

Global Hadamard form for the Green function in Schwarzschild spacetime

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The retarded Green function of a wave equation on a 4-dimensional curved background spacetime is a (generalized) function of two spacetime points and diverges when these are connected by a null geodesic. The Hadamard form makes explicit the form of this divergence but only when one of the points is in a normal neighborhood of the other point. In this paper we derive a representation for the retarded Green function for a scalar field in Schwarzschild spacetime which makes explicit its *complete* singularity structure beyond the normal neighborhood. We interpret this representation as a sum of Hadamard forms, the summation being taken over the number of times the null wavefront has passed through a caustic point; the sum of Hadamard forms applies to the nonsmooth contribution to the full Green function, not only the singular contribution. (The term nonsmooth applies modulo the causality-generating step functions that must appear in the retarded Green function.) The singularity structure is determined using two independent approaches, one based on a Bessel function expansion of the Green function, and another that exploits a link between the Green functions of Schwarzschild spacetime and Plebański-Hacyan spacetime (the latter approach also yields another representation for the *full* Schwarzschild Green function, not just for its nonsmooth part). Our representation is not valid in a neighborhood of caustic points. We deal with these points by providing a separate representation for the Green function in Schwarzschild spacetime which makes explicit its (different) singularity structure at caustics of this spacetime.

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I. INTRODUCTION

A fundamental object for the study of linear field perturbations of a curved spacetime is the retarded Green function (GF) of the wave equation satisfied by the perturbation. Heuristically, the GF may be viewed as the value of the field at a spacetime point resulting from the propagation of an ‘impulsive’ source at a base point. The global—not just local—behavior of the GF is useful, for example, for determining the evolution of initial data via a Kirchhoff integral [1], for determining the self-force acting on a particle that is moving on a background spacetime via the MiSaTaQuWa equation [2] and for determining the probability of a quantum particle detector being excited by a field emitted by another detector [3,4].

Based on the seminal work by Hadamard [5], an analytic expression is known for the GF, $G_R(x, x')$, which is valid within a normal neighborhood $\mathcal{N}(x)$ of the base spacetime point x [i.e., a region $\mathcal{N}(x)$ containing x such that every

$x' \in \mathcal{N}(x)$ is connected to x by a unique geodesic which lies in $\mathcal{N}(x)$]. In the case of a scalar field on a $(3+1)$ -dimensional spacetime, this Hadamard form is [6]

$$G_R(x, x') = [U_{4d}(x, x')\delta(\sigma_{4d}) + V_{4d}(x, x')\theta(-\sigma_{4d})]\theta_+(x, x'), \quad (1.1)$$

where δ and θ are, respectively, the Dirac-delta and Heaviside distributions, U_{4d} and V_{4d} are smooth biscalars, and $\theta_+(x, x')$ equals 1 if x' lies to the causal future of x and equals 0 otherwise. Here, $\sigma_{4d} = \sigma_{4d}(x, x')$ is Synge’s world function, i.e., one-half of the squared distance along the (unique) geodesic connecting x and x' . Equation (1.1) explicitly shows that, in a $(3+1)$ -dimensional spacetime, the GF has a Dirac-delta divergence at points $x' \in \mathcal{N}(x)$ that are connected to x via a null geodesic. The first term, $U_{4d}\delta$, and the second term, $V_{4d}\theta$, on the right-hand side of Eq. (1.1) are usually called the ‘direct’ and ‘tail’ parts, respectively.

It is further known [7–9] that, outside a normal neighborhood, the GF continues to diverge when the two spacetime points are connected via a null geodesic. The explicit form of the singularity *outside* a normal neighborhood, however, was not known within General Relativity

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until recently. Using a variety of methods, it has been shown [10–16] that the *global* form of the ‘leading’ singularity of the GF generally has the following fourfold structure in Schwarzschild, Kerr and other background spacetimes:

$$\begin{aligned} \delta(\sigma_{4d}) &\rightarrow \text{PV}(1/\sigma_{4d}) \rightarrow -\delta(\sigma_{4d}) \rightarrow -\text{PV}(1/\sigma_{4d}) \\ &\rightarrow \delta(\sigma_{4d}) \rightarrow \dots, \end{aligned} \quad (1.2)$$

where the first term corresponds to the direct part in Eq. (1.1) and PV denotes the Cauchy principal value distribution. This change in the character of the singularity is essentially due to the null wavefront of the field perturbation passing through a caustic point (i.e., a space-time point where neighboring null geodesics are focused). This is indicated in Eq. (1.2) by the arrow ‘ \rightarrow ’. Thus the *leading* singularity in $G_R(x, x')$ has the form $\delta(\sigma_{4d}(x, x'))$ [respectively, $\text{PV}(1/\sigma_{4d})$, $-\delta(\sigma_{4d})$, $-\text{PV}(1/\sigma_{4d})$] when there is a null geodesic from x to x' that has passed through $4n$ (respectively, $4n+1$, $4n+2$, $4n+3$) caustics, where n is a non-negative integer. The four-fold structure of the GF in Schwarzschild spacetime is beautifully illustrated with numerical animations in [17] and has been used in [11,18,19] in order to provide an insight into the origin of the self-force and in [3] and [20] in order to account for interesting features in respectively, the communication and entanglement between quantum particle detectors.

The above fourfold singularity structure of the GF, however, only represents its ‘leading’ singularity. In [14], we proved that there is also a ‘subleading’ discontinuity in the case of a black hole toy-model spacetime (Plebański-Hacyan spacetime, $\mathbb{M}_2 \times \mathbb{S}_2$, abbreviated as PH below) which displays another fourfold structure,

$$\begin{aligned} \theta(-\sigma_{4d}) &\rightarrow -\ln|\sigma_{4d}| \rightarrow -\theta(-\sigma_{4d}) \rightarrow \ln|\sigma_{4d}| \\ &\rightarrow \theta(-\sigma_{4d}) \rightarrow \dots, \end{aligned} \quad (1.3)$$

where the first term corresponds to the tail part in Eq. (1.1). To the best of our knowledge, such ‘subleading’ fourfold structure has not yet been shown on an actual black hole spacetime. In this paper we derive explicit forms for both the leading and subleading discontinuities of the GF of the massless scalar wave equation on Schwarzschild spacetime. Furthermore, although we have written the above global four-fold structures in terms of a world function σ_{4d} , this object is only well-defined in a geodesically convex domain; a region in which all pairs of points are connected by a unique geodesic. In this paper we write the fourfold structures in Schwarzschild spacetime in terms of a globally well-defined generalization of the world function $\hat{\sigma}$ of a spacetime that is conformally related to Schwarzschild.

Our derivation is underpinned by a simple conformal transformation of the Schwarzschild metric to a direct product spacetime, $\mathcal{M}_2 \times \mathbb{S}_2$, where \mathbb{S}_2 is the two-sphere

and \mathcal{M}_2 is a 2-dimensional Lorentzian spacetime (containing the time and radial variables). In a separate paper [21] we have shown that \mathcal{M}_2 is a causal domain (i.e., is geodesically convex, and obeys a certain causality condition) [6], which implies that its corresponding world function is valid globally. This allows us to write a *global* representation of the GF in Schwarzschild spacetime. This involves a sum over angular modes, indexed by the multipolar number ℓ , of *globally well-defined* 2-dimensional Green functions G_ℓ in the 2-dimensional spacetime \mathcal{M}_2 [see Eq. (2.24) below] multiplied by Legendre polynomials. When we use a series representation of the Riemann function \mathcal{U}_ℓ associated with G_ℓ (that is, \mathcal{U}_ℓ is the coefficient of the Heaviside step function in the 2-dimensional Green function) we obtain a form for the GF in Schwarzschild which makes explicit its *complete* fourfold singularity structure in terms of the distributions in Eqs. (1.2) and (1.3)) (with the argument σ_{4d} replaced by its globalized companion). The series representation of \mathcal{U}_ℓ is obtained by applying a theorem of Zauderer [22], and yields an expression in terms of an infinite series involving Bessel functions with coefficients coming from the Hadamard series (i.e., a series in powers of the world function) of the Riemann function of a certain ‘background’ 2-dimensional wave equation. The Bessel series converges whenever the Hadamard series converges. We then expand for large- ℓ ; resumming yields an expression for the GF in Schwarzschild. This representation for the GF in Schwarzschild is valid whenever the Hadamard series of the 2-dimensional Riemann function U associated with the background 2-dimensional wave equation convergence. This method is essentially the same as the one we used in Sec. V [14]. Our large- ℓ expansion is also in the same spirit as [12,16], although these works used a further separation in the time variable (via a Fourier transform) and expansion of the GF in terms of the so-called quasinormal modes,¹ instead of the Green function in the 2-dimensional spacetime \mathcal{M}_2 that we employ.

Furthermore: In any spacetime dimension, one can write the biscalars in the Hadamard form as a Hadamard series. The coefficients of the distributions in our representation for the GF in Schwarzschild are given in terms of the coefficients in the Hadamard series for the Riemann function U in \mathcal{M}_2 . Therefore, our representation of the GF has a direct geometrical interpretation in terms of geodesics in this 2-dimensional causal domain. Our representation for the GF naturally takes the form of a sum of expansions about each of the null geodesics in Schwarzschild. In effect, we give an extension of the local Hadamard form for the GF in Schwarzschild spacetime, valid beyond normal neighborhoods, which may be characterized as a ‘sum of Hadamard forms’.

¹The contribution from the branch cut that the GF has in the complex-frequency plane (see, e.g., [1,18,23]) was thus neglected in [12,16].

This representation for the GF is, however, not valid in a neighborhood of caustic points, for which the angle separation γ is equal to 0 or π [specifically, this global representation is valid for $\gamma \in (0, \gamma_0)$, where $\gamma_0 = 2(\sqrt{2} - 1)\pi \simeq 0.828\pi$]. Indeed, it is known [14,15] that on caustic points in a spherically symmetric spacetime the above fourfold structure does not generally apply and is instead a twofold structure. However, to the best of our knowledge, the analytical form of the singularity of the GF on caustics is not known in Schwarzschild spacetime and we derive its precise form here. We thus provide a *complete* description of the singularity structure of the GF.

As an illustration of the usefulness and the properties of the mentioned Bessel function representation, we calculate and plot (in Fig. 1) its leading order term (that is after carrying out an ℓ -mode decomposition but before carrying the full expansion for large- ℓ) and compare it with an ‘exact’ calculation of the GF obtained using an independent method developed in [18].

As mentioned above, the singularity structure and ‘sum of Hadamard forms’ is valid whenever the Hadamard series of a certain wave equation on \mathcal{M}_2 converges. In this paper we further present numerical studies which provide strong evidence that convergence holds on large domains of \mathcal{M}_2 . It is an open question as to whether or not convergence holds throughout \mathcal{M}_2 ; convergence would imply global-in-time validity of our results.

Specifically, we compare our analytical expressions for the global divergences of the GF against an ‘exact’ semianalytical calculation of the full GF. We find remarkable agreement for points x' timelike-separated from x up to distances which are quite ‘far’ from it. This indicates that the region where the Hadamard form converges uniformly is either equal to the whole of \mathcal{M}_2 or, at least, that it includes regions that correspond to extending well beyond the normal neighborhood of a point in the 4-dimensional Schwarzschild spacetime. Furthermore, for points x' ‘near’ x , our expression in terms of geometrical quantities in \mathcal{M}_2 for the direct divergence in the Hadamard form (1.1) has been successfully used in the works [3,24,25] to greatly facilitate the practical calculation of the GF. Thus, these works corroborate our useful expression for the direct divergence.

