Gravitational collapse of scalar and vector fields

Karim Mosani,[1,*] Koushiki,[1,†] Pankaj S. Joshi,[1,‡] Jay Verma Trivedi,[1,§] and Tapobroto Bhanja[2,∥]
[1]*International Centre for Space and Cosmology, School of Arts and Sciences, Ahmedabad University, Ahmedabad-380009 (Guj), India*
[2]*International Center for Cosmology and PDPIAS, Charotar University of Science and Technology, Anand-388421 (Guj), India*

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We study here the unhindered gravitational collapse of spatially homogeneous (SH) scalar fields ϕ with a potential $V_s(\phi)$, as well as vector fields $\tilde{A}$ with a potential $V_v(B)$ where $B = g(\tilde{A}, \tilde{A})$ and g is the metric tensor. We show that in both cases, classes of potentials exist that give rise to black holes or naked singularities. The strength of the naked singularity when it occurs is examined, and these are found to be gravitationally strong in the sense of Tipler, for a wide class of respective potentials (for both scalar and vector fields). We match the collapsing scalar or vector field with a generalized Vaidya spacetime outside. We highlight that full generality is maintained within the domain of SH scalar or vector field collapse.

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I. INTRODUCTION

The contraction of a matter field under its own gravitational influence is called gravitational collapse. In 1939, Oppenheimer and Snyder [1], and independently in 1938, Datt [2], developed the first solution of Einstein's field equations [the Oppenheimer-Snyder-Datt (OSD) model] depicting the gravitational collapse of a massive star. They considered a rather specific case of spatially homogeneous (SH) dust collapse (by spatial homogeneity, we mean homogeneous on a three-dimensional spacelike orbit with a six-dimensional isometry group G_6 corresponding to the spacetime [3]). Such a matter field undergoes gravitational collapse that ends up in a *singularity*. These spacetime singularities are hidden behind an *event horizon*, not visible to any observer, resulting in a *black hole* as the outcome of a continual collapse.

Extending the above special scenario, in 1969, Penrose proposed what is now known as the cosmic censorship hypothesis (CCH) [4]. The weaker version of the hypothesis states that all singularities of gravitational collapse are hidden within a black hole and hence, cannot be seen by a distant observer (a globally naked singularity cannot exist). The strong version of the hypothesis states that no past inextendable nonspacelike causal curves exist between the singularity and any point in the spacetime manifold. In other words, a causal geodesic with a well-defined positive tangent "at" the singularity does not exist (that is, a locally naked singularity does not exist). The supporting argument for the validity of the strong CCH is the desirability of the spacetime manifold to be globally hyperbolic. Global hyperbolicity implies the existence of Cauchy surfaces embedded in the total manifold, thereby making general relativity a deterministic and predictable theory [5–7].

The singularity theorems of Hawking and Penrose [6,8] do not imply that singularities are necessarily hidden from an external observer under all possible circumstances. In fact, singularity theorems take the causality condition as one of the axioms to start with to prove the existence of incomplete past (future) directed causal curves. Additionally, the OSD model that motivated cosmic censorship is a special case. Joshi and Malafarina [9] showed that any arbitrarily small neighborhood of the initial data giving rise to OSD collapse contains initial data corresponding to collapse evolution, giving rise to a singularity with the following property: one could trace outgoing past singular causal geodesics. This means that the end state of OSD collapse is unstable under small perturbations in initial data. Moreover, one can show the formation of naked singularities (global and local) as an end state of gravitational collapse from suitable, physically reasonable initial data for various matter fields [10,11]. This implies that the initial conditions must be fine-tuned for the cosmic censorship conjecture to hold.

In such a context, an important question one can ask is as follows: What will be the end state of an unhindered gravitational collapse of a fundamental matter field, such as a scalar field or a vector field? The answer to this question has been achieved up to a certain extent. A real scalar field is a map defined on a smooth manifold as $\phi: \mathcal{M} \rightarrow \mathbb{R}$ with a suitable continuity condition. Christodoulou showed that in

*kmosani2014@gmail.com
†koushiki.malda@gmail.com
‡pankaj.joshi@ahduni.edu.in
§jay.verma2210@gmail.com
∥tapobroto.bhanja@gmail.com

the case of gravitational collapse of a massless scalar field ϕ [the scalar-field Lagrangian is $\mathcal{L}_{\phi} = (-1/2)g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi$], the set of initial data giving rise to a naked singularity as an end state has positive codimension in the entire initial dataset [12,13]. This means that the initial dataset corresponding to naked singularity has a zero measure in the total initial dataset. In other words, naked singularity in such cases is unstable under arbitrarily small perturbations in the initial data.

One can have a massless scalar field with a potential $V_s(\phi)$ that is still a fundamental matter field. A massive scalar field will then be a particular case of a massless scalar field with a specific potential of the form $V_s(\phi) = (1/2)\mu^2\phi^2$, where μ is the mass term. Goswami and Joshi [14] showed the example of the gravitational collapse of a massless SH scalar field with a certain potential $V_s(\phi)$ that ends up in a naked singularity. Mosani *et al.* [15] conducted a similar investigation for a massless scalar field with a two-dimensional analog of the Mexican-hat-shaped Higgs field potential and found out that the end state of such unhindered scalar-field collapse is a naked singularity.

In addition to scalar fields (as fundamental matter fields), we also have vector fields as fundamental matter fields in nature and thus it becomes an intriguing problem to study the gravitational collapse of vector fields. Geometrically, vector fields on a smooth manifold $\mathcal{M}$ can be thought of as sections on the tangent bundle $\pi: T\mathcal{M} \to \mathcal{M}$, where π is a continuous surjection. A section is a smooth map $\sigma: \mathcal{M} \to T\mathcal{M}$ such that $\pi \circ \sigma$ is an identity map on $\mathcal{M}$. From a particle physics point of view, the fundamental nature of a vector field is different from that of a scalar field. The importance of vector fields here can be gauged from the fact that vector fields are the mediators (or propagators) of the basic three forces (interactions): QED, weak and strong (e.g., photon, a massless vector field, is the mediator of electromagnetic force in QED, etc.). A massless vector field with a potential function $V_v(B)$ is again a fundamental matter field. A massive vector field will then be a particular case of a massless vector field with a specific potential of the form $V_v(B) = (1/2)\mu^2 B$, where μ is the mass term. Garfinkle *et al.* [16] studied the collapse of a massive vector field and numerically obtained the critical initial conditions. To our knowledge, much analytical work has not been done in investigating the causal structure of the end-state spacetime of the unhindered gravitational collapse of matter fields that are vector fields.

Although the CCH is sometimes discussed abstractly, a general consensus is that it should be examined for our universe within the context of cosmological expansion (apart from the exception of the big bang singularity, which is known to be a naked singularity). Here, in order to give a full solution to Einstein field equations, we glue the interior spacetime of the collapsing cloud to the exterior generalized Vaidya spacetime. One could then argue that examining the final states of gravitational collapse and existence or otherwise of black holes or naked singularity in this manner would be a mere mathematical exercise and might not have implications for the physical expanding universe we live in. However, as is known, in view of the complexity of the real physical universe, in order to understand certain phenomena or properties of the universe, one often first studies an ideal scenario by isolating the problem under consideration. The first gravitational collapse model of Oppenheimer and Snyder [1] is one such example. Nevertheless, in [17–19] the authors examined the nature of singularities forming in the presence of the cosmological constant and formed due to gravitational collapse. It was seen that depending on the nature of the regular initial data from which the collapse develops, black holes or naked singularities form as collapse final states. Also the recent observations of the central compact object of our galaxy by the EHT group [20] has indicated that a particular naked singularity spacetime, namely the Joshi-Malafarina-Narayan (JMN)-1 model [21] maybe one of the best black hole mimickers for the SgrA*. This provides a good motivation to understand the formation of these singularities in various scenarios by looking at it as an isolated problem related to possible final states of gravitational collapse of a massive matter cloud.

