Exterior field of neutron stars: The singularity structure of vacuum and electrovac solutions

V. S. Manko,[1] I. M. Mejía,[2] C. I. Ramos,[1] and E. Ruiz[3]

[1]*Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 Ciudad de México, Mexico*

[2]*Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, 07738 Ciudad de México, Mexico*

[3]*Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca, 37008 Salamanca, Spain*

(Received 31 March 2023; accepted 8 August 2023; published 21 August 2023)

In the present paper we study the singularity structure of the exterior field of neutron stars with the aid of the four-parameter exact solution of the Einstein-Maxwell equations. The complete analysis of this problem in the generic case becomes possible due to the implementation of the novel analytical approach to the resolution of the singularity condition, and it shows the absence of the ring singularities off the symmetry axis in the positive mass case, as well as the possibility of the removal of the ring singularity by a strong magnetic field in the negative mass case. The solution takes an extraordinarily simple form in the equatorial plane, very similar to that of the Kerr solution, which makes it most suitable for astrophysical applications as the simplest model of a rotating magnetized deformed mass and hence of the exterior of a neutron star. It also provides a nontrivial example confirming a recent claim that the φ component of the electromagnetic four-potential has features inconsistent with the intrinsic properties of the electrovac metric, while the magnetic field is represented correctly by the t component of the dual electromagnetic four-potential.

DOI: 10.1103/PhysRevD.108.044042

I. INTRODUCTION

The singularity structure of the Kerr solution [1] describing the exterior field of the "most elementary" rotating astrophysical objects is well known and quite simple: the black-hole branch of the solution possesses a ring singularity covered by the regular event horizon, while in the hyperextreme sector of the solution, characterized by the absence of the event horizon, the ring singularity becomes "naked", i.e., visible to a distant observer. The simplicity of one of the basic characteristics of the Kerr spacetime is intrinsically and uniquely[1] defined by a specific set of the relativistic Geroch-Hansen multipole moments [3–5], concisely given by the formula $m_n = m(ia)^n$, $n = 0, 1, 2, \ldots$, the parameter m denoting the mass of the source and a the angular momentum per unit mass [4]. The above m_n are coefficients in the expansion of the function

$$\xi(z) = \frac{1 - e(z)}{1 + e(z)} = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}, \quad e(z) = \frac{z - m - ia}{z + m - ia}, \qquad (1)$$

when $z \to \infty$, and the knowledge of these m_n, calculated on the symmetry axis, is sufficient to construct the corresponding metric in the whole space by means of Sibgatullin's integral method [6,7]. The fact that all m_n are functions of m and a only is to some extent reflected in the "no-hair" theorem [8] according to which the mass and angular momentum fully characterize the Kerr black hole spacetime.

Next to black holes (BHs), neutron stars (NSs) are second densest (and simplest) astrophysical objects in nature, and the exterior field of NSs requires at least one more arbitrary physical parameter, the mass quadrupole moment [9], which takes into account the deformation of the source. An extensive study of the NS models with the aid of the analytical and numerical approaches carried out in recent decades [10–18] has eventually led to a remarkable discovery that the first few lowest multipole moments in fact determine entirely the geometry around NSs, which constitutes the essence of the Yagi *et al.* "NS no-hair conjecture" [19]. The latter conjecture in turn naturally singles out the six-parameter equatorially symmetric two-soliton solution [20] of the Einstein-Maxwell equations (henceforth referred to as MMR) as a generic analytical model for the exterior geometry of a NS, which also includes the parameters of electric charge and magnetic

[1]The claim about nonuniqueness of the multipole moments of the Kerr solution recently made in the paper [2] is wrong because of the uniqueness of the expansion (1) below.

dipole moment. For the pure vacuum specialization [21,22] of the solution [20] it has been shown [23] that the explicit form of the multipole moments higher than quadrupoles can be read off from the degeneration condition of the determinant L_n involved in the description of the axis data defining the general N-soliton vacuum metric [24]; the expressions for the electromagnetic multipoles are obtainable from the expansions of the electromagnetic potential via standard procedures [25–29].

One may expect that the singularity structure of the exact solutions representing NSs, which are defined by the Yagi *et al.* NS no-hair conjecture, is reacher compared to that of Kerr due to the presence of some additional parameters and a more complicated form of the respective metrical fields. Indeed, already the well-known Tomimatsu-Sato $\delta = 2$ (TS2) solution [30], which is a particular case of the double-Kerr solution of Kramer and Neugebauer [31] and of the MMR spacetime and hence could in principle describe the exterior of a specific NS, is endowed with a massless ring singularity outside the symmetry axis accompanied by a region with causality violation [32], and its origin was attributed in the paper [33] to the presence of negative mass. Moreover, the appearance of ring singularities in various particular 2-soliton spacetimes has been routinely analyzed in a number of papers [13,34–38], where it has been shown, in particular, that in the binary configurations of Kerr sources the constituent with negative Komar [39] mass develops a massless ring singularity outside the symmetry axis which is needed to prevent disintegration of that constituent, remembering that the single Schwarzschild and Kerr sources of negative mass are known to be unstable [40,41]. Such ring singularities can also be present when the two constituents have positive masses, in which case they do not allow the dynamical nonregular evolution of the joint stationary limit surfaces [42,43].

It may be observed that up to now the ring singularities arising in binary systems and in the NS exterior have been usually analyzed numerically because of the complexity of the explicit algebraic expressions involved in the analysis. In this respect, it turns out surprising that the singularity problem, as will be shown in the present paper, can be solved analytically in the case of the four-parameter subfamily of the MMR metric—a NS solution elaborated and studied a few years ago [43] that we are going to rewrite in the extended parameter space in a manner slightly different from the $e = 0$ specialization of the electrovac metric considered in [29]. Remarkably, this will permit us to establish the absence of the massless ring singularities outside the symmetry axis when the NS solution has positive mass, and also to see that the ring singularity emerging in the negative mass case can be removed by a strong magnetic field. Additionally, we shall obtain a very simple form of the NS solution in the equatorial plane and discuss the difference between the singularities of the metric coefficients of the solution and those of the corresponding component A_φ of the electromagnetic four-potential.

Our paper is organized as follows. In the next section we will consider the extended version of the solution [43] in which the free parameters correspond to arbitrary relativistic multipole moments, as well as the conditions defining the singularities of the solution. In Sec. III we solve analytically the condition for ring singularities on and off the equatorial plane in the general case of that solution, and compare the singularity structure of its extreme vacuum limiting case with that of the well-known Tomimatsu-Sato $\delta = 2$ metric. The results obtained are discussed in Sec. IV, where in particular we present a very simple form of the NS solution in the equatorial plane, and briefly comment on a better description of the magnetic field by the t component of the dual electromagnetic potential than by the φ component of the usual electromagnetic four-potential.