We also provide another, separate representation for the GF in Schwarzschild spacetime. This other representation also starts from the Bessel series but, instead of carrying out a large- ℓ expansion of its summands, we essentially express the full ℓ -sum of the ℓ -dependent factor in the summands as integrals of the GF in PH spacetime. This new representation not only is of value in itself (in that it offers a way of calculating the GF in Schwarzschild partly via the GF in PH, which is much easier to calculate) but also we use it to derive the global singularity structure of the GF in Schwarzschild in an alternative way from that mentioned above (namely, via a large- ℓ expansion of the

modes of the GF in Schwarzschild), thus offering a check of our results.

For the reader who is only interested in our main results rather than the details of the calculation, we here note our main equations. Equation (4.84) [see also (3.19)] is our sum of Hadamard forms for the GF in Schwarzschild spacetime, which explicitly shows the full structure of the divergences when the points are connected by a null geodesic. As mentioned, this form is not valid at caustics, where $\gamma = 0$ or π . We address this in Eq. (5.28), which gives the explicit full form of the divergences of the GF at caustics in the case $\gamma = 0$; Eq. (5.32) gives the structure in the case of antipodal points, $\gamma = \pi$. Finally, Eq. (3.10) is the expression for the GF in Schwarzschild in terms of integrals of the GF in PH, and (3.17) is a version of it after using the Hadamard form for the GF in PH.

The rest of this paper is organized as follows. In Sec. II we define the retarded GF in Schwarzschild spacetime and express it in terms of GFs in the 2D conformal spacetime. We introduce the key background 2-dimensional wave equation, and show how all 2-dimensional GFs derive from this seed equation. More precisely, the 2-dimensional Riemann functions mentioned above can be written as a series of Hadamard coefficients of this seed equation, scaled by Bessel functions $J_k((\ell + \frac{1}{2})\sigma)$, $k \geq 0$ where ℓ is the multipole index and σ is the globally defined world function of the 2-dimensional conformal Schwarzschild spacetime. This follows from a theorem of Zauderer ([22]; see also Theorem 6.4.2 of [6]), and we will refer to this as the Bessel expansion of the Riemann (or Green) function. This global representation of the 4-dimensional GF is then analyzed in the following sections, with the aim of determining its global (singularity) structure in a way that relates to the underlying causal structure of the spacetime.

In Sec. III, we show how the global-retarded GF of Schwarzschild spacetime may be derived by a sequence of iterated integrals of the corresponding GF of PH spacetime. This yields our first representations of the global singularity structure and of the ‘sum of Hadamard forms’ of the retarded GF of Schwarzschild spacetime.

In Sec. IV we apply a large- ℓ expansion of the Bessel functions arising in the Bessel series obtained in Sec. II to determine the global singularity structure of the retarded GF G_R on Schwarzschild spacetime, and to determine the ‘sum over Hadamard forms’ expansion of the nonsmooth contribution to G_R . This approach provides more detail on the distributions that contribute to this sum, relative to the approach of the previous section. In Sec. V, we derive the twofold singularity structure in the case of caustics in Schwarzschild spacetime. In Sec. VI we provide numerical evidence for our results. We conclude in Sec. VII with some comments and suggestions for possible applications of our results. In Appendix A, as an illustration of the multipolar modes of the GF, we calculate these in two simple $(3 + 1)$ -dimensional spacetimes; flat spacetime and Nariai

spacetime. In Appendix B we consider the zero mass ($M = 0$) limit of the results of this paper (taking us from Schwarzschild spacetime to Minkowski spacetime). This allows us to consider the question of convergence of the key 2-dimensional Hadamard series, and to draw some links between GFs on 2-dimensional anti-de Sitter spacetime AdS_2 , PH spacetime, and results relating to the representation of Legendre functions in terms of series of Bessel functions. Finally, in Appendix C, we show how our results for the Schwarzschild GF could be used to calculate the regularized self-field (i.e., the regularized value of the scalar field created by a scalar point charge evaluated on the location of the charge itself)—see [25]. This self-field is relevant to self-force calculations of radiation reaction.

Throughout this paper we use geometric units $c = G = 1$ and metric signature $(- + + +)$.

II. GREEN FUNCTION ON SCHWARZSCHILD SPACETIME

The perturbations by a massless scalar field Φ of a background spacetime \mathcal{M} satisfy a scalar (Klein-Gordon) wave equation. The retarded Green function $G_R(x, x')$ satisfies the inhomogeneous wave equation with a Dirac-delta distribution source, together with a boundary condition ensuring that $G_R(x, x')$ vanishes if x' is not in the causal future $J^+(x)$ of the point x , where $x, x' \in \mathcal{M}$. Thus,

$$\square G_R(x, x') = -4\pi\delta_4(x, x'), \quad G_R = 0 \quad \text{if } x' \notin J^+(x), \quad (2.1)$$

where $\square = \nabla_\alpha \nabla^\alpha$ is the d'Alembertian operator, $\delta_4(x, x') \equiv \frac{\delta_4(x-x')}{\sqrt{-g(x)}}$ and g is the determinant of the metric of the background spacetime.

In the Introduction we gave the Hadamard form for the retarded Green function (GF) in a $(3+1)$ -dimensional spacetime, Eq. (1.1). As mentioned, the great advantage of the Hadamard form is that it makes explicit the form of the singularity of the GF. This follows from the fact that Synge's world function $\sigma_{4d}(x, x')$ is positive/zero/negative if x and x' are, respectively, spacelike/null/timelike separated. The main disadvantage of the Hadamard form is that it is only valid in a normal neighborhood of x , i.e., for $x' \in \mathcal{N}(x)$. In many situations, however, it is very valuable to know the GF globally. In order to obtain a global representation for the GF in Schwarzschild spacetime where the form of its singularities becomes explicit, we will make use of the spherical symmetry of the spacetime.

We shall use the usual time coordinate of the Schwarzschild exterior (generated by the timelike Killing vector) and the tortoise radial coordinate $r_* \in (-\infty, +\infty)$ so that the line element has the form

$$ds^2 = -f(dt^2 - dr_*^2) + r^2 d\Omega_2^2, \quad (2.2)$$

where $f = f(r) \equiv 1 - 2M/r$ and $d\Omega_2^2$ is the standard line element of the unit 2-sphere. The area radius r and tortoise coordinate r_* are related by

$$\frac{dr}{dr_*} = f. \quad (2.3)$$

In these coordinates, the wave equation for G_R reads

$$\begin{aligned} -f^{-1} \frac{\partial^2 G_R}{\partial t^2} + f^{-1} \frac{\partial^2 G_R}{\partial r_*^2} + \frac{2}{r} \frac{\partial G_R}{\partial r_*} + \frac{1}{r^2} \nabla^2 G_R \\ = -\frac{4\pi}{r^2 f} \delta_2(x^A - x'^A) \delta_{\mathbb{S}_2}(x^a, x'^a), \end{aligned} \quad (2.4)$$

where ∇^2 is the Laplacian operator on the unit 2-sphere, $x^A = (t, r_*)$ are coordinates on the Lorentzian 2-space (i.e., the 2-space that arises by factoring the 4-dimensional spacetime by the action of the $SO(3)$ that generates the spherical symmetry) and $x^a = (\theta, \phi)$ are coordinates on the unit 2-sphere.

At this point, it is usual to rescale the field by a factor r ; this removes the first-order derivative from the wave equation. There is also a geometrical interpretation of this step. The appropriate rescaling amounts to making a conformal transformation of the metric,

$$d\hat{s}^2 \equiv r^{-2} ds^2 = d\hat{s}_2^2 + d\Omega_2^2, \quad (2.5)$$

where

$$d\hat{s}_2^2 \equiv -\frac{f}{r^2} (dt^2 - dr_*^2). \quad (2.6)$$

We will refer to the spacetime with line element (2.5) as the *conformal Schwarzschild spacetime*, which we shall denote as $\hat{\mathcal{M}}$. In its turn, we will refer to the 2-dimensional spacetime with line element (2.6) as the *2D conformal space*, and denote it by \mathcal{M}_2 . By general properties of Green functions in conformally related spacetimes, we can write [6,26]

$$G_R = \frac{1}{r \cdot r'} \hat{G}_R(x, x'), \quad (2.7)$$

where $\hat{G}_R(x, x')$ is the retarded Green function for the conformally invariant wave equation on $\hat{\mathcal{M}}$. The conformally invariant wave equation of a general 4-dimensional spacetime with metric tensor g is

$$\square_g \Phi - \xi R \Phi = 0, \quad \xi = \frac{1}{6}, \quad (2.8)$$

where R is the Ricci scalar corresponding to the metric g . Using this rescaling, we find

$$-\frac{\partial^2 \hat{G}_R}{\partial t^2} + \frac{\partial^2 \hat{G}_R}{\partial r_*^2} + \frac{f}{r^2} \left(\nabla^2 - \frac{2M}{r} \right) \hat{G}_R = -4\pi \delta_2(x^A - x^{A'}) \delta_{\mathbb{S}_2}(x^a, x^{a'}), \quad (2.9)$$

where we have used the value $R = 12M/r$ of the Ricci scalar of $\hat{\mathcal{M}}$.

In addition to rendering the wave equation more tractable, the conformal rescaling in (2.5) introduces a very useful simplification of the world function. The direct product structure of the metric induced by (2.5) yields

$$\hat{\sigma} = \sigma(x^A, x^{A'}) + \frac{1}{2} \gamma^2. \quad (2.10)$$

Here, $\hat{\sigma}$ is the world function of the conformal Schwarzschild spacetime $\hat{\mathcal{M}}$, $\sigma(x^A, x^{A'})$ is the world function of the 2-dimensional Lorentzian spacetime \mathcal{M}_2 and $\gamma \in [0, \pi]$ is the geodesic distance on the unit 2-sphere,

$$\gamma(x^a, x^{a'}) = \text{proper distance along the shortest path from } x^a \text{ to } x^{a'} \text{ on } \mathbb{S}_2. \quad (2.11)$$

In a previous paper [21], we proved that \mathcal{M}_2 is a *causal domain*. This means, in particular, that each pair of points of this 2D spacetime are joined by a unique geodesic. As a consequence, σ is defined globally on \mathcal{M}_2 , in contrast with the usual situation in four dimensions; e.g., there is no base point p of conformal Schwarzschild spacetime for which the corresponding maximal normal neighborhood is the whole spacetime, and consequently $\hat{\sigma}$ is not defined globally on this spacetime. This technical point underpins the present paper, where we use a 2 + 2 approach to determine certain global properties of the GF on Schwarzschild spacetime.

There is an immediate payoff in terms of understanding the global causal structure of Schwarzschild spacetime. The world function σ of the 2-D spacetime \mathcal{M}_2 satisfies

$$\nabla_A \sigma \nabla^A \sigma = 2\sigma, \quad (2.12)$$

with the initial conditions $\lim_{x' \rightarrow x} \sigma(x, x') = 0$ and $\lim_{x' \rightarrow x} \nabla_A \sigma(x, x') = g_{AB}(x)$. As noted, there is a unique geodesic connecting any given pair of points in \mathcal{M}_2 . Furthermore, any geodesic of the 4D spacetime $\hat{\mathcal{M}}$ decomposes as a geodesic on \mathcal{M}_2 and a geodesic on \mathbb{S}_2 . That is, if $I \subseteq \mathbb{R}$ is an interval and

$$c: I \rightarrow \hat{\mathcal{M}}: s \mapsto x^\alpha(s) = (x^A(s), x^a(s)), \quad (2.13)$$

is a geodesic on $\hat{\mathcal{M}}$, then

$$\begin{aligned} c_1: I &\rightarrow \mathcal{M}_2: s \mapsto x^A(s), \\ c_2: I &\rightarrow \mathbb{S}_2: s \mapsto x^a(s), \end{aligned}$$

are geodesics on \mathcal{M}_2 and \mathbb{S}_2 respectively (the converse statement also holds). We refer to c_1 as the projection of the geodesic c onto \mathcal{M}_2 ; there is a unique c_1 for a given geodesic c of $\hat{\mathcal{M}}$. Now consider any pair of points $x^\alpha = (x^A, x^a), x^{\alpha'} = (x^{A'}, x^{a'})$ of $\hat{\mathcal{M}}$. As proven in [21], there is a unique geodesic of \mathcal{M}_2 connecting x^A and $x^{A'}$. When x^a and $x^{a'}$ are neither antipodal points nor the same point (for which we would have $\gamma = \pi$ and $\gamma = 0$, respectively), there is a countably infinite family of geodesics of \mathbb{S}_2 connecting x^a and $x^{a'}$, corresponding to multiple circuits of the appropriate great circle of the sphere. By lifting these geodesics from \mathcal{M}_2 and \mathbb{S}_2 , we see that any pair of points of $\hat{\mathcal{M}}$ are connected by a countably infinite family of geodesics. Each of these geodesics projects to the same geodesic of \mathcal{M}_2 . This resolves the question of the existence and multiplicity of geodesics on $\hat{\mathcal{M}}$.