From such a perspective, in the present work, we would like to investigate the gravitational collapse and the nature and formation of singularities for scalar and vector fields with potential. As indicated, such fields are often considered to be of much interest due to their fundamental nature. This is mainly because these fields constitute fundamental matter Lagrangian. Due to this, they are sometimes argued to be more basic and closer to being physically realistic. At the same time, for the sake of simplicity, we consider here spatially homogeneous matter clouds and we leave the investigation of a more general inhomogeneous scenario for a future work. It is worth noting that even if the universe may appear to be homogeneous on rather large scales, spatial homogeneity need not constitute an astrophysical scale or need not be considered to be a physically realistic model, as evidenced in many structure-formation studies and various numerical structure-formation simulations. In that sense our exercise may be considered of a toy model nature, despite the fields considered being physically realistic as above. Here, SH scalar fields and vector fields correspond to the matter field with stress-energy tensor of a perfect fluid, governed by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

In this paper, in both the massless SH scalar-field as well as vector-field cases, we show that there are broad classes of potentials for which the configuration collapses and ends up in either a black hole or a naked singularity depending on the potential function chosen. We approach the causality investigation problem of scalar-field as well as vector-field collapse in a unified way, so to speak. As far as general relativity is concerned, it does not discriminate between whether a scalar field or a vector field seeds the matter field. The matter field is entirely identified by a rank 2 tensor field, which is the stress-energy tensor. As far as SH perfect fluid is concerned, one can identify a given matter field by the functional form of the equation of state parameter $\omega(a)$, where a is the scale factor of the collapsing cloud. We derive relevant equations of collapsing SH scalar field $\phi(a)$ and vector field $\tilde{A}(a)$ in the subsections of Sec. II. The main body of Sec. II contains discussions and relevant relations regarding the gravitational collapse of SH perfect fluids. In Sec. III, we smoothly join the interior collapsing perfect fluid with an external generalized Vaidya spacetime. In Sec. IV, we investigate the causal structure of the spacetime (condition of obtaining a naked singularity) at the end of the collapse of the interior perfect fluid that is either a scalar field ϕ with potential V_s or a vector field $\tilde{A}$ with potential V_v. We also depict a few examples of well-known scalar fields and vector fields. In Sec. V, we derive the criteria for the singularity, thus obtained in the end, to be strong of Tipler's type. In the last section, we highlight the key points of the investigation. Here we use the geometrized units $8\pi G = c = 1$ throughout.

II. INTERIOR COLLAPSING MATTER FIELD

Consider a gravitational collapse of a SH perfect fluid. The components of the stress-energy tensor in the coordinate basis $\{dx^\mu \otimes \partial_\nu | 0 \leq \mu, \nu \leq 3\}$ of the comoving coordinates (t, x, y, z) are given by

$$T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p). \tag{1}$$

The spacetime geometry is governed by the flat ($k = 0$) FLRW metric

$$ds^2 = -dt^2 + a^2 d\Sigma^2, \tag{2}$$

where $d\Sigma^2 = dx^2 + dy^2 + dz^2$. Here $a = a(t)$ is the scale factor such that $a(0) = 1$ and $a(t_s) = 0$, where t_s is the time of formation of the singularity. $R = R(t, r)$ is the physical radius of the collapsing cloud and can be written as

$$R(t, r) = ra(t), \tag{3}$$

where r is the radial spherical coordinate. For a FLRW spacetime, Eq. (2), we have

$$\rho = \frac{3\dot{a}^2}{a^2}, \tag{4}$$

and

$$p = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}. \tag{5}$$

The overhead dot denotes the partial time derivative of a. Equation (4) can be rewritten to obtain the dynamics of the collapse as

$$\dot{a} = -\sqrt{\frac{\rho(a)}{3}}a. \tag{6}$$

Differentiating the above equation once again gives us

$$\ddot{a} = \frac{1}{3}a\left(\frac{a\rho_{,a}}{2} + \rho\right). \tag{7}$$

Integrating Eq. (6), we obtain the time curve, which is

$$t(a) = \int_a^1 \sqrt{\frac{3}{\rho}}\frac{da}{a}. \tag{8}$$

The dynamics of the scale factor $a(t)$ is, thus, the inverse of the lhs of the above equation. The time of formation of the singularity $t_s = t(0)$ is

$$t_s = \int_0^1 \sqrt{\frac{3}{\rho}}\frac{da}{a}. \tag{9}$$

Now, let us consider a particular matter field $\hat{T}$ from a set of all the possible SH perfect fluids. Choosing such an element means choosing a specific functional form of the equation of state parameter,

$$\omega(a) = \frac{p}{\rho}. \tag{10}$$

Using Eqs. (4), (5), and (10), we can express the density of the matter field with the equation of state parameter ω as

$$\rho = \rho_0 \exp\left(\int_a^1 \frac{3(1+\omega(a))}{a} da\right). \tag{11}$$

A SH perfect fluid is a fundamental matter field since it can be derived by a fundamental matter Lagrangian. In the following two subsections, we will describe two distinct ways of obtaining such a matter field.

A. Scalar-field collapse

We prove that any SH perfect fluid is equivalent to a SH scalar field $\phi(a)$ with a suitable potential $V_s(a)$, as far as the gravitational collapse is concerned. If $\phi(a)$ is invertible, then the following statement holds: Any SH perfect fluid is gravitationally equivalent to a SH scalar field ϕ with a suitable potential $V_s(\phi)$.

Consider a real scalar field defined on the manifold $\mathcal{M}$ as

$$\phi\colon \mathcal{M} \to \mathbb{R}. \tag{12}$$

The Lagrangian of a massless scalar field ϕ with potential $V_s(a)$ is given by

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_s(\phi). \tag{13}$$

The stress-energy tensor is obtained from the Lagrangian $\mathcal{L}_\phi$ as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}}. \tag{14}$$

