(13\lambda^2+61\lambda+36)\lambda^5}{r^2} + \frac{(92\lambda^3+197\lambda^2-942\lambda-1032)\lambda^4}{r^3}\right.\\
&+\frac{(-107\lambda^4+665\lambda^3+4582\lambda^2+726\lambda-7272)\lambda^3}{2r^4} - \frac{3(118\lambda^4+138\lambda^3-1133\lambda^2-3180\lambda+1872)\lambda^2}{r^5}\\
&-\frac{18(84\lambda^3-137\lambda^2+317\lambda+90)\lambda}{r^7} - \frac{3(291\lambda^4+455\lambda^3+2211\lambda^2-5106\lambda+1080)\lambda}{r^6}\\
&\left.+\frac{54(115\lambda^2-1138\lambda+918)}{r^9} - \frac{9(443\lambda^3+2319\lambda^2-8670\lambda+2808)}{2r^8} + \frac{648(42\lambda-97)}{r^{10}} + \frac{25920}{r^{11}}\right].\end{aligned} \tag{C2}$$

APPENDIX D: SUPERPOTENTIAL AT SECOND ORDER IN THE SPIN

In Sec. III, we showed how to the derive the superpotential that generates the first-order correction to the Regge-Wheeler and Zerilli equations. Here, we proceed to show the analogous calculation at second order in the spin. First of all, it is convenient to split W^{ℓ}_2 and κ_2 into five parts, such as

$$W_2^\ell = \tilde{W}_2^\ell + \bar{W}_2^\ell \rho^2 + m^2\left(\frac{\hat{W}_2^\ell}{\rho^2} + \check{W}_2^\ell + \widehat{W}_2^\ell \rho^2\right), \quad \text{(D1)}$$

$$\kappa_2 = \tilde{\kappa}_2 + \bar{\kappa}_2\rho^2 + m^2\left(\frac{\hat{\kappa}_2}{\rho^2} + \check{\kappa}_2 + \widehat{\kappa}_2\rho^2\right), \quad \text{(D2)}$$

as well as splitting the tortoise coordinate function at second order as

$$f_2 = \tilde{f}_2 + m^2\left(\frac{\hat{f}_2 + f_1^2}{\rho^2} + \check{f}_2\right). \quad \text{(D3)}$$

Let us insert this expansion in Eqs. (55) and (56) and take quadratic terms in the spin, assuming that W_0^ℓ, W_1^ℓ, f_1, and κ_1 are specified as in Sec. III. By setting to zero the term proportional to $\rho^2 m^0$ in Eq. (55), we obtain

$$\bar{W}_2^\ell = \bar{k}_2 + \frac{(\lambda-6)(2\lambda+1)}{3(\lambda+2)(4\lambda+5)r(\lambda r+3)} + \frac{2}{3rf_0(\lambda+3)} + \frac{9(\lambda^2+\lambda-6) + (-4\lambda^3+34\lambda+36)r^2}{3(\lambda+2)(\lambda+3)(4\lambda+5)r^2(\lambda r+3)}. \quad \text{(D4)}$$

Analogously to the case linear in a, we find that the $\rho^2 m^0$ term of Eq. (56) vanishes for this choice of the tortoise coordinate.

$$\tilde{f}_2 = \frac{\kappa_0\bar{\kappa}_2}{2} + \frac{1}{r^2 f_0} + \frac{f_0(\lambda+5)}{r(4\lambda+5)} + \left(\frac{\kappa_0}{2} + \beta_0^2 W_0^\ell\right)\bar{W}_2^\ell - \frac{3}{2}\frac{\mathrm{d}\bar{W}_2^\ell}{\mathrm{d}r_{*,0}}. \quad \text{(D5)}$$

We now move to the term proportional to $\rho^0 m^0$, and Eqs. (55) and (56) are satisfied simultaneously when

$$\tilde{W}_2^\ell = \tilde{k}_2\left[1 + \frac{\beta_0^2 W_0^\ell}{\kappa_0}\right] + \frac{1}{6r^3 f_0(\lambda+3)} + \frac{1}{(5+4\lambda)(3+r\lambda)^3}\left[\frac{(8\lambda^3+52\lambda^2+95\lambda+54)\lambda^2}{6(\lambda+3)} - \frac{54(\lambda-2)}{(\lambda+2)r^6} - \frac{9(31\lambda^2-111\lambda+326)}{4(\lambda+2)r^5} - \frac{3(47\lambda^3-368\lambda^2+843\lambda-1074)}{4(\lambda+2)r^4} + \frac{-100\lambda^4+1567\lambda^3-3045\lambda^2+7836\lambda-1116}{12(\lambda+2)r^3} - \frac{4\lambda^6-148\lambda^5-159\lambda^4-393\lambda^3-4095\lambda^2-252\lambda-324}{6(\lambda+2)(\lambda+3)r^2} + \frac{(12\lambda^5+34\lambda^4+281\lambda^3+1029\lambda^2+618\lambda+216)\lambda}{6(\lambda+2)(\lambda+3)r}\right], \quad \text{(D6)}$$

$$\bar{k}_2 = \bar{k}_1\frac{\lambda(\lambda-10)}{5+4\lambda}, \qquad \bar{\kappa}_2 = -\bar{k}_2 + \tilde{k}_2\left(\frac{6}{\kappa_0}\right)^2, \qquad \tilde{\kappa}_2 = -\tilde{k}_2. \quad \text{(D7)}$$

We can see from the previous equation that the constant $\tilde{k}_2$ that comes from the integration of $\tilde{W}_2^\ell$ is not fully specified by the equation. We now solve the terms proportional to m^2 in the Darboux relations. Let us start by checking the relation (55) for the term proportional to $\rho^2 m^2$, for which we infer

$$\widehat{W}_2^\ell = \widehat{k}_2 + \frac{1}{(3+r\lambda)^3}\left[\frac{-23\lambda^4+116\lambda^3+590\lambda^2+352\lambda+72}{3(\lambda+2)^3(4\lambda+5)} + \frac{2(2\lambda+1)\lambda^2 r^2}{3(\lambda+2)(4\lambda+5)} + \frac{-17\lambda^2+64\lambda+304}{(\lambda+2)^3(4\lambda+5)r^2} + \frac{2(-\lambda^4+17\lambda^3+78\lambda^2+80\lambda+24)\lambda r}{3(\lambda+2)^3(4\lambda+5)} + \frac{-28\lambda^3+38\lambda^2+320\lambda+48}{(\lambda+2)^3(4\lambda+5)r}\right], \quad \text{(D8)}$$

and again $\widehat{k}_2$ is an integration constant. From requiring consistency with Eq. (56) we must fix

$$\check{f}_2 = \frac{\kappa_0\widehat{\kappa}_2}{2} + \frac{f_0(\lambda-10)}{r^2(\lambda+2)(5+4\lambda)} - \frac{\beta_0^2}{2}\bar{W}_1^\ell + \left(\frac{\kappa_0}{2} + \beta_0^2 W_0^\ell\right)\widehat{W}_2^\ell - \frac{3}{2}\frac{\mathrm{d}\widehat{W}_2^\ell}{\mathrm{d}r_{*,0}}. \quad \text{(D9)}$$

Then, repeating the procedure for the terms proportional to $m^2\rho^0$, we obtain

$$\check{W}_2^\ell = \check{k}_2 + \frac{(\widehat{k}_2 + \widehat{\kappa}_2)\kappa_0 W_0^\ell}{2r} + \beta_0^2\frac{\widehat{\kappa}_2 W_0^{\ell 2}}{2r^2} + \frac{1}{(3+r\lambda)^4}\left[-\frac{2}{9}\lambda^5 r^2 + \frac{3(80\lambda^2-217\lambda-1186)}{2(\lambda+2)^3(4\lambda+5)r^6} + \frac{(284\lambda^6+1051\lambda^5+332\lambda^4-5822\lambda^3-11860\lambda^2-2120\lambda+720)\lambda^2}{36(\lambda+2)^3(4\lambda+5)} + \frac{811\lambda^3-6416\lambda^2-9686\lambda+32652}{4(\lambda+2)^3(4\lambda+5)r^5}\right.$$

$$+\frac{397\lambda^4-3527\lambda^3-782\lambda^2+21828\lambda-14568}{2(\lambda+2)^3(4\lambda+5)r^4}+\frac{527\lambda^5-3322\lambda^4+3296\lambda^3+22592\lambda^2-36296\lambda+4176}{4(\lambda+2)^3(4\lambda+5)r^3}$$
$$+\frac{776\lambda^6-1123\lambda^5+10464\lambda^4+25902\lambda^3-39724\lambda^2+18600\lambda+432}{12(\lambda+2)^3(4\lambda+5)r^2}+\frac{(8\lambda^4-46\lambda^3-202\lambda^2-156\lambda+9)\lambda^3 r}{9(\lambda+2)(4\lambda+5)}$$
$$\left.+\frac{(164\lambda^6+518\lambda^5+2379\lambda^4+3384\lambda^3-2974\lambda^2+3432\lambda+216)\lambda}{6(\lambda+2)^3(4\lambda+5)r}\right], \tag{D10}$$