Then, by conformal invariance of null geodesics, a null geodesic connects x and x' in Schwarzschild spacetime if and only if a null geodesic connects the corresponding points of the conformal Schwarzschild spacetime. This holds if and only if $\hat{\sigma}_k = 0$, for some $k \in \mathbb{Z}$, where

$$\hat{\sigma}_k \equiv \sigma + \frac{1}{2}(\gamma + 2k\pi)^2. \quad (2.14)$$

In this formula, $(\gamma + 2\pi k)^2$ is the square of the total proper distance elapsed along the projection of the geodesic onto the 2-sphere. For $k \geq 0$, the geodesic crosses $2k = 2|k|$ caustics at $\gamma = 0$ and $\gamma = \pi$. For $k < 0$, the geodesic has crossed $2|k| - 1$ caustics. Thus, the number of caustics crossed is given by

$$C_k \equiv \begin{cases} 2k, & k \geq 0; \\ 2|k| - 1, & k < 0. \end{cases} \quad (2.15)$$

We note that when the separation on \mathcal{M}_2 is timelike or null, we may write (2.14) as

$$\hat{\sigma}_k = -\frac{1}{2}\eta^2 + \frac{1}{2}(\gamma + 2k\pi)^2, \quad k \in \mathbb{Z}, \quad (2.16)$$

where η is the geodesic distance (along causal geodesics) on \mathcal{M}_2 (so that $\sigma = -\frac{1}{2}\eta^2$ and, particular, it is proper time in \mathcal{M}_2 for timelike separations and is zero for null separations). This ‘globalizes’ the world function $\hat{\sigma}$ on the 4D conformal Schwarzschild spacetime; $\hat{\sigma}_k$ is one-half of the square of the geodesic distance between any two causally-related points in conformal Schwarzschild spacetime along a timelike or null geodesic which has passed through C_k points with $\gamma = 0$ or $\gamma = \pi$ (both of which are caustics in the case of a null geodesic).

Returning to the Green functions, we separate the angle variables in the usual way via a multipolar decomposition and write

$$\hat{G}_R(x, x') = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) G_{\ell}(x^A, x^{A'}) P_{\ell}(\cos \gamma). \quad (2.17)$$

Here, P_{ℓ} are Legendre polynomials and G_{ℓ} satisfy the partial differential equation (PDE) for a Green function on the 2D conformal space,

$$\square_2 G_{\ell} - V_{\ell}(r_*) G_{\ell} = -4\pi \frac{r^2}{f} \delta_2(x^A - x^{A'}), \quad (2.18)$$

where \square_2 is the d'Alembertian operator of the 2D conformal space and the potential is

$$V_{\ell}(r_*) \equiv \ell(\ell+1) + \frac{2M}{r}. \quad (2.19)$$

The boundary conditions that the Green function G_{ℓ} obey must be such that when they are introduced in Eq. (2.17), and use is made of Eq. (2.7), the resulting Green function G_R is the *retarded* Green function of Schwarzschild spacetime. In fact, $G_{\ell}(x^A, x^{A'})$ must be equal to the retarded Green function of Eq. (2.18) since² it satisfies the same PDE [Eq. (2.18)] and it obeys the defining boundary conditions of a retarded Green function, as we now show. From Eq. (2.17) and the orthogonality properties of the Legendre polynomials it follows that

$$G_{\ell}(x^A, x^{A'}) = 2\pi \int_{-1}^{+1} d(\cos \gamma) P_{\ell}(\cos \gamma) \hat{G}_R(x, x'). \quad (2.20)$$

Now, from Eq. (2.14), $\sigma > 0$ implies $\hat{\sigma}_k > 0$, $\forall k \in \mathbb{Z}$. Since the retarded Green function $\hat{G}_R(x, x')$ of conformal Schwarzschild is zero if $x' \notin J^+(x)$, it follows that G_{ℓ} is zero if either $\sigma > 0$ or $\Delta t \equiv t - t' < 0$, which is the defining boundary condition of the retarded Green function in \mathcal{M}_2 . In order to illustrate the causality properties of the G_{ℓ} , in Appendix A we calculate these modes for two simple 4D spacetimes; flat and Nariai spacetimes.

In coordinates (t, r_*) , Eq. (2.18) takes the familiar form

$$-\frac{\partial^2 G_{\ell}}{\partial t^2} + \frac{\partial^2 G_{\ell}}{\partial r_*^2} - \frac{f}{r^2} V_{\ell} G_{\ell} = -4\pi \delta_2(x^A - x^{A'}). \quad (2.21)$$

Thus, the structure of G_{ℓ} , and hence \hat{G}_R and G_R can be probed using a variety of PDE techniques. In particular, there is a large body of work that exploits the amenability of (2.21) to a Fourier transform in the time coordinate. We will take an alternative approach that remains in the 2-dimensional setting and that applies (in particular) a result of Zauderer ([22]; see also Theorem 6.4.2 of [6]). This result—which we shall give explicitly below—provides an expansion of G_{ℓ} in terms of Bessel functions and

²We remind the reader that the properties described in the text define the *unique* retarded Green function—see Corollary 6.3.1 in [6].

of Hadamard coefficients for a ‘background’ wave equation. The Green function G for the background equation satisfies

$$PG = -4\pi \frac{r^2}{f} \delta_2(x^A - x^{A'}), \quad (2.22)$$

where

$$P \equiv \square_2 + \frac{1}{4} \left(1 - \frac{8M}{r} \right), \quad (2.23)$$

so that (2.18) can be written in the perturbative (but exact) form

$$(P - L^2)G_{\ell} = -4\pi \frac{r^2}{f} \delta_2(x^A - x^{A'}), \quad (2.24)$$

where $L \equiv \ell + 1/2$.

The Green function of the background wave equation (2.22) obeying retarded boundary conditions can be written in the 2-dimensional Hadamard form,

$$G(x^A, x^{A'}) = 2\pi \theta(t - t') \theta(-\sigma) U(x^A, x^{A'}), \quad (2.25)$$

where the biscalar $U(x^A, x^{A'})$ is the solution of the homogeneous equation $PU = 0$ satisfying boundary conditions derived from Eqs. (2.26) and (2.28) below (U is the so-called *Riemann function* of the operator P on \mathcal{M}_2 —see [7] and Sec. 6.2 of [6]). This biscalar exists and is uniquely defined on $\mathcal{M}_2 \times \mathcal{M}_2$, and for any point $p = x^A \in \mathcal{M}_2$, there is a neighborhood $\Omega_p \subset \mathcal{M}_2$ such that $\forall x^{A'} \in \Omega_p$ we can write (we note that we are using the conventions of [6]),

$$U(x^A, x^{A'}) = \sum_{k=0}^{\infty} U_k(x^A, x^{A'}) \frac{(-2\sigma)^k}{k!}, \quad (2.26)$$

where the series is uniformly convergent on Ω_p (see Theorem 6.2.1 [6]); this is known as the *Hadamard series*. The numerical evidence in Sec. VI provides a strong indication that Ω_p extends well beyond the region corresponding to the maximal normal neighborhood of a point of the 4-dimensional spacetime that projects to p . Furthermore, in Appendix B, we consider convergence of the series in the limiting case of zero mass ($M = 0$).

Using (2.25), (2.26) and (2.16), we have

$$G(x^A, x^{A'}) = 2\pi \theta(t - t') \theta(-\sigma) \sum_{k=0}^{\infty} U_k(x^A, x^{A'}) \frac{\eta^{2k}}{k!}. \quad (2.27)$$

The Hadamard coefficients U_k , $k \geq 0$, in \mathcal{M}_2 satisfy the following recurrence relations in the form of transport equations along the unique geodesic from x^A to $x^{A'}$ [6, 27–29]. In two dimensions, U_0 is the square root of the so-called van Vleck determinant $\Delta_{2d} = \Delta_{2d}(x^A, x^{A'})$,

$$U_0 = \Delta_{2d}^{1/2} \Leftrightarrow 2\sigma^A \nabla_A U_0 = (2 - \square_2 \sigma) U_0, \quad \lim_{x' \rightarrow x} U_0 = 1, \quad (2.28)$$

and the $U_k, k \geq 1$, are determined by solving the transport equations³

$$2\sigma^A \nabla_A U_k + (\square_2 \sigma + 2(k-1)) U_k = \frac{1}{2} P U_{k-1}, \quad k \geq 1. \quad (2.29)$$

Regularity at $\sigma = 0$ fixes constants of integration, so that the U_k are uniquely determined. Here, the operator P denotes the operator of Eq. (2.22) satisfied by G . We note that σ and U_0 depend only on the spacetime geometry, but the coefficients $U_k, k \geq 1$ depend on the details of the wave operator P . We also note that we have calculated σ and Δ_{2d} in \mathcal{M}_2 numerically in [21]. The 2D Hadamard coefficients U_k play an important part in the analysis of the GF in 4D Schwarzschild spacetime that we shall carry out in Sec. IV.

We wish to derive a globally valid form for G_R on Schwarzschild spacetime and its full singularity structure. As seen in Eqs. (2.7) and (2.17), this can be written as a sum of 2-dimensional Green functions,

$$G_R(x, x') = \frac{1}{2\pi r \cdot r'} \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) G_{\ell}(x^A, x^{A'}) P_{\ell}(\cos \gamma), \quad (2.30)$$

where the G_{ℓ} satisfy Eq. (2.24).

In Sec. IV below, we show how we can use a large- ℓ expansion to (i) determine the full-singularity structure of G_R and (ii) express the nonsmooth contribution to G_R as a ‘sum of Hadamard forms’. The resulting expressions are valid on regions of the form

$$\hat{\Omega}_P = \{Q \in \mathcal{M}_2 \times \mathbb{S}_2 : q \in \Omega_P \text{ and } \gamma(x^a, x^{a'}) \in [0, \gamma_0)\}, \quad (2.31)$$

where $P = (p, x^a) \in \mathcal{M}_2 \times \mathbb{S}_2$ and $Q = (q, x^{a'}) \in \mathcal{M}_2 \times \mathbb{S}_2$. This arises through the application of a theorem due to Zauderer [22], which is cited (with an alternative proof) as Theorem 6.4.2 of [6]. This result gives the following form for the retarded Green function G_{ℓ} on \mathcal{M}_2 of the operator $P - L^2$ of (2.24):

$$G_{\ell} = 2\pi \theta(-\sigma) \theta(\Delta t) \mathcal{U}_{\ell}(x^A, x^{A'}), \quad \mathcal{U}_{\ell}(x^A, x^{A'}) \equiv \sum_{k=0}^{\infty} U_k \left(\frac{2\eta}{L} \right)^k J_k(L\eta), \quad (2.32)$$

³We thank David Q. Aruquipa for identifying a sign error in an earlier version of Eq. (2.29).

where the J_k are Bessel functions, \mathcal{U}_{ℓ} is the Riemann function for the 2D wave equation (2.24) and the coefficients $U_k = U_k(x^A, x^{A'})$ are the 2D Hadamard coefficients of Eq. (2.26). These coefficients and the series Eq. (2.32) are defined globally on \mathcal{M}_2 . This result in (2.32) is *not* perturbative; it holds for all $L \in \mathbb{C} \setminus \{0\}$, and yields (2.27) in the limit $L \rightarrow 0$. According to Zauderer’s theorem, the series in Eq. (2.32) converges uniformly in the region Ω_p of any $p \in \mathcal{M}_2$ on which the Hadamard series Eq. (2.26) converges uniformly.