The density (ρ_s) and the isotropic pressure (p_s) are subsequently expressed in terms of the time derivative of the scalar field and its potential as

$$\rho_s = \frac{1}{2}\dot{\phi}^2 + V_s, \tag{15}$$

and

$$p_s = \frac{1}{2}\dot{\phi}^2 - V_s. \tag{16}$$

The overhead dot denotes the time derivative of the functions. From Eqs. (15) and (16), and from using the chain rule $\dot{\phi} = \phi_{,a}\dot{a}$, we get

$$\rho_s + p_s = \phi_{,a}^2\dot{a}^2. \tag{17}$$

We now equate $\rho_s = \rho$ and $p_s = p$. Using Eqs. (5) and (17), along with replacing $\dot{a}$ and $\ddot{a}$ using Eqs. (6) and (7), one obtains the expression of density as a function of a as

$$\rho_s = \rho_0 \exp\left(\int_a^1 a\phi_{,a}^2 da\right). \tag{18}$$

From Eqs. (15) and (16), we get

$$p_s = \rho_s - 2V_s. \tag{19}$$

Using Eq. (6) in Eq. (17), we get

$$\rho_s\left(1 - \frac{\phi_{,a}^2 a^2}{3}\right) + p_s = 0. \tag{20}$$

Using Eqs. (19) and (20), we get

$$V_s(\phi) = \rho_s\left(1 - \frac{\phi_{,a}^2 a^2}{6}\right). \tag{21}$$

Using Eqs. (17) and (6) in Eq. (5), one obtains

$$\frac{\rho_{s,a}}{\rho_s} = -\frac{\phi_{,a}^2}{a}. \tag{22}$$

We have, using Eq. (10), Eqs. (15) and (16),

$$V_s = \frac{\rho_s}{2}(1-\omega). \tag{23}$$

Now from Eqs. (21) and (23), we have

$$\phi(a)_{,a} = \pm\frac{\sqrt{3(1+\omega(a))}}{a}. \tag{24}$$

Integrating the above equation, one obtains

$$\phi(a) = \pm\int_a^1 \frac{\sqrt{3(1+\omega(a))}}{a} da + c. \tag{25}$$

From Eqs. (11) and (21) we have

$$V_s(a) = \rho_0\left(\frac{1-\omega(a)}{2}\right)\exp\left(\int_a^1 \frac{3(1+\omega(a))}{a} da\right). \tag{26}$$

Hence, we prove that given the functional form of the equation of state parameter $\omega(a)$, one can obtain the corresponding scalar field $\phi(a)$ given by Eq. (25) with potential $V_s(a)$ given by Eq. (26). As long as $\phi(a)$ is invertible [or, in other words, a bijective map from $(0,1] \to \mathbb{R}$)], we obtain $a(\phi)$, at least in principle, using which, we get $V_s(\phi)$.

Alternatively, given a scalar field $\phi(a)$, one can obtain the corresponding perfect fluid $\hat{T}$ [or the $\omega(a)$ by which it is identified], using Eq. (24).

On the other hand, we can also start with a given scalar-field potential $V(\phi)$. One can use Eqs. (18) and (21) to obtain the ordinary nonlinear differential equation

$$\mathcal{H}\left(a, \phi, \frac{d\phi}{da}, \frac{d^2\phi}{da^2}\right) = 0, \tag{27}$$

that can be solved in principle, to obtain $\phi(a)$, and later obtain $\omega(a)$ using Eq. (24). Hence, given a scalar-field potential $V_s(\phi)$, one can obtain the corresponding $\hat{T}$ [identified by $\omega(a)$] in the above manner.

B. Vector-field collapse

We prove that any SH perfect fluid is equivalent to a SH vector field $\tilde{A}(a)$ with a suitable potential $V_v(a)$, as far as the gravitational collapse is concerned. If $B(a)$ is invertible, then the following statement holds: Any SH perfect fluid is gravitationally equivalent to a SH vector field $\tilde{A}$ with a suitable potential $V_v(B)$ [where $B = g(\tilde{A}, \tilde{A})$].

Consider a vector field

$$\tilde{A}\colon \mathcal{M} \to T\mathcal{M}, \tag{28}$$

with potential $V(B)$. For a fixed $p \in \mathcal{M}$, $\tilde{A}(p) = A_\mu dx^\mu$, where $A_\mu = (A_0, A_i)$, $1 < i < 3$ (in the comoving Cartesian coordinate basis). Here $B = g^{\alpha\beta}A_\alpha A_\beta$. We consider a SH pure vector field: $A_0 = 0$ and $A_i = A \in \mathbb{R}\ \forall\, i \in (1,2,3)$. For such a vector field, $B = 3A^2/a^2$.

The Lagrangian of a massless vector field $\tilde{A}$ with potential $V_v(B)$ is given by

$$\mathcal{L}_{\tilde{A}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V_v(B). \tag{29}$$

F is a two-form called the field strength and can be written in terms of wedge product as $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$. The field strength is the exterior derivative of the vector field $\tilde{A}$, i.e., $F = d\tilde{A}$. The components are written as $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$.

The stress-energy tensor is obtained from the Lagrangian $\mathcal{L}_{\tilde{A}}$ as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\tilde{A}})}{\delta g^{\mu\nu}}. \tag{30}$$

This gives us

$$T_{\mu\nu} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} - V_v(B)g_{\mu\nu} + F_{\mu\alpha}F_\nu{}^\alpha + 2V_v'A_\mu A_\nu. \tag{31}$$

The overhead prime denotes the ordinary derivative with respect to B. The density and the isotropic pressure are subsequently expressed in terms of the time derivative of the vector-field component and its potential as

$$\rho_v = \frac{3}{2}\frac{\dot{A}^2}{a^2} + V_v(B), \tag{32}$$

and

$$p_v = \frac{1}{2}\frac{\dot{A}^2}{a^2} - V_v(B) + 2V_v'\frac{A^2}{a^2}. \tag{33}$$

We now equate $\rho_v = \rho$ and $p_v = p$. From Eqs. (32) and (4), we obtain

$$V_v = \rho_v\left(1 - \frac{1}{2}A_{,a}^2\right). \tag{34}$$

Substituting for $\rho(a)$ from Eq. (11), we obtain

$$V_v = \rho_0 \exp\left(\int_a^1 \frac{3(1+\omega(a))}{a}da\right)\left(1 - \frac{1}{2}A_{,a}^2\right). \tag{35}$$

On differentiating Eq. (34) with respect to B, we obtain

$$V_v' = \frac{\rho_{v,a}\left(1 - \frac{A_{,a}^2}{2}\right) - \rho_v A_{,a}A_{,aa}}{\frac{6A^2}{a^2}\left(\frac{A_{,a}}{A} - \frac{1}{a}\right)}. \tag{36}$$

Using Eqs. (33), (4), and (5), we obtain

$$\frac{a\rho_{v,a}}{3} + \rho_v\left(1 + \frac{1}{6}A_{,a}^2\right) = V_v - 2V_v'\frac{A^2}{a^2}. \tag{37}$$

Substituting for V_v and V_v' from Eqs. (34) and (36), and also substituting for $\rho_{,a}$ [by differentiating Eq. (11)] in Eq. (37), we obtain a second-order nonlinear differential equation