II. THE EXTENDED 4-PARAMETER SOLUTION FOR THE NS EXTERIOR AND THE SINGULARITY CONDITION

We remind that the NS solution [43] is determined by the axis values of the Ernst complex potentials [44] $\mathcal{E}$ and Φ of the form

$$\mathcal{E}(\rho = 0, z) \equiv e_+(z) = \frac{(z - m_1)(z - m_2) - ia(m_1 + m_2) + a^2 - \mu^2}{(z + m_1)(z + m_2) + ia(m_1 + m_2) + a^2 - \mu^2},$$
$$\Phi(\rho = 0, z) \equiv f_+(z) = \frac{i\mu(m_1 + m_2)}{(z + m_1)(z + m_2) + ia(m_1 + m_2) + a^2 - \mu^2}, \qquad (2)$$

where the parameters m_1, m_2, a, and μ take arbitrary real values.

One can easily see that the mass parameters m_1 and m_2 in (2) occur only in the combinations $m_1 + m_2$ and $m_1 m_2$, which suggests that the extension of the parameter set can be achieved by allowing for m_1 and m_2, in addition to the real values they can take, to be also a pair of complex conjugate quantities. In this way it turns out possible to introduce, instead of m_1 and m_2, the total mass m and the mass quadrupole moment Q of the source as arbitrary real parameters of the solution. Indeed, since $m = m_1 + m_2$ and $Q = -m(m_1 m_2 + a^2 - \mu^2)$ [43], then the substitution formulas take the form

$$m_1 = (m+d)/2, \qquad m_2 = (m-d)/2,$$
$$d = \sqrt{m^2 + 4(\kappa + a^2 - \mu^2)}, \qquad \kappa \equiv Q/m, \tag{3}$$

where the quantity d may take either real or pure imaginary values, depending on the interrelations of the arbitrary parameters. Then the axis data (2) assume the form

$$e_+(z) = \frac{z(z-m) - \kappa - ima}{z(z+m) - \kappa + ima},$$
$$f_+(z) = \frac{im\mu}{z(z+m) - \kappa + ima}, \tag{4}$$

and one can see that these define a four-parameter subfamily of the solution [20] when the parameters b, q in [20] are restricted to $b = a$, $q = 0$, while the constants k, c are chosen there in the form $k = \kappa + a^2$ and $c = m\mu$. So, the four roots α_i of the algebraic equation

$$e_+(z) + \bar{e}_+(z) + 2f_+(z)\bar{f}_+(z) = 0, \tag{5}$$

entering the final formulas of the solution, are given by

$$\alpha_1 = -\alpha_2 = \sigma_+, \qquad \alpha_3 = -\alpha_4 = \sigma_-,$$
$$\sigma_\pm = \sqrt{\frac{1}{4}(m \pm d)^2 - a^2 + \mu^2}, \tag{6}$$

where $\sigma_\pm$ are not restricted to only the real values and can take complex values too. The corresponding Ernst potentials of the solution [43] hence can be rewritten as

$$\begin{aligned}
&\mathcal{E} = (A-B)/(A+B), \qquad \Phi = C/(A+B),\\
&A = 2\sigma_+\sigma_-[(m^2+d^2)(R_+ + R_-)(r_+ + r_-) - 2(m^2-d^2)(R_+R_- + r_+r_-)] - [(m+d)^2\sigma_-^2 + (m-d)^2\sigma_+^2]\\
&\quad \times (R_+ - R_-)(r_+ - r_-) + 4imad[\sigma_+(R_+ + R_-)(r_+ - r_-) - \sigma_-(R_+ - R_-)(r_+ + r_-)],\\
&B = 4md\{\sigma_+\sigma_-[(m+d)(r_+ + r_-) - (m-d)(R_+ + R_-)] + ia[\sigma_+(m+d)(r_+ - r_-) - \sigma_-(m-d)(R_+ - R_-)]\},\\
&C = 4im\mu d[\sigma_-(m+d)(R_+ - R_-) - \sigma_+(m-d)(r_+ - r_-)],\\
&R_\pm = \sqrt{\rho^2 + (z \pm \sigma_+)^2}, \qquad r_\pm = \sqrt{\rho^2 + (z \pm \sigma_-)^2},
\end{aligned} \tag{7}$$

and the extended metric functions from the line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2 \tag{8}$$

have the form

$$\begin{aligned}
&f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{256 d^4 |\sigma_+|^2 |\sigma_-|^2 R_+R_-r_+r_-}, \qquad \omega = -\frac{\mathrm{Im}[G(\bar{A}+\bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}},\\
&G = -2zB + 2md\{\sigma_+[(m-d)^2 + 2\mu^2](R_+ + R_-)(r_+ - r_-) - \sigma_-[(m+d)^2 + 2\mu^2](R_+ - R_-)(r_+ + r_-)\\
&\quad - 4imad(R_+ - R_-)(r_+ - r_-) - 4\sigma_-[(m-d)\sigma_+^2 - m\mu^2](R_+ - R_-) + 4\sigma_+[(m+d)\sigma_-^2 - m\mu^2](r_+ - r_-)\\
&\quad - 4ia\sigma_+\sigma_-[(m-d)(R_+ + R_-) - (m+d)(r_+ + r_-)]\},\\
&I = 2im\mu\{[(m-d)\sigma_+^2 + (m+d)\sigma_-^2](R_+ - R_-)(r_+ - r_-) - 2\sigma_+\sigma_-[m(R_+ + R_-)(r_+ + r_-)\\
&\quad - 2(m+d)R_+R_- - 2(m-d)r_+r_-] - 2iad[\sigma_+(R_+ + R_-)(r_+ - r_-) - \sigma_-(R_+ - R_-)(r_+ + r_-)]\\
&\quad + 2d\sigma_+\sigma_-[(3m+d)(R_+ + R_-) - (3m-d)(r_+ + r_-) + 4md] + 4imad[\sigma_-(R_+ - R_-) - \sigma_+(r_+ - r_-)]\},
\end{aligned} \tag{9}$$

where $|x|^2$ means $x\bar{x}$. Note also that formulas for two nonzero components of the electromagnetic four-potential remain the same as in [43]:

$$A_t = -\mathrm{Re}\left(\frac{C}{A+B}\right), \qquad A_\varphi = \mathrm{Im}\left(\frac{I - zC}{A+B}\right). \tag{10}$$

The Kerr solution is contained in the above formulas (7)–(9) as the particular case $\mu = 0$, $\kappa = -a^2$.