In the next sections we proceed to exploit, in different ways, Eq. (2.30) with Eq. (2.32) in order to determine properties of the GF in Schwarzschild spacetime.

III. GREEN FUNCTIONS IN SCHWARZSCHILD AND IN PLEBAŃSKI-HACYAN SPACETIME $\mathbb{M}_2 \times \mathbb{S}_2$

A. General form

We note that the GF in Schwarzschild spacetime given by (2.30) and (2.32) bears an interesting and useful relationship to the GF G_R^{PH} of Plebański-Hacyan (PH) spacetime $\mathbb{M}_2 \times \mathbb{S}_2$ (where \mathbb{M}_2 is 2-dimensional Minkowski spacetime) for a scalar field with $m^2 + 2\xi = 1/4$, where m is the mass of the field. See Eq. (134) in Ref. [14]⁴ for an expression for the latter GF,

$$G_R^{\text{PH}}(\eta_{\text{PH}}, \gamma) = \theta(-\sigma_{\mathbb{M}_2}) \theta(\Delta t) \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \times P_{\ell}(\cos \gamma) J_0(L\eta_{\text{PH}}) \\ = \theta(\eta_{\text{PH}}) \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) P_{\ell}(\cos \gamma) J_0(L\eta_{\text{PH}}), \quad (3.1)$$

where the world function of the \mathbb{M}_2 factor of PH spacetime is given by $\sigma_{\mathbb{M}_2} = -\eta_{\text{PH}}^2/2$ and η_{PH} is the geodesic distance in \mathbb{M}_2 . [In the last equality in (3.1)—and throughout the paper—we adopt the convention that proper time along future-directed timelike geodesics and the time coordinate t are cosynchronous]. We see that this expression for the GF in PH is obtained from (2.30) by removing the conformal factor $1/(r \cdot r')$, setting $U_0 = 1$ and $U_k = 0, k \geq 1$ in (2.32), and replacing η by η_{PH} . We will use this fact in Sec. V to provide some basic checks on our caustic results in Schwarzschild spacetime.

We can in fact take the connection between the GF in Schwarzschild spacetime and the GF in PH further and we next write the former as a sum of η integrals of the latter. By inserting Eq. (2.32) into Eq. (2.30) and swapping the order of the ℓ - and k -summations, we obtain

⁴We note that in the last expression in Eq. (134) in [14] there is a missing factor $\theta(-\sigma_{\mathbb{M}_2})$.

$$G_R(x, x') = \frac{\theta(\eta)}{r \cdot r'} \sum_{k=0}^{\infty} (2\eta)^k U_k(x^A, x^{A'}) G_k^{\text{PH}}(\eta, \gamma), \quad (3.2)$$

where we have defined

$$G_k^{\text{PH}}(\eta, \gamma) \equiv \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) P_{\ell}(\cos \gamma) \frac{J_k(L\eta)}{L^k}, \quad (3.3)$$

and the prefactor $\theta(-\sigma)\theta(\Delta t) = \theta(\eta)$ has been taken outside as a common factor in (3.2). We note that the GF in PH is equal to $G_R^{\text{PH}}(\eta_{\text{PH}}, \gamma) = \theta(-\sigma_{\mathbb{M}_2})\theta(\Delta t)G_0^{\text{PH}}(\eta_{\text{PH}}, \gamma)$ (this is simply the identification mentioned above between GF's in Schwarzschild and in PH under $U_0 = 1$ and $U_k = 0, k \geq 1$). In order to relate G_k^{PH} for all $k \in \mathbb{Z}_{\geq 0}$ to the GF in PH, we use Eq. (10.6.6) in [30] to first obtain,

$$\left(\int d\eta \eta \right)^k F(\eta) \equiv \int_0^{\eta} d\eta_1 \eta_1 \left(\int_0^{\eta_1} d\eta_2 \eta_2 \left(\cdots \left(\int_0^{\eta_{k-1}} d\eta_k \eta_k F(\eta_k) \right) \cdots \right) \right), \quad (3.7)$$

for an arbitrary function $F(\eta)$ for which the integrals exist. Since the only η dependence of G_k^{PH} in (3.3) is via the $J_k(L\eta)$, it readily follows that

$$\begin{aligned} G_k^{\text{PH}}(\eta, \gamma) &= \eta^{-k} \int d\eta \eta^k G_{k-1}^{\text{PH}}(\eta, \gamma) \\ &= \eta^{-k} \left(\int d\eta \eta \right)^k G_0^{\text{PH}}(\eta, \gamma). \end{aligned} \quad (3.8)$$

We then have

$$G_R(x, x') = \frac{\theta(\eta)}{r \cdot r'} \sum_{k=0}^{\infty} 2^k U_k(x^A, x^{A'}) \left(\int d\eta \eta \right)^k G_0^{\text{PH}}(\eta, \gamma). \quad (3.9)$$

Finally, we note that the GF in PH is $G_R^{\text{PH}}(\eta_{\text{PH}}, \gamma) = \theta(\eta_{\text{PH}})G_0^{\text{PH}}(\eta_{\text{PH}}, \gamma)$. It is clear from the limits of integration in (3.7) that $\eta > 0$ is equivalent to $\eta_k > 0$. Thus, we can write the GF in Schwarzschild in terms of the GF in PH as

$$G_R(x, x') = \frac{1}{r \cdot r'} \sum_{k=0}^{\infty} 2^k U_k(x^A, x^{A'}) \left(\int d\eta \eta \right)^k G_R^{\text{PH}}(\eta, \gamma). \quad (3.10)$$

We have thus expressed the GF in Schwarzschild as a suggestive sum of integrals of the GF in $\mathbb{M}_2 \times \mathbb{S}_2$, weighted by the 2-dimensional Hadamard biscalars U_k .

In the next two subsections, we will use two different results for the GF in PH which we obtained in [14] in order to make some further progress with Schwarzschild's G_R . Specifically, we will use the Hadamard form for G_R^{PH} in the

$$\frac{1}{L^k} \left(\frac{1}{\eta} \frac{d}{d\eta} \right)^k (\eta^k J_k(L\eta)) = J_0(L\eta), \quad (3.4)$$

$$\frac{1}{\eta} \frac{d}{d\eta} (\eta^k J_k(L\eta)) = L\eta^{k-1} J_{k-1}(L\eta). \quad (3.5)$$

From these, it follows that

$$\begin{aligned} \frac{J_k(L\eta)}{L^k} &= \eta^{-k} \int_0^{\eta} d\eta' (\eta')^k \frac{J_{k-1}(L\eta')}{L^{k-1}} \\ &= \eta^{-k} \left(\int d\eta \eta \right)^k J_0(L\eta), \end{aligned} \quad (3.6)$$

where we have used the shorthand notation for the iterated integral

first subsection and the globally valid singularity structure of G_R^{PH} [obtained using (3.3)] in the second subsection.

B. Hadamard form

The GF in PH also admits, of course, its own Hadamard form, valid within normal neighborhoods of points in PH,

$$G_R^{\text{PH}}(\eta_{\text{PH}}, \gamma) = \theta(\Delta t) (\Delta_{\text{PH}}^{1/2} \delta(\sigma_{\text{PH}}) + V_{\text{PH}} \theta(-\sigma_{\text{PH}})), \quad (3.11)$$

where $\sigma_{\text{PH}} = (-\eta_{\text{PH}}^2 + \gamma^2)/2$, Δ_{PH} and V_{PH} are, respectively, the world function, the van Vleck determinant and the Hadamard tail biscalar in PH. In [14] we managed to calculate, in closed form, Δ_{PH} (which is actually equal to the van Vleck determinant in the two-sphere \mathbb{S}_2) as well as the first two terms in the Hadamard series for V_{PH} , while the latter biscalar was calculated numerically in [31] for any pair of points. Specifically, we found that

$$\Delta_{\text{PH}} = \frac{\gamma}{\sin \gamma}, \quad V_{\text{PH}}(\eta_{\text{PH}}, \gamma) = \sum_{n=0}^{\infty} \nu_n(\gamma) \sigma_{\text{PH}}^n, \quad (3.12)$$

with (in the case $m^2 + 2\xi = 1/4$)

$$\nu_0(\gamma) = \frac{1}{8} \Delta_{\text{PH}}^{1/2} \left(\frac{1}{\gamma^2} - \frac{\cot \gamma}{\gamma} \right) \quad (3.13)$$

and

$$\nu_1 = \Delta_{\text{PH}}^{1/2} \frac{2\gamma^2 - 3 \csc^2(\gamma) [6\gamma^2 + 2\gamma \sin(2\gamma) + 5 \cos(2\gamma) - 5]}{256\gamma^4}. \quad (3.14)$$

The higher orders ν_n , $n > 0$, can in principle be obtained from ν_{n-1} via a recurrence relation. These coefficients are closely related to the Hadamard coefficients of a certain wave equation in 2-dimensional anti-de Sitter spacetime AdS_2 (see Appendix B), and to the coefficients that arise in an expansion of the Legendre polynomials in terms of

Bessel functions. This expansion is due to Szego [32]; see (4.23) below. We next introduce the Hadamard form (3.11) and (3.12) for G_R^{PH} in (3.10), after replacing η_{PH} by η , and so $\sigma_{\text{PH}} = (-\eta_{\text{PH}}^2 + \gamma^2)/2$ by $\hat{\sigma} = (-\eta^2 + \gamma^2)/2$. Using the integral results, valid for all $k > 0$,

$$\left(\int d\eta \eta \right)^k \theta(\Delta t) \delta(\hat{\sigma}) = \left(\int d\eta \eta \right)^k \theta(\eta) \delta(\hat{\sigma}) = \frac{(-1)^{k+1} \theta(\eta - \gamma)}{(k-1)!} \hat{\sigma}^{k-1}, \quad (3.15)$$

and

$$\left(\int d\eta \eta \right)^k \theta(\Delta t) \theta(-\hat{\sigma}) \hat{\sigma}^n = \left(\int d\eta \eta \right)^k \theta(\eta) \hat{\sigma}^n = \frac{(-1)^k \theta(\eta - \gamma) n!}{(k+n)!} \hat{\sigma}^{k+n}, \quad (3.16)$$

it readily follows that, separating out the $k = 0$ term,

$$G_R(x, x') = \frac{1}{r \cdot r'} \left\{ \theta(\Delta t) U_0(x^A, x'^A) (\Delta_{\text{PH}}^{1/2} \delta(\hat{\sigma}) + V_{\text{PH}}(\eta, \gamma) \theta(-\hat{\sigma})) \right. \\ \left. + \theta(\eta - \gamma) \sum_{k=1}^{\infty} (-1)^k 2^k U_k(x^A, x'^A) \hat{\sigma}^{k-1} \left(-\frac{\Delta_{\text{PH}}^{1/2}}{(k-1)!} + \sum_{n=0}^{\infty} \frac{n!}{(k+n)!} \nu_n(\gamma) \hat{\sigma}^{n+1} \right) \right\}, \quad (3.17)$$

where we note that $\theta(\eta - \gamma) = \theta(-\hat{\sigma}) \theta(\Delta t)$ and $V_{\text{PH}}(\eta, \gamma) = \sum_{n=0}^{\infty} \nu_n(\gamma) \hat{\sigma}^n$. The first term $\theta(\Delta t) U_0 \Delta_{\text{PH}}^{1/2} \delta(\hat{\sigma}) / (r \cdot r')$ in (3.17) is, of course, just equal [see Eqs. (8) and (17) in [25]] to the direct part $U_{4d} \theta(\Delta t) \delta(\sigma_{4d})$ in the Hadamard form for the GF in Schwarzschild. We have thus effectively written a Hadamard form for Schwarzschild's G_R where the Hadamard tail V_{4d} is manifestly written as a series in $\hat{\sigma}$, with coefficients that depend on quantities defined in 2-dimensional manifolds; \mathcal{M}_2 in the case of U_k and \mathbb{S}_2 in the case of $\nu_n(\gamma)$. This is a manifest advantage over the standard Hadamard series of V_{4d} in σ_{4d} because one can thus apply available machinery for calculating Hadamard coefficients in 4-dimensional Schwarzschild to these easier 2-dimensional cases. As for the calculational machinery, Hadamard coefficients may be calculated, for example, by solving transport equations [29]. Alternatively, one might prefer to obtain the coefficients analytically in terms of small coordinate-distance expansions (see Ref. [33] in

Schwarzschild and Ref. [25] for explicit expansions of U_0 and η in \mathcal{M}_2).