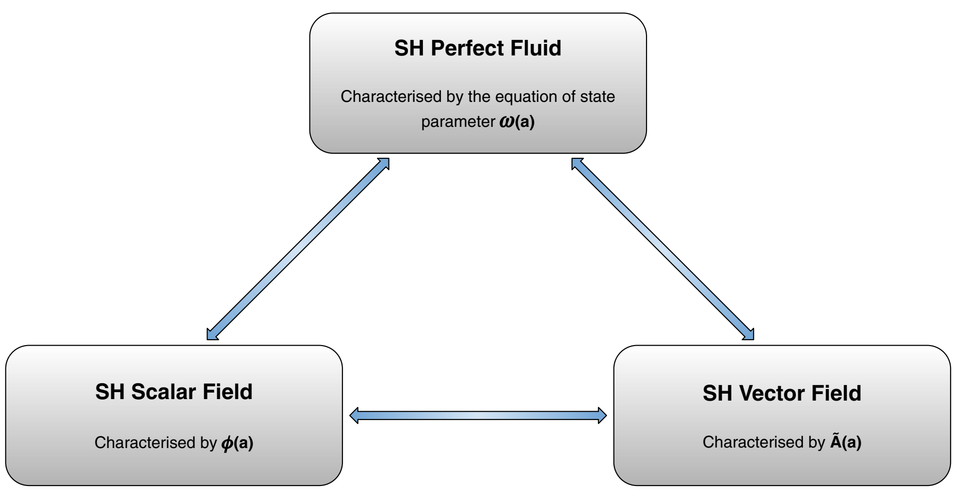


FIG. 1. A SH perfect fluid (governed by a flat FLRW spacetime metric) is completely characterized by the equation of state parameter $\omega(a)$, Eq. (10), of the matter field. This matter field is obtained from fundamental matter Lagrangian. Hence, the same matter field is also characterized by a SH scalar field $\phi(a)$, Eq. (25), or its potential $V_s(a)$, Eq. (26) [$V_s(\phi)$ if $\phi(a)$ is invertible]. Similarly, it can also be characterized by a SH vector field $\tilde{A}(a)$, Eq. (38), or its potential $V_v(a)$, Eq. (35) [$V_v(B)$ if $B(a)$ is invertible]. This schematic diagram depicts the equivalence between the gravitational collapse of SH perfect fluid, scalar field and vector field. By spatial homogeneity, we mean homogeneous on a three-dimensional spacelike orbit with a six-dimensional isometry group G_6 corresponding to the spacetime [3].

$$\mathcal{G}\left(a,\omega,A,\frac{dA}{da},\frac{d^2A}{da^2}\right)=0, \tag{38}$$

where $\mathcal{G}$ is

$$\mathcal{G}=\frac{d^2A}{da^2}-\frac{2}{A}\left(\frac{dA}{da}\right)^2+\frac{1}{2a}(1-3\omega)\frac{dA}{da}+\frac{3}{A}(1+\omega). \tag{39}$$

For a fixed $\omega(a)$, solving this differential equation with two initial conditions gives us $A(a)$, and consequently, the vector field $\tilde{A}$.

Hence, we prove that given the functional form of the equation of state parameter $\omega(a)$, one can obtain the corresponding vector field $\tilde{A}$ using Eq. (38), and consequently, the vector-field potential $V_v(a)$ using Eq. (35). Now, from the functional form $A(a)$, we obtain $B(a)$. As long as $B(a)$ is invertible [or, in other words, a bijective map from $(0,1]\to\mathbb{R}$),] we obtain $a(B)$, at least in principle, using which, we get $V_v(B)$.

Alternatively, given a vector field $\tilde{A}(a)$, one can obtain the corresponding perfect fluid $\hat{T}$ [or the $\omega(a)$ by which it is identified], using Eq. (38).

On the other hand, we can also start with a given vector-field potential $V_v(B)$. One can differentiate Eq. (35), and do some rearrangements to obtain

$$\omega\left(a,A,\frac{dA}{da},\frac{d^2A}{da^2}\right)$$

as

$$\omega=\frac{2AV'}{aV}\left(\frac{A}{a}-\frac{dA}{da}\right) -\frac{a}{3}\frac{dA}{da}\frac{d^2A}{da^2}\left(1-\frac{1}{2}\left(\frac{dA}{da}\right)^2\right)^{-1}-1. \tag{40}$$

Substituting Eq. (40) in Eq. (38), we obtain

$$\tilde{\mathcal{G}}\left(a,A,\frac{dA}{da},\frac{d^2A}{da^2}\right)=0. \tag{41}$$

In principle, this differential equation can be solved to obtain $A(a)$, which, when substituted in Eq. (40), gives us $\omega(a)$. Hence, given a vector-field potential $V(B)$, one can obtain the corresponding $\hat{T}$ [identified by $\omega(a)$] in the above manner.

III. EXTERIOR GENERALIZED VAIDYA SPACETIME

The collapsing vector-field spacetime ($g^-_{\mu\nu}$) can be joined smoothly with the exterior generalized Vaidya spacetime ($g^+_{\mu\nu}$) so that their union forms a valid solution of Einstein's field equations. The interior FLRW and the exterior generalized Vaidya spacetime [22] are respectively given as

$$ds_-^2=-dt^2+a(t)^2dr^2+r_b^2a(t)^2d\Omega^2, \tag{42}$$

and

$$ds_+^2=-\left(1-\frac{2\mathcal{M}(\mathcal{R},v)}{\mathcal{R}}\right)dv^2-2dvd\mathcal{R}+\mathcal{R}^2d\Omega^2. \tag{43}$$

Here, v is the retarded null coordinate, $\mathcal{R}$ is the generalized Vaidya radius, and r_b is the value of the radial coordinate r corresponding to the matching hypersurface, or in other words, the radial coordinate of the outermost shell of the collapsing scalar-/vector-field cloud. The matter field corresponding to the generalized Vaidya spacetime is a combination of *type I* and *type II*, such that the components of the stress-energy tensor written in the orthonormal basis appear as

$$T_{ab}=\begin{pmatrix}\frac{\bar{\epsilon}}{2}+\epsilon & \frac{\bar{\epsilon}}{2} & 0 & 0\\ \frac{\bar{\epsilon}}{2} & \frac{\bar{\epsilon}}{2}-\epsilon & 0 & 0\\ 0 & 0 & \mathcal{P} & 0\\ 0 & 0 & 0 & \mathcal{P}\end{pmatrix}. \tag{44}$$

$\epsilon=\mathcal{P}=0$ and $\bar{\epsilon}\neq 0$ correspond to the usual Vaidya spacetime as a special case. $\bar{\epsilon}=0$ and $\epsilon\neq 0$ correspond to a subclass of *type I* matter field. The generalized Vaidya solution encompasses many known Einstein field equation solutions. Matching the first and second fundamental forms for the interior and exterior metric on Σ gives the following equations:

$$\mathcal{R}(t)=R(t,r_b)(=r_ba(t)), \tag{45}$$

$$F(t,r_b)=2\mathcal{M}(\mathcal{R},v), \tag{46}$$

$$\left(\frac{dv}{dt}\right)_\Sigma=\frac{1+\dot{\mathcal{R}}}{1-\frac{F(t,r_b)}{\mathcal{R}}}, \tag{47}$$

and

$$\mathcal{M}(\mathcal{R},v)_{,\mathcal{R}}=\frac{F(t,r_b)}{2\mathcal{R}}+\mathcal{R}\ddot{\mathcal{R}}. \tag{48}$$

Here, $F=R\dot{R}^2$ is the Misner-Sharp mass function of the collapsing spherical SH perfect fluid. Using the relation (45), we can relate the generalized Vaidya mass with the density of the interior collapsing SH spherical perfect fluid cloud as

$$\mathcal{M}=\frac{\rho}{6}\mathcal{R}^3. \tag{49}$$

Using Eq. (7), differentiation of Eq. (25) with respect to a, and Eq. (49), in Eq. (48) we get

$$\mathcal{M}_{,\mathcal{R}} = \frac{3\mathcal{M}}{\mathcal{R}}\left(1 + \frac{(1+\omega(a))r_b^2}{\mathcal{R}^2}\right), \tag{50}$$

integrating which we obtain

$$\mathcal{M}(\mathcal{R}, v) = \mathcal{M}_1(v)\exp\left(\int \frac{3}{\mathcal{R}}\left(1 + \frac{(1+\tilde{\omega}(\mathcal{R}))r_b^2}{\mathcal{R}^2}\right)d\mathcal{R}\right). \tag{51}$$

Here $\mathcal{M}_1(v)$ is a constant of integration and is a function of null coordinate v, and

$$\tilde{\omega}(\mathcal{R}) = \omega\left(\frac{\mathcal{R}}{r_b}\right).$$

Equation (51) gives us the expression of the generalized Vaidya mass function of the exterior generalized Vaidya spacetime, in terms of interior collapsing perfect fluid equation of state parameter ω, to ensure smooth matching at the matching hypersurface.