as well as

$$\hat{\mathscr{f}}_2=\frac{1}{4}\frac{\mathrm{d}^2\check{\mathscr{f}}_2}{\mathrm{d}r_{*,0}^2}+\left[W_0^\ell(\kappa_0+\beta_0^2W_0^\ell)+\frac{3}{2}\frac{\mathrm{d}W_0^\ell}{\mathrm{d}r_{*,0}}\right]\check{\mathscr{f}}_2+\frac{\kappa_0\check{\kappa}_2}{2}+\mathscr{f}_1\left[\frac{2}{r^3}+\frac{3}{2}\frac{\mathrm{d}\bar{W}_1^\ell}{\mathrm{d}r_{*,0}}\right]+\left(\frac{\kappa_0}{2}+\beta_0^2W_0^\ell\right)\check{W}_2^\ell-\frac{3}{2}\frac{\mathrm{d}\check{W}_2^\ell}{\mathrm{d}r_{*,0}}$$
$$-\bar{W}_1^\ell\tilde{W}_1^\ell+\frac{r(\lambda-10)+12}{2r^5(\lambda+2)}+\mathscr{f}_0\frac{3(61\lambda^2+74\lambda-180)+2(25\lambda^3+45\lambda^2+96\lambda+320)r}{(\lambda+2)^3(5+4\lambda)r^6}. \tag{D11}$$

Finally, from the terms proportional to $m^2\rho^{-2}$ we obtain

$$\tilde{W}_2^\ell=\frac{(\check{k}_2+\check{\kappa}_2)\kappa_0}{36}(\kappa_0+2\beta_0^2W_0^\ell)+\frac{\mathscr{f}_0}{(3+r\lambda)^5}\left[\frac{(\lambda+10)^2\lambda^5}{36(\lambda+2)r}+\frac{(\lambda^3+60\lambda^2+302\lambda+624)\lambda^4}{12(\lambda+2)^2r^2}\right.$$
$$+\frac{(7\lambda^3+322\lambda^2+158\lambda+1202)\lambda^3}{6(\lambda+2)^2r^3}+\frac{(21\lambda^3+1618\lambda^2-1184\lambda+1532)\lambda^2}{4(\lambda+2)^2r^4}+\frac{3(9\lambda^2+5777\lambda-3426)\lambda}{4(\lambda+2)^2r^6}$$
$$\left.+\frac{3(8\lambda^3+2299\lambda^2-1910\lambda+400)\lambda}{4(\lambda+2)^2r^5}-\frac{27(8\lambda-259)}{2(\lambda+2)^2r^8}+\frac{9(1337\lambda-360)}{2(\lambda+2)^2r^7}-\frac{162}{(\lambda+2)^2r^9}\right], \tag{D12}$$

$$\tilde{\kappa}_2=(\check{k}_2+\check{\kappa}_2)\left(\frac{\kappa_0}{6}\right)^2,\qquad \overset{\frown}{k}_2=\bar{k}_1\frac{60+34\lambda+5\lambda^2}{(\lambda+2)^2(5+4\lambda)},\qquad \overset{\frown}{\kappa}_2=\left(\frac{6}{\kappa_0}\right)^2\check{k}_2-\bar{k}_1\frac{80+60\lambda+13\lambda^2}{(\lambda+2)^2(5+4\lambda)}. \tag{D13}$$

With this, we completely specify the function W_2^ℓ, upon choosing the integration constants. Indeed, we see again how the constants $\check{k}_2$ and $\check{\kappa}_2$ are not specified by the validity of the Darboux transformation.

APPENDIX E: TRANSFORMATION THEORY AT SECOND ORDER IN THE SPIN

We now sketch the form of the calculations carried out to obtain the second-order spin correction functions $\mathscr{f}\mathscr{f}_2$, T_2, β_2, and $\tilde{R}_2\equiv R_2-Q_2$ and constant K_2, which appear in the transformations defined in Eqs. (89), (91), and (96). We find it convenient to perform the following decomposition in powers of ρ and m,

$$F_2=\sum_{i=0}^{2}F_{2,i}\rho^i+m^2\sum_{i=-2}^{2}\bar{F}_{2,i}\rho^i, \tag{E1}$$

where F_2 stands for any of the functions $\mathscr{f}\mathscr{f}_2$, T_2, β_2, and $\tilde{R}_2$ (as well as R_2) and constant K_2. Moreover, in order to express the following formulas in a compact form, it is useful to split the Regge-Wheeler and the Zerilli equations potentials, the Teukolsky potential, and the reduced separation constant as[4]

$$V_2=V_{2,0}+V_{2,2}\rho^2+m^2\left(\frac{\bar{V}_{2,-2}}{\rho^2}+\bar{V}_{2,0}+\bar{V}_{2,2}\right), \tag{E2}$$

$$Q_2=\sum_{i=0}^{2}Q_{2,i}\rho^i+m^2\sum_{i=-2}^{2}\bar{Q}_{2,i}\rho^i, \tag{E3}$$

$$\lambda_2^{(s)}=\tilde{\lambda}_2+m^2\bar{\lambda}_2. \tag{E4}$$

Finally, we make extensive use of the superpotential W_2 and the tortoise coordinate $\mathscr{f}_2$ as defined in Eqs. (D1) and (D3), respectively. We proceed to solve the relations $\mathscr{R}_1=0$, $\mathscr{R}_2=0$, $\mathscr{R}_4=0$, and $\mathscr{R}_5=0$ at each order in ρ and m. From the term proportional to ρ^3m^0 we find

[4]Note that the slightly different definition of Q_2 compared with that given in Eq. (79) comes from making the m^2 dependency in $\lambda_2^{(s)}$ explicit.

$$\ell\ell_{2,2} = \beta_{2,2} = \tilde{R}_{2,2} = 0. \tag{E5}$$

We can manipulate the terms proportional to $\rho^2 m^0$ into three algebraic relations and one first-order differential equation, whose solutions are

$$2\ell\ell_{2,1} = T_{2,2} + \tilde{T}_0 \frac{V_{2,2}}{V_0} + \frac{1}{V_0}\frac{\mathrm{d}V_{2,2}}{\mathrm{d}r_{*,0}}, \tag{E6}$$

$$2\beta_{2,1} = K_{2,2} - \frac{1}{W_0}\left(\frac{Q_{2,2}V_0}{Q_0} + V_{2,2}\right) - \beta_0 T_{2,2}, \tag{E7}$$

$$2\tilde{R}_{2,1} = Q_0 T_{2,2} + \left(\tilde{T}_0 + 2\frac{3-2r}{r^2}\right) Q_{2,2} - \frac{\mathrm{d}Q_{2,2}}{\mathrm{d}r_{*,0}}, \tag{E8}$$

$$T_{2,2} = \frac{2\lambda - 5}{r(5+4\lambda)} - \frac{\tilde{\lambda}_2}{r} - (3+\beta_0)\bar{W}_2 + t_{2,2}, \tag{E9}$$

where $t_{2,2}$ is an integration constant. The structure of the equations is maintained at order $\rho^1 m^0$, for which we obtain

$$2\ell\ell_{2,0} = T_{2,1} + \frac{Q_0 W_0}{V_0}\beta_{2,1} + \frac{1}{V_0}\frac{\mathrm{d}V_0\ell\ell_{2,1}}{\mathrm{d}r_{*,0}}, \tag{E10}$$

$$2\beta_{2,0} = K_{2,1} - \beta_0 T_{2,1} - \tilde{T}_0\beta_{2,1} - \frac{V_0}{W_0}\left(\frac{R_{2,1}}{Q_0} + \ell\ell_{2,1}\right), \tag{E11}$$

$$2\tilde{R}_{2,0} = Q_0 T_{2,1} + \tilde{T}_0 Q_{2,1} + 2\frac{3-2r}{r^2}R_{2,1} + Q_0 W_0 \beta_{2,1} - \frac{\mathrm{d}R_{2,1}}{\mathrm{d}r_{*,0}}, \tag{E12}$$

$$2T_{2,1} = \frac{2}{r^3}\left(2r - \frac{1}{f_0 r}\right) - \kappa_0\bar{\kappa}_2 + \frac{2f_0\tilde{\lambda}_2}{r^2} + T_{2,2}\tilde{T}_0 - V_{2,2} - Q_{2,2} + 2\tilde{f}_2 + t_{2,1}, \tag{E13}$$

where $t_{2,1}$ is another integration constant. We find that the equations at order $\rho^0 m^0$ are consistent only by selecting $\tilde{f}_2$ as given in Eq. (D5). Explicitly, we obtain