C. Singularity structure

Let us here look at the singularity structure that expression (3.10) for G_R yields. Specifically, we shall calculate G_R^{disc} , defined as the contribution to G_R for which the coefficient of $\theta(\eta)$ in (3.9) does not take the form of a continuous function on the spacetime. We obtained the equivalent discontinuous contribution to $G_R^{\text{PH}}(\eta_{\text{PH}}, \gamma)$ in Eq. (156) in Ref. [14]. By carrying out the η -integrations of $G_R^{\text{PH}}(\eta, \gamma)$ as required in (3.10), the integrals of the terms in Eq. (156) in Ref. [14] containing θ 's and \log 's of the world function yield continuous terms. Therefore, the only other discontinuity in (3.10) arises from the one integral ($k = 1$) of the terms in Eq. (156) in Ref. [14] containing δ 's and PV's of the world function,

$$\theta(\eta) \int_0^\eta d\eta' \eta' \delta(\hat{\sigma}_n(\eta', \gamma)) = \theta(\eta - \gamma - 2\pi n), \quad \forall n \in \mathbb{Z}_{\geq 0}, \\ \theta(\eta) \int_0^\eta d\eta' \eta' \text{PV}(\hat{\sigma}_{-n}(\eta', \gamma)) = \theta(\eta) \left(-\ln(\hat{\sigma}_{-n}(\eta, \gamma)) + \ln\left(\frac{(2\pi n - \gamma)^2}{2}\right) \right), \quad \forall n \in \mathbb{Z}_{\geq 1}, \quad (3.18)$$

where $\hat{\sigma}_n = \hat{\sigma}_n(\eta, \gamma)$ is given in (2.16).⁵ That is, the discontinuity G_R^{disc} arising from (3.10) is given by its $k = 0$ summand as per Eq. (156) of Ref. [14] together with the discontinuous terms in (3.18) (and so not including the $2 \ln(2\pi - \gamma)$ within the $k = 1$ summand); in the next Sec. IV A we prove that the k -sum does not bring in any new discontinuities. After some calculations, we find

$$\begin{aligned} r \cdot r' \cdot G_R^{\text{disc}} = & \theta(\eta) \left[\frac{U_0}{\sqrt{\eta \sin \gamma}} \sum_{n=0}^{\infty} (-1)^n \delta(\eta - (\gamma + 2\pi n)) + \frac{U_0}{\pi \sqrt{\sin \gamma}} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2\pi n - \gamma}} \text{PV} \left(\frac{1}{\eta + (\gamma - 2\pi n)} \right) \right. \\ & + \frac{1}{8\pi \sqrt{\sin \gamma}} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2\pi n - \gamma}} \left(U_0 \left(\frac{1}{2\pi n - \gamma} + \cot \gamma \right) + 16(2\pi n - \gamma) U_1 \right) \ln((2\pi n - \gamma) - \eta) \\ & \left. + \frac{1}{8\sqrt{\sin \gamma}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2\pi n + \gamma}} \left(U_0 \left(\frac{1}{2\pi n + \gamma} - \cot \gamma \right) + 16(\gamma + 2\pi n) U_1 \right) \theta(\eta - (\gamma + 2\pi n)) \right]. \end{aligned} \quad (3.19)$$

This expression for the discontinuities in G_R should be compared against the terms in (4.74) and (4.76) below. By focusing on the behavior at the discontinuities and so allowing the evaluation at the discontinuities themselves of coefficients of the discontinuities (i.e., as indicated below Eq. (154) in Ref. [14]), it can be easily checked that the asymptotics at the discontinuities given by (3.19) agree with those given by (4.74) and (4.76).

IV. GLOBAL ANALYSIS OF THE SCHWARZSCHILD GREEN FUNCTION USING A LARGE- ℓ EXPANSION

In this section, we will use an asymptotic expansion for Bessel functions to obtain a representation for the retarded Green function of Schwarzschild spacetime given in (2.30). Bessel functions enter via Zauderer's theorem as expressed in (2.32), and via a representation of the Legendre polynomials as sums of Bessel functions [see Eq. (4.23) below]. We then sum over the mode index ℓ to identify the global singularity structure of G_R , and to determine the nonsmooth contribution to G_R as a sum over Hadamard forms. Here, the distinction between smooth and nonsmooth is made modulo the presence of an overall factor of $\theta(-\sigma)\theta(\Delta t)$ in G_R : that is, the distinction between smooth, nonsmooth and singular behavior is made with respect to the coefficient of this term. The results of this section are contingent on convergence of the background Hadamard series (2.26), and so apply on the region Ω_p of \mathcal{M}_2 on which convergence holds.

A. The large- ℓ expansion

The large- ℓ expansion of Eq. (2.30) is underpinned by the following large-argument expansion for Bessel functions [Eq. (8.451) of Ref. [34]]:

⁵For $n \in \mathbb{Z}_{\geq 0}$, it is $\hat{\sigma}_n = \sigma_n^{\text{even}}$ and $\hat{\sigma}_{-n} = \sigma_n^{\text{odd}}$, where $\sigma_n^{\text{even/odd}}$ are defined in Eqs. (157) and (158) of Ref. [14].

$$\begin{aligned} J_k(z) = & \sqrt{\frac{2}{\pi z}} \left[\sum_{m=0}^N E_m \left(z - \frac{\pi}{2} k - \frac{\pi}{4} \right) \frac{a_{k,m}}{(2z)^m} + R_{k,N}(z) \right], \\ & |\arg(z)| < \pi, \end{aligned} \quad (4.1)$$

where $N \in \mathbb{N}$ is odd, and

$$\begin{aligned} E_m(x) \equiv & \frac{e^{im\pi/2}}{2} (e^{ix} + (-1)^m e^{-ix}) = \cos \left(x + \frac{\pi}{2} m \right), \\ a_{k,m} \equiv & \frac{\Gamma(k + m + \frac{1}{2})}{m! \Gamma(k - m + \frac{1}{2})}, \end{aligned} \quad (4.2)$$

and the remainder term has the form

$$R_{k,N}(z) = \cos \left(z - \frac{\pi}{2} k - \frac{\pi}{4} \right) R_1 - \sin \left(z - \frac{\pi}{2} k - \frac{\pi}{4} \right) R_2, \quad (4.3)$$

with the bounds

$$|R_1| < \left| \frac{\Gamma(k + N + \frac{3}{2})}{(2z)^{N+1} (N+1)! \Gamma(k - N - \frac{1}{2})} \right|, \quad N > k - \frac{3}{2}, \quad (4.4)$$

$$|R_2| < \left| \frac{\Gamma(k + N + \frac{5}{2})}{(2z)^{N+2} (N+2)! \Gamma(k - 2N - \frac{3}{2})} \right|, \quad N \geq k - \frac{5}{2}. \quad (4.5)$$

A key feature of the expansion (4.1) is that it does not apply to all orders; for each positive integer choice of N in (4.1), we must cut off the sum in (2.32) at the appropriate value of k as determined by Eqs. (4.4) and (4.5). We note that, in Eq. (2.32), the argument is $z = L\eta$, and so large $\ell = L - \frac{1}{2}$ corresponds to large z (given a fixed $\eta > 0$).

Using Eq. (4.1), we can write down a corresponding equation for the 2D Green function, based on Eq. (2.32).

This equation holds for any odd $N \geq 1$, and we extract features of the Green function by taking a limit $N \rightarrow \infty$. We write

$$\mathcal{U}_\ell(x^A, x^{A'}) = \mathcal{U}_\ell^{(N)} + \check{R}_{\ell,N} + \mathcal{U}_\ell^{(N,\infty)}, \quad (4.6)$$

where

$$\begin{aligned} \mathcal{U}_\ell^{(N)} &\equiv \sum_{j=0}^N \frac{(2\eta)^j U_j}{L^j} \sqrt{\frac{2}{\pi L \eta}} \left(\sum_{m=0}^N E_m \left(L\eta - \frac{\pi}{2} j - \frac{\pi}{4} \right) \right. \\ &\quad \times \left. \frac{a_{j,m}}{(2L\eta)^m} \right), \end{aligned} \quad (4.7)$$

$$\check{R}_{\ell,N} \equiv \sum_{j=0}^N \frac{(2\eta)^j U_j}{L^j} \sqrt{\frac{2}{\pi L \eta}} R_{j,N}(L\eta), \quad (4.8)$$

$$\mathcal{U}_\ell^{(N,\infty)} \equiv \sum_{j=N+1}^{\infty} \frac{(2\eta)^j U_j}{L^j} J_j(L\eta). \quad (4.9)$$

In the discussion below, we will refer to $\mathcal{U}_\ell^{(N)}$ as the *finite sum*, to $\check{R}_{\ell,N}$ as the *remainder* and to $\mathcal{U}_\ell^{(N,\infty)}$ as the *infinite tail*.

Our next step is to rewrite these three terms as sums of ascending powers of L^{-1} , and to then sum over ℓ to determine the contribution of each term to Eq. (2.30). Our key conclusion is that we can take the limit $N \rightarrow \infty$ to obtain the following results: in the limit, only the finite sum $\mathcal{U}_\ell^{(N)}$ contributes to the singular and nonsmooth parts of $G_R(x, x')$, the remainder term $\check{R}_{\ell,N}$ contributes only to the smooth part and the infinite tail $\mathcal{U}_\ell^{(N,\infty)}$ does not contribute to the GF.

