Massive particle surfaces

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A novel generalization of photon surfaces to the case of massive charged particles is given for spacetimes with at least one isometry, including stationary ones. A related notion of *glued* massive particle surfaces is also defined. These surfaces join worldlines parametrized by a family of independent conserved quantities and naturally arise in integrable spacetimes. We describe the basic geometric properties of such surfaces and their relationship to slice-reducible Killing tensors, illustrating all concepts with a number of examples. Massive particle surfaces have potential applications in the context of uniqueness theorems, Penrose inequalities, integrability, and the description of black-hole shadows in streams of massive charged particles or photons in a medium with an effective mass and charge.

I. INTRODUCTION

A new observation window for studying extremely strong gravity near black holes, opened by the Event Horizon Telescope collaboration [1,2], has stimulated the development of new theoretical tools in the field of gravity based on the study of compact two-dimensional surfaces in space and the corresponding three-dimensional hypersurfaces in spacetime, in which photons can travel forever [3–7]. Well-known examples are the photon sphere in the Schwarzschild metric [8,9] or spherical surfaces in Kerr spacetime, on which nonequatorial spherical photon orbits wind [10]. The corresponding hypersurfaces in spacetime can be identified in terms of the geometric theory of submanifolds as three-dimensional umbilical or partially umbilical hypersurfaces [11,12].

The umbilical surface has the proportionality property of the induced metric tensor and the extrinsic curvature tensor, while the partially umbilical condition limits the equality of the first and second fundamental forms not for all tangent vectors but only for some subspace of the tangent space. In both cases, one can describe such surfaces without resorting directly to the geodesic equations. These timelike surfaces capturing photons have been called fundamental photon surfaces [11]. Their general properties are interesting not only for understanding the shadows of black holes [13–21] but also as theoretical tools for studying the stability of spacetimes, hidden symmetries [22–24] expressed by exact and conformal Killing tensors of the second rank [25,26], Penrose-type inequalities for regions of strong gravity [27–31], uniqueness theorems [32–41],

the impact of dark energy [42], etc. They can exist not only around black holes but also around wormholes or naked singularities, and they often open a way to distinguish these different ultracompact objects. The spatial sections of such surfaces are expected to possess restrictions on their area [43] due to the Penrose-type inequalities that can provide an analytical approach to exploring the compactness of gravitating objects.

A natural generalization of such characteristic surfaces is to consider timelike orbits of massive particles, as suggested in our previous paper [44] and in parallel in Ref. [45] with slightly different perspectives. Note that this also describes the properties of photons in a medium, such as plasma, which may have variable effective mass [46] and charge [47,48]. Surfaces that capture massive particles [10,49] share some common features with photon surfaces, but there are also important differences. The main difference with photons is that massive particles can move at an arbitrary speed, while photons have a fixed speed. For this reason, photon surfaces are more rigid and form a one-parameter family described by their radius only, while in Kerr metric the spherical orbits of massive particles form a *two-parameter family*: they can be parametrized by their radius and the value of the Carter constant [10]. Another important difference is that all spherical orbits of photons in Kerr are unstable, while timelike spherical Kerr geodesics in different parameter ranges can be both unstable and stable. As a result, a direct formal generalization of photon surfaces on the massive particle surfaces is insufficient and requires further ramifications. This is the purpose of this article, in which we introduce a new concept of "glued" surfaces, covering their internal splitting in the parameter space.

The article is organized as follows. In Sec. II, we briefly discuss the equations of motion for charged massive particles in spacetimes with several Killing vectors.

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We also cover the geometric aspects of particle velocity space, conventions of hypersurface geometry, and motivation for further generalization of massive particle surfaces. In Sec. III, we provide a more general and strict definition of massive particle surfaces, present a key theorem, and discuss the physical and geometric properties of these surfaces. We analyze the resulting geometric constraints, resolve them explicitly, determine the degree of arbitrariness of these surfaces, and discuss surfaces with all Killing vectors tangent to them. In Sec. IV, we define and analyze the effect of gluing of massive particle surfaces, a characteristic feature of integrable systems such as Kerr-like geometries. We examine the phenomenon of maximum gluing and its important geometric consequences. In Sec. V we discuss the close connection between massive particle surfaces and gluing phenomenon in relation to the integrability of dynamical systems and the existence of slice-reducible conformal Killing tensor fields of rank two [25,26]. Finally, in Sec. VI, we apply the developed formalism to various important examples, such as the Kerr metric, Zipoy-Voorhees solution, and the dyonic Kerr-Newman solution. The Appendix contains proofs of some statements formulated in the main part of the paper. We use symmetrization and antisymmetrization with weight 1, e.g., $A_{[\alpha\beta]} = A_{\alpha\beta} - A_{\beta\alpha}$.

II. SETUP

A. Killing vectors

Let M be a Lorentzian manifold of dimension n with metric tensor $g_{\alpha\beta}$ and Levi-Civita connection ∇_α. We define electromagnetic potential A_α and electromagnetic field tensor $F_{\alpha\beta} = \nabla_{[\alpha}A_{\beta]}$ in M. The worldline γ^α of a test particle with charge q and mass m in this geometry obeys the following equations of motion:

$$\dot{\gamma}^\alpha \nabla_\alpha \dot{\gamma}^\beta = qF^\beta{}_\lambda \dot{\gamma}^\lambda, \qquad \dot{\gamma}^\alpha \dot{\gamma}_\alpha = -m^2, \qquad (1)$$

where $\dot{\gamma}^\alpha = d\gamma^\alpha/ds$ is a four-velocity of the particle and s is an affine parameter. One can also use these equations for photons in a medium that can be described by an effective mass m and charge q [46–48], which can be a function of coordinates.

Consider a metric $g_{\alpha\beta}$ and an electromagnetic potential A_α that share the same symmetry with respect to Killing vector fields $k_a{}^\alpha$, where the Latin index a enumerates the Killing vectors [26]. Each Killing vector field satisfies Killing equations $\nabla_{(\alpha}k_{a\beta)} = 0$. This implies the existence of conserved quantities Q_a defined as (see Ref. [44] for details)

$$Q_a \equiv k_{a\alpha}(\dot{\gamma}^\alpha + qA^\alpha). \qquad (2)$$

It is also useful to consider two terms in the expression (2) separately, introducing the "kinetic" and "potential" components of the conserved quantity $Q_a = q_a + qp_a$:

$$p_a \equiv k_{a\alpha}A^\alpha, \qquad q_a \equiv Q_a - qp_a. \qquad (3)$$

In the general case, q_a and p_a are not conserved separately. The potential component is a predefined function for given $k_a{}^\alpha$ and A^α. The kinetic component q_a can be considered as a secondary quantity which is a certain function of coordinates for fixed conserved quantities Q_a. On the other hand, it can be represented as a scalar product of the Killing vector $k_a{}^\alpha$ with some properly normalized timelike (for $m \neq 0$, null for $m = 0$) vector v^α:

$$k_{a\alpha}v^\alpha = q_a = Q_a - qp_a, \qquad v_\alpha v^\alpha = -m^2. \qquad (4)$$

Any possible worldline of a particle with fixed Q_a, mass m, and charge q passing through a given point in spacetime has a four-velocity coinciding with some vector v^α subject to the constraint (4).

And vice versa, for any vector v^α obeying constraint (4) there is a worldline of a particle with fixed Q_a, m, and q, such that its four-velocity coincides with v^α. In other words, there is a bijective map between the set of all v^α constrained by (4) and the set of all charged particle worldlines passing through the given point with fixed Q_a, m, and q (see details in Ref. [11] for the massless case).

In what follows, we will assume that the Gram matrix $G_{ab} = k_a{}^\beta k_{b\beta}$ of Killing vectors $k_a{}^\alpha$ is nondegenerate.[1] In particular, all Killing vectors are linearly independent and the inverse matrix G^{ab} is well defined by the condition $G_{ac}G^{cb} = \delta_b^a$. According to intermediate results (A24) and (A27) of Theorem A.2 from the Appendix, the general solution of system (4) has the form

$$v^\alpha = k_a{}^\alpha G^{ab}q_b + u^\alpha, \qquad k_{a\alpha}u^\alpha = 0,$$
$$u^\alpha u_\alpha = -m^2 - G^{ab}q_a q_b. \qquad (5)$$

Vectors u^α are arbitrary vectors with a fixed norm that are orthogonal to all Killing vectors $k_a{}^\alpha$. The tangent space splits into a direct sum of the subspace generated by the Killing vectors and their orthogonal complement. Since the Gram matrix G_{ab} is not degenerate, these subspaces can have Lorentzian or Euclidean signatures (but none of them are null). If the orthogonal complement has a Euclidean signature, then it follows from (5) that the solution exists only if

$$G^{ab}q_a q_b \leq -m^2. \qquad (6)$$

If the orthogonal complement has a Lorentzian signature or consists of one timelike vector, then there are no restrictions [44]. In the classical Hamiltonian approach,

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[1] This condition can be relaxed by using pseudoinverse matrices. However, this could make the present work unnecessarily technical, so we will not consider the Gram matrix to be pseudoinverse.

inequality (6) follows from the effective potential descriptions [13].

If the orthogonal complement to all Killing vectors has a Euclidean signature, the particle with the given conserved quantities can move only in a certain domain of the spacetime given by inequality (6). In order to describe this domain, we introduce the following definition:

Definition 2.1: The $\mathcal{Q}$ domain of motion $\mathcal{M}_{\mathcal{Q}} \subseteq M$ is a maximal subset of M where the system (4) has a solution.

From the physical point of view, the motion along the worldlines with given values of the conserved quantities is possible only in a $\mathcal{Q}$ domain of motion $\mathcal{M}_{\mathcal{Q}}$. A boundary $\partial\mathcal{M}_{\mathcal{Q}}$ of the $\mathcal{Q}$ domain serves as a set of turning points for the corresponding worldlines. The boundaries correspond to saturation of the inequality (6) if the latter is applicable. Thus, $u^{\alpha} = 0$ and worldlines γ are tangent to the Killing vectors at the boundaries. In the particular case $m = 0$, it is easy to check that if $\mathcal{M}_{\mathcal{Q}}$ is connected, then it is nothing but the "causal ρ region" defined in Ref. [11] for null geodesics.

B. Hypersurface geometry

Let S be a timelike hypersurface without boundaries with normal unit vector n^{α}. The main geometric properties of S are specified by the induced metric $h_{\alpha\beta}$ and the second fundamental form $\chi_{\alpha\beta}$:

$$h_{\alpha\beta} = g_{\alpha\beta} - n_{\alpha}n_{\beta}, \qquad \chi_{\alpha\beta} \equiv h_{\alpha}^{\lambda}h_{\beta}^{\rho}\nabla_{\lambda}n_{\rho}. \quad (7)$$

The Gauss decomposition for two vector fields v^{α}, u^{α} tangent to S (i.e., $v^{\alpha}n_{\alpha} = u^{\alpha}n_{\alpha} = 0$) reads

$$v^{\alpha}\nabla_{\alpha}u^{\beta} = v^{\alpha}\mathcal{D}_{\alpha}u^{\beta} - \chi_{\alpha\gamma}v^{\alpha}u^{\gamma}n^{\beta}, \quad (8)$$

where $\mathcal{D}_{\alpha}$ is Levi-Civita connection on S, defined as [26,44]

$$\mathcal{D}_{\alpha}T^{\beta\cdots}_{\gamma\cdots} = h_{\alpha}^{\lambda}h_{\rho}^{\beta}\ldots h_{\gamma}^{\tau}\ldots\nabla_{\lambda}T^{\rho\cdots}_{\tau\cdots}, \qquad T^{\beta\cdots}_{\gamma\cdots} = h_{\rho}^{\beta}\ldots h_{\gamma}^{\tau}\ldots T^{\rho\cdots}_{\tau\cdots}. \quad (9)$$

An important observation is that the Gauss decomposition (8) allows us to reduce the problem of analyzing the behavior of worldlines (1) to the analysis of hypersurface fundamental forms (7). To illustrate this idea, consider a photon surface as an example [3]. A photon surface is a hypersurface S in M such that any null geodesic initially tangent to S will remain tangent to it. In other words, any null geodesics tangent to S are entirely in S. Applying the Gauss decomposition from Eq. (8) to the geodesic equation $\dot{\gamma}^{\alpha}\nabla_{\alpha}\dot{\gamma}^{\beta} = 0$ for an arbitrary null geodesic γ, we find

$$0 = \dot{\gamma}^{\alpha}\nabla_{\alpha}\dot{\gamma}^{\beta} = \dot{\gamma}^{\alpha}\mathcal{D}_{\alpha}\dot{\gamma}^{\beta} - \chi_{\alpha\gamma}\dot{\gamma}^{\alpha}\dot{\gamma}^{\gamma}n^{\beta} \Rightarrow \dot{\gamma}^{\alpha}\mathcal{D}_{\alpha}\dot{\gamma}^{\beta} = 0,$$

$$\chi_{\alpha\beta}\dot{\gamma}^{\alpha}\dot{\gamma}^{\beta} = 0. \quad (10)$$

Keeping in mind the correspondence between geodesic and tangent vectors from the previous analysis, we find out that for all null $v^{\alpha} \in TS$, $v^{\alpha}h_{\alpha\beta}v^{\beta} = 0$ the second fundamental form of the photon surface satisfies

$$\chi_{\alpha\beta}v^{\alpha}v^{\beta} = 0. \quad (11)$$

From Theorem A.1 in the Appendix follows that Eq. (11) holds for any null v^{α} if and only if the photon surface S is totally umbilical [3], i.e., its second fundamental form is proportional to the induced metric:

$$\chi_{\alpha\beta} = \frac{\chi}{n-1}h_{\alpha\beta}, \quad (12)$$

where $\chi = \chi_{\alpha}{}^{\alpha}$ is an arbitrary scalar function. Condition (12) is formulated in terms of the geometric properties of the surface only and does not refer to geodesic equations explicitly. This formulation is an effective way to analyze nonintegrable dynamical systems [11,12,23] and to study general theoretical problems such as Penrose inequalities [27–29], uniqueness theorems [32–41], and hidden symmetries [22–26].