$$T_{2,0} = \frac{2r-1}{f_0 r^4} - \frac{\beta_0 W_0}{f_0 r^2} - 2\tilde{f}_2\left(4\frac{3-2r}{r^2} + \tilde{T}_0\right) - \frac{1}{Q_0}\left[2\frac{3-2r}{r^2}R_{2,0} + Q_{2,0}\tilde{T}_0\right] - \beta_{2,0}W_0 + \frac{1}{Q_0^2}\frac{\mathrm{d}\tilde{f}_2 Q_0^2}{\mathrm{d}r_{*,0}} - \frac{1}{Q_0}\frac{\mathrm{d}R_{2,0}}{\mathrm{d}r_{*,0}}, \tag{E14}$$

$$K_{2,2} = 2\beta_0 t_{2,2} - \frac{4\lambda(\lambda-10)}{5+4\lambda}, \tag{E15}$$

$$K_{2,1} = 2\beta_0 t_{2,1} + \kappa_0 t_{2,2} + \frac{36\beta_0\tilde{k}_2}{\kappa_0}, \tag{E16}$$

$$K_{2,0} = \kappa_0 t_{2,1} + 2\beta_0^2\tilde{k}_2. \tag{E17}$$

In the first-order case, the integration constants $t_{2,1}$ and $t_{2,2}$ are not specified by the transformation. We now move to the terms proportional to m^2. The procedure follows the same lines as the previous one. From the terms proportional to $\rho^3 m^2$ we obtain

$$\bar{\ell}_{2,2} = \bar{\beta}_{2,2} = \bar{\tilde{R}}_{2,2} = 0. \tag{E18}$$

Then, from the terms proportional to $\rho^2 m^2$ we obtain

$$2\bar{\ell}_{2,1} = \bar{T}_{2,2} + \frac{\mathrm{d}}{\mathrm{d}r_{*,0}}\left(\frac{\bar{V}_{2,2}}{V_0} + \frac{1}{2}\frac{V_{1,2}^2}{V_0^2}\right) - \frac{Q_0 W_0}{V_0^2}(\bar{V}_{2,2}V_0 + V_{1,2}^2), \tag{E19}$$

$$2\bar{\beta}_{2,1} = \bar{K}_{2,2} - \frac{\bar{Q}_2 V_0 - Q_{2,1}V_{2,1}}{Q_0 W_0} + \frac{\bar{V}_{2,2}}{W_0} - \beta_0\bar{T}_{2,2}, \tag{E20}$$

$$2\bar{\tilde{R}}_{2,1} = Q_0\bar{T}_{2,2} + \left(\tilde{T}_0 + 2\frac{3-2r}{r^2}\right)\bar{Q}_{2,2} - T_{1,1}Q_{1,1} - \frac{\mathrm{d}Q_{2,2}}{\mathrm{d}r_{*,0}}, \tag{E21}$$

$$\bar{T}_{2,2} = \bar{t}_{2,2} - \frac{2(\lambda-10)}{r(\lambda+2)(5+4\lambda)} - \frac{\bar{\lambda}_2}{r} - (3+\beta_0)\widehat{W}_2. \tag{E22}$$

From the terms proportional to $\rho^1 m^2$ we obtain

$$2\bar{\ell}_{2,0} = \bar{T}_{2,1} + \frac{Q_0 W_0}{V_0}\left(\bar{\beta}_{2,1} + \frac{V_{1,2}}{V_0}\beta_{1,1}\right) + \frac{1}{V_0}\frac{\mathrm{d}V_0\bar{\ell}_{2,1}}{\mathrm{d}r_{*,0}} - \ell\ell_{1,1}\frac{\mathrm{d}}{\mathrm{d}r_{*,0}}\frac{V_{1,2}}{V_0}, \tag{E23}$$

$$2\bar{\beta}_{2,0} = \bar{K}_{2,1} - \beta_0\bar{T}_{2,1} - \tilde{T}_0\bar{\beta}_{2,1} + T_{1,2}\beta_{1,1} - \frac{V_0}{W_0}\left[\ell\ell_{1,1}\left(\frac{V_{1,2}}{V_0} + \frac{Q_{2,1}}{Q_0}\right) - \bar{\ell}_{2,1}\right] + \frac{R_{1,1}V_{1,2} - \bar{R}_{2,1}V_0}{W_0 Q_0}, \tag{E24}$$

$$2\bar{\tilde{R}}_{2,0} = Q_0\bar{T}_{2,1} + \bar{Q}_{2,1}\tilde{T}_0 - Q_{1,2}T_{1,1} - Q_{1,1}T_{1,2} + 2\frac{3-2r}{r^2}\bar{R}_{2,1} + Q_0 W_0\bar{\beta}_{2,1} - \frac{\mathrm{d}\bar{R}_{2,1}}{\mathrm{d}r_{*,0}}, \tag{E25}$$

$$2\bar{T}_{2,1} = \bar{t}_{2,1} - \kappa_0 \overset{\frown}{\kappa}_2 + \frac{2f_0 \bar{\lambda}_2}{r^2} + \bar{T}_{2,2}\tilde{T}_0 - \frac{T_{1,2}^2}{2} - \bar{V}_{2,2} - \bar{Q}_{2,2} + 2\overset{\frown}{f}_2. \tag{E26}$$

From the terms proportional to $\rho^0 m^2$ we obtain

$$2\overline{\ell}_{2,-1} = \bar{T}_{2,0} + (T_{1,2} - 2\ell\ell_{1,1})f_1 + \overset{\frown}{f}_2\tilde{T}_0 + \frac{Q_0 W_0}{V_0}\left[\bar{\beta}_{2,0} + \frac{V_{1,2}}{V_0}\beta_{1,1} - \beta_0\left(\frac{V_{1,2}V_{1,0}}{V_0^2} + \frac{\bar{V}_{2,0}}{V_0}\right)\right] - \ell\ell_{1,0}\frac{\mathrm{d}}{\mathrm{d}r_{*,0}}\frac{V_{1,2}}{V_0} + \frac{\mathrm{d}}{\mathrm{d}r_{*,0}}\frac{V_{1,2}V_{1,0}}{V_0^2} + \frac{\mathrm{d}}{\mathrm{d}r_{*,0}}\frac{V_{2,0}}{V_0} + \frac{1}{V_0}\frac{\mathrm{d}V_0\overline{\ell}_{2,0}}{\mathrm{d}r_{*,0}}, \tag{E27}$$

$$2\bar{\beta}_{2,-1} = \bar{K}_{2,0} - \beta_0\bar{T}_{2,0} - \tilde{T}_0\bar{\beta}_{2,0} + T_{1,2}\beta_{1,0} + T_{1,1}\beta_{1,1} + \frac{Q_{1,2}V_{1,0} + R_{1,0}V_{1,2} - Q_0\bar{V}_{2,0} - \bar{R}_{2,0}V_0}{W_0 Q_0} + \frac{V_0}{W_0}\left[\ell\ell_{1,0}\left(\frac{V_{1,2}}{V_0} + \frac{Q_{1,2}}{Q_0}\right) + \frac{R_{1,1}}{Q_0}\ell\ell_{1,1} - \overline{\ell}_{2,0} + f_1\left(\frac{V_{1,2}}{V_0} - \frac{Q_{1,2}}{Q_0}\right) - \overset{\frown}{f}_2\right], \tag{E28}$$

$$2\bar{\bar{R}}_{2,-1} = Q_0\bar{T}_{2,0} - Q_{1,0}T_{1,2} - Q_{1,2}T_{1,0} - Q_{1,1}T_{1,1} + \bar{Q}_{2,0}\tilde{T}_0 + (Q_{1,2}\tilde{T}_0 - Q_0T_{1,2} - 2\tilde{R}_{1,1})f_1 + Q_0\overset{\frown}{f}_2 + 2\frac{3-2r}{r^2}\bar{R}_{2,0} + Q_0W_0\bar{\beta}_{2,0} - Q_0W_0^2\frac{\mathrm{d}}{\mathrm{d}r_{*,0}}\frac{\overset{\frown}{f}_2}{W_0^2} - Q_{1,2}\frac{\mathrm{d}f_1}{\mathrm{d}r_{*,0}} - \frac{\mathrm{d}\bar{R}_{2,0}}{\mathrm{d}r_{*,0}}, \tag{E29}$$

whereas the expression for $\bar{T}_{2,0}$ is extremely long and uninformative and we do not display it. Then, by setting to zero the terms proportional to $\rho^{-1}m^2$ we obtain

$$2\overline{\ell}_{2,-2} = \bar{T}_{2,-1} + (T_{1,1} - 2\ell\ell_{1,0})f_1 + \frac{Q_0W_0}{V_0}\left(\bar{\beta}_{2,-1} + \frac{V_{1,0}}{V_0}\beta_{1,1}\right) - \ell\ell_{1,1}\frac{\mathrm{d}}{\mathrm{d}r_{*,0}}\frac{V_{1,0}}{V_0} + \frac{1}{V_0}\frac{\mathrm{d}V_0\overline{\ell}_{2,-1}}{\mathrm{d}r_{*,0}}, \tag{E30}$$