Turning now to the analysis of the singularity structure of our 4-parameter solution outside the symmetry axis, we begin by remarking that the ring singularities are roots of the algebraic equation

$$A + B = 0, \tag{11}$$

and these are located on the stationary limit surface (SLS) $f = 0$ in the pure vacuum case ($\mu = 0$), or outside the SLS when the electromagnetic field is present, similar to the case of the Kerr-Newman solution [45] endowed with negative mass (see, e.g., Ref. [46]). We find it plausible to first analyze the appearance of ring singularities in the equatorial ($z = 0$) plane, their habitual location in the spacetimes with reflection symmetry [47–50]. This, on the one hand, will help the reader understand a special character of the singularity structure of the solution (7) and, on the other hand, will simplify the consideration of the general case where, as will be seen later on, the singularities off the equatorial plane can only occur on the symmetry axis. The reader will also see that one of the conditions defined by Eq. (11), namely the condition (14) below, should be excluded from the subsequent singularity analysis due to the degeneration of our solution on the equatorial plane.

As a preliminary, we first observe that in the equatorial plane

$$R_+ = R_- = \sqrt{\rho^2 + \sigma_+^2}, \quad r_+ = r_- = \sqrt{\rho^2 + \sigma_-^2}, \tag{12}$$

and the condition (11) takes the form

$$[(m-d)R_- - (m+d)r_-][(m+d)R_- - (m-d)r_- + 2md]\sigma_+\sigma_- = 0. \tag{13}$$

The above condition will be fulfilled if one of the factors on the left-hand side of (13) takes zero value. In what follows we exclude the cases $\sigma_+ = 0$ and $\sigma_- = 0$ as leading to the double roots of Eq. (5) and hence to indetermination of the potentials $\mathcal{E}$ and Φ in (7). The remaining two conditions to investigate are therefore

$$(m-d)R_- - (m+d)r_- = 0, \tag{14}$$

and

$$(m+d)R_- - (m-d)r_- + 2md = 0. \tag{15}$$

We mention that out of these two it is the condition (14) that might give rise to the ring singularity outside the symmetry axis independently of the sign of the mass parameter m, the location of the singularity being defined by

$$\rho_S = \sqrt{a^2 - \mu^2}, \qquad z_S = 0, \qquad a^2 > \mu^2. \tag{16}$$

In particular, the above formula is verified by the singularity shown in Fig. 3 of [43] for the parameter choice $m = 2.5$, $a = 0.5$, $\mu = 0.25$. To see that (14) is verified by (16) for positive and negative values of m, it is sufficient to substitute (16) into (12) and then check that $R_- = (m+d)/2$, $r_- = (m-d)/2$ for $m \geq |d| > 0$ ($d^2 > 0$) and $R_- = -(m+d)/2$, $r_- = -(m-d)/2$ for $m \leq -|d| < 0$ ($d^2 > 0$) satisfy identically (14).

It is remarkable, however, that after a considerable effort spent on identifying in a rigorous way of all the occasions when the singularity (16) is present in the solution in the positive mass case, we have eventually discovered, to our big surprise, that the condition (14) is not in fact involved in the analysis of the singularity problem (and hence should be excluded from the further consideration) because the complex potentials (7) degenerate in the equatorial plane to the expressions

$$\mathcal{E} = \frac{(m+d)R_- - (m-d)r_- - 2md}{(m+d)R_- - (m-d)r_- + 2md}, \qquad \Phi = 0, \tag{17}$$

due to appearance, at $z = 0$, of the factor $(m-d)R_- - (m+d)r_-$ both in the denominator and numerator of $\mathcal{E}$. As a result, the singularity problem in the equatorial plane considerably simplifies and reduces to analyzing the condition (15) alone. In what follows we shall demonstrate that although the latter condition can also give rise to a singular point defined by (16), this point (ρ_S, z_S) is a solution of (15) for the negative values of m only.

III. SOLVING THE SINGULARITY PROBLEM IN AND OFF THE EQUATORIAL PLANE

Fortunately, the analysis of the condition (15), whose roots, if exist, will define the genuine ring singularities, is straightforward. Indeed, solving (15) for r_-, we get

$$r_- = \frac{R_-(m+d) + 2md}{m-d}, \tag{18}$$

and this must be substituted into the equality

$$R_-^2 - r_-^2 - \sigma_+^2 + \sigma_-^2 = 0, \tag{19}$$

which is a direct consequence of (12), thus leading to the equation

$$-\frac{md(2R_- + m + d)^2}{(m-d)^2} = 0, \tag{20}$$

with the obvious solution

$$R_- = -\frac{1}{2}(m+d). \tag{21}$$

Then the substitution of (21) into (18) yields

$$r_- = -\frac{1}{2}(m-d), \tag{22}$$

so that (21) and (22) define the location of the singularity in the equatorial plane.

Our further estimations must take into account that all four square roots $R_\pm$ and $r_\pm$ entering the solution (7) are understood as the positive branch of these functions: $\mathrm{Re}(R_\pm) > 0, \mathrm{Re}(r_\pm) > 0$. We should also bear in mind that the quantity d can assume (positive) real or pure imaginary values, i.e., distinguish between the cases $d^2 > 0$ and $d^2 < 0$. Then the former case implies that the inequalities

$$m + d < 0 \quad \text{and} \quad m - d < 0 \tag{23}$$

must be satisfied simultaneously, whence we get immediately

$$m < 0. \tag{24}$$

On the other hand, the latter case $d^2 < 0$ also leads directly to the condition (24) by applying the above mentioned criterion on the square roots to the expressions (21) and (22). Therefore, we can conclude that our solution does not develop singularities in the equatorial plane for the positive values of m.

The value of ρ in the $z = 0$ plane at which the singularity occurs in the negative mass case is readily obtainable from (12), (21), and (22), and it is given by the above formula (16) for ρ_S and z_S. It suggests that negative mass of the source itself is not yet a guarantee for the formation of a ring singularity. Indeed, it is clear that for the values $\mu^2 > a^2$ of the magnetic dipole parameter μ the singularity does not arise, which means that the magnetic field plays a stabilizing role in the NS solution.

Another restriction on the singularity in the $m < 0$ case follows from the first inequality in (23):

$$d < -m \quad \Rightarrow \quad \kappa < \mu^2 - a^2, \tag{25}$$

which means in particular that the oblate configurations of the negative mass source in our solution, corresponding to positive κ, do not develop a ring singularity in the equatorial plane. Recalling that the Kerr solution with negative mass is always accompanied by a massless ring singularity outside the symmetry axis [33], we can draw a conclusion that NSs might probably be more stable negative mass configurations than the $m < 0$ Kerr source due to presence of an arbitrary mass quadrupole moment.