In our previous paper [44] we generalized this geometric result to the case of worldlines of massive electrically charged particles. To do this, we used explicit spacetime symmetries described by a single Killing vector field. This allows us to give a new definition of massive particle surfaces at fixed energy and prove the key theorem about their geometric description. However, as in the case of photon surfaces, the surfaces defined in Ref. [44] for massive charged particles exist in static or spherically symmetrical spacetimes but not in stationary spacetimes with rotation. Nevertheless, in Ref. [10] different spherical orbits are shown to fill the whole spheres in Kerr metric. The novel feature is that different worldlines lying in the same sphere have different values of conserved energy. In fact, the energy integral on spherical orbits in Kerr depends on two quantities: their radius and the Carter constant associated with the second-rank Killing tensor. In order to describe this situation in more general terms, here we introduce a generalization of static massive particle surfaces of [44] in the same fashion as photon surfaces in static spacetimes were generalized to fundamental photon surfaces in the stationary ones in [11].

Recall that fundamental photon surfaces satisfy Eq. (11) only for a subset of null v^{α} constrained by a linear condition. For Kerr-like metrics, this constraint extracts such vectors v^{α} that correspond to the fixed impact parameter $\rho = -v_{\alpha}k^{\alpha}_{(\varphi)}/v_{\beta}k^{\beta}_{(t)}$, where $k^{\alpha}_{(t)}$, $k^{\alpha}_{(\varphi)}$ are timelike and azimuthal Killing vectors. First, we will introduce a more general definition of massive particle surfaces that satisfy condition (11) only for v^{α} corresponding to worldlines with fixed conserved quantities corresponding to a set of Killing vectors. And then we will show that the same

hypersurface can correspond not just to one value of conserved quantities but to some family of conserved quantities with a nonlinear constraint.

III. MASSIVE PARTICLES SURFACES

A. Definition and theorem

The first step toward our goal is to give a more general definition for massive particle surfaces [44] suitable for spacetimes with a larger number of isometries. Let S be a timelike hypersurface without boundaries. Define projections of Killing vectors $k_a{}^\beta$ as follows

$$\kappa_a{}^\alpha \equiv h_\beta^\alpha k_a{}^\beta, \qquad (13)$$

with a Gram matrix

$$\mathcal{G}_{ab} = \kappa_a{}^\alpha \kappa_{b\alpha} = G_{ab} - (k_{a\alpha}n^\alpha)(k_{b\beta}n^\beta). \qquad (14)$$

As before, we will assume that the matrix $\mathcal{G}_{ab}$ is non-degenerate and the matrix $\mathcal{G}^{ab}$ is its inverse. If some Killing vectors or their projections become linearly dependent or have a singular point, then we exclude them from the set.

We denote the subspace spanned by projections of Killing vectors as $\mathbf{K} \subseteq TS$ and its orthogonal complement as $\mathbf{K}^\perp$, i.e., $\mathbf{K} \oplus \mathbf{K}^\perp = TS$. Their dimensions are abbreviated as[2]

$$n_K \equiv \dim \mathbf{K}, \quad n_\perp \equiv \dim \mathbf{K}^\perp, \quad n_K + n_\perp + 1 = n. \qquad (15)$$

Using the inverse Gram matrix $\mathcal{G}^{ab}$, one can define projectors to the subspaces $\mathbf{K}$ and $\mathbf{K}^\perp$ as

$$K_\alpha^\beta = \kappa_{a\alpha}\mathcal{G}^{ab}\kappa_b{}^\beta, \qquad U_\alpha^\beta = h_\alpha^\beta - K_\alpha^\beta. \qquad (16)$$

Note that worldlines with a fixed set of conserved quantities Q_a cannot touch an arbitrary point on the timelike surface S. Indeed, for the worldline to touch S, the tangent vectors v^α must satisfy the conditions of Eq. (4). According to the intermediate results (A24) and (A27) of Theorem A.2 from the Appendix, the general solution of system (4) with the additional condition $v^\alpha n_\alpha = 0$ has the form

$$v^\alpha = \kappa_a{}^\alpha \mathcal{G}^{ab} q_b + u^\alpha, \qquad n_\alpha u^\alpha = 0, \qquad k_{a\alpha} u^\alpha = 0,$$

$$u^\alpha u_\alpha = -m^2 - \mathcal{G}^{ab} q_a q_b. \qquad (17)$$

Thus, as before, if the subspace $\mathbf{K}^\perp$ has a Euclidean signature, then the worldlines can only touch points of the surface S such that

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[2]In general, the dimensions can change from point to point. But we assume that $n_\perp \geq 1$ and $n_K \geq 1$.

$$\mathcal{G}^{ab} q_a q_b \leq -m^2. \qquad (18)$$

If $\mathbf{K}^\perp$ has a Lorentzian signature or consists of a single timelike vector, then there are no restrictions. Since it is pointless to look for massive particle surfaces among surfaces that worldlines cannot even touch, we suggest the following:

Definition 3.1: The $\mathcal{Q}$-touched hypersurface $\mathcal{S}_\mathcal{Q} \subset S$ is a maximal subset of S where the system (4) with the additional condition $v^\alpha n_\alpha = 0$ has a solution.

As in the case of the $\mathcal{Q}$ domain of motion, the definition of $\mathcal{Q}$-touched hypersurface ensures that worldlines can touch the surface $\mathcal{S}_\mathcal{Q}$ but cannot touch $S\backslash\mathcal{S}_\mathcal{Q}$. In particular, the worldline cannot exit $\mathcal{S}_\mathcal{Q}$ tangentially through its boundary $\partial\mathcal{S}_\mathcal{Q}$ (consisting of points, where $\mathcal{G}^{ab} q_a q_b = -m^2$). Instead, the worldline can only leave $\mathcal{S}_\mathcal{Q}$ by moving in the normal direction from the surface. It is clear that any $\mathcal{Q}$-touched hypersurface is inside a $\mathcal{Q}$ domain, i.e., $\mathcal{S}_\mathcal{Q} \subseteq \mathcal{M}_\mathcal{Q}$. Indeed, by definition, the domain $M\backslash\mathcal{M}_\mathcal{Q}$ does not contain vectors satisfying the constraints (4). Using Eq. (14), we can see that if all Killing vectors are tangent to the surface, then $q_a\mathcal{G}^{ab}q_b = q_a G^{ab}q_b$. This implies that $\partial\mathcal{S}_\mathcal{Q}$ is a subset of $\partial\mathcal{M}_\mathcal{Q}$. Now we are ready to select the massive particle surface among the $\mathcal{Q}$-touched hypersurfaces.

Definition 3.2: A massive particle surface is a $\mathcal{Q}$-touched hypersurface $\mathcal{S}_\mathcal{Q}$ such that, for every point $p \in \mathcal{S}_\mathcal{Q}$ and every vector $v^\alpha|_p \in T_p\mathcal{S}_\mathcal{Q}$ such that $v^\alpha\kappa_{a\alpha}|_p = q_a$ and $v^\alpha v_\alpha|_p = -m^2$, there exists a worldline γ of M for a particle with mass m, electric charge q, and the integrals of motion Q_a associated with the Killing vectors $k_a{}^\alpha$ such that $\dot{\gamma}^\alpha(0) = v^\alpha|_p$ and $\gamma \subset \mathcal{S}_\mathcal{Q}$.

Simply put, any worldline with a given set of conserved quantities Q_a that touches $\mathcal{S}_\mathcal{Q}$ (at least at one point) belongs to $\mathcal{S}_\mathcal{Q}$ entirely. Note that this definition and its implications remain valid for photons in a medium, such as plasma, with the only difference being that the effective photon mass may vary as a function of coordinates [46,47]. The key geometric properties of the massive particle surfaces are given in the following theorem:

Theorem 3.1: Let $\mathcal{S}_\mathcal{Q}$ be a smooth q-touched hypersurface. The following statements are equivalent:

(i) $\mathcal{S}_\mathcal{Q}$ is a massive particle surface for given q, m, and Q_a;

(ii) the second fundamental form satisfies

$$\chi_{\alpha\beta}v^\alpha v^\beta = -qn^\beta F_{\beta\lambda}v^\lambda, \qquad (19)$$

for all $p \in \mathcal{S}_\mathcal{Q}$ and $\forall v^\alpha \in T\mathcal{S}_\mathcal{Q}$ such that $v^\alpha v_\alpha = -m^2$ and $v^\alpha \kappa_{a\alpha} = q_a$;

(iii) the second fundamental form satisfies

$$\chi_{\alpha\beta} = \chi_\tau(h_{\alpha\beta} + \mathcal{H}^{ab}\kappa_{a\alpha}\kappa_{b\beta}) + \beta^a{}_{(\alpha}\kappa_{a\beta)}, \qquad (20a)$$

where χ_τ, $\mathcal{H}^{ab}$, and $\beta^a_{\ \alpha}$ $(\kappa_{a\alpha}\beta^{b\alpha} = 0)$ are some functions restricted by the following constraints

$$\chi_\tau(\mathcal{H}^{ab}q_a q_b - m^2) + q\mathcal{F}^a q_a = 0,$$
$$2q_a\beta^a_{\ \alpha} = -qf_\alpha, \qquad (20b)$$

and $\mathcal{F}^a$, f_α represent the following components of the field tensor $F_{\alpha\beta}$

$$\mathcal{F}^a = \mathcal{G}^{ab}\kappa_b^{\ \beta}n^\alpha F_{\alpha\beta}, \qquad f_\alpha = U_\alpha^\beta n^\gamma F_{\gamma\beta},$$
$$\text{i.e.,} \quad n^\beta F_{\beta\alpha} = \mathcal{F}^a\kappa_{a\alpha} + f_\alpha, \qquad \kappa_a^{\ \alpha}f_\alpha = 0; \quad (20c)$$

(iv) every worldline in $\mathcal{S}_Q$ with $\dot{\gamma}^\alpha\kappa_{a\alpha}|_p = q_a$ and $\dot{\gamma}^\alpha\dot{\gamma}_\alpha|_p = -m^2$ at some point $p \in \mathcal{S}_Q$ is a worldline in M.

Proof: The proofs that $(i) \Rightarrow (ii)$, $(iii) \Rightarrow (iv)$, and $(iv) \Rightarrow (i)$ are the same as the corresponding proofs from the theorem in Ref. [44]. Accordingly, we will generalize explicitly only the proof of $(ii) \Rightarrow (iii)$. We start from the following conditions on vectors v^α tangent to the hypersurface $\mathcal{S}_Q$:

$$h_{\alpha\beta}v^\alpha v^\beta = -m^2, \qquad v^\alpha\kappa_{a\alpha} = q_a. \qquad (21)$$

The aim is to answer the question of when are the solutions of this system in the set of solutions of the following system

$$\chi_{\alpha\beta}v^\alpha v^\beta = -qn^\beta F_{\beta\lambda}v^\lambda, \qquad (22)$$

without any additional linear constraints. In order to apply Theorem A.3 from the Appendix, we define the following matrices and vectors:

$$A = h_{\alpha\beta}, \qquad B = \kappa_{a\alpha}, \qquad f = q_a, \qquad w = 0,$$
$$d = -m^2, \qquad x = v^\alpha, \qquad (23a)$$

$$A' = \chi_{\alpha\beta}, \qquad B' = 0, \qquad f' = 0,$$
$$w' = \frac{1}{2}qn^\beta F_{\beta\lambda}, \qquad d' = 0. \qquad (23b)$$

All vectors and tensors with greek indices defined in (23) are tangent to $\mathcal{S}_Q$ by construction. Therefore, we can work with the matrix A as with the operator $T\mathcal{S}_Q \to T^*\mathcal{S}_Q$, but not as one acting on the entire spacetime M. In this case, the operator A is nondegenerate, since it is a nonsingular metric in $\mathcal{S}_Q$. The nondegeneracy of A is one of the requirements of Theorem A.3 from the Appendix. The identity operator $\mathbb{1}$ in $T\mathcal{S}_Q$ corresponds to h_α^β. In addition, we must define the matrices G, Π, and A_{pr} as follows [see Eqs. (A19), (A20), and (A28) for details]:

$$G = \mathcal{G}^{ab},$$
$$\Pi = h_\alpha^\beta - \kappa_{a\alpha}\mathcal{G}^{ab}\kappa_{b\gamma}h^{\gamma\beta} = U_\alpha^\beta,$$
$$A_{pr} = \Pi^T A\Pi = h_{\alpha\beta} - \kappa_{a\alpha}\mathcal{G}^{ab}\kappa_{b\beta} = U_{\alpha\beta}. \qquad (24)$$

Application of the theorem is possible if the kernel of the matrix B is spanned by solutions x for system (24). Recall that $\ker \kappa_{a\alpha} = \{x^\alpha: \kappa_{a\alpha}x^\alpha = 0\}$, i.e., the kernel of B in this case is the entire subspace of $\mathbf{K}^\perp$. This is equivalent to the requirement that projections of the solutions onto the orthogonal complement $u^\alpha = U_\beta^\alpha v^\beta$ span the entire kernel of the matrix $\kappa_{a\alpha}$. This requirement is valid at interior points of the hypersurface $\mathcal{S}_Q/\partial\mathcal{S}_Q$, since u^α are all vectors from $\mathbf{K}^\perp$ with fixed norm $u^\alpha u_\alpha = -m^2 - \mathcal{G}^{ab}q_a q_b$. The norm sign fits the $\mathbf{K}^\perp$ signature because $\mathcal{S}_Q$ is Q touched. Only on the boundaries $\partial\mathcal{S}_Q$ the norm becomes equal to zero, and this condition is violated. We will consider only interior points of $\mathcal{S}_Q\backslash\partial\mathcal{S}_Q$, but the result can be extrapolated to the entire hypersurface $\mathcal{S}_Q$ due to its smoothness.