$$2\bar{\beta}_{2,-2} = \bar{K}_{2,-1} - \beta_0\bar{T}_{2,-1} - \tilde{T}_0\bar{\beta}_{2,-1} + T_{1,1}\beta_{1,0} + T_{1,0}\beta_{1,1} + \frac{V_0}{W_0}\left[\ell\ell_{1,1}\left(\frac{R_{1,0}}{Q_0} + \frac{V_{1,0}}{V_0} + f_1\right) - \overline{\ell}_{2,-1}\right] + \frac{V_0}{W_0Q_0}\left[R_{1,1}\left(\frac{V_{1,0}}{V_0} + \ell\ell_{1,0} - f_1\right) - \bar{R}_{2,-1}\right], \tag{E31}$$

$$2\bar{\bar{R}}_{2,-2} = Q_0\bar{T}_{2,-1} - Q_{1,0}T_{1,1} - Q1,1T_{1,0} + \bar{Q}_{2,-1}\tilde{T}_0 + (Q_{1,1}\tilde{T}_0 - Q_0T_{1,1} - 2\tilde{R}_{1,0})f_1 + 2\frac{3-2r}{r^2}\bar{R}_{2,-1} + Q_0W_0\bar{\beta}_{2,-1} - R_{1,1}\frac{\mathrm{d}f_1}{\mathrm{d}r_{*,0}} - \frac{\mathrm{d}\bar{R}_{2,-1}}{\mathrm{d}r_{*,0}}, \tag{E32}$$

as well as $\bar{T}_{2,-1}$ whose expression is extremely long and uninformative and we do not display it. Finally, from the terms proportional to $\rho^{-2}m^2$ we find

$$\bar{K}_{2,2} = 2\beta_0\bar{t}_{2,2} - \frac{4\lambda(60 + 34\lambda + 5\lambda^2)}{(\lambda+2)^2(5+4\lambda)}, \tag{E33}$$

$$\bar{K}_{2,1} = 2\beta_0\bar{t}_{2,1} + \kappa_0\bar{t}_{2,2} + \frac{36\beta_0\check{k}_2}{\kappa_0} + 4t_2\lambda - \frac{\beta_0 t_2^2}{2}, \tag{E34}$$

$$\bar{K}_{2,0} = 2\beta_0\bar{t}_{2,0} + \kappa_0\left(\bar{t}_{2,1} - \frac{t_2^2}{4}\right) + 18\check{k}_2 + 4t_1\lambda - \beta_0 t_1 t_2, \tag{E35}$$

$$\bar{K}_{2,-1} = 2\beta_0\bar{t}_{2,-1} - \frac{\beta_0 t_1^2}{2} + \kappa_0\left[\bar{t}_{2,0} + \beta_0\left(\check{k}_2 + \check{\kappa}_2 - \frac{t_1t_2}{2}\right)\right], \tag{E36}$$

$$\bar{K}_{2,-2} = \kappa_0\left[\bar{t}_{2,-1} - \frac{t_1^2}{4} + \frac{\kappa_0}{2}(\check{k}_2 + \check{\kappa}_2)\right], \tag{E37}$$

and the expression for $\bar{T}_{2,-2}$ is extremely long and uninformative and we do not display it. As expected, the integration constants $\bar{t}_i$ can be freely chosen. With this last calculation we completely specified the transformation theory at second order in the spin.

APPENDIX F: COEFFICIENTS OF THE PERTURBED EINSTEIN EQUATIONS

In this section, we provide the coefficients that appear in Eqs. (11), (12a), (12b), (15a), and (15b). We start by showing all the coefficients of the first group:

$$\begin{aligned} A^{(1)}_{0,\ell} = {}& \frac{H_2\rho^2}{2} + \frac{H_1\rho(3-4r)}{2r^2} - f_0\frac{H_0\ell(\ell+1)}{2r^2} + \frac{f_0(4r-1)H_0'}{4r^2} + f_0\frac{H_2'}{4r^2} + \frac{f_0^2H_0''}{2} - \rho f_0 H_1' - f_0\frac{K'}{2r^2} + K\rho^2 \\ & + \frac{iam(2r-1)H_1}{2r^5} + a^2\left[\frac{H_0}{2r^4}\left(m^2 - \frac{1}{2f_0r^4} - \frac{2r^2+r-1}{r^3}\right) + \frac{H_2(2\rho^2r^4-6r+5)}{4f_0r^6} - \frac{\rho H_1'}{r^2} + \frac{f_0H_1\rho}{2r^4}\right. \\ & \left. + \frac{(r^2+4r-1)H_0'}{4r^6} + \frac{H_2'}{2r^4} + \frac{f_0H_0''}{2r^2} - \frac{K'}{4r^4} - \frac{K(2\rho^2r^4-2r+1)}{4f_0r^6}\right], \end{aligned} \tag{F1a}$$

$$A^{(1)}_{1,\ell} = -a\frac{2h_0\ell(\ell+1)}{r^5} + \frac{ima^2}{r^5}\left[\frac{h_0\rho(r-2)}{f_0r} + f_0h_1' + \frac{h_1(8-7r)}{r^2}\right], \tag{F1b}$$

$$\begin{aligned} A^{(1)}_{2,\ell} = a^2\Bigg[&-\frac{H_2(\rho^2r^4-7r+5)}{2r^6} + \frac{\rho f_0H_1'}{r^2} + \frac{H_1\rho(4r^2-6r+1)}{2r^5} + f_0H_0\left(\frac{\ell^2+\ell}{2r^4} + \frac{3}{4r^6}\right) \\ & - \frac{f_0(4r^2-10r+1)H_0'}{4r^5} - \frac{5f_0H_2'}{4r^4} - \frac{f_0^2H_0''}{2r^2} + \frac{(9r-10)K'}{4r^5} - \frac{K(2\rho^2r^5-4\rho^2r^4+14r^2-33r+20)}{4f_0r^7}\Bigg], \end{aligned} \tag{F1c}$$

$$B^{(1)}_{1,\ell} = \frac{a}{2r^5}\left[-h_1\rho + (1-2r)h_0' - \frac{h_0(r-2)}{f_0r^2}\right] + ima^2\left[\frac{h_0\rho}{f_0r^4} - \frac{h_1}{2r^6}\right] \qquad B^{(1)}_{2,\ell} = \frac{a^2}{2r^5}\left[H_0\left(3-\frac{4}{r}\right) + H_2\right], \tag{F1d}$$

$$\begin{aligned} A^{(2)}_{0,\ell} = {}& \frac{K\rho(2r-3)}{2r^2} - \frac{H_2\rho f_0}{r} - \frac{H_1f_0\ell(\ell+1)}{2r^2} + \rho f_0K' + \frac{ima}{2r^3}\left(\frac{H_0+H_2}{2r} - H_1\rho + f_0H_0'\right) \\ & + \frac{a^2}{r^2}\left[H_1\left(\frac{m^2}{2r^2} - \frac{f_0}{r^3}\right) - \frac{3H_0\rho}{4r^3} + \frac{H_2\rho(1-2r)}{4r^2} - \frac{\rho K'}{2} + \frac{K\rho}{2f_0r^2}\right], \end{aligned} \tag{F1e}$$

$$A^{(2)}_{1,\ell} = -a\frac{f_0h_1\ell(\ell+1)}{r^5} + \frac{ima^2}{r^4}\left(\frac{h_1\rho f_0}{r} - \frac{4h_0}{r^2} + f_0h_0'\right), \tag{F1f}$$

$$A^{(2)}_{2,\ell} = \frac{a^2}{r^3}\left[\frac{3H_0\rho}{4r^2} + f_0H_1\frac{r\ell(\ell+1)+6}{2r^2} + H_2\rho\left(\frac{3}{2} - \frac{9}{4r}\right) + \rho\left(1-\frac{r}{2}\right)K' + \frac{3K\rho}{2r}\right], \tag{F1g}$$

$$B^{(2)}_{1,\ell} = \frac{a}{r^3}\left[h_1\left(\frac{\rho^2}{2} - \frac{f_0}{r^2}\right) + \frac{h_0\rho}{r} - \frac{\rho h_0'}{2}\right] + \frac{ima^2}{2r^4}\left(h_1\rho - \frac{h_0}{f_0r^2} + h_0'\right) \qquad B^{(2)}_{2,\ell} = a^2f_0\frac{H_1(r+1)}{r^5}, \tag{F1h}$$