For the exterior matter field to satisfy the weak-energy condition, $\bar{\epsilon}$ and ϵ should be non-negative [22]. These inequalities, in turn, put restrictions on the generalized Vaidya mass function as

$$\mathcal{M}_{,v} \leq 0, \quad \text{and} \quad \mathcal{M}_{,\mathcal{R}} \geq 0. \tag{52}$$

Using Eqs. (50) and (51) in the above two relations, we obtain

$$\mathcal{M}_{1,v} \leq 0, \tag{53}$$

and

$$\left(\frac{1+\omega(a)}{a^2}\right) \geq 0. \tag{54}$$

The inequality (54) is always satisfied if the interior collapsing matter field obeys the weak-energy condition. Hence, Eq. (53) is the only restriction on the generalized Vaidya mass function for the exterior spacetime to obey at least the weak-energy condition.

Now, we have a complete solution of Einstein's field equations consisting of an interior collapsing SH scalar/vector field (with some potential) and the exterior generalized Vaidya solution, matched smoothly at the matching hypersurface. The free functions are categorically the potential function [$V_s(\phi)$ in case of scalar-field collapse, and $V_v(B)$ in case of vector-field collapse], and the component of generalized Vaidya mass function $\mathcal{M}_1(v)$, the latter one restricted by the inequality (53). It is evident that the choice of $\mathcal{M}_1(v)$ does not affect the causal structure of the spacetime obtained as an end state of unhindered gravitational collapse. Of course, instead of considering the potential function $V_s(\phi)$ [or $V_v(B)$] as a free function, one could also consider any one of the remaining functions: $\omega(a)$, $\rho(a)$, $\phi(a)$ [or $A(a)$], $V_s(a)$ [or $V_v(a)$] as a free function, without any trouble. In the next section, we study the end state of this class of global dynamical spacetime identified by any one of the free functions.

IV. CAUSAL STRUCTURE AND STRENGTH OF THE SINGULARITY

Once the singularity is formed as an end state of gravitational collapse of the interior scalar (vector) field with potential $V_s(\phi)$ [$V_v(B)$], one can investigate whether or not causal geodesics can escape the singularity. Additionally, one can investigate whether or not such singularity is gravitationally strong in the sense of Tipler. The following two subsections discuss these two properties.

A. Causal structure of the singularity

We say that a singularity formed due to unhindered gravitational collapse is naked if there exists a family of outgoing causal curves whose past endpoint is the singularity. In the future, these curves can either reach a faraway observer or fall back to the singularity. The singularities are then termed globally naked and locally naked, respectively. Whether or not the singularity is naked essentially depends on the geometry of trapped surfaces as the collapse evolves. Trapped surfaces are two-surfaces in the spacetime on which not only the ingoing congruence but also the outgoing congruence necessarily converge. Convergence or otherwise of the outgoing null geodesic congruence is determined by the behavior of its expansion scalar, which we denote here as $\theta_l(t, r)$. It is expressed in terms of the metric coefficients, in comoving spherical coordinates, as

$$\theta_l = \frac{2}{R}\left(1 - \sqrt{\frac{\rho R^2}{3}}\right). \tag{55}$$

The region in which $\theta_l < 0$ is called the trapped region. The boundary of the trapped region, given by $\theta_l = 0$, is called the apparent horizon. If the neighborhood of the singular center is surrounded by a trapped region since before the time of formation of the singularity t_s, then it is covered, and we get a black hole. Hence, the necessary condition for singular null geodesic congruence to escape the singularity is the absence of a trapped region, which is ensured by the condition $\theta_l(t_s, r) > 0$ for such congruence. The absence of trapped region in the neighbourhood of the singularity $(t, r) = (t_s, 0)$ is ensured by the following inequality:

$$\lim_{t\to t_s} \frac{\rho R^2}{3} \leq \lim_{a\to 0} \frac{\rho(a) r_b^2 a^2}{3} < 1. \tag{56}$$

The inequality (56) is definitely satisfied if

TABLE I. Four examples of spatially homogeneous scalar fields that collapse to form a singularity that is either hidden (black hole or BH) or (naked singularity or NS). In the fourth example, $\mu = -\frac{16}{3}\lambda$. The first three types end up in a strong singularity in the sense of Tipler.

Massless scalar field	$V_s(\phi) = 0$	$\phi(a) = c \pm \sqrt{6}\log a$	Strong	BH
Homogeneous dust ($\omega = 0$)	$V_s(\phi) \propto \exp(\sqrt{3}\phi)$	$\phi(a) = c \pm \sqrt{3}\log a$	Strong	BH
Goswami/Joshi [14] ($\omega = -\frac{2}{3}$) (SF1)	$V_s(\phi) \propto \exp \phi$	$\phi(a) = c \pm \log a$	Strong	NS
Two-dimensional analog of Mexican hat [15] (SF2)	$V_s(\phi) = \frac{1}{2}\mu\phi^2 + \lambda\phi^4$	$\phi(a) = \pm 2\sqrt{2}\sqrt{c - \log a}$	Weak	NS

TABLE II. Four examples of spatially homogeneous vector fields that collapse to form a singularity that is either hidden within a black hole (BH) or is naked (NS). The ones mentioned in the third and the fourth row are newly constructed vector fields from known scalar fields (mentioned in the third [14] and the fourth [15] row of Table 1, respectively) by exploiting the gravitational equivalence depicted in Fig. 1. The corresponding vector-field component $A(a)$ for each case is plotted in Figs. 2 and 3. The first three types end up in a gravitationally strong singularity in the sense of Tipler.

Massless vector field	$V_v(B) = 0$	Strong	BH
Massive vector field	$V_v(B) = -\frac{1}{2}\mu^2 B$	Strong	BH
VF1	$V_v(a)$ as in Fig. (3)	Strong	NS
VF2	$V_v(a)$ as in Fig. (3)	Weak	NS

$$\lim_{a\to 0}\rho(a) < \frac{1}{a^2}. \qquad (57)$$

For

$$\lim_{a\to 0}\rho(a) = \frac{k}{a^2},$$

for some $k \in \mathbb{R}^+$, the inequality is satisfied only for $r_b < \sqrt{3/k}$. Rewriting the inequality (57) in terms of the equation of state parameter $\omega(a)$ using Eq. (11), one obtains

$$\lim_{a\to 0}\rho_0 a^2 \exp\left(\int_a^1 \frac{3(1+\omega(a))}{a}\right) da < 1. \qquad (58)$$

If a collapsing matter field with the equation of state parameter $\omega(a)$ satisfies the inequality (58), then it will end up in a naked singularity [14]. In the case of otherwise, the final outcome is a black hole.

Hence, as decided by the above inequality, we get a class of SH matter fields that include scalar and vector fields, identified by the functional form $\omega(a)$, that goes to either the black hole or naked-singularity final state as an end state of unhindered gravitational collapse.