Applying the theorem, we get the form of $\chi_{\alpha\beta}$

$$\chi_{\alpha\beta} = \chi_\tau(h_{\alpha\beta} + \kappa_{a\alpha}\mathcal{H}^{ab}\kappa_{b\beta}) + \beta^a_{\ (\alpha}\kappa_{\beta)a}, \qquad (25)$$

where we use χ_τ, $\beta^a_{\ \alpha}$, and $\chi_\tau\mathcal{H}^{ab}$ instead of an arbitrary function λ, matrices S_1 and S_2 from the theorem formulation, respectively. The theorem states that the following three conditions must hold

$$0 = \kappa_{a\alpha}\beta^{b\alpha},$$
$$0 = \frac{q}{2}U_\alpha^\beta n^\gamma F_{\gamma\beta} + \beta^a_{\ \alpha}q_a = \frac{q}{2}f_\alpha + \beta^a_{\ \alpha}q_a,$$
$$0 = \chi_\tau(q_a\mathcal{H}^{ab}q_b - m^2) + qq_a\mathcal{G}^{ab}\kappa_{b\alpha}h^{\alpha\beta}n^\gamma F_{\gamma\beta}$$
$$= \chi_\tau(q_a\mathcal{H}^{ab}q_b - m^2) + qq_a\mathcal{F}^a, \qquad (26)$$

where the projections $\mathcal{F}^a$, f_α are defined in (20c). This completes the proof of the theorem.

This theorem is a direct generalization of an analogous result from Ref. [44] to the case of an arbitrary number of isometries. The basic geometric description of the massive particle surface is given by statement *(iii)* of the theorem. Let us try to clarify the geometric meaning of this definition. First of all, Eq. (20a) projected onto $\mathbf{K}$ and $\mathbf{K}^\perp$ gives the identities

$$\kappa_a^{\ \alpha}\chi_{\alpha\beta}\kappa_b^{\ \beta} = \chi_\tau(\mathcal{H}_{ab} + \mathcal{G}_{ab}), \qquad (27a)$$

$$\kappa_a^{\ \alpha}\chi_{\alpha\beta}U_\lambda^\beta = \mathcal{G}_{ab}\beta^b_{\ \lambda}, \qquad (27b)$$

$$U_\gamma^\alpha\chi_{\alpha\beta}U_\lambda^\beta = \chi_\tau U_\gamma^\alpha h_{\alpha\beta}U_\lambda^\beta, \qquad (27c)$$

where $\mathcal{H}_{ab} = \mathcal{G}_{ac}\mathcal{H}^{cd}\mathcal{G}_{db}$. From Eq. (27c) follows that the massive particle surface is partially umbilical [11] in $\mathbf{K}^\perp$. A partially umbilical surface is defined by the condition that

the extrinsic curvature is the same along certain directions, such as the subspace $\mathbf{K}^\perp$ in our case. When there is only one such direction ($n^\perp = 1$), the condition for partially umbilical surface degenerates, fixing the function χ_τ. One can extract $\mathcal{H}_{ab}$ and $\beta^a{}_\alpha$ from Eqs. (27a) and (27b)

$$\mathcal{H}_{ab} = \frac{1}{\chi_\tau}\kappa_a{}^\alpha \kappa_b{}^\beta \chi_{\alpha\beta} - \mathcal{G}_{ab}, \tag{28a}$$

$$\beta^a{}_\lambda = G^{ab}\kappa_b{}^\alpha U^\beta_\lambda \chi_{\alpha\beta}, \tag{28b}$$

where we assume that χ_τ is not zero.

B. Extended massive particle surface

It is convenient to ignore condition (18) and consider the hypersurface at first sight without it. To do that, we introduce an *extended massive particle surface* $\mathcal{S}_e$. Namely, the massive particle surfaces $\mathcal{S}_Q$ are defined as a part of some surface S. The geometry of the complement part of the surface $S \backslash \mathcal{S}_Q$ is not fixed in any way, since none of the worldlines with the corresponding Q_a can touch these points. The most natural extension of $\mathcal{S}_Q$ seems to be an entirely smooth one without boundaries. We require $S = \mathcal{S}_e$ to possess the second fundamental form satisfying Eq. (20a) with conditions (20b) in all its points. Of course, in the general case, such an extension may not exist. Altogether, the definition of the extended massive particle surface is the following definition.

Definition 3.3: An extended massive particle surface is a hypersurface $\mathcal{S}_e$ with no boundaries, such that for every point $p \in \mathcal{S}_e$, the second fundamental form takes the form (20a) subject to the imposed conditions (20b).

Obviously, any $\mathcal{Q}$-touched part of the extended massive particle surface is an ordinary massive particle surface. However, $\mathcal{S}_e$ has a stiffer geometry in the complement part $S \backslash \mathcal{S}_Q$. In our opinion, this restriction will be extremely useful for the analysis of both ordinary surfaces of massive particles and general issues such as integrability and Penrose inequalities. In particular, boundary-less $\mathcal{S}_e$ surfaces require less technical detail and suggest more topological applications, making them a particularly useful object of study.

C. Constraints

Conditions (20b) are invariant under the set of transformations

$$q_a \to \lambda q_a, \qquad q \to \lambda q, \qquad m^2 \to \lambda m^2. \tag{29}$$

In particular, for the null geodesics, there is an invariance $q_a \to \lambda q_a$ associated with the conformal invariance of photon worldlines. Note that any set Q_a can have its own unique hypersurface. However, as we will see later, the hidden symmetries of space can lead to gluing different

surfaces together and forming complete hypersurface S without boundaries.

Conditions (20b) impose some restrictions on the components of the second fundamental form $\chi_{\alpha\beta}$. However, not all components are generally rigidly fixed, and some of them can be completely arbitrary. To explicitly reveal the role of constraints (20b), we define the vector τ^α (similarly to the covector τ_α, the dual vector of the impact parameter ρ^α in Ref. [11]) in the subspace $\mathbf{K}$ as

$$\tau^\alpha = h^{\alpha\beta}\kappa_{\alpha\beta}\mathcal{G}^{ab}q_b, \qquad \tau^2 \equiv \tau^\alpha\tau_\alpha = q_a\mathcal{G}^{ab}q_b. \tag{30}$$

If $\mathbf{K}$ is Euclidean, then τ^2 is strictly positive for any nonzero q_a. Otherwise, it is nonzero due to the inequality (6) for $m \neq 0$. Then we define the projection T^α_β along the direction τ^α and the projector P^α_β onto the orthogonal complement τ^α in $\mathbf{K}$:

$$T^\alpha_\beta = \tau^{-2}\tau^\alpha\tau_\beta, \qquad P^\alpha_\beta = K^\alpha_\beta - T^\alpha_\beta. \tag{31}$$

The second and third terms in Eq. (20a) can be further projected using (31) as follows:

$$\chi_{\alpha\beta} = \chi_\tau h_{\alpha\beta} + z\tau_\alpha\tau_\beta + \sigma_{\alpha\beta} + \tau_{(\alpha}z_{\beta)}, \tag{32}$$

where z^α and $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ have the following projection properties

$$K^\alpha_{\alpha'}z_\alpha = 0, \qquad T^\alpha_{\alpha'}\sigma_{\alpha\beta}T^\beta_{\beta'} = 0, \qquad T^\alpha_{\alpha'}\sigma_{\alpha\beta}U^\beta_{\beta'} = 0,$$
$$U^\alpha_{\alpha'}\sigma_{\alpha\beta}U^\beta_{\beta'} = 0. \tag{33}$$

Plugging Eqs. (28) and (32) and the identity $q_a = \kappa_{a\alpha}\tau^\alpha$ in the constraints (20b), we get the following independent of $\sigma_{\alpha\beta}$ results:

$$z = \tau^{-4}(\chi_\tau m^2 - q\mathcal{F}^a\kappa_{a\alpha}\tau^\alpha), \qquad z_\alpha = -\frac{1}{2}\tau^{-2}qf_\alpha. \tag{34}$$

Finally, we can reformulate statement *(iii)* in Theorem 3.1 without additional restrictions as follows:

(iii) the second fundamental form can be represented as

$$\chi_{\alpha\beta} = \chi_\tau h_{\alpha\beta} + \tau^{-4}(\chi_\tau m^2 - q\mathcal{F}^a\kappa_{a\gamma}\tau^\gamma)\tau_\alpha\tau_\beta$$
$$+ \sigma_{\alpha\beta} - \frac{1}{2}\tau^{-2}q\tau_{(\alpha}f_{\beta)}, \tag{35}$$

where χ_τ is some function, the vector τ^α is defined in Eq. (30), an arbitrary symmetric matrix $\sigma_{\alpha\beta}$ is constrained by the conditions Eq. (33), while $\mathcal{F}^a$, f_α represent the components of the $F_{\alpha\beta}$ field tensor defined in (20c).

The number of unrestricted components in $\chi_{\alpha\beta}$ is placed in $\sigma_{\alpha\beta}$. In total $\sigma_{\alpha\beta}$ has $n(n-1)/2$ components restricted by projectors. There are 1, $n_\perp$, and $n_\perp(n_\perp + 1)/2$ independent constraints in each condition on $\sigma_{\alpha\beta}$ from (33), respectively.

By subtracting the number of constraints from the total number of components in $\sigma_{\alpha\beta}$, we get $(n_K - 1)(2n - n_K)/2$ unrestricted components. In the case of $n_K = 1$ there are no unrestricted components at all, and the outer geometry of the hypersurface is uniquely determined by the inner geometry, as expected [44]. For null geodesics, we simply get

$$\chi_{\alpha\beta} = \chi_\tau h_{\alpha\beta} + \sigma_{\alpha\beta}. \qquad (36)$$

Thus, the hypersurface is umbilical for subspace $\{T^\alpha_\beta, U^\alpha_\beta\}$ as expected for fundamental photon surfaces [11].

D. One Killing vector

In the case of only one Killing vector field k^α, all components of the second fundamental form turn out to be strictly fixed (in this subsection, we omit the indices a, b, since there is only one direction). Indeed, Eq. (20b) implies $(\mathcal{E}_k = -q_k)$

$$\mathcal{H} = \frac{m^2}{\mathcal{E}_k^2} + \frac{q}{\chi_\tau \mathcal{E}_k}\mathcal{F}, \qquad \beta_\alpha = \frac{q}{2\mathcal{E}_k}f_\alpha, \qquad (37)$$

Substituting this back to Eq. (20a), the second fundamental form is read

$$\chi_{\alpha\beta} = \chi_\tau \left(h_{\alpha\beta} + \frac{m^2}{\mathcal{E}_k^2}\kappa_\alpha\kappa_\beta \right) + \frac{q}{2\mathcal{E}_k}n^\rho F_{\rho(\alpha}\kappa_{\beta)}. \quad (38)$$

This is exactly the same property *(iii)* from the theorem in Ref. [44]. Thus, the new massive particle surfaces are in fact a direct generalization of similar surfaces from Ref. [44] for spacetimes admitting a larger number of isometries.

E. Orthogonal Killing vector

If there exists a Killing vector $k^\alpha_\perp$ that is everywhere orthogonal to the hypersurface $\mathcal{S}_Q$, then $\mathcal{S}_Q$ is a totally geodesic hypersurface, i.e., $\chi_{\alpha\beta} = 0$. This follows from the Killing equation $\nabla_{(\alpha}k_{\perp\beta)} = 0$ and the symmetry of the product $h^\lambda_\alpha h^\rho_\beta$:

$$\chi_{\alpha\beta} \equiv h^\lambda_\alpha h^\rho_\beta \nabla_\lambda n_\rho = h^\lambda_\alpha h^\rho_\beta \nabla_\lambda(\mathfrak{K}k_{\perp\rho})$$
$$= h^\lambda_\alpha h^\rho_\beta [\mathfrak{K}\nabla_\lambda k_{\perp\rho} + k_{\perp\rho}\nabla_\lambda\mathfrak{K}] = 0, \qquad (39)$$

where the normal vector is expressed through the Killing vector $n^\alpha = \mathfrak{K}k^\alpha_\perp$. The worldline γ can touch the hypersurface $\mathcal{S}_Q$ if and only if $q_\perp = 0$. Since the condition $\chi_{\alpha\beta} = 0$ results in $\chi_\tau = 0$, $\beta_a^\alpha = 0$, constraints (20b) can be simplified to the conditions for the Lorentz force acting on particles lying on the surface

$$q\mathcal{F}^a q_a = 0, \qquad qf_\alpha = 0, \qquad (40)$$

i.e., either particles are neutral $q = 0$, or $\mathcal{F}^a q_a = 0, f_\alpha = 0$.