$$A^{(3)}_{0,\ell} = \frac{H_1\rho}{2r^2} - H_2\left(\frac{\rho^2}{2} + f_0\frac{\ell(\ell+1)}{2r^2}\right) - \frac{3f_0H_0'}{4r^2} - \frac{(4r-3)f_0H_2'}{4r^2} - \frac{f_0^2H_0''}{2} + \rho f_0 H_1' + \frac{(4r-3)f_0K'}{2r^2} + f_0^2K'' + \frac{\mathrm{i}am}{r^3}\left(\frac{H_1}{2r^2} - H_2\rho + f_0H_1'\right) + \frac{a^2}{r^2}\left[H_2\left(\frac{m^2}{2r^2} + \frac{2(\rho^2+4)r^2 - 10r + 3}{4f_0r^4}\right) - \frac{\rho H_1'}{r^2} + \frac{H_1\rho(-2r^3+r^2-1)}{2f_0r^4} - \frac{f_0H_2'}{2r} + \frac{(2r^3-r^2-8r+9)H_0'}{4r^4} + \frac{f_0H_0''}{2r^2} + \frac{H_0(22r^2-36r+15)}{4f_0r^6} + \frac{(3-4r)K'}{4r^2} - \frac{f_0K''}{2} - \frac{K(8r^2-10r+3)}{4f_0r^4}\right], \tag{F2a}$$

$$A^{(3)}_{1,\ell} = \frac{\mathrm{i}ma^2}{r^4}(h_0\rho + f_0^2h_1'), \tag{F2b}$$

$$A^{(3)}_{2,\ell} = \frac{a^2}{r^3}\left[H_1\rho\left(f_0 - \frac{1}{2r^2}\right) + H_2\left(\frac{\rho^2}{2} + f_0\frac{r(\ell^2+\ell-4)+3}{2r^2}\right) - \frac{f_0(2r^2+4r-9)H_0'}{4r^2} + \frac{f_0(6r-5)H_2'}{4r} + \frac{f_0^2H_0''}{2} + \frac{3f_0H_0(8r-5)}{4r^3} - \rho f_0H_1' + \frac{(4r^2-11r+6)K'}{4r^2} - \frac{f_0(r-2)K''}{2} + \frac{K(8r^3+2r^2-21r+12)}{4f_0r^4}\right], \tag{F2c}$$

$$B^{(3)}_{1,\ell} = \frac{a}{r^3}\left[\frac{h_1\rho}{2r^2} + \frac{(6r-5)h_0'}{2r^2} - \frac{h_0(12r^2-17r+6)}{2f_0r^4} + \rho f_0h_1' - f_0h_0''\right] + \frac{\mathrm{i}ma^2}{r^4}\left[\frac{h_1}{2r^2} + f_0h_1'\right], \tag{F2d}$$

$$B^{(3)}_{2,\ell} = \frac{f_0a^2}{2r^4}(H_0 + 3H_2), \tag{F2e}$$

$$A^{(4)}_{0,\ell} = \frac{H_0\ell(\ell+1)}{2r^2} - \frac{H_2(\ell^2+\ell+4)}{2r^2} - \frac{f_0(H_0'+H_2')}{r} + \frac{2H_1\rho}{r} + \frac{(4r-3)K'}{r^2} + f_0K'' - K\left(\frac{\rho^2}{f_0} + \frac{\ell^2+\ell-2}{r^2}\right) + \frac{\mathrm{i}ma}{r^3}\left(\frac{2H_1}{r} - \frac{K\rho}{f_0}\right) + \frac{a^2}{r^2}\left[\frac{\ell^2+\ell+2}{2f_0r^2}\left(K - \frac{H_0}{r^2} - H_2\right) + \frac{H_1\rho(4r+3)}{2r^2} + \frac{(3+r-4r^2)H_0'}{4r^3} - \frac{(8r^2-r+1)H_2'}{4r^3} + \frac{H_0(2r+3)}{2r^4} - \frac{2H_2}{r^3} + \frac{(6r^2+r+1)K'}{2r^3} + K'' + K\left(\frac{3r-2}{f_0r^4} + \frac{\rho^2}{f_0^2r^2}\right)\right], \tag{F2f}$$

$$A^{(4)}_{1,\ell} = \frac{2ah_0\ell(\ell+1)}{(r-1)r^4} + \frac{2\mathrm{i}ma}{r^5}\left[\frac{2h_1(r^2-2r+2)}{r^2} - \frac{h_0\rho(r+1)}{f_0} + \left(r - \frac{1}{r}\right)h_1'\right], \tag{F2g}$$

$$A^{(4)}_{2,\ell} = \frac{a^2}{r^2}\left[\frac{r(\ell^2+\ell+6)+16}{2r^3}H_2 - \frac{H_1\rho(4r+3)}{2r^2} - \frac{r(\ell^2+\ell+12)+3}{2r^4}H_0 + \frac{f_0(4r+3)H_0'}{4r^2} + \frac{f_0(8r-1)H_2'}{4r^2} + \frac{(-6r^2+r-1)K'}{2r^3} - f_0K'' + \frac{K}{f_0}\left(\frac{r^2(\ell^2+\ell-6) - 2r(\ell^2+\ell+11)+20}{2r^4} + \frac{\rho^2}{r}\right)\right], \tag{F2h}$$

$$B^{(4)}_{1,\ell} = \frac{a}{r^4}\left[-h_1\rho + \frac{h_0(r\ell^2+r\ell+2)}{f_0r^2} + h_0'\right] - \frac{\mathrm{i}ah_1m}{r^6} \qquad B^{(4)}_{2,\ell} = \frac{a^2}{r^5}\left[H_0(-r-2) + H_2(2r+1) + \frac{K(2-4r)}{r-1}\right]. \tag{F2i}$$

Then, from the second group

$$\alpha^{(1)}_{0,\ell} = H_2\rho - \frac{H_1}{r^2} - f_0H_1' + K\rho + \frac{a^2}{r^2}\left[\frac{H_1(2r^2-r+1)}{2r^3} + \frac{H_2\rho}{f_0} - H_1' - \frac{K\rho}{2f_0}\right], \tag{F3a}$$

$$\alpha^{(1)}_{1,\ell} = \frac{a}{r^3}\left[\frac{h_1(7r-8)}{r^2} - 2f_0h_1'\right] \qquad \alpha^{(1)}_{2,\ell} = \frac{a^2}{r^2}\left[-H_2\rho - \frac{H_1(2r^2-7r+1)}{2r^3} + f_0H_1' - \frac{K\rho(r-2)}{2f_0r}\right], \tag{F3b}$$

$$\tilde{\alpha}^{(1)}_{2,\ell} = \frac{a^2}{2r^2}\left[\frac{H_1(2r^2+r-1)}{r^3} - \frac{K\rho}{f_0}\right], \tag{F3c}$$

$$\beta^{(1)}_{0,\ell} = \frac{2f_0 h_1 \rho}{r} + \frac{h_0(r\ell^2 + r\ell - 2)}{r^3} + \rho f_0 h_1' - f_0 h_0'' \qquad \beta^{(1)}_{1,\ell} = \frac{a}{r^3}(H_2 - 2K), \tag{F3d}$$

$$\beta^{(1)}_{2,\ell} = \frac{a^2}{r^2}\left[\frac{1+10r-4r^2}{2r^3}h_1\rho + \frac{2r^2(\ell^2+\ell+12) - r^3\ell(\ell+1) - 27r - 2}{2(r-1)r^4}h_0 - \frac{(2r^2+r+1)h_0'}{2r^3} - \rho f_0 h_1' + f_0 h_0''\right], \tag{F3e}$$

$$\tilde{\beta}^{(1)}_{2,\ell} = \frac{a^2}{2r^5}\left[h_1\rho + \frac{h_0(r^3(-\ell)(\ell+1) - 4r^2 + r - 2)}{(r-1)r} + (2r^2 + r - 1)h_0'\right], \tag{F3f}$$

$$\eta^{(1)}_{1,\ell} = \frac{a h_1 \ell(\ell+1)}{2r^4} \qquad \eta^{(1)}_{2,\ell} = \frac{a^2}{r^2}\left[\frac{H_1(6r-8)}{r^3} + H_2\rho\left(\frac{1}{r}+1\right) - \frac{2K\rho}{r} - 2\frac{f_0 H_1'}{r}\right], \tag{F3g}$$

$$\zeta^{(1)}_{1,\ell} = \frac{a}{r^2}\left[-2H_1\rho - \frac{H_0\ell(\ell+1)}{2r} + \frac{5f_0H_0'}{2} - \frac{1}{2}f_0H_2' + \frac{2H_2}{r} - \frac{K'}{r} + \frac{K(\rho^2r^3 - 2r + 2)}{(r-1)r}\right], \tag{F3h}$$

$$\zeta^{(1)}_{2,\ell} = \frac{a^2h_0(r-3)\ell(\ell+1)}{(r-1)r^5} \qquad \xi^{(1)}_{1,\ell} = \frac{aH_0}{2r^3} \qquad \xi^{(1)}_{2,\ell} = -\frac{a^2h_0}{r^5} \qquad \chi^{(1)}_{1,\ell} = \frac{a}{r^3}\left(\frac{h_1}{2r} - \frac{h_0\rho}{f_0}\right), \tag{F3i}$$

$$\alpha^{(2)}_{0,\ell} = H_1\rho + \frac{(H_0 - H_2)(2r-3)}{2r^2} + \frac{H_2}{r^2} + f_0(K' - H_0') + \frac{a^2}{r^3}\left[\frac{K(2r-1)}{2f_0r} - \frac{3H_0}{2r^2} - \frac{H_1\rho}{f_0r} + \frac{H_0'}{r} - \frac{(H_0+H_2)(2r+1)}{4r} - \frac{rK'}{2}\right], \tag{F3j}$$