In the case of scalar-field collapse, the restriction (58) on $\omega(a)$ gives us a restriction on the scalar field $\phi(a)$ using Eq. (25), and the scalar-field potential function $V_s(a)$ using Eq. (26). Hence, obtaining a class of $\omega(a)$ is gravitationally equivalent to obtaining a class of scalar-field potentials $V_s(a)$ that goes to the naked singularity as an end state of unhindered gravitational collapse. Moreover, suppose $\phi(a)$ is a bijective map from $(0, 1] \to \mathbb{R}$. In that case, obtaining a class of $\omega(a)$ is gravitationally equivalent to obtaining a class of scalar-field potentials $V_s(\phi) = V_s(a(\phi))$ that goes

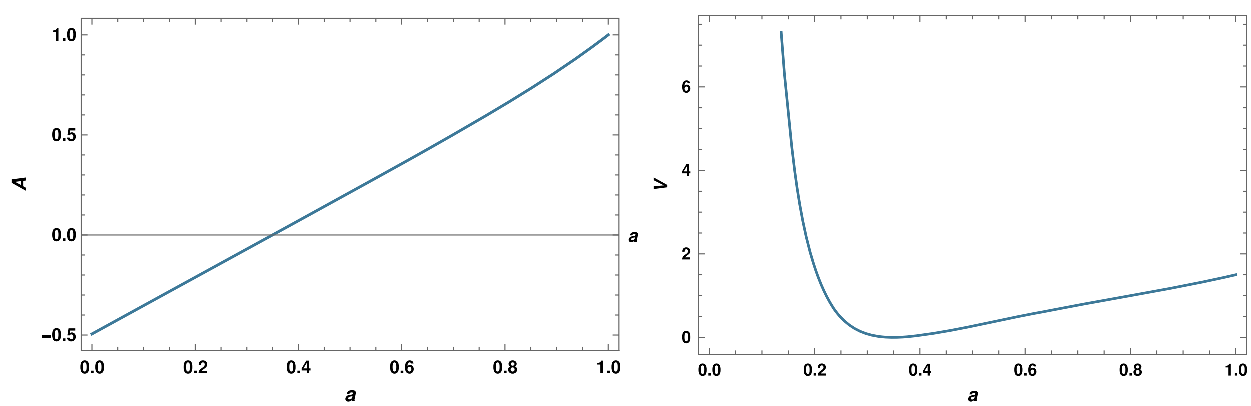


FIG. 2. The dynamics of the vector-field component $A(a)$ in the case of the massive ($\mu = 1$) vector field $\tilde{A}$ (left panel) and its potential $V_v(a)$ (right panel). First we obtain $\omega(a, A, \frac{dA}{da}, \frac{d^2A}{da^2})$ by substituting $V_v(B) = -\frac{1}{2}\mu^2 B$ in Eq. (40). Substituting for $\omega(a, A, \frac{dA}{da}, \frac{d^2A}{da^2})$ in Eq. (41) and solving the differential equation with initial conditions $A(1) = 1$ and $A'(1) = 2$, we obtain $A(a)$. Consequently, substituting $V_v(B) = -\frac{1}{2}\mu^2 B$, and the obtained $A(a)$ in Eq. (40), we obtain $\omega(a)$. With further substitution of $\omega(a)$ in Eq. (35), we obtain $V_v(a)$.

to the naked singularity as an end state of gravitational collapse.

Similarly, in the case of vector-field collapse, the restriction (58) on $\omega(a)$ gives us a restriction on the vector field $\tilde{A}$ [or more specifically, a restriction on the vector-field component $A(a)$] obtained by solving the differential equation (38), and the vector-field potential function $V_v(a)$ obtained by substituting $A(a)$ and ρ from Eq. (11), in Eq. (34). Hence, obtaining a class of $\omega(a)$ is gravitationally equivalent to obtaining a class of vector-field potential $V_v(a)$ that goes to the naked singularity as an end state of unhindered gravitational collapse. Moreover, suppose $A(a)$ is a bijective map from $(0,1] \to \mathbb{R}$. In that case, obtaining a class of $\omega(a)$ is gravitationally equivalent to obtaining a class of vector-field potential $V_v(A) = V_v(a(A))$ that goes to the naked singularity as an end state of unhindered gravitational collapse.

In Tables I and II, we discuss examples of such scalar-field collapse and vector-field collapse that end up in either a black hole or a naked singularity. Exploiting the equivalence between SH perfect fluids, scalar fields with potential $V_s(a)$, and vector fields with potential $V_v(a)$, we construct two examples of collapsing vector fields with potential out-of-known examples of collapsing scalar fields