F. Tangent Killing vectors

Of great interest is the class of massive particle surfaces that have the same set of symmetries as the original spacetime. It is well known that surfaces are invariant under the action of some isometry group if the corresponding Killing vector field is tangent to S. Thus, we assume that the Killing vectors touch the massive particle surface (not necessarily all the spacetime Killing vectors, but at least those that span the **K** subspace), and they are all linearly independent. If they are not linearly independent, then remove the redundant vectors from the considered set of Killing vectors. The mixed components of the second fundamental form can be rewritten as

$$U^\alpha_\gamma \kappa^\beta_a \chi_{\alpha\beta} = U^\alpha_\gamma \kappa^\beta_a \nabla_\alpha n_\beta = -U^\alpha_\gamma n^\lambda \nabla_\alpha \kappa_{a\lambda}. \qquad (41)$$

Combining with Eq. (28b), we get

$$\beta^a_{\ \alpha} = -\mathcal{G}^{ab}n^\lambda U^\beta_\alpha \nabla_\beta \kappa_{b\lambda}. \qquad (42)$$

Substituting this expression into the linear constraint in Eq. (20b), we get a constraint on some components of $F_{\alpha\beta}$

$$U^\beta_\gamma n^\alpha (qF_{\alpha\beta} - 2q_a \mathcal{G}^{ab}\nabla_\beta \kappa_{b\alpha}) = 0. \qquad (43)$$

Before proceeding to the analysis of the components of the second fundamental form tangent to the Killing vectors, we derive several identities. From the Killing equations, it follows

$$n^\beta(\kappa_a^{\ \alpha}\nabla_\alpha\kappa_{b\beta} + \kappa_b^{\ \alpha}\nabla_\alpha\kappa_{a\beta}) = -n^\beta(\kappa_a^{\ \alpha}\nabla_\beta\kappa_{b\alpha} + \kappa_b^{\ \alpha}\nabla_\beta\kappa_{a\alpha})$$
$$= -n^\beta\nabla_\beta(\kappa_a^{\ \alpha}\kappa_{b\alpha}) = -n^\beta\nabla_\beta\mathcal{G}_{ab}. \qquad (44)$$

Using the symmetry of the second fundamental form $(\chi_{\alpha\beta} = \chi_{\beta\alpha})$, we get the expression for the components of the second fundamental form tangent to the Killing vectors

$$\kappa_a^{\ \alpha}\kappa_b^{\ \beta}\chi_{\alpha\beta} = \kappa_a^{\ \alpha}\kappa_b^{\ \beta}\nabla_\alpha n_\beta = -n^\beta\kappa_a^{\ \alpha}\nabla_\alpha\kappa_{b\beta}$$
$$= -\frac{1}{2}n^\beta(\kappa_a^{\ \alpha}\nabla_\alpha\kappa_{b\beta} + \kappa_b^{\ \alpha}\nabla_\alpha\kappa_{a\beta})$$
$$= \frac{1}{2}n^\beta\nabla_\beta\mathcal{G}_{ab}. \qquad (45)$$

Comparing Eq. (28a) and (45), we get the expression for the tensor $\mathcal{H}_{ab}$

$$\mathcal{H}_{ab} = \frac{1}{2\chi_\tau} n^\beta \nabla_\beta \mathcal{G}_{ab} - \mathcal{G}_{ab} \quad \text{or}$$

$$\mathcal{H}^{ab} = -\frac{1}{2\chi_\tau} n^\beta \nabla_\beta \mathcal{G}^{ab} - \mathcal{G}^{ab}, \tag{46}$$

Quadratic constraint (20b) becomes the master equation [12,25] for massive particle surface

$$\frac{1}{2} n^\beta \nabla_\beta \mathcal{G}^{ab} q_a q_b + \chi_\tau (\mathcal{G}^{ab} q_a q_b + m^2) - q \mathcal{F}^a q_a = 0. \tag{47}$$

Thus the second fundamental form of such a massive particle surface has no arbitrary components, i.e., tensor $\sigma_{\alpha\beta}$ in expression (35) turns out to be completely fixed.

IV. GLUED PARTICLE SURFACES

The surfaces defined in the previous section usually capture worldlines only for certain values of the integrals of motion. However, in a Kerr-type spacetime that allows spherical orbits of constant radius, each sphere captures worldlines defined by a one-parameter family of integrals of motion [10]. Therefore we find it useful to introduce the "glued" massive particle surfaces as follows:

Definition 4.1: A glued particle surface $\mathcal{S}_g$ of the order g is a hypersurface, which is an extended massive particle surface for a family of Q_a of dimension $g \geq 1$ for fixed mass m and electric charge q.

Less formally, a glued particles surface $\mathcal{S}_g$ is a continuous stack of extended massive particle surfaces, representing the same hypersurface. Obviously, $\mathcal{S}_g$ contains all the corresponding massive particle surfaces (not extended), and the order of its gluing is the same for all points of $p \in \mathcal{S}_g$.

To find out whether the extended massive particle surface is glued, we calculate the variation of constraints (20b) with respect to Q_a

$$(2\chi_\tau \mathcal{H}^{ab} q_b + q \mathcal{F}^a) \delta Q_a = 0, \qquad \beta^a{}_\alpha \delta Q_a = 0. \tag{48}$$

The massive particle surface is glued if there exists a nontrivial solution for the constant δQ_a on the surface. The dimension of the space of constant solutions δQ_a determines the order of the glued surface.

We can also approach the problem from a different angle. Namely, consider some timelike surface S without boundaries, partially umbilical everywhere, i.e., the second fundamental form takes form (20). At each point of such a surface, one can calculate the matrices $\mathcal{H}^{ab}$ and $\beta^a{}_\alpha$ and consider the conditions from the Eq. (20b) as a system of quadratic and linear equations in unknown real variables Q_a. Clearly, the dimension of this family determines the possible *maximal gluing order* g_{max}. If the hyperplane in space of Q_a determined by the linear constraint intersects the quadric hypersurface determined by the quadratic equation, then the

maximal gluing order is $g_{max} = n_K - 1 - \text{rank}B$. If the hyperplane is only tangent to the hypersurface along some directions, then $g_{max} < n_K - 1 - \text{rank}B$.

The solution Q_a corresponds to the extended massive particle surface $\mathcal{S}_e$ if and only if it satisfies condition (20b) at all points $p \in \mathcal{S}_e$. Thus, we can apply any Lie derivative to condition (20b) while keeping Q_a unchanged. The dimension of the subset of such solutions Q_a, satisfying the conditions on the entire extended massive particle surface, determines the actual order of gluing $g \leq g_{max}$. If the number of such solutions is countable, then the surface is not glued.

We rewrite system (20b) in the form given in Eq. (A40) from Theorem A.3 as follows:

$$Q_a \mathcal{H}^{ab} Q_b + 2 \mathcal{J}^a Q_a = \mathcal{M}^2, \qquad \beta^a{}_\alpha Q_a = j_\alpha, \tag{49}$$

where we introduce the following notations

$$\mathcal{J}^a = q \left(\frac{1}{2\chi_\tau} \mathcal{F}^a - \mathcal{H}^{ab} p_b \right), \qquad j_\alpha = q(p_a \beta^a{}_\alpha - f_\alpha/2), \tag{50a}$$

$$\mathcal{M}^2 = m^2 + q^2 p_a \left(\frac{1}{\chi_\tau} \mathcal{F}^a - \mathcal{H}^{ab} p_b \right). \tag{50b}$$

The transition to matrix notation in Theorem A.3 is achieved by the following identification

$$\boldsymbol{x} = Q_a, \qquad A = \mathcal{H}^{ab}, \qquad B = \beta^a{}_\alpha, \qquad \boldsymbol{w} = \mathcal{J}^a,$$
$$d = \mathcal{M}^2, \qquad \boldsymbol{f} = j_\alpha, \tag{51}$$

The primed system (A41) can be represented as the Lie derivative of condition (49):

$$A' = \mathcal{D}_\gamma \mathcal{H}^{ab}, \qquad B' = \mathcal{D}_\gamma \beta^a{}_\alpha, \qquad \boldsymbol{w}' = \mathcal{D}_\gamma \mathcal{J}^a,$$
$$d' = \mathcal{D}_\gamma \mathcal{M}^2, \qquad \boldsymbol{f}' = \mathcal{D}_\gamma j_\alpha. \tag{52}$$

For maximum gluing $g = g_{max}$, any solution of system (49) must also satisfy the primed system, and we can apply Theorem A.3, which will impose additional conditions on the surface geometry. A complete general analysis of an arbitrary glued particle surface using Theorem A.3 is voluminous, since it depends on various factors, such as the kernels of the A and B matrices, their intersection, the relation between the vectors $\mathcal{J}^a$ and these kernels, the $\mathcal{M}^2$ sign, etc. Therefore, in this paper we will consider only the simple case $\beta^a{}_\alpha = 0$ to demonstrate the applicability of the formalism, deferring a more complete analysis to future work.

A. Example: Case $\beta^a{}_\alpha = 0$

The case of maximum gluing $g = g_{\max} = n_K - 1$ without the linear constraint $\beta^a{}_\alpha = 0$ offers a simpler analysis but yields fruitful results. First of all, we obtain that $f_\alpha = 0$, $B = B' = 0$, $\boldsymbol{f} = \boldsymbol{f}' = 0$, and the pseudoinverse matrix G can be set to zero. This follows from Theorem A.3 that

$$A' = \lambda A, \qquad \boldsymbol{w}' = \lambda \boldsymbol{w}, \qquad d' = \lambda d \qquad (53)$$

without any additional constraints. Note that an arbitrary function λ has a vector index because the primed quantities contain the derivative $\mathcal{D}_\tau$ along any direction tangent to the hypersurface. Hence, by expressing these equations in terms of the derivative, we obtain

$$\mathcal{D}_\tau \mathcal{H}^{ab} = \lambda_\tau \mathcal{H}^{ab}, \qquad (54a)$$

$$\mathcal{D}_\tau \mathcal{J}^a = \lambda_\tau \mathcal{J}^a, \qquad (54b)$$

$$\mathcal{D}_\tau \mathcal{M}^2 = \lambda_\tau \mathcal{M}^2. \qquad (54c)$$

In the chargeless massive case, Eq. (54b) holds automatically, from Eq. (54c) follows $\lambda_\tau = 0$, and the only condition is $\mathcal{D}_\tau \mathcal{H}^{ab} = 0$. In the null case $q = m = 0$, Eqs. (54b) and (54c) holds automatically, and the only condition is (54a). It is worth mentioning with no detail that one can similarly apply Theorem A.3 to the neutral case $q = 0$ with arbitrary $\beta^a{}_\alpha$, getting the same result that either $\lambda_\tau = 0$ or $m = 0$. One can find from Eq. (50a) that $\mathcal{M}^2 = m^2 + q p_a(\mathcal{J}^a + q\mathcal{F}^a/2\chi_\tau)$, which can be used to combine Eq. (54) into new ones (we assume that the mass m is constant)

$$q\mathcal{H}^{ab}\mathcal{D}_\tau p_b = q\mathcal{D}_\tau\left(\frac{1}{2\chi_\tau}\mathcal{F}^a\right) - q\lambda_\tau\left(\frac{1}{2\chi_\tau}\mathcal{F}^a\right), \quad (55a)$$

$$q^2 \mathcal{F}^a \mathcal{D}_\tau p_a = \lambda_\tau \chi_\tau m^2, \qquad (55b)$$

which can replace Eqs. (54b) and (54c).

Generally, Eq. (54) can be represented as $\mathcal{D}_\tau X = \lambda_\tau X$, where $X \in \{\mathcal{H}^{ab}, \mathcal{J}^a, \mathcal{M}^2\}$ is a scalar with respect to connection $\mathcal{D}_\tau$. Let us calculate the commutator $[\mathcal{D}_\tau, \mathcal{D}_\sigma]$ acting on any of these Xs. Since it is a scalar, the commutator must be zero $[\mathcal{D}_\tau, \mathcal{D}_\sigma]X = 0$. On the other hand, from Eq. (54) it follows that

$$0 = [\mathcal{D}_\tau, \mathcal{D}_\sigma]X = (\mathcal{D}_\tau\lambda_\sigma - \mathcal{D}_\sigma\lambda_\tau)X \Rightarrow \mathcal{D}_\tau\lambda_\sigma - \mathcal{D}_\sigma\lambda_\tau = 0, \qquad (56)$$

which ensures that λ_τ can be expressed locally as

$$\lambda_\tau = -\mathcal{D}_\tau \ln \Lambda, \qquad (57)$$

where Λ is some positive function. In this case, Eq. (54) can be represented as

$$\mathcal{D}_\tau(\Lambda\mathcal{H}^{ab}) = 0, \qquad (58a)$$

$$\mathcal{D}_\tau(\Lambda\mathcal{J}^a) = 0, \qquad (58b)$$

$$\mathcal{D}_\tau(\Lambda\mathcal{M}^2) = 0. \qquad (58c)$$

From Eq. (58c) follows that one can set $\Lambda = M^{-2}$ up to some multiplicative function that is constant on $\mathcal{S}_g$. This leaves us with only two first equations

$$\mathcal{H}^{ab} = M^2 C^{ab}, \qquad \mathcal{J}^a = M^2 C^a, \qquad (59)$$

where C^{ab}, C^a are constant in $\mathcal{S}_g$. Equation (59) means that the original quadratic equation (49) is constant on the surface up to some arbitrary multiplicative function. In the final section of the article, we show by a number of examples that in Kerr-like geometries all the discussed gluing conditions are indeed satisfied. But first, let us try to understand the deeper cause of the gluing phenomenon and its consequences.