$$\alpha^{(2)}_{1,\ell} = \frac{a}{r^3}\left[-h_1\rho + \frac{h_0(8-9r)}{r-r^2} - h_0'\right], \tag{F3k}$$

$$\alpha^{(2)}_{2,\ell} = \frac{a^2}{r^3}\left[-H_1\rho + \frac{H_0(-2r^2-5r+6)}{4r^2} + \frac{(r-1)H_0'}{r} + H_2\left(\frac{3}{2} - \frac{1}{4r}\right) + \left(1 - \frac{r}{2}\right)K' + \frac{K(2r^2-7r+4)}{2(r-1)r}\right], \tag{F3l}$$

$$\tilde{\alpha}^{(2)}_{2,\ell} = \frac{a^2}{r^2}\left[\frac{H_0(2r^2+r-6)}{4r^3} + \frac{H_2(2r+1)}{4r^2} - \frac{K'}{2}\right], \tag{F3m}$$

$$\beta^{(2)}_{0,\ell} = h_1\left(\rho^2 + \frac{f_0\lambda}{r^2}\right) + \frac{2h_0\rho}{r} - \rho h_0' + \frac{a^2}{r^4}\left[\frac{h_0\rho(2r^2+r+4)}{2-2r} - h_1\left(\frac{\rho^2}{f_0} + \frac{f_0}{r} + \frac{\lambda+2}{2}\right) + \frac{\rho h_0'}{f_0}\right], \tag{F3n}$$

$$\beta^{(2)}_{1,\ell} = \frac{aH_1}{r^3}\beta^{(2)}_{2,\ell} = \frac{a^2}{r^3}\left[\rho h_0' + \frac{h_0\rho(-2r^2+r+4)}{2(r-1)r} - h_1\left(\rho^2 + \frac{r^2(\ell^2+\ell-8) - 2r(\ell^2+\ell+1) + 10}{2r^3}\right)\right], \tag{F3o}$$

$$\tilde{\beta}^{(2)}_{2,\ell} = \frac{a^2}{r^4}\left[h_1\left(\frac{f_0}{r} - \frac{1}{2}\ell(\ell+1)\right) + h_0\rho\left(\frac{3}{2f_0} + r\right)\right] \qquad \eta^{(2)}_{1,\ell} = \frac{a\ell(\ell+1)}{2r^3}\left[3h_0' - 3h_1\rho + \frac{h_0(6-7r)}{(r-1)r}\right], \tag{F3p}$$

$$\eta^{(2)}_{2,\ell} = \frac{a^2}{r^3}\left[\frac{H_0(8r-9)}{r^2} + 2H_1\rho r + \left(\frac{1}{r} - r\right)H_0' + H_2\left(\frac{1}{r} - 2\right) - 2K' + K\left(\frac{6}{r} + \frac{1}{r-1} + 2\right)\right], \tag{F3q}$$

$$\zeta^{(2)}_{1,\ell} = \frac{a}{r}\left[\frac{3(H_0 - H_2 + 2K)\rho}{2r} - \frac{H_1\ell(\ell+1)}{2r^2} + \rho K'\right] \qquad \zeta^{(2)}_{2,\ell} = -\frac{a^2h_1(r-2)f_0\ell(\ell+1)}{r^5}, \tag{F3r}$$

$$\xi^{(2)}_{1,\ell} = \frac{aH_1}{2r^3} \qquad \xi^{(2)}_{2,\ell} = -\frac{a^2h_1f_0}{r^4} \qquad \chi^{(2)}_{1,\ell} = \frac{a}{r^3}\left[-h_1\rho + h_0\left(\frac{2}{r} + \frac{1}{r-1}\right) - h_0'\right], \tag{F3s}$$

and finally, from the third group,

$$j_{0,\ell} = H_0 - H_2 - \frac{a^2}{f_0r^2}\left(\frac{H_0}{r^2} + H_2\right) \qquad j_{2,\ell} = \frac{a^2}{r^2}\left(H_2 - \frac{H_0}{r}\right), \tag{F4a}$$

$$k_{0,\ell} = 2\left(\frac{h_1}{r^2} - \frac{h_0\rho}{f_0} + f_0 h_1'\right) + \frac{2a^2}{r^2}\left[\frac{h_1(1-2r)}{2r^2} + \frac{h_0\rho}{f_0^2 r^2} + h_1'\right] \qquad k_{2,\ell} = \frac{2a^2}{r^3}\left[\frac{h_0\rho}{f_0} - f_0 r h_1' + h_1\left(1 - \frac{7}{2r}\right)\right], \tag{F4b}$$

$$f_{1,\ell} = -\frac{2aK\rho}{f_0 r} \qquad f_{2,\ell} = \frac{4a^2}{r^3}\left[h_1\left(1 + \frac{7}{2r} - \frac{4}{r^2}\right) + \frac{h_0\rho}{f_0} - f_0 h_1'\right], \tag{F4c}$$

$$g_{1,\ell} = \frac{3a}{r^2}\left[2h_0' - 2h_1\rho - h_0\left(\frac{2\ell(\ell+1)}{3f_0 r} + \frac{4}{r}\right)\right] \qquad g_{2,\ell} = \frac{2a^2}{r^2}\left[K\left(-\frac{2}{r} + \frac{1}{r-1} - 1\right) - H_2 f_0\right], \tag{F4d}$$

$$h_{2,\ell} = \frac{a^2 h_1(2r+1)\ell(\ell+1)}{r^4}, \tag{F4e}$$

$$\tilde{h}_{2,\ell} = \frac{a^2}{r^2}\left[\frac{H_0(9-2r)}{r^2} - H_1\rho + \frac{f_0(H_0' + H_2')}{2} - \frac{2H_2(r+2)}{r} + \left(4r + \frac{1}{r} + 1\right)K' - \frac{K(r^2\lambda + 4)}{f_0 r^2}\right]. \tag{F4f}$$

APPENDIX G: DECOUPLING COEFFICIENTS, SECOND ORDER IN THE SPIN

In this appendix we provide the decoupling coefficients that enter at second order in the spin in Eqs. (36)–(39). We find it useful to define $\Gamma = (\lambda-4)(\lambda+2)r^2 + 6(\lambda+1)r + 9$ and $\Lambda = 3 + \lambda r$,

$$\begin{aligned} d_0^\ell = {} & 4\rho^2 \frac{3(\lambda-1) + (\lambda-4)(\lambda+2)r}{(5+4\lambda)\Gamma} + \frac{1}{(\lambda+2)(5+4\lambda)f_0\Gamma^2}\Big[-42\lambda^6 + 395\lambda^5 + 464\lambda^4 - 1240\lambda^3 - 8000\lambda^2 - 28976\lambda \\ & + \frac{27(13\lambda^3 - 125\lambda^2 + 290\lambda - 304)}{r^4} + \frac{9(43\lambda^4 - 260\lambda^3 + 597\lambda^2 - 1363\lambda - 2122)}{r^3} \\ & + \frac{3(47\lambda^5 - 483\lambda^4 + 485\lambda^3 - 1430\lambda^2 + 708\lambda + 10312)}{r^2} + (\lambda^2 - 2\lambda - 8)^2(29\lambda^2 + 53\lambda + 170)r \\ & + \frac{486(14 - 5\lambda)}{r^5} + \frac{17\lambda^6 - 428\lambda^5 + 1222\lambda^4 + 2856\lambda^3 + 10004\lambda^2 + 26084\lambda + 7144}{r} - 27968\Big] \\ & - 6f_0 m^2 \frac{(\lambda-16)r + 21}{(\lambda+2)^2\rho^2 r^7} + \rho^2 m^2 \frac{-6(2\lambda^2 + \lambda - 12) - 2(\lambda-4)(\lambda+2)(2\lambda+7)r}{(\lambda+2)^2(5+4\lambda)\Gamma} \\ & + \frac{m^2}{\Gamma^2}\Big[\frac{34\lambda^6 - 735\lambda^5 + 2940\lambda^4 + 7276\lambda^3 - 14592\lambda^2 + 10704\lambda + 39328}{2(\lambda+2)^2(4\lambda+5)} - \frac{486(19\lambda^2 + 33\lambda - 34)}{(\lambda+2)^3(4\lambda+5)r^4} \\ & + \frac{27(26\lambda^4 - 891\lambda^3 - 2190\lambda^2 + 216\lambda - 464)}{2(\lambda+2)^3(4\lambda+5)r^3} + \frac{9(86\lambda^5 - 1321\lambda^4 - 2146\lambda^3 + 6316\lambda^2 + 1072\lambda - 11216)}{2(\lambda+2)^3(4\lambda+5)r^2} \\ & + 2(\lambda+2)(\lambda-4)^2 r^2 - \frac{9(3\lambda^5 - 30\lambda^4 + 22\lambda^3 + 136\lambda^2 + 16\lambda + 960)r}{(\lambda+2)(4\lambda+5)} \\ & + \frac{3(94\lambda^6 - 1195\lambda^5 + 170\lambda^4 + 17688\lambda^3 + 19528\lambda^2 + 4960\lambda + 22016)}{2(\lambda+2)^3(4\lambda+5)r}\Big], \end{aligned} \tag{G1}$$