A. The extreme case

We find it very instructive to compare the singularity structure of the extreme case of the solution (7) with that of the Tomimatsu-Sato $\delta = 2$ spacetime [30] which was historically the first nontrivial example of an asymptotically flat stationary axisymmetric solution constructed within the framework of the Ernst formalism [51]. Written in the prolate ellipsoidal coordinates (x, y), the Ernst potential of the TS2 solution has the form

$$\begin{aligned}
&\mathcal{E} = (A - B)/(A + B),\\
&A = p^2x^4 + q^2y^4 - 1 - 2ipqxy(x^2 - y^2),\\
&B = 2px(x^2 - 1) - 2iqy(1 - y^2),\\
&x = \frac{1}{2k_0}(r_+ + r_-), \qquad y = \frac{1}{2k_0}(r_+ - r_-), \qquad r_\pm = \sqrt{\rho^2 + (z \pm k_0)^2},
\end{aligned} \tag{26}$$

where the real constants p and q are subject to the constraint $p^2 + q^2 = 1$, and k_0 is an arbitrary real positive parameter. The total mass of this solution is $M = 2k_0/p$, so, with account of the positivity of k_0, the positive or negative values of M are determined by the positive or negative values of p, respectively. As had been pointed out by Tomimatsu and Sato themselves, their solution (for $q \neq 0$) always has a ring singularity in the equatorial plane ($y = 0$), the exact location of which in the positive-mass case being given by the formula [33]

$$x_0 = \frac{1}{2p}\left(\chi - 1 + \sqrt{3 - \chi^2 - \frac{2}{\chi}(1 - 2p^2)}\right), \qquad y_0 = 0, \qquad \chi \equiv \sqrt{1 + (2p)^{2/3}(p^2 - 1)^{1/3}}, \tag{27}$$

while for the locus of the singularity in the negative-mass case we refer the reader to Ref. [33].

On the other hand, the special case $\sigma_+ = \sigma_-$, or $d = 0$, corresponding to a pair of equal α's in (6) was worked out in [43], and in the absence of the magnetic field ($\mu = 0$) it is defined by the Ernst potential of the form

$$
\begin{aligned}
&\mathcal{E} = (A - B)/(A + B),\\
&A = m^4(x^4 - 1) + a^4(x^2 - y^2)^2 - 2m^2ax[ax(x^2 - y^2) - 2i\sigma y(1 - y^2)],\\
&B = 2m\{m^2[\sigma x(x^2 - 1) + iay(1 - y^2)] - a\sigma(x^2 - y^2)(ax - i\sigma y)\},\\
&x = \frac{1}{2\sigma}(r_+ + r_-), \qquad y = \frac{1}{2\sigma}(r_+ - r_-), \qquad r_\pm = \sqrt{\rho^2 + (z \pm \sigma)^2}, \qquad \sigma = \sqrt{m^2 - a^2},
\end{aligned} \tag{28}
$$

where the parameter m is now half the m that appears in the general solution (7). It was shown in [52] that the subextreme branch of the solution (28) belongs to the Kinnersley-Chitre family of the vacuum spacetimes [53].

It is easy to check that, similar to the generic case, the Ernst potential (28) considerably simplifies in the equatorial plane, taking the form

$$\mathcal{E} = \left(\frac{\sigma x - m}{\sigma x + m}\right)^2, \tag{29}$$

and hence the singularity occurs at

$$x = -m/\sigma, \tag{30}$$

so that the ring singularity outside the symmetry axis requiring $x > 1$ (i.e. $\rho > 0$) can only be developed by a negative mass. It is worth noting that in the case of the TS2 solution a similar simplification of the Ernst potential does not take place, and we have

$$\mathcal{E}_{\mathrm{TS}}(y = 0) = \frac{p^2x^4 - 2px(x^2 - 1) - 1}{p^2x^4 + 2px(x^2 - 1) - 1}, \tag{31}$$

that is the quartic polynomials in x both in the numerator and denominator of $\mathcal{E}$, like in the general formulas (26).

Fortunately, the singularity problem of the extreme solution (28) can be solved analytically in the entire space too due to exceptional factorization properties of this solution. In the generic case, the system of equations to be solved is

$$\mathrm{Re}(A + B) = 0, \qquad \mathrm{Im}(A + B) = 0, \tag{32}$$

and the first equation, as it follows from (28), takes the form

$$(\sigma^2x^2 + a^2y^2 - m^2)[(\sigma x + m)^2 + a^2y^2] = 0, \tag{33}$$

while the second equation yields

$$2amy[(\sigma x + m)(\sigma x - 2my^2 + m) + a^2y^2] = 0. \tag{34}$$

Actually, we must be only interested in the first factor in (33) and the last factor in (34) since the condition $y = 0$ leads to the "equatorial" case already considered above, whereas the second factor on the left-hand side of (33) is a positive quantity for $y \neq 0$. Moreover, if in addition, instead of equating to zero the last factor in (34), we shall consider the difference of that factor with the first factor in (33), then the system to be solved becomes eventually composed of the following two equations:

$$\sigma^2x^2 + a^2y^2 - m^2 = 0, \qquad (1 - y^2)(\sigma x + m) = 0, \tag{35}$$

and besides the solution $y = 0$, $x = -m/\sigma$ involving the equatorial plane, we arrive at the off-plane solutions

$$y = 1, \qquad x = \pm 1 \quad \text{and} \quad y = -1, \qquad x = \pm 1. \tag{36}$$

Apparently, these solutions do not represent the ring singularities outside the symmetry axis.

B. The general case

It is remarkable that the generic analysis of the ring singularities outside the symmetry axis admits in the case of the solution (7) a purely analytical treatment. However, since it does not lead to the new singularities in addition to the already considered in the equatorial plane, in what follows we shall restrict ourselves to the description of the general scheme of our novel approach to the resolution of the condition (11), omitting numerous mathematical details which do not give us an extra physical information. As a sort of a compensation, we shall also consider a curious case of a line singularity on the symmetry axis covered by our general scheme for a special relation of the parameters of the solution.