First, we consider the infinite tail contribution to (2.30). For any finite N , this term is continuous. We apply the following large-order asymptotic relation for Bessel functions [30]:

$$J_k(z) \sim \frac{1}{k!} \left(\frac{z}{2} \right)^k, \quad k \rightarrow \infty. \quad (4.10)$$

Then for large N , the infinite tail has the following asymptotic behavior,

$$\mathcal{U}_\ell^{(N,\infty)} \equiv \sum_{j=N+1}^{\infty} \frac{(2\eta)^j U_j}{L^j} J_j(L\eta) \sim \mathcal{G}^{(N,\infty)}, \quad N \rightarrow \infty, \quad (4.11)$$

where

$$\mathcal{G}^{(N,\infty)} \equiv \sum_{j=N+1}^{\infty} U_j \frac{\eta^{2j}}{j!}. \quad (4.12)$$

From (2.27), we recognize this as the tail of the infinite series which generates the background Green function G ; note that

the approximation is independent of L . It follows that, in the region where the Hadamard series Eq. (2.27) converges, the infinite tail satisfies

$$\mathcal{U}_\ell^{(N,\infty)}(x^A, x^{A'}) \rightarrow 0, \quad N \rightarrow \infty, \quad (4.13)$$

and that for each x^A , the convergence is uniform in L and $x^{A'}$. Summing over ℓ yields the contribution of the infinite tail to G_R in Eq. (2.30). This corresponds to

$$\begin{aligned} &\sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \mathcal{U}_\ell^{(N,\infty)} P_\ell(\cos \gamma) \\ &\sim \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \mathcal{G}^{(N,\infty)} P_\ell(\cos \gamma), \quad N \rightarrow \infty, \end{aligned} \quad (4.14)$$

and we have

$$\begin{aligned} &\sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \mathcal{G}^{(N,\infty)} P_\ell(\cos \gamma) \\ &= \mathcal{G}^{(N,\infty)} \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) P_\ell(\cos \gamma) = \mathcal{G}^{(N,\infty)} \sum_{k \in \mathbb{Z}} \delta(\gamma + 2k\pi), \end{aligned} \quad (4.15)$$

where we have used the fact that $\mathcal{G}^{(N,\infty)}$ is independent of ℓ . This contribution is a large- N error term which, by Eq. (4.13), vanishes in the limit as $N \rightarrow \infty$. *This term does not contribute to the physical Green function.*

Next, we consider the contribution of the remainder term (4.8) to (2.30).

The quantities $R_{k,N}$ introduced in (4.3) are subject to the bounds of Eqs. (4.4) and (4.5), and so it follows that, for a fixed $\eta > 0$,

$$\check{R}_{\ell,N} = O(\ell^{-N-3/2}), \quad \ell \rightarrow \infty. \quad (4.16)$$

The corresponding contribution to (2.30) then satisfies

$$\left| \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \check{R}_{\ell,N} P_\ell(\cos \gamma) \right| \leq \sum_{\ell=0}^{\infty} O(\ell^{-N-1}), \quad (4.17)$$

where we have used $P_\ell(\cos \gamma) = O(\ell^{-1/2})$, $\ell \rightarrow +\infty$, $\gamma \in (0, \pi)$ (recall that $N \geq 1$). Since the series in (4.17) comprises a series of continuous functions, the upper bound stated allows us to apply the Weierstrass M -test and deduce that this contribution must be both finite and continuous. Furthermore, by differentiating the summands on the left-hand side of (4.8), applying recursion formulas for Bessel functions and Legendre polynomials along with the relations (4.1), (4.3) and (4.8), we obtain a bound equivalent to (4.17) for any first-order partial derivative. (Recall that the Hadamard coefficients $U_k(x^A, x^{A'})$ are smooth functions of their arguments.) There is a loss of one power of ℓ on the right-hand side, so that, in general, ℓ^{-N-1} must be replaced by ℓ^{-N} . This process can be

repeated iteratively, and it shows that we can always choose N large enough so that derivatives of the remainder terms of arbitrarily high order converge to a continuous function. Hence, *the remainder term does not contribute to the nonsmooth part of the retarded Green function*.⁶

Next, we consider the contribution of the finite sum (4.7) to the Green function (2.30). Our aim is to show that only this term contributes nonsmooth (including singular) terms to the Green function. To see this, it is necessary to include again all contributions to \mathcal{U}_ℓ ; the finite sum, the remainder, and the infinite tail. We can do this by rewriting the asymptotic expansion of the Bessel functions in the following way: for any (odd) $N \geq 1$, we have

$$J_k(z) = \sqrt{\frac{2}{\pi z}} \sum_{m=0}^{N+2} E_m \left(z - \frac{\pi}{2} k - \frac{\pi}{4} \right) \frac{a_{k,m}}{(2z)^m}, \quad (4.18)$$

where E_m is defined as above, $a_{k,m} = a_{k,m}$ for $0 \leq m \leq N$, and

$$a_{k,N+1} = b_{k,N+1}(z) \frac{\Gamma(k + N + \frac{3}{2})}{(N+1)! \Gamma(k - N - \frac{1}{2})}, \quad (4.19)$$

$$a_{k,N+2} = b_{k,N+2}(z) \frac{\Gamma(k + N + \frac{5}{2})}{(N+2)! \Gamma(k - N - \frac{3}{2})}, \quad (4.20)$$

where the quantities $b_{k,N+1}(z)$ and $b_{k,N+2}(z)$ are for each k and N , bounded functions of z —in fact, bounded by unity [this is equivalent to the bounds on R_1 and R_2 given in (4.4) and (4.5) above]. Then for each odd $N \geq 1$, we can write

$$\begin{aligned} \mathcal{U}_\ell &= \sqrt{\frac{2}{\pi \eta L}} \sum_{k=0}^N \left(\frac{(2\eta)^k U_k}{L^k} \sum_{m=0}^{N+2} E_m \left(L\eta - \frac{\pi}{2} k - \frac{\pi}{4} \right) \frac{a_{k,m}}{(2\eta)^m} L^{-m} \right) + \mathcal{U}_\ell^{(N,\infty)} \\ &= \sqrt{\frac{2}{\pi \eta L}} \left(\sum_{k=0}^N E_k \left(L\eta - \frac{\pi}{4} \right) \hat{\alpha}_k(x^A, x^{A'}) L^{-k} + \sum_{k=N+1}^{2N+2} \bar{\alpha}_k(x^A, x^{A'}; L) L^{-k} \right) + \mathcal{U}_\ell^{(N,\infty)}, \end{aligned} \quad (4.21)$$

where the coefficients $\hat{\alpha}_k(x^A, x^{A'})$, $k = 0, \dots, N$ and $\bar{\alpha}_k(x^A, x^{A'}; L)$, $k = N+1, \dots, 2N+2$ arise by formally collecting like powers of L , treating $b_{k,N+1}(L\eta)$ and $b_{k,N+2}(L\eta)$ as though they were independent of L . This procedure is well-defined, and yields coefficients $\bar{\alpha}_k(x^A, x^{A'}; L)$ that are “weakly” dependent on L ; these coefficients have the form of linear combinations of functions of η multiplied by the zero order [i.e., $O(L^0) = O(1)$] terms $b_{k,N+1}, b_{k,N+2}$. In particular, these cannot contribute positive integer powers of L that would “contaminate” the coefficients $\hat{\alpha}_k$ for $k \leq N$; that is, these coefficients are completely determined by $a_{m,k} = a_{m,k}$.

$$\hat{\alpha}_k(x^A, x^{A'}) = \sum_{m=0}^k (-1)^m a_{m,k-m} \frac{U_m(x^A, x^{A'})}{(2\eta)^{k-2m}}. \quad (4.22)$$

Note that these coefficients are independent of L . Recall that the functions E_k are trigonometric, and so are $O(1)$. As above, the term $\mathcal{U}_\ell^{(N,\infty)}$ in (4.21) is the infinite-tail contribution (4.9).

We can derive a corresponding expansion for the Legendre polynomials, based on the following expansion in terms of Bessel functions (see [32] and volume 2, page 58, Eq. (15) of Ref. [35]):

$$P_\ell(\cos \gamma) = \left(\frac{\gamma}{\sin \gamma} \right)^{1/2} \sum_{j=0}^{\infty} V_j(\gamma) \frac{J_j(L\gamma)}{L^j}, \quad (4.23)$$

where the coefficients $V_j(\gamma)$ are elementary functions which are regular for $\gamma \in [0, \pi)$. See Appendix B for more details. In particular,

$$V_0(\gamma) = 1, \quad V_1(\gamma) = \frac{1}{8} \left(\cot \gamma - \frac{1}{\gamma} \right). \quad (4.24)$$

The series (4.23) is uniformly convergent in any interval of the form $[0, \gamma_0 - \epsilon]$ with $\epsilon > 0$ and $\gamma_0 = 2(\sqrt{2} - 1)\pi \simeq 0.828\pi$ [32]. Note, in particular, that the expansion is not valid at $\gamma = \pi$. An alternative approach to the calculation of the coefficients V_k is given in Appendix B.

Using the representation (4.18) for $J_k(\gamma)$, we can write

$$\begin{aligned} P_\ell(\cos \gamma) &= \left(\frac{\gamma}{\sin \gamma} \right)^{1/2} \sqrt{\frac{2}{\pi L \gamma}} \left(\sum_{k=0}^N E_k \left(L\gamma - \frac{\pi}{4} \right) \hat{\beta}_k(\gamma) L^{-k} \right. \\ &\quad \left. + \sum_{k=N+1}^{2N+2} \bar{\beta}_k(\gamma; L) L^{-k} \right) + P_\ell^{(N,\infty)}, \end{aligned} \quad (4.25)$$

where the $\bar{\beta}_k(\gamma; L)$ are weakly dependent on L in the sense defined above and

⁶This is true for a fixed $\eta > 0$, which, for causal separation, merely excludes the coincidence limit (i.e., $\eta, \gamma \rightarrow 0$) which is covered by the Hadamard form Eq. (1.1).

$$\hat{\beta}_j(\gamma) = \sum_{m=0}^j (-1)^m a_{m,j-m} \frac{V_m(\gamma)}{(2\gamma)^{j-m}}. \quad (4.26)$$

The infinite tail term $P_\ell^{(N,\infty)}$ is defined analogously to (4.9),

$$P_\ell^{(N,\infty)} \equiv \left(\frac{\gamma}{\sin \gamma} \right)^{1/2} \sum_{j=N+1}^{\infty} V_j(\gamma) \frac{J_j(L\gamma)}{L^j}. \quad (4.27)$$

Note that both $\mathcal{U}_\ell^{(N,\infty)}$ and $P_\ell^{(N,\infty)}$ are zero in the limit as $N \rightarrow \infty$; these terms correspond to tails of infinite series that converge in the domain under consideration. That is,

$$\lim_{N \rightarrow \infty} \sqrt{\frac{2}{\pi\eta L}} \left(\sum_{k=0}^N E_k \left(L\eta - \frac{\pi}{4} \right) \hat{\alpha}_k(x^A, x^{A'}) L^{-k} + \sum_{k=N+1}^{2N+2} \bar{\alpha}_k(x^A, x^{A'}; L) L^{-k} \right) = \mathcal{U}_\ell, \quad (4.28)$$

$$\lim_{N \rightarrow \infty} \left(\frac{\gamma}{\sin \gamma} \right)^{1/2} \sqrt{\frac{2}{\pi L \gamma}} \left(\sum_{k=0}^N E_k \left(L\gamma - \frac{\pi}{4} \right) \hat{\beta}_k(\gamma) L^{-k} + \sum_{k=N+1}^{2N+2} \bar{\beta}_k(\gamma; L) L^{-k} \right) = P_\ell. \quad (4.29)$$

Multiplying and collecting inverse powers of L (again, in a well-defined manner), we can write

$$\begin{aligned} \left(\ell + \frac{1}{2} \right) \mathcal{U}_\ell P_\ell &= \frac{1}{\pi \sqrt{\eta \sin \gamma}} \left[\sum_{k=0}^{2N} \left(\sum_{j=\max\{0, k-N\}}^{\min\{k, N\}} \left(E_k \left(L\eta + L\gamma - \frac{\pi}{2} \right) + (-1)^{j-k} E_k(L\eta - L\gamma) \right) \hat{\alpha}_j \hat{\beta}_{k-j} \right) L^{-k} + O(L^{-N-1}) \right] \\ &= \frac{1}{\pi \sqrt{\eta \sin \gamma}} \left[\sum_{k=0}^N \left(\sum_{j=0}^k \left(E_k \left(L\eta + L\gamma - \frac{\pi}{2} \right) + (-1)^{j-k} E_k(L\eta - L\gamma) \right) \hat{\alpha}_j \hat{\beta}_{k-j} \right) L^{-k} + O(L^{-N-1}) \right], \end{aligned} \quad (4.30)$$

where (we recall) N is an odd integer with $N \geq 1$. The functional dependence of the sinusoidal functions E_k on terms proportional to $\eta \pm \gamma$ in (4.30) is worth noting. These combinations in the arguments of the E_k are the seeds that will later give rise to the functional dependence of the GF on the global world function $\hat{\sigma}_k$ [see (2.16)] in the expressions for the singularity structure and the expression as a sum of Hadamard forms [see Eq. (4.84) below].