$$\begin{aligned} d_1^\ell = {} & \frac{1}{f_0^2(5+4\lambda)\Gamma}\Big[-6(\lambda^3 + \lambda^2 + 8\lambda - 22) - \frac{18(2\lambda^2 + 3\lambda - 14)}{(\lambda+2)r^3} - \frac{6(2\lambda^3 + 9\lambda^2 - 9\lambda + 34)}{(\lambda+2)r^2} \\ & - 2(\lambda+2)(\lambda^2 + \lambda - 20)r - \frac{18(\lambda^3 + 3\lambda^2 + 25\lambda + 34)}{(\lambda+2)r}\Big] - \frac{18m^2}{(\lambda+2)^2\rho^2 r^5} \\ & \times \frac{m^2}{f_0(5+4\lambda)\Gamma}\Big[\frac{4\lambda^4 - 13\lambda^3 + 114\lambda^2 + 164\lambda - 512}{(\lambda+2)^2} + \frac{-99\lambda^2 + 180\lambda + 972}{(\lambda+2)^3 r^2} - 2(\lambda^2 - 14\lambda + 40)r \\ & + \frac{6(2\lambda^4 - 7\lambda^3 + 24\lambda^2 + 184\lambda + 148)}{(\lambda+2)^3 r}\Big], \end{aligned} \tag{G2}$$

$$
\begin{aligned}
d_K^\ell = &\frac{f_0}{r} d_H^\ell + \frac{\rho}{f_0 \Lambda(5+4\lambda)}\left[\frac{\lambda(\lambda^2+7\lambda+6)}{2r^2} - \frac{3(59\lambda^2-293\lambda+330)}{4(\lambda+2)r^6} + \frac{-11\lambda^3+488\lambda^2-1179\lambda+690}{(4\lambda+8)r^5}\right.\\
&\left. - \frac{\lambda(4\lambda^3+45\lambda^2-7\lambda+30)}{2(\lambda+2)r^3} + \frac{6\lambda^4+51\lambda^3-565\lambda^2+232\lambda-228}{4(\lambda+2)r^4} - \frac{72(\lambda-1)}{(\lambda+2)r^7}\right]\\
&+ \frac{m^2\rho}{\Lambda^2(5+4\lambda)}\left[\frac{3(20\lambda^2+263\lambda+662)}{(\lambda+2)^3 r^6} + \frac{9(27\lambda^3+468\lambda^2+724\lambda-1264)}{4(\lambda+2)^3 r^5} + \frac{\lambda^2-3\lambda^3}{2r}\right.\\
&+ \frac{\lambda(25\lambda^4+596\lambda^3-554\lambda^2-5488\lambda-744)}{4(\lambda+2)^3 r^3} + \frac{61\lambda^4+1238\lambda^3+707\lambda^2-6272\lambda-108}{2(\lambda+2)^3 r^4}\\
&\left. + \frac{\lambda(\lambda^5+15\lambda^4-143\lambda^3-514\lambda^2-148\lambda+24)}{2(\lambda+2)^3 r^2}\right] - \frac{m^2}{\rho}\frac{\lambda(\lambda+4)r^2+3}{8(\lambda+2)r^9\Lambda^2}[(\lambda-2)\lambda r^2+(5\lambda-6)r+8], \qquad \text{(G3)}
\end{aligned}
$$

$$
\begin{aligned}
d_H^\ell = &\frac{\rho^3}{f_0^2\Lambda(5+4\lambda)}\left[5(\lambda-5) + \frac{48}{\lambda+2} + \frac{24}{r^2}\left(\frac{3}{\lambda+2}-1\right) - 2(2\lambda-3)r + \frac{3}{r}\left(15-3\lambda-\frac{40}{\lambda+2}\right)\right]\\
&+ \frac{\rho}{f_0^2\Lambda^3(5+4\lambda)}\left[\frac{(56\lambda^3+197\lambda^2-244\lambda-60)\lambda^2}{2(\lambda+2)} - \frac{27(107\lambda^2-717\lambda+634)}{4(\lambda+2)r^5}\right.\\
&+ \frac{3(44\lambda^3+4681\lambda^2-11139\lambda+8214)}{4(\lambda+2)r^4} + \frac{437\lambda^4+2395\lambda^3-21018\lambda^2+31950\lambda-11124}{4(\lambda+2)r^3}\\
&+ \frac{80\lambda^5-561\lambda^4-4787\lambda^3+16494\lambda^2-9744\lambda+1440}{(4\lambda+8)r^2} - \frac{1080(\lambda-1)}{(\lambda+2)r^6} - (7\lambda^2+22\lambda+12)\lambda^2 r\\
&\left. + \frac{(-39\lambda^4+11\lambda^3+1027\lambda^2-771\lambda+150)\lambda}{(\lambda+2)r}\right] - \frac{m^2\rho^3}{f_0\Lambda^3(5+4\lambda)(\lambda+2)}\left[2\lambda^2(2\lambda+1)r^3\right.\\
&+ \frac{3(53\lambda^3-16\lambda^2-436\lambda+48)}{(\lambda+2)^2} + \frac{\lambda(7\lambda^4+28\lambda^3+72\lambda^2+160\lambda+48)r^2}{(\lambda+2)^2} + \frac{6(32\lambda^2+5\lambda-226)}{(\lambda+2)^2 r}\\
&\left. + \frac{2(29\lambda^4+31\lambda^3-83\lambda^2+176\lambda+36)r}{(\lambda+2)^2}\right] + \frac{m^2\rho}{f_0\Lambda^4(5+4\lambda)(\lambda+2)}\left[(5\lambda^2+14\lambda-4)\lambda^3 r^2\right.\\
&+ \frac{(4\lambda^5-349\lambda^4-2817\lambda^3+2792\lambda^2+21940\lambda+4944)\lambda^2}{2(\lambda+2)^2} + \frac{9(-28\lambda^2+899\lambda+2990)}{(\lambda+2)^2 r^5}\\
&+ \frac{3(521\lambda^3+19916\lambda^2+24140\lambda-90288)}{4(\lambda+2)^2 r^4} + \frac{3(743\lambda^4+10952\lambda^3-22402\lambda^2-117840\lambda+58104)}{4(\lambda+2)^2 r^3}\\
&+ \frac{3(247\lambda^5+1062\lambda^4-28714\lambda^3-52712\lambda^2+85688\lambda+2880)}{4(\lambda+2)^2 r^2}\\
&- \frac{(21\lambda^5+90\lambda^4-416\lambda^3-1540\lambda^2-624\lambda+48)\lambda^2 r}{(\lambda+2)^2}\\
&\left. + \frac{(51\lambda^5-2508\lambda^4-34338\lambda^3-19904\lambda^2+149336\lambda+14400)\lambda}{4(\lambda+2)^2 r}\right] + \frac{m^2}{f_0\Lambda^5(\lambda+2)^2\rho}\left[\frac{135(7\lambda-34)}{r^8}\right.\\
&- \frac{(7\lambda^4-168\lambda^3-644\lambda^2+944\lambda+1536)\lambda^3}{8r^2} - \frac{3(163\lambda^3+86\lambda^2+1532\lambda-1032)\lambda}{4r^5}\\
&- \frac{(275\lambda^4+586\lambda^3-952\lambda^2-5856\lambda+1728)\lambda}{4r^4} - \frac{9(165\lambda^2+1876\lambda-3564)}{8r^7}\\
&- \frac{3(553\lambda^3-1236\lambda^2-2340\lambda+3024)}{8r^6} + \frac{-\frac{129\lambda^6}{8}+34\lambda^5+\frac{901\lambda^4}{2}+304\lambda^3-504\lambda^2}{r^3} + \frac{1728}{r^9}\\
&\left. + \frac{(\lambda^3-3\lambda^2-34\lambda-24)\lambda^4}{r}\right], \qquad \text{(G4)}
\end{aligned}
$$

$$s^{\ell}_{1,0} = 2\rho^4 r(\ell+3)\frac{r^2\ell(\ell^3+5\ell^2-\ell-21) - r(\ell^4+5\ell^3-2\ell^2-25\ell+21) - \ell^2-4\ell+21}{\ell(\ell+1)^4(\ell+2)(r\ell(\ell+3)+3)^2}$$
$$+\rho^2(\ell+3)\frac{16r^3\ell^2(\ell+3)+r^2\ell(\ell^3-19\ell^2-57\ell+75)+r(-\ell^4+3\ell^3+14\ell^2-91\ell+27)-5\ell^2+16\ell-27}{2r^2\ell(\ell+1)^3(r\ell(\ell+3)+3)^2}$$
$$+(\ell+2)\frac{2r^3\ell^2(\ell+3)^2-r^2\ell(2\ell^3+13\ell^2+16\ell-15)+r(\ell^3-2\ell^2-16\ell+3)+\ell-3}{4r^5(\ell+1)(r\ell(\ell+3)+3)^2}, \tag{G5}$$