Obviously, the system of algebraic equations defining the singularities on and off the equatorial plane that must be solved by us is composed of the conditions for vanishing real and pure imaginary parts of (11), i.e.,

$$
\begin{aligned}
&2\sigma_+\sigma_-\{[(m^2 + d^2)(R_+ + R_-)(r_+ + r_-) - 2(m^2 - d^2)\\
&\quad\times(R_+R_- + r_+r_-)] + 2md[(m + d)(r_+ + r_-)\\
&\quad-(m - d)(R_+ + R_-)]\} - [(m + d)^2\sigma_-^2 + (m - d)^2\sigma_+^2]\\
&\quad\times(R_+ - R_-)(r_+ - r_-) = 0,
\end{aligned} \tag{37}
$$

and

$$\sigma_+(r_+ - r_-)(R_+ + R_- + m + d) - \sigma_-(R_+ - R_-) \times (r_+ + r_- + m - d) = 0. \quad (38)$$

Note that one also can easily arrive at the system (37) and (38) by merely exploring the equatorial symmetry of the solution (7), which implies that if the point (ρ_S, z_S) satisfies Eq. (11) then the point $(\rho_S, -z_S)$ will satisfy this equation too.

Though the resolution of the above system may look impossible analytically since the dependence in it on the coordinates (ρ, z) is not direct but through the square roots $R_\pm$ and $r_\pm$, we still have been able to find a way out of this unpleasant situation by considering that $R_\pm$ and $r_\pm$ enter Eqs. (37) and (38) as independent variables, which requires to supplement the latter equations with two additional conditions/constraints that must verify $R_\pm$ and $r_\pm$ as functions of ρ and z. Thus, using the definition (7) of $R_\pm$ and $r_\pm$, we can express ρ and z, say, in terms of $R_\pm$ as

$$\rho^2 = R_-^2 - \frac{(R_-^2 - R_+^2 + 4\sigma_+^2)^2}{16\sigma_+^2}, \qquad z = \frac{R_+^2 - R_-^2}{4\sigma_+}, \quad (39)$$

and then, with the aid of $r_\pm$, we get the following two extra conditions

$$(\sigma_+ - \sigma_-)R_+^2 + (\sigma_+ + \sigma_-)R_-^2 - 2\sigma_+(r_-^2 + \sigma_+^2 - \sigma_-^2) = 0,$$
$$(\sigma_+ - \sigma_-)R_-^2 + (\sigma_+ + \sigma_-)R_+^2 - 2\sigma_+(r_+^2 + \sigma_+^2 - \sigma_-^2) = 0, \quad (40)$$

which complement the system (37) and (38). Of course, once the particular values of the functions $R_\pm$ and $r_\pm$ defining the singularities are found, the corresponding (ρ_S, z_S) should be obtained by means of formulas (39).

The advantage of solving the system of four equations (37), (38), (40) for $R_\pm$ and $r_\pm$ instead of directly solving the two equations (37) and (38) for (ρ, z) turns out significant, as now we can obtain from (37) and (38), via standard substitutions, the following four analytical solutions for $R_\pm$, $r_\pm$:

Solution 1.

$$R_\pm = \frac{1}{2}[h(m-d)(\pm 2k\sigma_+ - 1) - m - d],$$
$$r_\pm = \frac{1}{2}[h(m+d)(\pm 2k\sigma_- - 1) - m + d]. \quad (41)$$

Solution 2.

$$R_\pm = -\frac{1}{2}(m+d)[h(\pm 2k\sigma_+ - 1)(4k^2\sigma_-^2 - 1) + 1],$$
$$r_\pm = -\frac{1}{2}(m-d)[h(\pm 2k\sigma_- - 1)(4k^2\sigma_+^2 - 1) + 1]. \quad (42)$$

Solution 3.

$$R_\pm = \frac{1}{2}[\pm 2h(m-d)\sigma_+ - m - d],$$
$$r_\pm = \frac{1}{2}[\pm 2h(m+d)\sigma_- - m + d]. \quad (43)$$

Solution 4.

$$R_\pm = -\frac{1}{2}(m+d)(\pm 2h\sigma_- + 1),$$
$$r_\pm = -\frac{1}{2}(m-d)(\pm 2h\sigma_+ + 1). \quad (44)$$

The above solutions contain the arbitrary (real or complex) constants h and k, which in our scheme would be better regarded as arbitrary variables, whose particular form should be found from the constraint equations (40). The evaluation of these h and k in each case is straightforward and does not represent difficulties. Thus, the substitution of (41) and (42) into (40) yields the following four solutions of the general system (37), (38), and (40):

$$\text{(I)} \qquad R_+ = R_- = -\frac{1}{2}(m+d), \quad r_+ = r_- = -\frac{1}{2}(m-d);$$
$$\text{(II)} \qquad R_+ = R_- = \frac{1}{2}(m+d), \quad r_+ = r_- = \frac{1}{2}(m-d);$$
$$\text{(III)} \qquad R_+ = -R_- = -\frac{1}{2}(m+d)\epsilon, \quad (\epsilon^2 = 1) \quad r_+ = r_- = -\frac{1}{2}(m-d);$$
$$\text{(IV)} \qquad R_+ = R_- = -\frac{1}{2}(m+d), \quad r_+ = -r_- = -\frac{1}{2}(m-d)\epsilon, \quad (\epsilon^2 = 1)$$

where (I) is a corollary of either (41) or (42), while (II)–(IV) of exclusively (42), and all of these lead to the same singular point (16) located in the equatorial plane. However, the solutions (III) and (IV) are not admissible because they do not ensure the positivity of the real parts of all $R_\pm$, $r_\pm$, and in view of the degeneration of the Ernst potentials (7) in the equatorial plane the solution (II) must be also discarded as satisfying the condition (14) and not satisfying the condition (15). The remaining solution (I) has already been considered at the beginning of this section and it coincides with the formulas (21) and (22) leading to the singularity (16) in the negative mass case $m < 0$.

On the other hand, the substitution of the solutions (43) and (44) into the constraints (40) does not permit to fix h, the constraints being satisfied only at the concrete parameter choice $\mu^2 = a^2$ independently of the value of h. Although the latter choice, as will be seen later on, determines a singularity on the symmetry axis (while we are interested in the singularities off the symmetry axis), we still find it instructive to study this curious particular case, common to the two solutions, as a nontrivial application of the general scheme. When $\mu^2 = a^2$, the expressions of $\sigma_\pm$ simplify to the form

$$\sigma_\pm = \frac{1}{2}\epsilon_\pm(m \pm d), \qquad \epsilon_\pm^2 = 1, \tag{45}$$

so that the corresponding $R_\pm$ and $r_\pm$ of the solution (43) become

$$R_\pm = \frac{1}{2}(m+d)[\pm\epsilon_+ h(m-d) - 1],$$
$$r_\pm = \frac{1}{2}(m-d)[\pm\epsilon_- h(m+d) - 1], \tag{46}$$

or, equivalently, after expressing m and d in terms of $\sigma_\pm$ with the aid of (45),

$$R_\pm = \sigma_+(\pm 2h\epsilon_-\sigma_- - \epsilon_+),$$
$$r_\pm = \sigma_-(\pm 2h\epsilon_+\sigma_+ - \epsilon_-). \tag{47}$$

Now we can identify the location of the singularity by substituting $R_\pm$ from (47) into (39), yielding

$$\rho = 0, \qquad z = -2h\epsilon_+\epsilon_-\sigma_+\sigma_-. \tag{48}$$

Using the explicit form (45) for $\sigma_\pm$ in the $\mu^2 = a^2$ case, (48) also rewrites as

$$\rho = 0, \qquad z = 2h\kappa, \tag{49}$$

which shows in particular that the singularity's locus reduces to the origin $\rho = z = 0$ in the case of vanishing κ. However, when $\kappa \neq 0$, one could think that the singularity extends along the whole symmetry axis, as h can take arbitrary real values, thus suggesting that the solution (7) might be not asymptotically flat at least in the special $\mu^2 = a^2$ case.