We *cannot* at this point take the limit $N \rightarrow \infty$ and neglect the $O(L^{-N-1})$ term in (4.30). However, as N can be arbitrarily large, when we sum the $O(L^{-N-1})$ term over ℓ , as we will see below, we obtain a smooth (i.e., C^∞) contribution to the full Green function. The sum to N terms in (4.30) yields the full nonsmooth contribution to the Green function, including the singular contribution. We proceed to calculate this nonsmooth contribution (and show that the contribution from the remainder term in (4.30) must be smooth [see the paragraph below following Eq. (4.42)]).

As already mentioned, the function $\mathcal{U}_\ell(x^A, x^{A'})$ in Eq. (2.32) is the Riemann function for the 2D wave equation (2.24). This is a smooth function on \mathcal{M}_2 , and so the anticipated nonsmoothness in Eq. (2.30) is due to the infinite sum; any cutoff at a finite $\ell = \ell_{\max}$ would yield a smooth quantity. Thus the nonsmooth nature of $G_R(x, x')$ arises from the large- ℓ behavior of Eq. (2.30), and so we used a large- ℓ expansion above to determine the nonsmooth contributions.

The large- ℓ expansion requires that we first subtract the $\ell = 0$ contribution to G_R , and so we define

$$G_R^{\ell \geq 1}(x, x') \equiv \frac{1}{r \cdot r'} \sum_{\ell=1}^{\infty} \left(\ell + \frac{1}{2} \right) \mathcal{U}_\ell(x^A, x^{A'}) P_\ell(\cos \gamma), \quad (4.31)$$

so that

$$G_R = \theta(-\sigma) \theta(\Delta t) \left(\frac{\mathcal{U}_0}{2r \cdot r'} + G_R^{\ell \geq 1} \right). \quad (4.32)$$

From the comments above, the only discontinuity in the first term on the right is due to the presence of the Heaviside distributions $\theta(-\sigma)$ and $\theta(\Delta t)$. So our aim now is to expand the summand in Eq. (4.31) for $G_R^{\ell \geq 1}$ in inverse powers of ℓ and to identify the nonsmooth contributions. We note that presence of the causal terms $\theta(-\sigma) \theta(\Delta t)$ in (4.32) implies that G_R is zero for $\eta(x^A, x^{A'}) < 0$. Thus, in the remainder of the paper we take $\eta \geq 0$, and so we set $\theta(-\sigma) \theta(\Delta t) = \theta(\eta) = 1$.

Next, we define the contribution to $G_R^{\ell \geq 1}(x, x')$ that arises by including the finite sum to N terms in (4.30), and then take the limit as $N \rightarrow \infty$. This results in the following expression (with the subscript “NS” for nonsmooth):

$$G_{R,NS}^{\ell \geq 1} = \frac{2}{\pi r \cdot r'} \frac{1}{\sqrt{\eta \sin \gamma}} \sum_{\ell=1}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\hat{\nu}_k}{\ell^k} \right), \quad \gamma \in (0, \gamma_0), \quad (4.33)$$

where

$$\hat{\nu}_k \equiv \sum_{j=0}^k \frac{(-1)^{k-j}}{2^{k-j}} \binom{k-1}{k-j} \nu_j, \quad (4.34)$$

$$\begin{aligned} \nu_j &\equiv E_j \left(\gamma L + \eta L - \frac{\pi}{2} \right) \nu_j^{(+)}(x^A, x^{A'}, \gamma) \\ &+ E_j(\gamma L - \eta L) \nu_j^{(-)}(x^A, x^{A'}, \gamma), \end{aligned} \quad (4.35)$$

$$\nu_j^{(+)} \equiv \frac{1}{2} \sum_{k=0}^j \hat{\alpha}_{j-k}(x^A, x^{A'}) \hat{\beta}_k(\gamma), \quad (4.36)$$

$$\nu_j^{(-)} \equiv \frac{1}{2} \sum_{k=0}^j (-1)^k \hat{\alpha}_{j-k}(x^A, x^{A'}) \hat{\beta}_k(\gamma), \quad (4.37)$$

and where $\hat{\alpha}_k$ and $\hat{\beta}_k$ are defined in (4.22) and (4.26), respectively. We note that in (4.34), we use the binomial coefficients

$$\binom{-1}{0} = \binom{0}{0} = 1, \quad \binom{j}{k} = 0 \quad \text{for all } k > j \geq 0. \quad (4.38)$$

In (4.33), the summation index ℓ appears only as an inverse power and in the phase functions E_j . This leads to the next step, which involves collecting like phases and then summing over ℓ ,

$$G_{R,NS}^{\ell \geq 1} = \frac{2}{\pi r \cdot r'} \frac{1}{\sqrt{\eta \sin \gamma}} \sum_{k=0}^{\infty} [\mathcal{A}_k(\gamma + \eta) V_k^{(+)} + \bar{\mathcal{A}}_k(\gamma + \eta) \bar{V}_k^{(+)} + \mathcal{A}_k(\gamma - \eta) V_k^{(-)} + \bar{\mathcal{A}}_k(\gamma - \eta) \bar{V}_k^{(-)}], \quad (4.39)$$

where

$$\mathcal{A}_k(x) \equiv \sum_{\ell=1}^{\infty} \frac{e^{i\ell x}}{\ell^k}, \quad (4.40)$$

$$V_k^{(+)} \equiv e^{i(\gamma+\eta)/2} \sum_{m=0}^k \frac{e^{i\pi(2k-m-1)/2}}{2^{k-m+1}} \binom{k-1}{k-m} \nu_m^{(+)}, \quad (4.41)$$

$$V_k^{(-)} \equiv e^{i(\gamma-\eta)/2} \sum_{m=0}^k \frac{e^{i\pi(2k-m)/2}}{2^{k-m+1}} \binom{k-1}{k-m} \nu_m^{(-)}. \quad (4.42)$$

We note that the $V_k^{(\pm)}$ are smooth functions on the conformal Schwarzschild spacetime which are independent of ℓ .

The distributions $\mathcal{A}_k, k \geq 0$ play a key role in our description of (4.39) as a sum of Hadamard forms. The basis of this structure is the observation that the \mathcal{A}_k are periodic in their argument. From the spacetime point of view, this periodicity is reflected in structural similarities in the form of (4.39) that repeat periodically as null geodesics emerging from the base point x execute multiple orbits around the black hole. The forms of the different $\mathcal{A}_k, k \geq 0$ are also crucial in identifying the singular and nonsmooth contribution to G_R . We note that inclusion of the remainder term from (4.30) yields terms with coefficients of the form \mathcal{A}_N , with N arbitrarily large. The sum over ℓ produces smooth functions—essentially trigonometric functions with arguments $\eta \pm \gamma$. This establishes the fact that $G_{R,NS}^{\ell \geq 1}$ includes all nonsmooth contributions to the retarded

Green function, and that the remainder term (4.8) ultimately contributes a smooth term to G_R .

B. Structure of the \mathcal{A}_k

We write

$$\mathcal{A}_k(x) = \mathcal{C}_k(x) + i\mathcal{S}_k(x), \quad (4.43)$$

where

$$\mathcal{C}_k(x) \equiv \sum_{\ell=1}^{\infty} \frac{\cos(\ell x)}{\ell^k}, \quad k \geq 0 \quad (4.44)$$

and

$$\mathcal{S}_k(x) \equiv \sum_{\ell=1}^{\infty} \frac{\sin(\ell x)}{\ell^k}, \quad k \geq 0. \quad (4.45)$$

We note the distributional result

$$\mathcal{A}'_k(x) = i\mathcal{A}_{k-1}(x), \quad k \geq 1, \quad (4.46)$$

and correspondingly

$$\mathcal{A}_k(x) = \zeta(k) + i \int_0^x \mathcal{A}_{k-1}(y) dy, \quad k \geq 2, \quad (4.47)$$

where $\zeta(k) = \mathcal{A}_k(0)$ is the Riemann zeta function, and so we have the regularity results

$$\mathcal{A}_k \in C^{k-2}(\mathbb{R}), \quad \mathcal{A}_k^{(k-1)} \in L^1_{\text{loc}}(\mathbb{R}), \quad k \geq 2. \quad (4.48)$$

For $k \geq 1$, we have

$$\mathcal{C}_{2k}(x) = \check{\mathcal{B}}_{2k}\left(\frac{x}{2\pi}\right), \quad 0 \leq x \leq 2\pi, \quad (4.49)$$

and

$$\mathcal{S}_{2k+1}(x) = \hat{\mathcal{B}}_{2k+1}\left(\frac{x}{2\pi}\right), \quad 0 \leq x \leq 2\pi, \quad (4.50)$$

where

$$\check{\mathcal{B}}_{2k}(x) \equiv \frac{(-1)^{k-1}}{2} \frac{(2\pi)^{2k}}{(2k)!} \mathcal{B}_{2k}(x), \quad x \in \mathbb{R}, \quad (4.51)$$

$$\hat{\mathcal{B}}_{2k+1}(x) \equiv \frac{(-1)^{k-1}}{2} \frac{(2\pi)^{2k+1}}{(2k+1)!} \mathcal{B}_{2k+1}(x), \quad x \in \mathbb{R}, \quad (4.52)$$

and where $\mathcal{B}_n(x)$, $n \geq 1$ is the n^{th} Bernoulli polynomial [30]. It follows from Fourier theory that for $k \geq 2$ even, $\mathcal{C}_k(x)$, $x \in \mathbb{R}$ is the periodic continuation to the real line of the corresponding Bernoulli polynomial, and likewise for $\mathcal{S}_k(x)$, $x \in \mathbb{R}$ with $k \geq 3$ odd. To see this, we consider the following theorem (see e.g., Theorem 14.29 of [36]):

Theorem 1 If $f: \mathbb{R} \rightarrow \mathbb{R}$ is periodic on \mathbb{R} with period 2π , has bounded variation on $[0, 2\pi]$ and is continuous on the closed interval I , then the Fourier series of f converges uniformly to f on I .

Applying Theorem 1 then allows us to write \mathcal{C}_{2k} and \mathcal{S}_{2k+1} as the periodic continuation of the polynomials appearing in (4.51) and (4.52) respectively:

$$\mathcal{C}_{2k}(x) = \sum_{\ell=1}^{\infty} \frac{\cos(\ell x)}{\ell^{2k}} = \sum_{n \in \mathbb{Z}} \theta(x - 2n\pi) \theta(2(n+1)\pi - x) \check{\mathcal{B}}_{2k}\left(\frac{x - 2n\pi}{2\pi}\right), \quad k \geq 1, x \in \mathbb{R}, \quad (4.53)$$

and

$$\mathcal{S}_{2k+1}(x) = \sum_{\ell=1}^{\infty} \frac{\sin(\ell x)}{\ell^{2k+1}} = \sum_{n \in \mathbb{Z}} \theta(x - 2n\pi) \theta(2(n+1)\pi - x) \hat{\mathcal{B}}_{2k+1}\left(\frac{x - 2n\pi}{2\pi}\right), \quad k \geq 1, x \in \mathbb{R}. \quad (4.54)$$

This accounts for approximately half of the terms in (4.39). To deal with the terms \mathcal{C}_{2k+1} and \mathcal{S}_{2k} , $k \geq 1$, we must first recall some facts established in [14] regarding \mathcal{A}_0 and \mathcal{A}_1 .