V. RELATION TO KILLING TENSORS

The existence of glued surfaces of massive particles in the Kerr metric is closely related to the existence of hidden spacetime symmetries described by the Killing tensor of the second rank. For many physically relevant spacetimes, the Killing tensor of the second rank is slice-reducible with respect to some foliation, so that the Killing vectors are slice tangent, $k_a^\alpha = \kappa_a^\alpha$ (see [26] for details). According to Ref. [26], the slice-reducible Killing tensor of the second rank (with not totally geodesic slices) has the form

$$\mathcal{K}_{\alpha\beta} = \alpha g_{\alpha\beta} + \gamma^{ab}\kappa_{a\alpha}\kappa_{b\beta} + e^\Psi n_\alpha n_\beta, \qquad (60)$$

where α, γ^{ab}, Ψ are functions obeying some differential equations presented in Ref. [26]. In particular, the functions α and γ^{ab} are constant along hypersurfaces, i.e., $\mathcal{D}_\tau \alpha = 0$ and $\mathcal{D}_\tau \gamma^{ab} = 0$.

A number of restrictions are also automatically applied to slices. In particular, they must be partially umbilical, i.e., the second fundamental form provides conditions (20a) and the mixed components are zero $\beta^a{}_\alpha = 0$ [see Eq. (21) in [26][3]]. Upon comparing Eq. (46) presented in this work and Eq. (34b) in Ref. [26], it is evident that in slices $\mathcal{D}_\tau\mathcal{H}^{ab} = \lambda_\gamma \mathcal{H}^{ab}$ where $\lambda_\gamma = -\mathcal{D}_\gamma \ln(\chi_\tau\varphi^3)$ (or $\Lambda = \chi_\tau\varphi^3$ up to a multiplicative constant). Here φ is the lapse function of the foliation defined as[4]

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[3]The symbol β^a_α corresponds to $\mathcal{X}_i^\beta$ in the reference.
[4]In original equations from Ref. [26] there is an additional ϵ which is equal to 1 for timelike hypersurfaces.

$$n^\alpha \nabla_\alpha n_\beta = -\mathcal{D}_\beta \ln \varphi. \qquad (61)$$

Thus, we can establish a connection between slice-reducible conformal Killing tensor and massive particle surfaces [26]. In order to find out whether the slice is a massive particle surface, one can differentiate Eq. (49) with respect to any direction tangent to the hypersurface and combine it with condition (54a), which holds automatically for the slice. As a result, we obtain only the following remaining condition

$$2(\mathcal{D}_\tau \mathcal{J}^a - \lambda_\tau \mathcal{J}^a)Q_a = \mathcal{D}_\tau \mathcal{M}^2 - \lambda_\tau \mathcal{M}^2, \qquad (62)$$

for some Q_a satisfying (49). In particular, the maximum gluing order is achieved under conditions (54b) and (54c) or equivalently (55) which resolve the condition (62).

Theorem 5.1: Let a timelike foliation of the manifold M generates slice-reducible conformal Killing tensor, i.e., satisfies all conditions of the Theorem 3.5 in Ref. [26]. If a slice S has no boundary and satisfies (55), it is an extended glued particle surface $\mathcal{S}_g$ of maximal order.

If the Killing tensor is exact rather than conformal, then we can use Eq. (46b) of Ref. [26] resulting in $\lambda_\tau = 0$, which implies that $\mathcal{H}^{ab}$, $\mathcal{J}^a$, and $\mathcal{M}^2$ are constant along the hypersurface, or equivalently

$$q\mathcal{H}^{ab}\mathcal{D}_\tau p_b = q\mathcal{D}_\tau \left(\frac{1}{2\chi_\tau} \mathcal{F}^a \right), \qquad q^2 \mathcal{F}^a \mathcal{D}_\tau p_a = 0. \quad (63)$$

In particular, for $q = 0$ the slices are always glued particle surfaces. This generalizes theorems from Refs. [25,26] that slices associated with a slice-reducible exact Killing tensor are fundamental photon surfaces to the case of massive particle surfaces with neutral particles.

Now, we can analyze the situation in an opposite direction, i.e., whether the existence of foliation with slices representing glued massive particle surfaces results in the existence of an exact Killing tensor of rank 2. If there is a foliation of glued massive particle surface of maximum order with $\beta^a{}_\alpha = 0$, Eq. (54a) corresponds to one of the necessary conditions for the existence of slice-reducible Killing tensor from Ref. [26] if the lapse function φ of the foliation satisfies the condition $\lambda_\gamma = -\mathcal{D}_\gamma \ln (\chi_\tau \varphi^3)$ [compare with conditions (34) in Ref. [26]]. If it additionally meets another condition

$$\mathcal{D}_\gamma(\varphi\chi_\tau - \varphi n^\alpha \nabla_\alpha \ln \varphi) = 0, \qquad (64)$$

we immediately get the evidence of the existence of the slice-reducible conformal Killing tensor of rank two.

If the spacetime possesses slice-reducible conformal (or exact) Killing tensor, following Eqs. (24a) and (29) from Ref. [26], then there are two differential equations

$$n^\alpha \nabla_\alpha \Psi = 2\chi_\tau, \qquad n^\alpha \nabla_\alpha (e^\Psi \mathcal{G}^{ab}) = n^\alpha \nabla_\alpha \gamma^{ab}. \quad (65)$$

These equations are useful to express $\mathcal{H}^{ab}$ in form (46) through the components of the Killing tensor

$$\mathcal{H}^{ab} = -\frac{e^{-\Psi}}{2\chi_\tau} n^\alpha \nabla_\alpha \gamma^{ab}. \qquad (66)$$

The presence of glued particle surfaces can serve as an indication of the existence of an exact Killing tensor. We will illustrate the potential of this representation for metrics such as Kerr and others in the following examples.

VI. EXAMPLES

A. Kerr metric

The metric for Kerr solution in the Boyer-Lindquist coordinates reads

$$ds^2 = -f(dt - \omega d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \Delta f^{-1} \sin^2 \theta d\phi^2, \qquad (67)$$

where

$$f = \frac{\Delta - a^2 \sin^2\theta}{\Sigma}, \qquad \Sigma = r^2 + a^2\cos^2\theta,$$

$$\omega = \frac{-2Mar \sin^2\theta}{\Delta - a^2 \sin^2\theta}, \qquad \Delta = r(r - 2M) + a^2. \quad (68)$$

The Kerr spacetime possesses two Killing vectors $k^\alpha_{(t)}\partial_\alpha = \partial_t$, $k^\alpha_{(\phi)}\partial_\alpha = \partial_\phi$, and an exact Killing tensor in the form (60) with

$$\alpha = -r^2, \qquad \gamma^{ab}\kappa_a^\alpha \kappa_b^\beta = -\Delta^{-1}S^\alpha S^\beta,$$

$$S^\alpha = (r^2 + a^2)\delta_t^\alpha + a\delta_\phi^\alpha, \qquad e^\Psi = \Sigma, \qquad (69)$$

which is slice reducible with respect to slices $r = \text{const}$ with the normal unit vector $n^\alpha = \sqrt{\Delta/\Sigma}\delta_r^\alpha$. The slices are tangent to the Killing vectors $\partial_t, \partial_\phi$. According to the result obtained in Ref. [26], Sec. V A, four-dimensional solutions with two commuting Killing vectors $n_K = n - 2$ have two different foliations orthogonal to each other that generate the same Killing tensor. The tensor (69) is indeed slice-reducible with respect to $\theta = \text{const}$ as well, possessing another representation of the form (60)

$$\alpha = -a^2\cos^2\theta, \qquad \gamma^{ab}\kappa_a^\alpha \kappa_b^\beta = S^\alpha S^\beta,$$

$$S^\alpha = a \sin\theta \delta_t^\alpha + \delta_\phi^\alpha / \sin\theta, \qquad e^\Psi = \Sigma \qquad (70)$$

with $n^\alpha = \sqrt{1/\Sigma}\delta_\theta^\alpha$. As we have shown, slices associated with a slice-reducible Killing tensor are glued massive particle surfaces for neutral particles. Since the Maxwell form is trivial $A_\alpha = 0$ in the Kerr solution, the electric charge q plays no role in the particles' motion. It motivates us to consider these two

cases. In the first case, $r = $ const, spacetime sections of the hypersurfaces are compact and the corresponding particles are bounded. In the second case $\theta = $ const, spacetime sections are not compact, and, as we will see, particles fly by the black hole in the scattering regime.

$$\chi_\tau = \chi_{\theta\theta}/h_{\theta\theta} = r\sqrt{\frac{\Delta}{\Sigma^3}}, \tag{71a}$$

$$\mathcal{H}^{ab} = \frac{1}{r\Delta^2}\begin{pmatrix} (a^2+r^2)(a^2(M+r)+r^2(r-3M)) & aM(a^2-r^2) \\ aM(a^2-r^2) & a^2(M-r) \end{pmatrix}. \tag{71b}$$

One can check that Eq. (66) does hold. Matrix $\mathcal{H}^{ab}$ is explicitly independent on θ, thus it provides the maximal gluing. Since the determinant of $\mathcal{H}^{ab}$ is negative

$$\det \mathcal{H}^{ab} = -\left(\frac{a}{r(r-2M)+a^2}\right)^2, \tag{72}$$

the constraint (20b) describes a hyperbola. Constraint (20b) and the constraint on the vector norm $\dot\gamma^2 = -m^2$ explicitly reads

$$(E(a^2+r^2)-aL_z)(Ea^2(M+r)+Er^2(r-3M)$$
$$+aL_z(r-M))-r\Delta^2 m^2 = 0, \tag{73a}$$

1. Spheres r = const

From the $\theta\theta$ component we get χ_τ, and comparing the rest of $\chi_{\alpha\beta}$ with $h_{\alpha\beta}$, we get the representation (20) of the second fundamental form with[5]

$$(r^2+a^2)(\mathcal{Q}-(L_z-aE)^2-r^2E^2)$$
$$-2M\mathcal{Q}r+r^2(L_z^2+\Delta m^2) = 0, \tag{73b}$$

where

$$E = -\dot\gamma^\alpha k_{(t)\alpha}, \qquad L_z = \dot\gamma^\alpha k_{(\phi)\alpha},$$
$$\mathcal{Q} = -\dot\gamma^\alpha\dot\gamma^\beta \mathcal{K}_{\alpha\beta}, \qquad q_a = \{-E, L_z\}. \tag{74}$$

There are two families of solutions (E^+, L_z^+) and (E^-, L_z^-) to Eq. (73a), where

$$E^\pm = \pm\frac{m^2(r(a^2+2r^2)-3Mr^2)+\mathcal{Q}(r-M)}{2r\sqrt{\Delta(m^2r^2+\mathcal{Q})}}, \tag{75a}$$

$$L_z^\pm = \pm\frac{m^2r(a^2r(r-3M)+a^4+Mr^3)-\mathcal{Q}(r^2(r-3M)+a^2(r+M))}{2ar\sqrt{\Delta(m^2r^2+\mathcal{Q})}}. \tag{75b}$$

Here, the Carter constant plays the role of the parameter of the one-dimensional family of the solutions. Thus, the order of the glued particle surface is 1. The region accessible for particles is determined by inequality (18), which takes the following form for Kerr solution:

$$\frac{f(L_z - E\omega)^2}{\Delta\sin^2\theta} - \frac{E^2}{f} \le -m^2. \tag{76}$$

The left-hand side diverges to $+\infty$ at the poles of the sphere $\theta = 0$ or π if $L_z \ne 0$. In this case, the massive particle surfaces represent belts cut out from the sphere.

The resulting extended surfaces of maximal order for massive particles are constructed to include sets of

spherical massive particle worldlines. In the case of charge-less particles, these surfaces contain spherical timelike geodesics discussed in Teo's paper [10]. Therefore, we have obtained the geometric definition of Teo's spheres. Our expressions coincide with Teo's results after shifting the Carter's constant $\mathcal{Q} = \mathcal{Q}_{\text{Teo}} + (L_z - aE)^2$. Teo has shown the existence of both stable and unstable orbits on the same sphere depending on the value of Q. The marginally stable orbit is determined by Carter's constant

$$\mathcal{Q} = \mathcal{Q}_{MS} \equiv m^2\frac{r^3(a^2-M^2)\pm\sqrt{M\Delta^3 r^3}}{r(r-M)^2-\Delta M}. \tag{77}$$

Thus, glued massive particle surfaces can contain particles with both stable and unstable worldlines. However, the stability of each individual surface in the family is well

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[5]We use the following index ordering: t, φ.

defined. One can approach the problem of worldline stability on massive particle surfaces in a general manner by utilizing calculations similar to those described in Ref. [44].