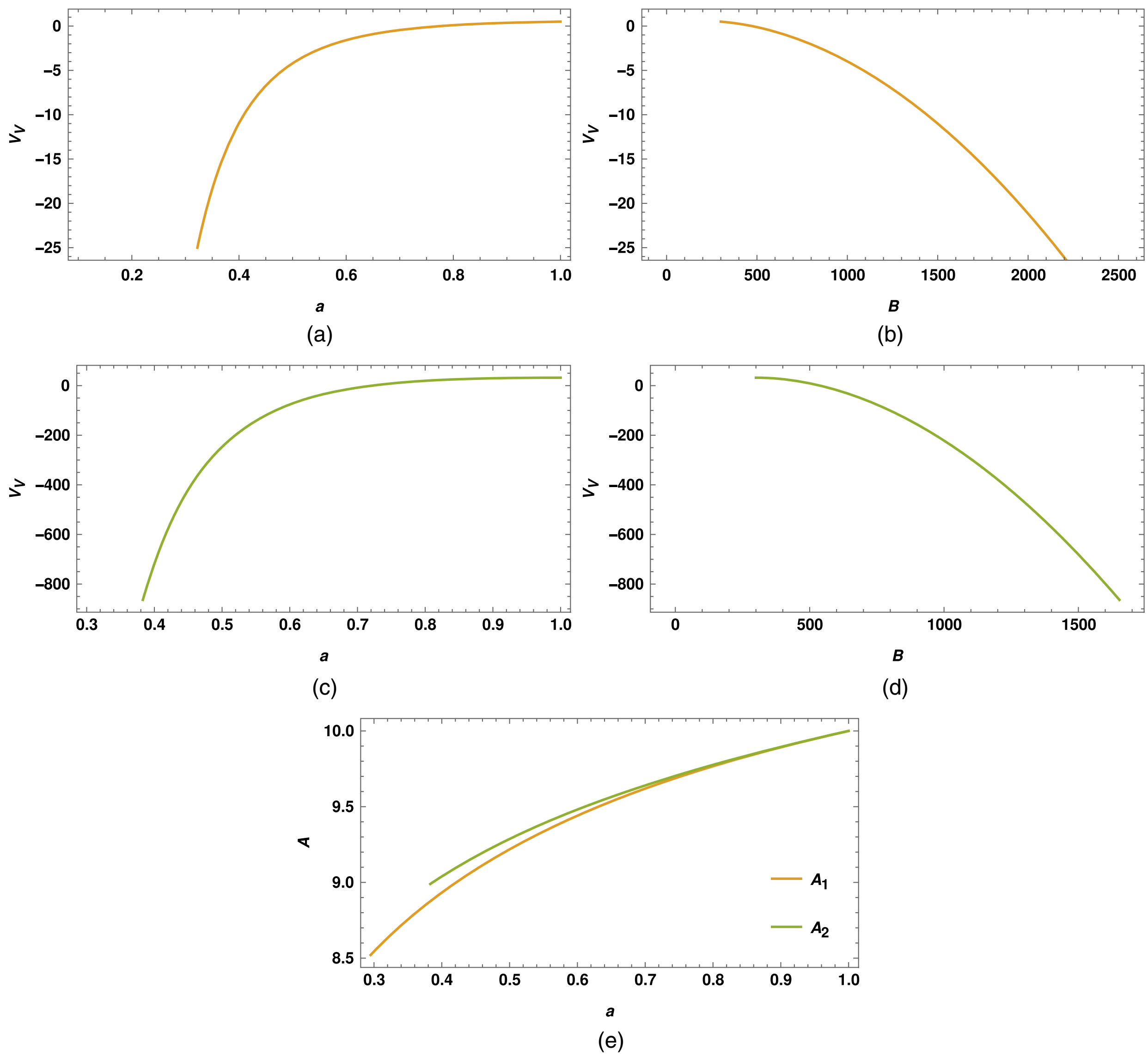


FIG. 3. (a), (c) Vector-field potentials $V_v(a)$ corresponding to newly constructed vector fields VF1 (orange) and VF2 (green), as mentioned in the third and fourth row of Table (II), respectively. (b), (d) The same vector-field potentials $V_v(B)$ as function of B. (e) The vector-field components $A(a)$ in both of these cases. In the latter example, $\mu = -8/3$ and $\lambda = 1$. First, we obtain $\omega_i(a)$, using Eq. (25) (Here $i \in 1, 2$ corresponds to VF1 and VF2 respectively). Then we obtain the vector-field components $A_i(a)$ by solving the differential equation (38) with initial conditions $A_i(1) = 10$ and $A_i'(1) = 1$. With further substitution of $\omega_i(a)$ and the obtained $A_i(a)$ in Eq. (35), we get $V_{v(i)}(a)$. Once $V_{v(i)}(a)$ is obtained, we obtain $V_{v(i)}(B)$.

with potentials, giving rise to the naked singularity as an end state.

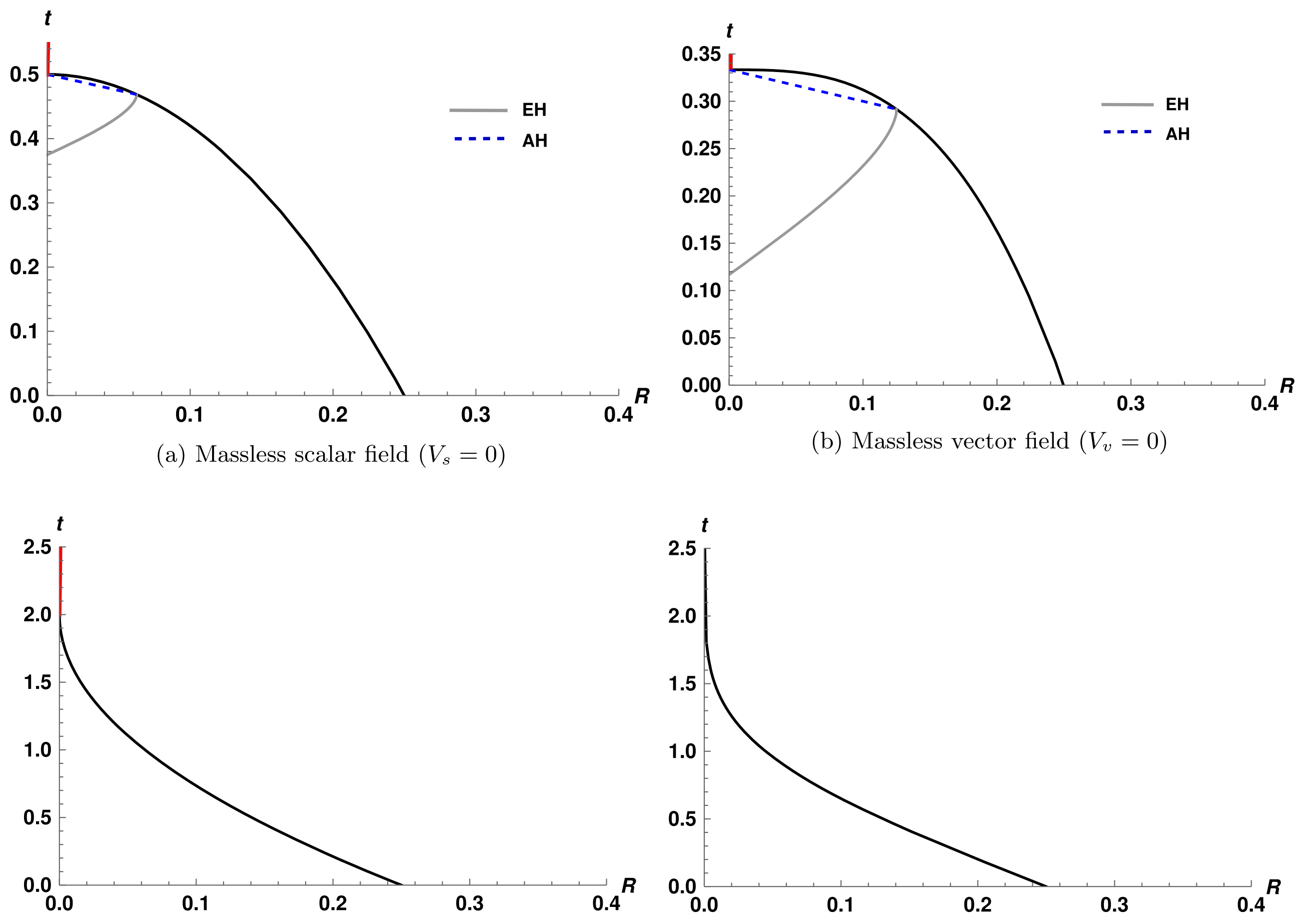
The first example of a collapsing vector field with potential $V_v(a)$ is constructed from the scalar field with potential mentioned in the third row of Table I [14]. The perfect fluid corresponding to such scalar-field example has an equation of state parameter $\omega(a) = -\frac{2}{3}$. The constructed collapsing vector field $\tilde{A} = (0, A, A, A)$ (in the comoving coordinate basis) has the property [dynamics of $A(a)$ and $V_v(a)$] as shown in Fig. 3. Refer to the third row of Table II.

The second example of a collapsing vector field with potential $V_v(a)$ is constructed from the scalar field with potential mentioned in the fourth row of Table I [15]. Such a scalar field has a two-dimensional analog of Mexican-hat-shaped potential. The constructed collapsing vector field $\tilde{A} = (0, A, A, A)$ (in the comoving coordinate basis) has the property [dynamics of $A(a)$ and $V_v(a)$] as shown in Fig. 3. Refer to the fourth row of Table II. The spacetime diagrams of some of the examples in Tables I and II are plotted in Fig. 4.