$$s^{\ell}_{1,1} = \frac{r^2\ell^4+(3r^2+29r+8)\ell+r(5r+3)\ell^3+r(7r+20)\ell^2+20}{2r^3\ell(\ell+1)^2(r\ell(\ell+3)+3)} - \rho^2\frac{r(\ell^4+7\ell^3+9\ell^2-7\ell+6)+3\ell^2+14\ell-1}{\ell(\ell+1)^3(\ell+2)(r\ell(\ell+3)+3)}, \tag{G6}$$

$$s^{\ell}_{1,K} = \frac{2f_0}{\ell+1}\left[4\rho^2\frac{\ell^3+7\ell^2+12\ell+2-r(\ell^3+5\ell^2+5\ell-3)}{r^4\ell(\ell+1)^2(\ell+2)} - \ell(\ell+3)\frac{r(\ell+2)-1}{r^7} + \frac{2\rho^4(\ell^3+5\ell^2-\ell-21)}{r\ell(\ell+1)^3(\ell+2)}\right], \tag{G7}$$

$$s^{\ell}_{1,H} = -\frac{(\ell+3)((r-1)r(r+3)\ell^2+r(r(r+4)-7)\ell-2r(r+2)+4)}{r^7(\ell+1)^2} + \frac{4\rho^4(\ell+3)(\ell(\ell+2)-7)}{\ell(\ell+1)^4(\ell+2)}$$
$$+\frac{\rho^2}{(\ell+1)^3}\left[\frac{\frac{6}{\ell}+2}{r^4} + \frac{2(\ell+3)(\ell(\ell(\ell(\ell+7)-3)-23)+2)}{r^2\ell(\ell+1)(\ell+2)} + \frac{2(\ell(\ell(\ell(3\ell+31)+103)+121)+14)}{r^3\ell(\ell+1)(\ell+2)}\right], \tag{G8}$$

$$t^{\ell}_{1,0} = -2\rho^4 r^3(\ell+3)\frac{r\ell^4+5r\ell^3-(r-1)\ell^2+(4-21r)\ell-21}{\ell(\ell+1)^4(\ell+2)(r\ell(\ell+3)+3)^2}\frac{\rho^2}{(\ell+1)^3(\ell+1)(r\ell(\ell+3)+3)^2}$$
$$\times\left[\frac{r^2\ell(\ell^3+5\ell^2-9\ell-29)(\ell+3)^2}{\ell+1} + \frac{3(5\ell+\frac{3}{\ell}+12)}{2r} + \frac{r(\ell^6+5\ell^5+6\ell^4+18\ell^3+21\ell^2-231\ell-396)}{\ell+1}\right.$$
$$\left.+\frac{17\ell^5+86\ell^4+154\ell^3+184\ell^2+21\ell-270}{2\ell(\ell+1)}\right]$$
$$+\frac{4r^3\ell^2(\ell+3)^2-2r^2\ell(\ell^3+7\ell^2+11\ell-3)+r(\ell^3+6\ell^2+11\ell-6)-3\ell+9}{4r^4(\ell+1)^2(r\ell(\ell+3)+3)^2}, \tag{G9}$$

$$t^{\ell}_{1,1} = 2\rho^2 r\frac{r(\ell^3+13\ell^2+31\ell+3)+\ell^2+8\ell+31}{f_0\ell(\ell+1)^4(\ell+2)(r\ell(\ell+3)+3)}$$
$$-\frac{r^2(4\ell^5+23\ell^4+27\ell^3-27\ell^2-35\ell+24)+4r^3\ell(\ell+1)^2(\ell+3)+r(17\ell^3+56\ell^2+7\ell-56)+12(\ell+1)}{2f_0r^3\ell(\ell+1)^3(\ell+2)(r\ell(\ell+3)+3)}, \tag{G10}$$

$$t^{\ell}_{1,K} = 2\rho^2(\ell+3)\frac{r(\ell^3+2\ell^2-7\ell-16)+6(\ell+2)}{r^3\ell(\ell+1)^3(\ell+2)} + 12\frac{1-r(\ell+2)}{r^6(\ell+1)^2(\ell+2)}, \tag{G11}$$

$$t^{\ell}_{1,H} = \frac{-2r^2\ell(\ell+2)(\ell+1)^2+r(2\ell^5+19\ell^4+56\ell^3+43\ell^2-16\ell+4)-\ell(\ell^3+6\ell^2-7\ell-36)}{(r-1)r^4\ell(\ell+1)^3(\ell+2)}$$
$$+2\rho^2(\ell+3)\frac{2r(\ell^2-6\ell-15)+\ell^2+6\ell+29}{(r-1)r\ell(\ell+1)^4(\ell+2)}, \tag{G12}$$

$$u^{\ell}_{0} = \frac{2(\ell+3)(r(\ell+2)((2r-1)\ell(\ell+3)(\ell+4)+6)+3(\ell+1))}{r^3(\ell+1)^2(\ell+2)(r\ell(\ell+3)+3)} + \frac{\rho^2(\ell+3)(\ell+4)(\ell+9)}{2(\ell+1)^2(\ell+2)(2\ell+3)}, \tag{G13}$$

$$u^{\ell}_{1} = \frac{4(\ell+3)(\ell+4)}{(r-1)(\ell+1)^2(\ell+2)}, \tag{G14}$$

$$u_K^\ell = \frac{r(2r^2(\ell+1)(\ell+3)(\ell+4)(4\ell+9) - r(\ell(\ell(2\ell(\ell+13)+99)+125)+26)+2\ell^2+\ell-21)+12}{2r^4(\ell+1)(\ell+2)(r(\ell+1)(\ell+4)+3)} + \rho^2 \frac{r(\ell+1)(\ell+3)(\ell+4)(\ell(\ell+9)+30)+\ell(\ell(3\ell+4)+19)+114}{2(\ell+1)^2(\ell+2)(2\ell+3)(r(\ell+1)(\ell+4)+3)}, \tag{G15}$$

$$u_H^\ell = \frac{\rho^2 r^2(8r(\ell+3)(\ell+4)-11\ell-35)}{(r-1)(\ell+1)^2(\ell+2)(r(\ell+1)(\ell+4)+3)} + \frac{1}{f_0(3+(\ell-3)\ell r)}\left[\frac{(\ell+4)(8\ell^2+41\ell+49)}{(\ell+1)(\ell+2)} - \frac{(4\ell^3+50\ell^2+182\ell+197)}{r(\ell^2+3\ell+2)} + \frac{3(17\ell+41)}{4r^3(\ell+1)^2(\ell+2)} + \frac{7\ell^3+48\ell^2+74\ell-3}{r^2(\ell+1)^2(\ell+2)}\right], \tag{G16}$$

$$v_0^\ell = 2\rho^2 f_0(\ell+3)\frac{\ell(2r(\ell+3)(\ell+4)-\ell+1)+18}{r(\ell+1)^2(\ell+2)(r\ell(\ell+3)+3)} + f_0(\ell+3)\frac{8r^2\ell(\ell+3)(\ell+4)-2r\ell(\ell+3)(3\ell+17)+24r+\ell(5\ell+19)-18}{2r^4(\ell+1)(r\ell(\ell+3)+3)}, \tag{G17}$$

$$v_1^\ell = \frac{(5r-2)(\ell+3)(\ell+4)}{r^3(\ell+1)(\ell+2)} + \frac{\rho^2(\ell+3)(\ell+4)(\ell+9)}{2(\ell+1)^2(\ell+2)(2\ell+3)}, \tag{G18}$$

$$v_K^\ell = \rho^2 f_0 \frac{r(\ell+1)(\ell+3)(\ell+4)+11\ell+35}{r(\ell+1)^2(\ell+2)(r(\ell+1)(\ell+4)+3)} + f_0(\ell+3)\frac{r(\ell+1)(4\ell+7)(r(\ell+4)-1)-8}{2r^4(\ell+1)(r(\ell+1)(\ell+4)+3)}, \tag{G19}$$

$$v_H^\ell = \rho^2 \frac{r(\ell+3)(\ell+4)(\ell(\ell(\ell+14)+17)-12)+\ell(\ell(3\ell+92)+431)+534}{2(\ell+1)^2(\ell+2)(2\ell+3)(r(\ell+1)(\ell+4)+3)} - \frac{-2r^2(\ell+1)(\ell+3)(\ell+4)(9\ell+17)+2r(\ell+1)(\ell(\ell(5\ell+56)+180)+163)+\ell(19\ell+124)+153}{4r^3(\ell+1)^2(r(\ell+1)(\ell+4)+3)}. \tag{G20}$$

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