The issue of the above singularity can be clarified after recalling that the roots $R_\pm$ and $r_\pm$ are defined under the positive branch criterion. Therefore, the values (47) which, after changing h to z by means of (48), take the form

$$R_\pm = \mp\,\epsilon_+(z \pm \sigma_+), \qquad r_\pm = \mp\,\epsilon_-(z \pm \sigma_-), \tag{50}$$

must be consistent with the axis values $R_\pm(0,z)$, $r_\pm(0,z)$ that follow from their definition (7), i.e.,

$$R_\pm(0,z) = \sqrt{(z\pm\sigma_+)^2}, \qquad r_\pm(0,z) = \sqrt{(z\pm\sigma_-)^2}. \tag{51}$$

Then, for instance, if $\sigma_\pm$ are real-valued and $\sigma_+ > \sigma_- > 0$, we get on the upper part of the symmetry axis ($z > \sigma_+$)

$$R_\pm(0,z) = z \pm \sigma_+, \qquad r_\pm(0,z) = z \pm \sigma_-, \tag{52}$$

and a simple inspection shows that there is no choice of $\epsilon_\pm$ for which $R_\pm$ and $r_\pm$ in (50) would fully coincide with $R_\pm(0,z)$ and $r_\pm(0,z)$ in (52). The same is true for the lower part of the symmetry axis ($z < -\sigma_+$), and also for the parts $\sigma_- < z < \sigma_+$ and $-\sigma_+ < z < -\sigma_-$. However, on the intermediate part $-\sigma_- < z < \sigma_-$ of the z-axis, formulas (51) assume the form

$$R_\pm(0,z) = \pm z + \sigma_+, \qquad r_\pm(0,z) = \pm z + \sigma_-, \tag{53}$$

and now the choice $\epsilon_+ = \epsilon_- = -1$ makes the expressions (50) and (53) identical, which means that the singularity defined by formulas (49) occupies exclusively the finite interval $(-\sigma_-, \sigma_-)$ of the symmetry axis. In addition, it is not difficult to see that this singularity is developed by the *negative* mass. Indeed, under the above choice $\epsilon_\pm = -1$ and our supposition $\sigma_\pm > 0$, formulas (45) imply that

$$m + d < 0, \qquad m - d < 0, \tag{54}$$

whence we immediately arrive at $m < 0$.

IV. DISCUSSION

The analysis carried out in the present paper shows that the singularity structure of the exterior field of NSs defined by the solution (7) is quite similar to that of black holes: no ring singularities are present outside the symmetry axis in the positive mass case, and a ring singularity located in the equatorial plane arises in the case of negative mass. To some extent, the singularity of NSs in the latter case has a more benign character than that of black holes because in the Kerr and Kerr-Newman solutions with negative mass the singularity is irremovable [33,46], while a strong magnetic field $\mu^2 > a^2$ in the solution (7) removes the ring singularity. This could be interpreted, bearing in mind the generic instability of the negative mass sources [40,41] (the singularities thus preserving the stationarity of the sources), as a stabilization effect exerted by the magnetic field on the massive sources of NSs. In other words, NSs carrying negative mass and magnetic dipole moment are more stable objects than the "black holes" of negative mass. Interestingly, the location of the ring singularity defined by formula (16) does not depend on the mass quadrupole parameter κ. It is also clear from our analysis that the angular momentum plays a key role in developing ring singularities in the solution (7), so that for instance the exterior field of a nonrotating NS is free of the ring singularities completely.

The analytical approach to the singularity problem of NSs developed in the present paper has allowed us to rectify some incorrect statements about the presence of the ring singularities made earlier in the literature [43] on the basis of the numerical analysis of the condition (11). As a matter of fact, the idea to resort to analytical study of the singularity problem came to us only after we were able to clearly realize that the numerical methods were failing in the vicinity of singular points and were often producing some exotic unrealistic results that could be erroneously taken for the genuine ring singularities.

An important outcome of our consideration is a surprisingly simple form of the neutron star metric (9) in the equatorial plane ($z = 0$). Thus, for the metric functions f and ω taking part in the analysis of the behavior of test particles and study of various phenomena occurring in this plane, formulas (9) give us the following simple expressions:

$$\begin{aligned} & f = \frac{A-B}{A+B}, \qquad \omega = -\frac{W}{A+B}, \\ & A = (m+d)r_+ - (m-d)r_-, \qquad B = 2md, \qquad W = 4mad, \\ & r_\pm = \sqrt{\rho^2 + \frac{1}{2}(m^2 + 2\kappa \pm md)}, \qquad d = \sqrt{m^2 + 4(\kappa + a^2 - \mu^2)}, \end{aligned} \tag{55}$$

which are by far simpler than the analogous expressions for f and ω obtained in the papers [11,22] (pure vacuum case) and [54] (the electrovac case); (55) also improve and generalize the "equatorial" formulas of the paper [52] for which the possibility of further simplifications was previously overlooked. Actually, the above formulas (55) are practically as simple as in the case of the Kerr solution due to the linear dependence of A on $r_\pm$ and constant values of B and W, the latter black hole case being contained in (55) just as the $d = m$ (i.e., $\kappa = -a^2$, $\mu = 0$) specialization. At the same time, compared to the Kerr case, Eqs. (55) represent much more generic sources, as they contain two additional independent parameters κ and μ that allow one to take account of an arbitrary quadrupole mass deformation and of the dipole magnetic field of neutron stars. As a curiosity, we find it worth noting that the constant object d entering (55) can take arbitrary real or pure imaginary values, leaving the functions f and ω to be real-valued in both cases.