B. Strength of the singularity

Generally, a singularity in the spacetime manifold is identified by the existence of at least one past/future incomplete geodesic. However, in the case of singularities forming as the end state of a gravitational collapse, apart from the existence of such incomplete geodesics, one expects an additional physical property as follows: An object approaching such singularity should be crushed to zero volume. We call such a singularity gravitationally *strong* in the sense of Tipler [23]. A precise definition of a strong singularity is as follows:

Consider a smooth spacetime manifold $(\mathcal{M}, g)$ and a causal geodesic $\gamma : [t_0, 0) \to \mathcal{M}$. Let λ be an affine



(a) Massless scalar field ($V_s = 0$)

(b) Massless vector field ($V_v = 0$)

(c) SF1/VF1

(d) SF2/VF2

FIG. 4. Spacetime diagram of the examples of spatially homogeneous scalar fields and vector fields mentioned in Tables I and II. The solid black curve in each of them represents the boundary of the collapsing cloud. Upper panel (a and b) [These diagrams represent the collapse of both vector and scalar fields for 0-potentials]: the singularity is not visible in both examples. Lower panel: (c) in the case of SF1/VF1, the singularity forms in a finite comoving time and is globally visible because of the absence of the apparent and event horizons and (d) In the case of SF2/VF2, the singularity forms in an infinite comoving time. However, an ultrahigh-density region is obtained in finite comoving time, which can be visible globally because of the absence of the apparent and event horizons.

parameter along this geodesic. Let $\xi_{(i)}$, ($0 \le i \le 2$ in the case of null geodesic, $0 \le i \le 3$ in the case of timelike geodesic) be the independent Jacobi vector fields. The wedge product of these Jacobi fields gives us the volume form $\mathcal{V} = \bigwedge \xi_{(i)}$. We say that a singularity is gravitationally strong in the sense of Tipler if this volume form vanishes as $\lambda \to 0$.

Clarke and Krolak [24] related the existence of a *Tipler* strong singularity with the growth rate of the curvature terms as follows: At least along one null geodesic with affine parameter λ (such that $\lambda \to 0$ as the singularity is approached), the following inequality

$$\lim_{\lambda\to 0} \lambda^2 R_{ij} K^i K^j > 0 \tag{59}$$

should hold for the singularity to be strong in the sense of Tipler. Here $K^i = \frac{dx^i}{d\lambda}$ are the tangents to the chosen null geodesic, and x^i is the coordinate system. This condition puts a lower bound on the growth of the curvature scalar. In the spherical coordinate system (t, r, θ, ϕ), the radial null geodesic equation reads

$$\frac{dt}{dr} = a. \tag{60}$$

Hence, we have the relation between the tangents K^t and K^r as

$$K^t = aK^r, \tag{61}$$

and subsequently, in terms of the affine parameter,

$$K^t = \frac{R}{\lambda}, \quad \text{and} \quad K^r = \frac{r}{\lambda}. \tag{62}$$

The inequality (59) can then be written in terms of ω as

$$\lim_{a\to 0}\left(r^2 (1+\omega)\rho_0 \exp\left(\int_a^1 \frac{3(1+\omega)}{a} da \right) \right) > 0. \tag{63}$$

Hence, the singularity formed due to the gravitational collapse of a scalar/vector field is strong in the sense of Tipler if the following inequality holds (assuming that the weak-energy condition is respected):

$$\lim_{a\to 0} \exp\left(\int_a^1 \frac{3(1+\omega)}{a} da \right) > 0. \tag{64}$$

Hence, [along with using the condition (58)] one can obtain a naked singularity that is strong in the sense of Tipler for that ω that satisfies the following constraint:

$$0 < \lim_{a\to 0} \exp\left(\int_a^1 \frac{3(1+\omega)}{a} da \right) < \mathcal{O}(a^{-2}). \tag{65}$$

This constraint gives us the class of SH collapsing matter fields that we identify by $\omega(a)$, which ends up in strong curvature naked singularity. Or in other words, we have a class of scalar-/vector-field potentials corresponding to the given scalar/vector field that collapses to a strong naked singularity. As an example, in Tables I and II, we mention the causal property and the strength of the singularity formed due to the gravitational collapse of four different scalar/vector fields.

V. CONCLUSIONS AND REMARKS

Following are the concluding remarks:

(1) Unlike the singularity theorems that provide rigorous proof of the existence of incomplete causal geodesics under rather generic conditions, one does not currently have proof or disproof of the cosmic censorship hypothesis. In fact, we need a mathematically rigorous formulation of this conjecture, which is not available currently, before we can prove or disprove it.

Under the situation at present, we can only speculate its validity or otherwise. Proposed counterexamples hence have great importance in understanding whether naked singularities, in fact, exist or not in our universe. Through such analysis of gravitational collapse models only, one could possibly hope to arrive at a suitable formulation of cosmic censorship. The collapse of inhomogeneous dust and the Vaidya null fluids were the first examples proposed to produce naked singularities. However, an important objection could be that, even if astrophysically interesting, they are not fundamental forms of matter [7,25]. One could then ask whether the collapse of matter configuration that is obtained from a fundamental matter Lagrangian ends up in a naked singularity. Scalar fields with potential and vector fields with potential are fundamental matter fields in this sense. Here we show that not just one particular choice of these fields but an entire class of such types could collapse and form a naked singularity as an end state. This basically divides the allowed class of potential functions into classes that take the unhindered collapse to a black hole or naked singularity.

(2) To achieve this, we show equivalence between SH
   (a) *Perfect fluid*: characterized by $\omega(a)$,
   (b) *Massless scalar field* ϕ: characterized by $\phi(a)$ or its potential $V_s(a)$ or $V_s(\phi)$ [if $\phi(a)$ is invertible], and
   (c) *Massless vector field* $\tilde{A}$: characterized by $A(a)$, or its potential $V_v(a)$, or $V_v(B)$ [if $B(a)$ is invertible].

as far as the gravitational collapse is concerned. This gravitational equivalence is described in subsections of Sec. II and depicted in Fig. 1. Now, if the functional form of $\omega(a)$ satisfies the inequality (58), then the singular null geodesic congruence, if at all there exists, does not get trapped as $a \to 0$. Hence, we have a class of functions $\omega(a)$ corresponding to a naked singularity as an end state of gravitational collapse. Now, because of the above equivalence, in the case of a SH scalar-field collapse, one then has a class of scalar-field function $\phi(a)$, or a class of scalar field potential $V_s(a)$, or a class of scalar-field potential in terms of ϕ, i.e. $V_s(\phi)$ [provided $\phi(a)$ is invertible], that corresponds to the naked singularity as an end state. Similarly, in the case of a SH vector-field collapse, one has a class of vector-field component function $A(a)$, or a class of vector-field potential $V_v(a)$, or a class of vector-field potential in terms of $B = g(\tilde{A}, \tilde{A})$, i.e. $V_s(B)$ [provided $B(a)$ is invertible], that corresponds to the naked singularity as an end state.

(3) A naked singularity formed due to gravitational collapse may or may not be relevant if it is not gravitationally strong in the sense of Tipler [23]. Here, we show a class of $\omega(a)$ that satisfies the inequalities (65) that corresponds to the formation of a strong curvature naked singularity. Using arguments similar to point No. 2 of this section, we have equivalently shown a class of scalar-field potential (in case of scalar-field collapse) and a class of vector-field potential (in case of vector-field collapse) that corresponds to a strong curvature naked singularity.

(4) For the sake of completion, we studied the global spacetime, consisting of the interior collapsing scalar/vector field and the exterior generalized Vaidya spacetime. The smooth matching demands a restriction on the free function, that is, the generalized Vaidya mass function, in terms of the property of the interior collapsing scalar/vector field. We have fulfilled this demand by deriving the expression of the generalized Vaidya mass in terms of the equation of state parameter of the interior collapsing field in Eq. (51).

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