A different point of view on glued massive particle surfaces is also useful. For each value of the momentum L_z at fixed energy E, there exists a unique massive particle surface, and the entire set of surfaces for all L_z forms a region similar to the photon region [50–53]. The structure of this region is the key for describing the shadow of a given black hole in a stream of massive charged particles on a given energy scale in future works.

2. Cones $\theta = const$

Similarly, for cones with constant θ we get

$$\chi_\tau = \chi_{rr}/h_{rr} = -\frac{a^2 \sin(2\theta)}{2\Sigma^{3/2}}, \quad \mathcal{H}^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1/a^2\sin^4\theta \end{pmatrix}. \tag{78}$$

One can check that Eq. (66) does hold. Conditions on the integrals of motion

$$\frac{L_z^2}{a^2 \sin^4 \theta} - E^2 + m^2 = 0, \tag{79a}$$

$$\frac{\mathcal{Q}'}{a^2 \cos^2 \theta} + \frac{L_z^2}{a^2 \sin^2 \theta} - E^2 + m^2 = 0 \tag{79b}$$

have the following solution:

$$L_z^2 = \mathcal{Q}'\tan^4\theta, \qquad E^2 = m^2 + \frac{\mathcal{Q}'}{a^2\cos^4\theta}, \tag{80}$$

where $\mathcal{Q}' = \mathcal{Q} + (L_z - aE)^2$ is a modified Carter constant. As a result, we find that $E^2 \geq m^2$. The condition for the accessible region has the same form (76). It is noteworthy that $\mathcal{H}^{ab}$ is nonexistent for $a = 0$ due to the fact that, as a approaches 0, all worldlines on the cone degenerate into radial geodesics without any gluing. By construction, all resulting $\theta = const$ massive particles cones contain conical orbits which are generally noncompact, corresponding to the minimum of the effective potential in the equations for θ from the geodesic equations [10]. One can expect the existence of compact conical orbits in the charged case in Kerr-Newman dyons (see Ref. [49] for an example of compact orbits between two closely spaced cones). While these noncompact orbits may not hold the same level of significance as compact spherical orbits in the formation of gravitational shadows or in the context of Penrose inequalities, they offer possibilities for investigating particle scattering problems.

From the expression of the second derivative of the effective potential for $u = \cos\theta$ (see Ref. [10])

$$V''(u) = -8a^2(E^2 - m^2)u^2 \leq 0 \tag{81}$$

follows that all massive particle surfaces $\theta = const \neq \pi/2$ are stable. In contrast to spherical surfaces, this property remains unaffected by Carter's constant $\mathcal{Q}$.

B. Zipoy-Voorhees solution

Zipoy-Voorhees solution is a vacuum nonspherical spacetime with the following metric:

$$ds^2 = -f^\delta dt^2 + k^2 f^{-\delta}\left(\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right) + (x^2-1)(y^2-1)d\phi^2\right), \qquad f = \frac{x-1}{x+1}, \tag{82}$$

with $x > 1$ and $y^2 \leq 1$. The Zipoy-Voorhees solution is known to lack photon surfaces, but it has fundamental photon surfaces. Thus, we expect that there are massive particle surfaces, which are not glued. We take surfaces $x = const$ as an ansatz to investigate whether this spacetime allows for spherical massive particle surfaces. Following a similar approach to previous examples, we obtain

$$\mathcal{H}^{ab} = \frac{1}{F_1}\begin{pmatrix} f^{-\delta}(F_1 - \delta(x^2-y^2)) & 0 \\ 0 & \frac{(1-\delta^2)xf^\delta}{k^2(x^2-1)} \end{pmatrix}, \tag{83}$$

where

$$F_1 = x^3 - \delta x^2 + x(\delta^2(1-y^2)-1) + \delta y^2. \tag{84}$$

Substituting (83) and integrals of motion $Q_a = q_a = \{-E, L_z\}$ into Eq. (20b), we find that the expression (which is quite voluminous) does not depend on y if

$$L_z^2 = \frac{E^2k^2(x^2-1)^2f^{-2\delta}}{1 - \delta x}. \tag{85}$$

The final solution is

$$E^2 = m^2 f^\delta \frac{1-\delta x}{2-\delta x}, \qquad L_z^2 = k^2 m^2 f^{-\delta}\frac{(x^2-1)^2}{2-\delta x}. \tag{86}$$

Condition (18) reads

$$\frac{m^2(x^2-y^2)}{(1-y^2)(2-\delta x)} \leq 0. \tag{87}$$

Along with this condition, we require non-negativeness of E^2 and L_z^2. However, this is not possible for timelike geodesics, but it is possible for tachyons $m^2 < 0$ only.

C. Kerr-Newman dyon

The dyonic Kerr-Newman solution is an electrovacuum solution with electric $\mathfrak{q}$ and magnetic $\mathfrak{p}$ charges. Its metric has the form (67) with redefined functions

$$\Delta = r(r-2M)+a^2+\mathfrak{e}^2, \quad \omega=\frac{-2a\sin^2\theta}{\Delta-a^2\sin^2\theta}(Mr-\mathfrak{e}^2/2) \tag{88}$$

and electromagnetic vector potential

$$A_\alpha dx^\alpha = \frac{\mathfrak{q}r+a\mathfrak{p}\cos\theta}{\Sigma}dt - \frac{\mathfrak{p}(a^2+r^2)\cos\theta+a\mathfrak{q}r\sin^2\theta}{\Sigma}d\phi, \tag{89}$$

where $\mathfrak{e}^2 = \mathfrak{q}^2 + \mathfrak{p}^2$. We can easily generalize our formalism to particles both with electric q and magnetic p charges as follows. First, we notice if the particle possesses a magnetic charge, then the equation of motion has the form of Eq. (1) with the following substitution:

$$F^\beta{}_\lambda \rightarrow \tilde{F}^\beta{}_\lambda = F^\beta{}_\lambda + \frac{p}{q}(\star F)^\beta{}_\lambda, \qquad (\star F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \tag{90}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor. Since tensor $F_{\alpha\beta}$ fulfills both Maxwell and Bianchi equations, its dual tensor $(\star F)_{\alpha\beta}$ can be represented with a vector potential $(\star F)_{\alpha\beta} = \partial_\alpha A_\beta^\star - \partial_\beta A_\alpha^\star$ as well. Thus, we can make the following substitution of the vector potential in our analysis

$$A_\alpha \rightarrow \tilde{A}_\alpha = A_\alpha + \frac{p}{q}A_\alpha^\star. \tag{91}$$

A detailed analysis of the motion of dyonic particles in the dyonic Kerr-Newman solution is presented in Ref. [49]. Due to the dual nature of dyons, the transformed potential $\tilde{A}_\alpha$ is the same as A_α up to substitution

$$\mathfrak{q} \rightarrow \mathfrak{q}+\mathfrak{p}p/q, \qquad \mathfrak{p} \rightarrow \mathfrak{p}-\mathfrak{q}p/q. \tag{92}$$

Performing the same steps as for previous examples, and making use of Eq. (50a), we have the following quantities for spheres $r = \text{const}$

$$\mathcal{H}^{ab} = \frac{1}{r\Delta^2}\begin{pmatrix} (a^2+r^2)[2r\Delta+(a^2+r^2)(M-r)] & a[a^2M-r(Mr-\mathfrak{e}^2)] \\ a[a^2M-r(Mr-\mathfrak{e}^2)] & a^2(M-r) \end{pmatrix}, \tag{93a}$$

$$\mathcal{J}^a = -\frac{(\mathfrak{p}p+\mathfrak{q}q)}{2r\Delta^2}\begin{pmatrix} r^4-4Mr^3+(2a^2+3\mathfrak{e}^2)r^2+a^2(a^2+\mathfrak{e}^2) \\ a(a^2+\mathfrak{e}^2-r^2) \end{pmatrix}, \tag{93b}$$

$$\mathcal{M}^2 = m^2 + \frac{(\mathfrak{p}p+\mathfrak{q}q)^2(r^2-Mr-\Delta)}{\Delta^2}, \tag{93c}$$

and for cones $\theta = \text{const}$

$$\mathcal{H}^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1/a^2\sin^4\theta \end{pmatrix}, \tag{94a}$$

$$\mathcal{J}^a = \frac{\mathfrak{q}p-\mathfrak{p}q}{2a^2\cos\theta\sin^4\theta}\begin{pmatrix} a\sin^4\theta \\ 1+\cos^2\theta \end{pmatrix}, \tag{94b}$$

$$\mathcal{M}^2 = m^2 + \left(\frac{\mathfrak{q}p-\mathfrak{p}q}{a\sin^2\theta}\right)^2. \tag{94c}$$

All quantities are constant along the surface, so spheres and cones achieve maximal gluing. It is notable that particle charges come only in the combination $\mathfrak{p}p+\mathfrak{q}q$ for spheres, and $\mathfrak{q}p-\mathfrak{p}q$ for cones. Making the first or the second combination equal to zero (i.e., $p=-q\mathfrak{q}/\mathfrak{p}$ or $p=q\mathfrak{p}/\mathfrak{q}$),

particles on the corresponding surfaces will not be affected by the electromagnetic field, since the electric and magnetic forces balance each other out. Particularly, if $p=\mathfrak{p}=0$, then conical surfaces are not affected, because the electromagnetic field exerts only the radial electric force, which has no influence on the motion normal to the cones. Similarly, if $p=\mathfrak{q}=0$, spheres are not affected, because the electric force is absent, and the magnetic force is tangent to the surface.

VII. DISCUSSION AND CONCLUSIONS

We have generalized the concept of photon surfaces and fundamental photon surfaces to the case of massive charged particles or photons in the medium [46,47] in general spacetimes with at least one isometry, among which are stationary ones. We introduced the notion of massive

particle surfaces as timelike hypersurfaces such that any worldline of a particle with mass m, electric charge q, and a set of conserved quantities Q_a, remains tangent to the surface forever, provided that the worldline initially was tangent to it. This concept encompasses the generalizations for both ordinary photon spheres and all their standard generalizations [30,31]. In general, such surfaces have boundaries, but the notion can be naturally extended to surfaces without boundaries, which can be useful for analyzing their topological properties and integrability problems. In particular, for the analysis of areal constraints, Riemannian Penrose inequalities, one can try to apply the standard approach of inverse mean curvature flow [30].

The new concepts introduced here represent a natural extension of our prior research, as presented in Refs. [11,44]. In particular, we have proved the key Theorem 3.1, which is a direct analog of Theorem 2.2 obtained in Ref. [3] for photon surfaces, and establishes a purely geometric definition of such surfaces. We have found that in the general case the second fundamental form $\chi_{\alpha\beta}$ is partially umbilical. The condition of the partially umbilical surface becomes degenerate when at least $n-2$ Killing vectors are involved, resulting in the fixation of the function χ_τ without any additional conditions. Also, we have obtained restrictions on the electromagnetic field tensor and a compact master equation.

We have also found that in spacetimes with hidden symmetries (Kerr-like spacetimes) described by slice-reducible conformal Killing fields of rank 2, massive particle surfaces can be found among the slices of the corresponding foliation. Moreover, such surfaces are glued together in a nontrivial way. We have defined the general concept of glued massive particle surfaces of arbitrary order. Glued surfaces capture worldlines from some continuous family of conserved quantities of dimension larger than or equal to one. We have reinforced these concepts with a number of examples demonstrating that the previously known features of the particle movement along worldlines are indeed successfully explained by the geometric properties of the glued surfaces. We have investigated the relationship between the gluing phenomenon and the formation of a slice-reducible conformal Killing tensor. Particularly, we have found that the existence of a foliation of glued massive particle surfaces of maximum order can serve as a good indication of the existence of the Killing tensor, and vice versa. However, a rigorous substantiation of this correspondence remains a part of further research. Despite this, we can already retrieve one of the necessary and sufficient integrability conditions for the existence of conformal Killing tensors.

If an exact Killing tensor of rank 2 is present in spacetime, then there is a quantity $\mathcal{H}^{ab}$ extracted from the second fundamental form, which is constant along the surface. If massive particle surfaces allow for charged particles, then there are two more constant quantities,

$\mathcal{J}^a$, and $\mathcal{M}^2$. Together they can be understood as invariants of the massive particle surface.

A number of examples are called to demonstrate the usefulness of the concept. We have considered the problem of the existence of massive particle surfaces in Kerr, Zipoy-Voorhees, and Kerr-Newman metrics. The obtained results can be generalized to particles coupled to background fields in various theories by replacing of m^2 and A_α with the corresponding expressions. The obtained results have potential applications in various areas, including Penrose inequalities, uniqueness theorems, integrability theory, and the description of massive particle dynamics in the presence of real supermassive objects, as well as for further photons in plasma.

APPENDIX: QUADRATIC FORMS WITH LINEAR CONSTRAINTS

Here we prove several useful theorems about quadratic forms with linear constraints.