It is also remarkable that the solution (7) can be considered as providing a nontrivial evidence in favor of the recent claim [55,56] that the component A_φ of the electromagnetic four-potential does not describe correctly the magnetic field of the Einstein-Maxwell spacetimes. Although in the aforementioned papers the claim was made about the asymptotically nonflat potential A_φ linked to the magnetic charge and possessing two semi-infinite singularities, so that the claim looks quite natural, the function A_φ of the solution (7), on the other hand, is defined by formula (10) and is asymptotically flat, thus presumably looking well-behaved and reflecting correctly the properties of the metric (9). However, our study of the singularities of the solution (7) in the equatorial plane makes it possible to establish a not very conspicuous inconsistency between the singularity structures of the metric (9) and the corresponding potential A_φ. Indeed, it follows from the formulas (7), (9), and (10) that in the equatorial plane the potential A_φ takes the form

$$A_\varphi = \frac{2m\mu(r_+ - r_- + d)}{(d-m)r_+ + (d+m)r_-}, \tag{56}$$

where $r_\pm$ and d are the same as in (55). The above formula (56) means that the simplification of the expression of A_φ at $z = 0$ occurs differently than in the case of the functions f and ω: while the formulas (55) have been obtained after canceling out the common factor $(d - m)r_+ + (d+m)r_-$ in the numerators and denominators of f and ω, the common factor leading to (56) is $(m+d)r_+ - (m-d)r_- + 2md$. As a result, the singularity structure of A_φ in (56) is defined by the roots of Eq. (14), whereas the singularities of f and ω in (55) emerge as the roots of Eq. (15). Therefore, the singularity structure of A_φ turns out to be inconsistent with that of the metric functions f and ω because, as we have already mentioned, the singularity (16) satisfies the condition (14) for the positive values of m too. The correct description of the magnetic field in the solution (7) is provided by the well-behaved t component $B_t = \mathrm{Im}(\Phi)$ of the dual electromagnetic four-potential B_μ which is devoid of the undesirable singularities of the potential A_φ since, according to (17), B_t vanishes in the equatorial plane.

As a final remark we would like to mention that the similarity in the singularity structures of the black-hole and NS solutions looks also extending to their multipole structures. Indeed, the complex moments m_n determining the mass and angular momentum distributions of the NS solution obtainable from (1) and (4) have a very simple form

$$m_{2n} = m\kappa^n, \qquad m_{2n+1} = ima\kappa^n, \quad n = 0, 1, 2, \ldots \tag{57}$$

quite comparable in simplicity with those of the Kerr solution, while for the electromagnetic moments q_n arising as coefficients in the expansion ($z \to \infty$)

$$\eta(z) = \frac{2f(z)}{1 + e(z)} = \sum_{n=0}^{\infty} \frac{q_n}{z^{n+1}}, \tag{58}$$

we get from (4) and (58) the expressions

$$q_{2n} = 0, \qquad q_{2n+1} = im\mu\kappa^n, \quad n = 0, 1, 2, \ldots \tag{59}$$

which even exceed in simplicity the corresponding q_n of the Kerr-Newman black hole whose q_{2n} are all nonzero.

ACKNOWLEDGMENTS

We are thankful to the referees for valuable comments and suggestions that helped us improve the presentation. One of us (V. S. M.) would like to thank Erasmo Gómez for technical computer assistance. This work was partially supported by CONAHCyT of Mexico, and by Grant PID2021-122938NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe".

[1] R. P. Kerr, Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics, Phys. Rev. Lett. **11**, 237 (1963).
[2] B. Bonga and H. Yang, Mimicking Kerr's multipole moments, Phys. Rev. D **104**, 084040 (2021).
[3] R. Geroch, Multipole moments. II. Curved space, J. Math. Phys. (N.Y.) **11**, 2580 (1970).
[4] R. O. Hansen, Multipole moments of stationary space-times, J. Math. Phys. (N.Y.) **15**, 46 (1974).
[5] G. Fodor, C. Hoenselaers, and Z. Perjés, Multipole moments of axisymmetric systems in relativity, J. Math. Phys. (N.Y.) **30**, 2252 (1989).
[6] N. R. Sibgatullin, *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields* (Springer, Berlin, 1991).
[7] V. S. Manko and N. R. Sibgatullin, Construction of exact solutions of the Einstein-Maxwell equations corresponding to a given behaviour of the Ernst potentials on the symmetry axis, Classical Quantum Gravity **10**, 1383 (1993).
[8] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[9] W. G. Laarakkers and E. Poisson, Quadrupole moments of rotating neutron stars, Astrophys. J. **512**, 282 (1999).
[10] G. B. Cook, S. L. Shapiro, and S. A. Teukolsky, Rapidly rotating neutron stars in general relativity: Realistic equations of state, Astrophys. J. **424**, 823 (1994).
[11] N. R. Sibgatullin and R. A. Sunyaev, Disc accretion in the gravitatiional field of a rapidly rotating neutron star with a rotationally induced quadrupole mass moment, Astron. Lett. **24**, 774 (1998).
[12] N. R. Sibgatullin and R. A. Sunyaev, Energy release during disk accretion onto a rapidly rotating neutron star, Astron. Lett. **26**, 699 (2000).
[13] V. S. Manko, E. W. Mielke, and J. D. Sanabria-Gómez, Exact solution for the exterior field of a rotating neutron star, Phys. Rev. D **61**, 081501R (2000).
[14] E. Berti and N. Stergioulas, Approximate matching of analytic and numerical solutions for rapidly rotating neutron stars, Mon. Not. R. Astron. Soc. **350**, 1416 (2004).
[15] N. Stergioulas and J. L. Friedman, Comparing models of rapidly rotating relativistic stars constructed by two numerical methods, Astrophys. J. **444**, 306 (1995).
[16] N. Stergioulas, Rotating stars in relativity, Living Rev. Relativity **6**, 3 (2003).
[17] G. Pappas and T. A. Apostolatos, Revising the Multipole Moments of Numerical Spacetimes and its Consequences, Phys. Rev. Lett. **108**, 231104 (2012).
[18] G. Pappas, What can quasi-periodic oscillations tell us about the structure of the corresponding compact objects?, Mon. Not. R. Astron. Soc. **422**, 2581 (2012).
[19] K. Yagi, K. Kyutoku, G. Pappas, N. Yunes, and T. A. Apostolatos, Effective no-hair relations for neutron stars and quark stars: Relativistic results, Phys. Rev. D **89**, 124013 (2014).
[20] V. S. Manko, J. Martín, and E. Ruiz, Six-parameter solution of the Einstein-Maxwell equations possessing equatorial symmetry, J. Math. Phys. (N.Y.) **36**, 3063 (1995).
[21] G. Pappas and T. A. Apostolatos, An all-purpose metric for the exterior of any kind of rotating neutron star, Mon. Not. R. Astron. Soc. **429**, 3007 (2013).
[22] V. S. Manko and E. Ruiz, Exterior field of slowly and rapidly rotating neutron stars: Rehabilitating spacetime metrics involving hyperextreme objects, Phys. Rev. D **93**, 104051 (2016).
[23] V. S. Manko and E. Ruiz, A note on hierarchy of universal relations for neutron stars in terms of multipole moments, Classical Quantum Gravity **36**, 147002 (2019).
[24] V. S. Manko and E. Ruiz, Extended multi-soliton solutions of the Einstein field equations, Classical Quantum Gravity **15**, 2007 (1998).
[25] W. Simon, The multipole expansion of stationary Einstein-Maxwell fields, J. Math. Phys. (N.Y.) **25**, 1035 (1984).
[26] C. Hoenselaers and Z. Perjés, Multipole moments of axisymmetric electrovacuum spacetimes, Classical Quantum Gravity **7**, 1819 (1990).
[27] T. P. Sotiriou and T. A. Apostolatos, Corrections and comments on the multipole moments of axisymmetric electrovacuum spacetimes, Classical Quantum Gravity **21**, 5727 (2004).