Theorem A.1: Let $\mathbf{Y} = \{y\}$ be a nonempty set of all solutions for the following quadratic equation

$$y^T Q y = d, \qquad (A1)$$

and similarly $\mathbf{Y}' = \{y'\}$ is a set of all solutions for

$$y^T Q' y = d', \qquad (A2)$$

where Q, Q' are symmetric nondegenerate $n_Q \times n_Q$ matrices. Then the set of all solutions for the first quadratic equation (A1) is contained in the set of solutions for the second quadratic equation (A2), $\mathbf{Y} \subseteq \mathbf{Y}'$, if and only if the primed quantities for the second system can be represented as

$$Q' = \lambda Q, \qquad d' = \lambda d, \qquad (A3)$$

where λ is an arbitrary scalar.[6]

Proof: Sufficient ($\Leftarrow$): Let y be an arbitrary solution of the first quadratic equation (A1). Then, conditions (A3) give

$$y^T Q' y = \lambda y^T Q y = \lambda d = d', \qquad (A4)$$

i.e., exactly the second quadratic equation (A2).

Necessary ($\Rightarrow$): Let all solutions y for the first quadratic equation (A1) be contained in the set of solutions for the second quadratic equation (A2). Let us introduce an orthonormal basis for Q in the following way:

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[6]Including $\lambda = 0$. In case $\lambda \neq 0$ we find that $\mathbf{Y} = \mathbf{Y}'$.

$$e_i^T Q e_j = -\delta_{ij}, \qquad e_{i'}^T Q e_{j'} = \delta_{i'j'}, \qquad e_i^T Q e_{i'} = 0. \quad \text{(A5)}$$

Consider a subset of the solutions of the system (A1) defined as follows:

$$y = a(e_i \cos\psi + e_j \sin\psi) + b(e_{i'} \cos\psi' + e_{j'} \sin\psi'),$$
$$-a^2 + b^2 = d. \quad \text{(A6)}$$

Writing quadratic form Q' in this orthonormal basis $Q'_{ij} \equiv e_i^T Q' e_j$, Eq. (A2) for chosen y gives

$$a^2 \left(\frac{Q'_{ii} + Q'_{jj}}{2} + Q'_{ij} \sin(2\psi) + \frac{Q'_{ii} - Q'_{jj}}{2} \cos(2\psi) \right)$$
$$+ b^2 \left(\frac{Q'_{i'i'} + Q'_{j'j'}}{2} + Q'_{i'j'} \sin(2\psi') + \frac{Q'_{i'i'} - Q'_{j'j'}}{2} \cos(2\psi') \right)$$
$$+ 2ab(Q'_{ii'} \cos\psi \cos\psi' + Q'_{ij'} \cos\psi \sin\psi'$$
$$+ Q'_{ji'} \sin\psi \cos\psi' + Q'_{jj'} \sin\psi \sin\psi') = d'.$$

So from the arbitrariness of ψ and ψ'

$$a^2 Q'_{ii} + b^2 Q'_{i'i'} = d', \qquad Q'_{ii} = Q'_{jj}, \qquad Q'_{i'i'} = Q'_{j'j'},$$
$$Q'_{ij} = Q'_{i'j'} = Q'_{ii'} = 0. \quad \text{(A7)}$$

Eliminating a from the first equation, we get

$$b^2 (Q'_{i'i'} + Q'_{ii}) = d' + dQ'_{ii}. \quad \text{(A8)}$$

From the arbitrariness of b follows

$$-Q'_{ii} = Q'_{i'i'} = d'/d \equiv \lambda \quad \text{(A9)}$$

in orthonormal basis for Q. So Q' must be proportional to Q with the same proportionality factor as d' and d:

$$Q' = \lambda Q, \qquad d' = \lambda d. \quad \text{(A10)}$$

Theorem A.2: Let $\mathbf{X} = \{x\}$ be a nonempty set of all solutions for the following quadratic equation with a system of linear constraints

$$x^T A x = d, \qquad B x = f, \quad \text{(A11)}$$

and similarly $\mathbf{X}' = \{x'\}$ is a set of all solutions for

$$x^T A' x + 2w'^T x = d', \qquad B' x = f', \quad \text{(A12)}$$

where A, A' are symmetric $n_A \times n_A$ matrices, B, B' are $n_B \times n_A$ matrices and n_B vector f is in the image of B, and A is nondegenerate. If all solutions $\mathbf{X}$ projected onto $\ker B$ with respect to the pseudoinner product A span the entire $\ker B$, then the following statement is fair. The entire set of solutions for the first system (A11) is contained in the set of solutions for the second system (A12), $\mathbf{X} \subseteq \mathbf{X}'$, if and only if the primed quantities for the second system can be represented as

$$A' = \lambda A + S_1 B + (S_1 B)^T + B^T S_2 B, \quad \text{(A13a)}$$

$$B' = S_3 B, \quad \text{(A13b)}$$

$$f' = S_3 f, \quad \text{(A13c)}$$

where λ is an arbitrary scalar, matrices S_1, S_2, S_3 are arbitrary matrices constrained by the following conditions

$$f^T S_2 f = d' - \lambda d - 2f^T GBA^{-1} w',$$
$$(\mathbb{1} - B^T GBA^{-1}) w' + S_1 f = 0, \qquad BA^{-1} S_1 = 0, \quad \text{(A14)}$$

where G is a pseudoinverse matrix of G defined from the following operator equation $(BA^{-1}B^T) GB = B$.

Proof: Sufficient ($\Leftarrow$): For an arbitrary solution x of system (A11), conditions (A13) and (A14) from the theorem give

$$x^T A' x + 2w'^T x = \lambda x^T A x + x^T S_1 B x + x^T (S_1 B)^T x + x^T B^T S_2 B x + 2w'^T x$$
$$= \lambda d + 2x^T S_1 f + f^T S_2 f + 2w'^T x$$
$$= d' + 2x^T S_1 f - 2f^T GBA^{-1} w' - 2f^T S_1^T x + 2w'^T A^{-1} B^T GB x$$
$$= d' - 2f^T GBA^{-1} w' + 2w'^T A^{-1} B^T Gf = d', \quad \text{(A15)}$$

and

$$B' x = S_3 B x = S_3 f = f'. \quad \text{(A16)}$$

Thus, every solution of the first system (A11) is a solution for the second system (A12).

Necessary ($\Rightarrow$): For simplicity we define the spaces U_x and U_f, so

$$w', x \in U_x, \qquad f \in U_f,$$
$$A, A' : U_x \mapsto U_x, \qquad B, B' : U_x \mapsto U_f. \quad \text{(A17)}$$

We can understand matrix A as a metric tensor in U_x. It defines the pseudoinner products $x_1^T A x_2$. We will understand orthogonality as $x_1^T A x_2 = 0$. Further, the subspace defined by the kernel of B will play an important role. We will define the kernel of matrix B and the orthogonal complement of the kernel $(\ker B)^\perp$ as

$$\ker B = \{y : By = 0\},$$
$$(\ker B)^\perp = \{x : x^T A y = 0, \ \forall y \in \ker B\}. \quad (A18)$$

We define a pseudoinverse matrix G from the following operator equation

$$(BA^{-1}B^T)GB = B. \quad (A19)$$

This matrix is useful to solve linear equations with matrix B. For example, since f is in the image of B, we have the following identity $(BA^{-1}B^T)Gf = f$. One can define G as an inverse matrix $(BA^{-1}B^T)^{-1}$ if the latter exists. Also, we can introduce a projector at the subspace $\ker B$ in the following way

$$\Pi = \mathbb{1} - A^{-1}B^T GB. \quad (A20)$$

Indeed, it is a projector, since it preserves the vector from the kernel $\Pi y = y$, and any vector $x \in (\ker B)^\perp$ orthogonal to the kernel is zero

$$\Pi(A^{-1}B^T t) = A^{-1}B^T t - A^{-1}(BA^{-1}B^T GB)^T t$$
$$= A^{-1}B^T t - A^{-1}B^T t = 0, \quad (A21)$$

where $(A^{-1}B^T t)$ can represent any vector from $(\ker B)^\perp$. To show this, we notice that A^{-1} is a full-rank matrix, therefore the image of the operator $A^{-1}B^T$ has the same dimension as the orthogonal complement of the kernel of B

$$\mathrm{rank}(A^{-1}B^T) = \mathrm{rank}B = \dim(\ker B)^\perp, \quad (A22)$$

and the result is orthogonal to any vector $y \in \ker B$

$$y^T A A^{-1}B^T t = (By)^T t = 0. \quad (A23)$$

Thus, the image of $(A^{-1}B^T)$ is $(\ker B)^\perp$.

First, vector x and matrices A', B' can be split into terms that are spanned onto the kernel $\ker B \subseteq U_x$ and its orthogonal complement $(\ker B)^\perp \subseteq U_x$:

$$x = z + y, \qquad A' = S_0 + S_1 B + (S_1 B)^T + B^T S_B B,$$
$$B' = S_4 + S_3 B, \quad (A24)$$

and we have imposed the following conditions

$$By = 0, \qquad z^T A y = 0, \qquad S_0 A^{-1}B^T = BA^{-1}S_0 = 0,$$
$$BA^{-1}S_1 = 0, \qquad S_4 A^{-1}B^T = 0. \quad (A25)$$

Matrices S_0 and S_B are symmetric. Constraints (A25) are orthogonality conditions with respect to the aforementioned pseudoinner product. The new linear operators act as follows:

$$S_0: \ U_x \mapsto U_x, \qquad S_1: \ U_x \mapsto U_f, \qquad S_4: \ U_x \mapsto U_f,$$
$$S_B, S_3: \ U_f \mapsto U_f. \quad (A26)$$

Substituting the definition of x into the linear system of Eq. (A11), we get

$$Bx = Bz = f \Rightarrow z = A^{-1}B^T Gf. \quad (A27a)$$

The term $A^{-1}B^T Gf$ can be understood as a particular solution to the linear system $Bx = f$, and y is a solution to the homogeneous part of the linear system $Bx = 0$. Substituting the definition of x and z into the quadratic equation in (A11), we get

$$x^T A x = f^T Gf + y^T A y = d \Rightarrow y^T A y = d - f^T Gf, \quad (A27b)$$

The vector y is in $\ker B$, thus we can use the projected operator A_{pr} in the quadratic condition

$$y^T A_{\mathrm{pr}} y = d - f^T Gf, \quad (A28)$$

where $A_{\mathrm{pr}} = \Pi^T A \Pi = A - B^T GB$ is a projected operator A onto $\ker B$. The operator A_{pr} has the following properties:
  (i) $A_{\mathrm{pr}} y = Ay$ for all $y \in \ker B$. Hence, A_{pr} acts on y the same way as A, and the quadratic form $y^T A_{\mathrm{pr}} y$ is identical to $y^T A y$ in $\ker B$.
 (ii) $A_{\mathrm{pr}} A^{-1}B^T = BA^{-1}A_{\mathrm{pr}} = 0$, so A_{pr} satisfies all conditions on S_0 in Eq. (A25).
The Eq. (A28) may have no solutions at all, which is not the case for the theorem.

Substituting x into the system (A12) and taking into account the solution for z, we get a pair of conditions

$$f^T S_B f + 2y^T S_1 f + y^T S_0 y + 2f^T GBA^{-1}w' + 2w'^T y = d', \quad (A29a)$$

$$S_4 y + S_3 f - f' = 0. \quad (A29b)$$

Our current aim is to find the form of the matrices S_0, S_1, S_3, S_4, and S_B that turns Eq. (A29) into an identity for all y satisfying (A27b). Since (A27b) is an even equation with respect to y, Eq. (A29) can be split by the parity with respect to y:

$$\boldsymbol{y}^T S_0 \boldsymbol{y} = d' - \boldsymbol{f}^T S_B \boldsymbol{f} - 2 \boldsymbol{f}^T G B A^{-1} \boldsymbol{w}', \qquad \text{(A30a)}$$

$$\boldsymbol{y}^T (S_1 \boldsymbol{f} + \boldsymbol{w}') = 0, \qquad \text{(A30b)}$$

$$\boldsymbol{f}' = S_3 \boldsymbol{f}, \qquad \text{(A30c)}$$

$$S_4 \boldsymbol{y} = 0. \qquad \text{(A30d)}$$

The matrix S_3 can be arbitrary. By the condition of the theorem, the set of all $\boldsymbol{y}$ satisfying condition (A27b) spans the entire $\ker B$. Thus, Eqs. (A30b) and (A30d) can be rewritten as

$$S_4 \Pi = 0, \qquad \Pi^T (\boldsymbol{w}' + S_1 \boldsymbol{f}) = 0. \qquad \text{(A31)}$$

Making use of conditions (A25), we get

$$S_4 = 0, \qquad (\mathbb{1} - B^T G B A^{-1}) \boldsymbol{w}' + S_1 \boldsymbol{f} = 0. \qquad \text{(A32)}$$