[28] G. Fodor, E. S. Costa Filho, and B. Hartmann, Calculation of multipole moments of axistationary electrovacuum spacetimes, Phys. Rev. D **104**, 064012 (2021).

[29] V. S. Manko, I. M. Mejía, and E. Ruiz, Metric of a rotating charged magnetized sphere, Phys. Lett. B **803**, 135286 (2020).

[30] A. Tomimatsu and H. Sato, New Exact Solution for the Gravitational Field of a Spinning Mass, Phys. Rev. Lett. **29**, 1344 (1972).

[31] D. Kramer and G. Neugebauer, The superposition of two Kerr solutions, Phys. Lett. **75A**, 259 (1980).

[32] G. W. Gibbons and R. A. Russel-Clark, Note on the Tomimatsu-Sato Solution of Einstein's Equations, Phys. Rev. Lett. **30**, 398 (1973).

[33] V. S. Manko, On the physical interpretation of $\delta = 2$ Tomimatsu-Sato solution, Prog. Theor. Phys. **127**, 1057 (2012).

[34] O. V. Manko, V. S. Manko, and J. D. Sanabria-Gómez, Remarks on the charged, magnetized Tomimatsu-Sato $\delta = 2$ solution, Gen. Relativ. Gravit. **31**, 1539 (1999).

[35] V. S. Manko, E. Ruiz, and J. D. Sanabria-Gómez, Extended multi-soliton solutions of the Einstein field equations: II. Two comments on the existence of equilibrium states, Classical Quantum Gravity **17**, 3881 (2000).

[36] J. A. Rueda, V. S. Manko, E. Ruiz, and J. D. Sanabria-Gómez, The double-Kerr equilibrium configurations involving one extreme object, Classical Quantum Gravity **22**, 4887 (2005).

[37] I. Cabrera-Munguia, V. S. Manko, and E. Ruiz, Remarks on the mass-angular momentum relations for two extreme Kerr sources in equilibrium, Phys. Rev. D **82**, 124042 (2010).

[38] V. S. Manko, E. Ruiz, and M. B. Sadovnikova, Stationary configurations of two extreme black holes obtainable from the Kinnersley-Chitre solution, Phys. Rev. D **84**, 064005 (2011).

[39] A. Komar, Covariant conservation laws in general relativity, Phys. Rev. **113**, 934 (1959).

[40] G. W. Gibbons, S. A. Hartnoll, and A. Ishibashi, On the stability of naked singularities with negative mass, Prog. Theor. Phys. **113**, 963 (2005).

[41] R. J. Gleiser and G. Dotti, Instability of the negative mass Schwarzschild naked singularity, Classical Quantum Gravity **23**, 5063 (2006).

[42] V. S. Manko and E. Ruiz, On a simple representation of the Kinnersley-Chitre metric, Prog. Theor. Phys. **125**, 1241 (2011).

[43] V. S. Manko and E. Ruiz, Simple metric for a magnetized, spinning, deformed mass, Phys. Rev. D **97**, 104016 (2018).

[44] F. J. Ernst, New formulation of the axially symmetric gravitational field problem. II, Phys. Rev. **168**, 1415 (1968).

[45] E. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating charged mass, J. Math. Phys. (N.Y.) **6**, 918 (1965).

[46] V. S. Manko and E. Ruiz, Singularities in the Kerr-Newman and charged $\delta = 2$ Tomimatsu-Sato spacetimes endowed with negative mass, Prog. Theor. Exp. Phys. **2013**, 103E01 (2013).

[47] P. Kordas, Reflection-symmetric, asymptotically flat solutions of the vacuum axistationary Einstein equations, Classical Quantum Gravity **12**, 2037 (1995).

[48] R. Meinel and G. Neugebauer, Asymptotically flat solutions to the Ernst equation with reflection symmetry, Classical Quantum Gravity **12**, 2045 (1995).

[49] L. A. Pachón and J. D. Sanabria-Gómez, Note on reflection symmetry in stationary axisymmetric electrovacuum spacetimes, Classical Quantum Gravity **23**, 3251 (2006).

[50] F. J. Ernst, V. S. Manko, and E. Ruiz, Equatorial symmetry/antisymmetry of stationary axisymmetric electrovac spacetimes, Classical Quantum Gravity **23**, 4945 (2006).

[51] F. J. Ernst, New formulation of the axially symmetric gravitational field problem, Phys. Rev. **167**, 1175 (1968).

[52] I. M. Mejía, V. S. Manko, and E. Ruiz, Simplest static and stationary vacuum quadrupolar metrics, Phys. Rev. D **100**, 124021 (2019).

[53] W. Kinnersley and D. M. Chitre, Symmetries of the stationary Einstein-Maxwell equations. IV. Transformations which preserve asymptotic flatness, J. Math. Phys. (N.Y.) **19**, 2037 (1978).

[54] V. S. Manko, A note on magnetic generalizations of the Kerr and Kerr-Newman solutions, Classical Quantum Gravity **34**, 177002 (2017).

[55] C. J. Ramírez-Valdez, H. García-Compeán, and V. S. Manko, Dyonic black holes in the theory of two electromagnetic potentials. I, Phys. Rev. D **107**, 064016 (2023).

[56] H. García-Compeán, V. S. Manko, and C. J. Ramírez-Valdez, Dyonic black holes in the theory of two electromagnetic potentials. II, Phys. Rev. D **107**, 064017 (2023).