The remaining condition is related with $\ker B$ only:

$$\boldsymbol{y}^T S_0 \boldsymbol{y} = d' - 2 \boldsymbol{f}^T G B A^{-1} \boldsymbol{w}' - \boldsymbol{f}^T S_B \boldsymbol{f}, \qquad \text{(A33)}$$

for all $\boldsymbol{y}$ satisfying (A28). The problem (A33) is solved in the simpler version of the theorem. According to Theorem A.1, we have

$$S_0 = \lambda A_{\mathrm{pr}}, \qquad d' - 2 \boldsymbol{f}^T G B A^{-1} \boldsymbol{w}' - \boldsymbol{f}^T S_B \boldsymbol{f} = \lambda (d - \boldsymbol{f}^T G \boldsymbol{f}), \qquad \text{(A34)}$$

where λ is an arbitrary scalar. The first condition defines the matrix S_0, and the second condition can be represented as

$$S_B = S_2 + \lambda G, \qquad \text{(A35)}$$

where $S_2 \colon U_f \mapsto U_f$ is an arbitrary linear operator such that $\boldsymbol{f}^T S_2 \boldsymbol{f} = d' - \lambda d - 2 \boldsymbol{f}^T G B A^{-1} \boldsymbol{w}'$. Combining all together, the second system must have the following form:

$$A' = \lambda A + S_1 B + (S_1 B)^T + B^T S_2 B, \qquad \text{(A36)}$$

$$B' = S_3 B, \qquad \text{(A37)}$$

$$\boldsymbol{f}' = S_3 \boldsymbol{f}, \qquad \text{(A38)}$$

where λ is an arbitrary scalar, $\boldsymbol{w}'$ and $S_{1,2,3}$ are arbitrary matrices satisfying the following conditions

$$\boldsymbol{f}^T S_2 \boldsymbol{f} = d' - \lambda d - 2 \boldsymbol{f}^T G B A^{-1} \boldsymbol{w}',$$
$$(\mathbb{1} - B^T G B A^{-1}) \boldsymbol{w}' + S_1 \boldsymbol{f} = 0, \qquad B A^{-1} S_1 = 0. \qquad \text{(A39)}$$

Theorem A.3: Let $\mathbf{X} = \{\boldsymbol{x}\}$ be a nonempty set of all solutions for the following quadratic equation with a system of linear constraints

$$\boldsymbol{x}^T A \boldsymbol{x} + 2 \boldsymbol{w}^T \boldsymbol{x} = d, \qquad B \boldsymbol{x} = \boldsymbol{f}, \qquad \text{(A40)}$$

and similarly $\mathbf{X}' = \{\boldsymbol{x}'\}$ is a set of all solutions for

$$\boldsymbol{x}^T A' \boldsymbol{x} + 2 \boldsymbol{w}'^T \boldsymbol{x} = d', \qquad B' \boldsymbol{x} = \boldsymbol{f}', \qquad \text{(A41)}$$

where A, A' are symmetric nondegenerate $n_A \times n_A$ matrices, B, B' are $n_B \times n_A$ matrices, and n_B vector $\boldsymbol{f}$ is in the image of B. If all solutions $\mathbf{X}$ projected to the $\ker B$ with respect to the pseudoinner product A span the entire $\ker B$, then the following statement is fair. The entire set of solutions for the first system (A11) is contained in the set of solutions for the second system (A12), $\mathbf{X} \subseteq \mathbf{X}'$, if and only if the primed quantities for the second system can be represented as

$$A' = \lambda A + S_1 B + (S_1 B)^T + B^T S_2 B, \qquad \text{(A42a)}$$

$$B' = S_3 B, \qquad \text{(A42b)}$$

$$\boldsymbol{f}' = S_3 \boldsymbol{f}, \qquad \text{(A42c)}$$

where λ is an arbitrary scalar, matrices S_1, S_2, S_3 are arbitrary matrices constrained by the following condition:

$$d' + (2 \boldsymbol{w}'^T - \boldsymbol{w}^T A^{-1} A') A^{-1} \boldsymbol{w} - \lambda (d + \boldsymbol{w}^T A^{-1} \boldsymbol{w}) = (\boldsymbol{f} + B A^{-1} \boldsymbol{w})^T S_2 (\boldsymbol{f} + B A^{-1} \boldsymbol{w}) + 2 (\boldsymbol{f} + B A^{-1} \boldsymbol{w})^T G B A^{-1} (\boldsymbol{w}' - A' A^{-1} \boldsymbol{w})$$
$$(\mathbb{1} - B^T G B A^{-1}) \boldsymbol{w}' = (\lambda + B^T (S_1^T + S_2 B) A^{-1}) \boldsymbol{w} - S_1 \boldsymbol{f},$$
$$B A^{-1} S_1 = 0. \qquad \text{(A43)}$$

Proof: Since A is nondegenerate, it is always possible to make a transformation translating the system (A40) into (A11) from Theorem 3.1. Such a transformation is

$$\boldsymbol{x} \to \boldsymbol{x} - A^{-1} \boldsymbol{w}. \qquad \text{(A44)}$$

After a careful substitution and identification of the translated quantities with original from Theorem 3.1, we immediately obtain Eq. (A42) with conditions (A43).

[1] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), First Sagittarius A* event horizon telescope results. I. The shadow of the supermassive black hole in the center of the Milky Way, Astrophys. J. Lett. **930**, L12 (2022).

[2] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), First M87 event horizon telescope results. I. The shadow of the supermassive black hole, Astrophys. J. Lett. **875**, L1 (2019).

[3] C. M. Claudel, K. S. Virbhadra, and G. F. R. Ellis, The geometry of photon surfaces, J. Math. Phys. (N.Y.) **42**, 818 (2001).

[4] G. W. Gibbons and C. M. Warnick, Aspherical photon and anti-photon surfaces, Phys. Lett. B **763**, 169 (2016).

[5] A. A. Shoom, Metamorphoses of a photon sphere, Phys. Rev. D **96**, 084056 (2017).

[6] L. M. Cao and Y. Song, Quasi-local photon surfaces in general spherically symmetric spacetimes, Eur. Phys. J. C **81**, 714 (2021).

[7] Y. Koga, T. Igata, and K. Nakashi, Photon surfaces in less symmetric spacetimes, Phys. Rev. D **103**, 044003 (2021).

[8] K. S. Virbhadra and G. F. R. Ellis, Schwarzschild black hole lensing, Phys. Rev. D **62**, 084003 (2000).

[9] K. S. Virbhadra and G. F. R. Ellis, Gravitational lensing by naked singularities, Phys. Rev. D **65**, 103004 (2002).

[10] E. Teo, Spherical orbits around a Kerr black hole, Gen. Relativ. Gravit. **53**, 10 (2021).

[11] K. V. Kobialko and D. V. Gal'tsov, Photon regions and umbilic conditions in stationary axisymmetric spacetimes: Photon regions, Eur. Phys. J. C **80**, 527 (2020).

[12] K. Kobialko and D. Gal'tsov, Photon regions in stationary axisymmetric spacetimes and umbilic conditions, in *Proceedings of the 16th Marcel Grossman Meeting on General Relativity*, edited by R. Ruffini and G. Vereshchagin (World Scientific, Singapore, 2023).

[13] P. V. P. Cunha and C. A. R. Herdeiro, Shadows and strong gravitational lensing: A brief review, Gen. Relativ. Gravit. **50**, 42 (2018).

[14] V. I. Dokuchaev and N. O. Nazarova, Silhouettes of invisible black holes, Usp. Fiz. Nauk **190**, 627 (2020).

[15] V. I. Dokuchaev and N. O. Nazarova, Visible shapes of black holes M87* and SgrA*, Universe **6**, 154 (2020).

[16] V. Perlick and O. Y. Tsupko, Calculating black hole shadows: Review of analytical studies, Phys. Rep. **947**, 1 (2022).

[17] T. Bronzwaer and H. Falcke, The nature of black hole shadows, Astrophys. J. **920**, 155 (2021).

[18] R. A. Konoplya and A. Zhidenko, Shadows of parametrized axially symmetric black holes allowing for separation of variables, Phys. Rev. D **103**, 104033 (2021).

[19] I. Bogush, D. Gal'tsov, G. Gyulchev, K. Kobialko, P. Nedkova, and T. Vetsov, Photon surfaces, shadows, and accretion disks in gravity with minimally coupled scalar field, Phys. Rev. D **106**, 024034 (2022).

[20] A. Grenzebach, *The Shadow of Black Holes: An Analytic Description* (Springer, Heidelberg, 2016).

[21] K. S. Virbhadra, Distortions of images of Schwarzschild lensing, Phys. Rev. D **106**, 064038 (2022).

[22] M. Cariglia, Hidden symmetries of dynamics in classical and quantum physics, Rev. Mod. Phys. **86**, 1283 (2014).

[23] V. Frolov, P. Krtous, and D. Kubiznak, Black holes, hidden symmetries, and complete integrability, Living Rev. Relativity **20**, 6 (2017).

[24] G. W. Gibbons, T. Houri, D. Kubiznak, and C. M. Warnick, Some spacetimes with higher rank Killing-Stackel tensors, Phys. Lett. B **700**, 68 (2011).

[25] K. Kobialko, I. Bogush, and D. Gal'tsov, Killing tensors and photon surfaces in foliated spacetimes, Phys. Rev. D **104**, 044009 (2021).

[26] K. Kobialko, I. Bogush, and D. Gal'tsov, Slice-reducible conformal Killing tensors, photon surfaces, and shadows, Phys. Rev. D **106**, 024006 (2022).

[27] T. Shiromizu, Y. Tomikawa, K. Izumi, and H. Yoshino, Area bound for a surface in a strong gravity region, Prog. Theor. Exp. Phys. **2017**, 033E01 (2017).

[28] X. H. Feng and H. Lu, On the size of rotating black holes, Eur. Phys. J. C **80**, 551 (2020).

[29] R. Q. Yang and H. Lu, Universal bounds on the size of a black hole, Eur. Phys. J. C **80**, 949 (2020).

[30] H. Yoshino, K. Izumi, T. Shiromizu, and Y. Tomikawa, Extension of photon surfaces and their area: Static and stationary spacetimes, Prog. Theor. Exp. Phys. **2017**, 063E01 (2017).

[31] H. Yoshino, K. Izumi, T. Shiromizu, and Y. Tomikawa, Transversely trapping surfaces: Dynamical version, Prog. Theor. Exp. Phys. **2020**, 023E02 (2020).

[32] C. Cederbaum and G. J. Galloway, Uniqueness of photon spheres in electro-vacuum spacetimes, Classical Quantum Gravity **33**, 075006 (2016).

[33] S. S. Yazadjiev, Uniqueness of the static spacetimes with a photon sphere in Einstein-scalar field theory, Phys. Rev. D **91**, 123013 (2015).

[34] S. Yazadjiev and B. Lazov, Uniqueness of the static Einstein–Maxwell spacetimes with a photon sphere, Classical Quantum Gravity **32**, 165021 (2015).

[35] S. Yazadjiev and B. Lazov, Classification of the static and asymptotically flat Einstein-Maxwell-dilaton spacetimes with a photon sphere, Phys. Rev. D **93**, 083002 (2016).

[36] M. Rogatko, Uniqueness of photon sphere for Einstein-Maxwell-dilaton black holes with arbitrary coupling constant, Phys. Rev. D **93**, 064003 (2016).

[37] C. Cederbaum, Uniqueness of photon spheres in static vacuum asymptotically flat spacetimes, arXiv:1406.5475.

[38] H. Yoshino, Uniqueness of static photon surfaces: Perturbative approach, Phys. Rev. D **95**, 044047 (2017).

[39] Y. Koga, Photon surfaces as pure tension shells: Uniqueness of thin shell wormholes, Phys. Rev. D **101**, 104022 (2020).

[40] S. Yazadjiev, Classification of static asymptotically flat spacetimes with a photon sphere in Einstein-multiple-scalar field theory, Phys. Rev. D **104**, 124070 (2021).

[41] C. Cederbaum and G. J. Galloway, Photon surfaces with equipotential time-slices, J. Math. Phys. (N.Y.) **62**, 032504 (2021).

[42] S. L. Adler and K. S. Virbhadra, Cosmological constant corrections to the photon sphere and black hole shadow radii, Gen. Relativ. Gravit. **54**, 93 (2022).

[43] K. S. Virbhadra, Compactness of supermassive dark objects at galactic centers, arXiv:2204.01792.

[44] K. Kobialko, I. Bogush, and D. Gal'tsov, Geometry of massive particle surfaces, Phys. Rev. D **106**, 084032 (2022).

[45] Y. Song and C. Zhang, Quasi-local studies of the particle surfaces and their stability in general spacetimes, Eur. Phys. J. C **83**, 50 (2023).

[46] V. Perlick and O. Y. Tsupko, Light propagation in a plasma on Kerr spacetime: Separation of the Hamilton-Jacobi equation and calculation of the shadow, Phys. Rev. D **95**, 104003 (2017).

[47] F. A. Asenjo, V. Munoz, and A. Valdivia, Relativistic mass and charge of photons in thermal plasmas through electromagnetic field quantization, Phys. Rev. E **81**, 056405 (2010).

[48] J. T. Mendonca, A. M. Martins, and A. Guerreiro, Field quantization in a plasma: Photon mass and charge, Phys. Rev. E **62**, 2989 (2000).

[49] E. Hackmann and H. Xu, Charged particle motion in Kerr-Newmann space-times, Phys. Rev. D **87**, 124030 (2013).

[50] A. Grenzebach, V. Perlick, and C. Lämmerzahl, Photon regions and shadows of Kerr-Newman-NUT black holes with a cosmological constant, Phys. Rev. D **89**, 124004 (2014).

[51] A. Grenzebach, V. Perlick, and C. Lämmerzahl, Photon regions and shadows of accelerated black holes, Int. J. Mod. Phys. D **24**, 1542024 (2015).

[52] D. V. Gal'tsov and K. V. Kobialko, Completing characterization of photon orbits in Kerr and Kerr-Newman metrics, Phys. Rev. D **99**, 084043 (2019).

[53] D. V. Gal'tsov and K. V. Kobialko, Photon trapping in static axially symmetric spacetime, Phys. Rev. D **100**, 104005 (2019).