In [14], we showed that [using the notation of (4.43) and with $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$]

$$\mathcal{C}_1(x) = -\ln|x| - \sum_{n \in \mathbb{Z}_0} \ln \left| 1 - \frac{x}{2n\pi} \right|, \quad (4.55)$$

$$\mathcal{S}_1(x) = -\frac{x}{2} + \pi \sum_{n \in \mathbb{Z}} \left[\theta(x - 2n\pi) - \frac{1}{2} \right], \quad (4.56)$$

with the distributional derivatives

$$\mathcal{C}_0(x) = \mathcal{S}'_1(x) = -\frac{1}{2} + \pi \sum_{n \in \mathbb{Z}} \delta(x - 2n\pi) \quad (4.57)$$

$$\mathcal{S}_0(x) = -\mathcal{C}'_1(x) = \sum_{n \in \mathbb{Z}} \text{PV} \left(\frac{1}{x - 2n\pi} \right). \quad (4.58)$$

We define

$$\mathcal{Z}_k(x) \equiv \begin{cases} i\mathcal{C}_k(x), & k \text{ odd;} \\ \mathcal{S}_k(x), & k \text{ even,} \end{cases} \quad k \geq 1, x \in \mathbb{R}. \quad (4.59)$$

Then

$$\mathcal{Z}'_k(x) = -i\mathcal{Z}_{k-1}(x), \quad k \geq 2, x \in \mathbb{R}, \quad (4.60)$$

and so

$$\mathcal{Z}_k(x) = \mathcal{Z}_k(0) - i \int_0^x \mathcal{Z}_{k-1}(t) dt, \quad k \geq 2, x \in \mathbb{R}. \quad (4.61)$$

We note the values

$$\mathcal{Z}_k(0) = \begin{cases} i\zeta(k), & k \text{ odd;} \\ 0, & k \text{ even.} \end{cases} \quad (4.62)$$

Integrating (4.60) with $k = 2$ using (4.55) then yields

$$\mathcal{Z}_2(x) = x(1 - \ln|x|) + \sum_{n \in \mathbb{Z}_0} \left[x + (2n\pi - x) \ln \left| 1 - \frac{x}{2n\pi} \right| \right] \in C^0(\mathbb{R}), \quad (4.63)$$

where we use $\mathbb{Z}_0 = \{n \in \mathbb{Z} : n \neq 0\}$. Integrating repeatedly yields the form below for $\mathcal{Z}_k, k \geq 2$. We state the result as a proposition: this is proven by showing that the given functions satisfy the sequence of initial value problems (IVPs)

$$\mathcal{Z}'_k(x) = -i\mathcal{Z}_{k-1}(x), \quad x \in \mathbb{R}, \quad \mathcal{Z}_k(0) = \begin{cases} i\zeta(k), & k \text{ odd;} \\ 0, & k \text{ even.} \end{cases} \quad k \geq 3. \quad (4.64)$$

It follows by an inductive argument that each IVP in this sequence has a unique solution (essentially by virtue of the continuity of the sequence of right-hand sides that emerges); this solution, which takes the form stated in Proposition 1, must be the required function \mathcal{Z}_k .

Proposition 1 Let $\mathcal{Z}_k(x), k \geq 2$ be as defined in (4.59). Then

$$\mathcal{Z}_k(x) = \underbrace{{}_0Q_k(x) + A_k x^{k-1} \ln|x|}_{n=0} + \sum_{n \in \mathbb{Z}_0} \left[{}_nQ_k(x) + A_k (x - 2n\pi)^{k-1} \ln \left| 1 - \frac{x}{2n\pi} \right| \right], \quad (4.65)$$

where A_k are constants and

$${}_0Q_k(x) = b_{k,0} + b_{k,1}x + \cdots + b_{k,k-1}x^{k-1}, \quad (4.66)$$

$${}_nQ_k(x) = {}_nB_{k,0} + {}_nB_{k,1}(x - 2n\pi) + \cdots + {}_nB_{k,k-1}(x - 2n\pi)^{k-1}, \quad n \in \mathbb{Z}_0 \quad (4.67)$$

are polynomials of degree $k-1$. The constant and the polynomial coefficients for the $n=0$ contribution are determined by the relations (valid for $k \geq 2$)

$$\begin{aligned} A_k &= \frac{(-i)^k}{(k-1)!}, \\ b_{k,j} &= \frac{(-i)^j}{j!} \mathcal{Z}_{k-j}(0), \quad 0 \leq j \leq k-2, \\ b_{k,k-1} &= -\frac{(-i)^k}{(k-1)!} (\gamma + \psi(k)), \end{aligned} \quad (4.68)$$

where γ is the Euler-Mascheroni constant and $\psi(k) = \Gamma'(k)/\Gamma(k)$ is the digamma function. The polynomial coefficients for the $n \in \mathbb{Z}_0$ contributions are determined by

$$\begin{aligned} {}_nB_{k,k-1} &= -\frac{(-i)^k}{(k-1)!} (\gamma + \psi(k)), \quad k \geq 3, \\ {}_nB_{k,j} &= \frac{(-i)^j}{j!} {}_nB_{k-j,0}, \quad 0 < j \leq k-2, \\ {}_nB_{k,0} &= -\sum_{j=1}^{k-1} (-2n\pi)^j {}_nB_{k,j}, \quad k \geq 3, \end{aligned} \quad (4.69)$$

with the initial values [read off from (4.63)]

$${}_nB_{2,0} = 2n\pi, \quad {}_nB_{2,1} = 1. \quad (4.70)$$

Note that, as anticipated, the functions $\mathcal{Z}_k(x), k \geq 2$ are smooth (i.e., infinitely differentiable) everywhere except at points of the form $x = 2n\pi, n \in \mathbb{Z}$. The term labeled “ $n=0$ ” is nonsmooth at $x=0$.

By way of summary, we note that the formulas (4.53)–(4.58) and (4.65) provide, via (4.43), the complete description of the functions/distributions \mathcal{A}_k that is required to determine (a) singularity structure of the GF and (b) the sum-of-Hadamard forms representation of GF. We turn now to the first of these.

C. The singularity of the GF

The singular contribution to $G_R^{\ell \geq 1}$ arises from the $k=0$ and $k=1$ terms of $G_{R,NS}^{\ell \geq 1}$ in (4.39). This follows from the regularity conditions (4.48), and from the argument in Sec. IV A above that the remainder term (4.8) and the infinite tail (4.9) do not contribute to the singularities of G_R . The result in this subsection is valid everywhere that both the Hadamard series (2.26) and Szego’s series (4.23) converge.

Collecting the relevant terms, we see that the ‘most’ singular contribution, the $k = 0$ term, is

$$G_R^{\ell \geq 1} \Big|_{k=0} = \frac{1}{\pi r \cdot r'} \frac{U_0(x^A, x^{A'})}{\sqrt{\eta \sin \gamma}} \left[\sin\left(\frac{\gamma + \eta}{2}\right) \mathcal{C}_0(\gamma + \eta) + \cos\left(\frac{\gamma + \eta}{2}\right) \mathcal{S}_0(\gamma + \eta) \right. \\ \left. + \cos\left(\frac{\gamma - \eta}{2}\right) \mathcal{C}_0(\gamma - \eta) + \sin\left(\frac{\gamma - \eta}{2}\right) \mathcal{S}_0(\gamma - \eta) \right]. \quad (4.71)$$

Some simplification is possible here using the distributional identity $f(x)\delta(x - a) = f(a)\delta(x - a)$. This yields

$$\sin\left(\frac{\gamma + \eta}{2}\right) \mathcal{C}_0(\gamma + \eta) = \sin\left(\frac{\gamma + \eta}{2}\right) \left[-\frac{1}{2} + \pi \sum_{n \in \mathbb{Z}} \delta(\gamma + \eta - 2n\pi) \right] \\ = -\frac{1}{2} \sin\left(\frac{\gamma + \eta}{2}\right), \quad (4.72)$$

and

$$\cos\left(\frac{\gamma - \eta}{2}\right) \mathcal{C}_0(\gamma - \eta) = \cos\left(\frac{\gamma - \eta}{2}\right) \left[-\frac{1}{2} + \pi \sum_{n \in \mathbb{Z}} \delta(\gamma - \eta - 2n\pi) \right] \\ = -\frac{1}{2} \cos\left(\frac{\gamma - \eta}{2}\right) + \pi \sum_{n \in \mathbb{Z}} (-1)^n \delta(\gamma - \eta - 2n\pi). \quad (4.73)$$

Thus,

$$G_R^{\ell \geq 1} \Big|_{k=0} = \frac{1}{\pi r \cdot r'} \frac{U_0(x^A, x^{A'})}{\sqrt{\eta \sin \gamma}} \left\{ -\frac{1}{2} \sin\left(\frac{\gamma + \eta}{2}\right) - \frac{1}{2} \cos\left(\frac{\gamma - \eta}{2}\right) \right. \\ \left. + \sum_{n \in \mathbb{Z}} \left[(-1)^n \pi \delta(\gamma - \eta - 2n\pi) + \cos\left(\frac{\gamma + \eta}{2}\right) PV\left(\frac{1}{\gamma + \eta - 2n\pi}\right) + \sin\left(\frac{\gamma - \eta}{2}\right) PV\left(\frac{1}{\gamma - \eta - 2n\pi}\right) \right] \right\}. \quad (4.74)$$

Corresponding to this equation for $k = 1$ we have

$$G_R^{\ell \geq 1} \Big|_{k=1} = -\frac{1}{8\pi r \cdot r'} \frac{1}{\sqrt{\eta \sin \gamma}} \left\{ \mathcal{V}(\gamma) \left[\cos\left(\frac{\gamma + \eta}{2}\right) \mathcal{C}_1(\gamma + \eta) - \sin\left(\frac{\gamma + \eta}{2}\right) \mathcal{S}_1(\gamma + \eta) \right] \right. \\ \left. + \mathcal{V}(-\gamma) \left[\sin\left(\frac{\gamma - \eta}{2}\right) \mathcal{C}_1(\gamma - \eta) - \cos\left(\frac{\gamma - \eta}{2}\right) \mathcal{S}_1(\gamma - \eta) \right] \right\} \quad (4.75)$$

$$= -\frac{1}{8\pi r \cdot r'} \frac{1}{\sqrt{\eta \sin \gamma}} \left\{ \mathcal{V}(\gamma) \left[-\cos\left(\frac{\gamma + \eta}{2}\right) \left(\log |\gamma + \eta| + \sum_{n \in \mathbb{Z}_0} \log \left| 1 - \frac{\gamma + \eta}{2n\pi} \right| \right) \right. \right. \\ \left. + \sin\left(\frac{\gamma + \eta}{2}\right) \left(\frac{\gamma + \eta}{2} - \pi \sum_{n \in \mathbb{Z}} \left[\theta(\gamma + \eta - 2n\pi) - \frac{1}{2} \right] \right) \right] \\ \left. + \mathcal{V}(-\gamma) \left[-\sin\left(\frac{\gamma - \eta}{2}\right) \left(\log |\gamma - \eta| + \sum_{n \in \mathbb{Z}_0} \log \left| 1 - \frac{\gamma - \eta}{2n\pi} \right| \right) \right. \right. \\ \left. + \cos\left(\frac{\gamma - \eta}{2}\right) \left(\frac{\gamma - \eta}{2} - \pi \sum_{n \in \mathbb{Z}} \left[\theta(\gamma - \eta - 2n\pi) - \frac{1}{2} \right] \right) \right] \right\}, \quad (4.76)$$

where

$$\mathcal{V}(\gamma) \equiv \left(\frac{1}{\eta} + \cot \gamma \right) U_0 + 16\eta U_1. \quad (4.77)$$

[For ease of notation, we omit the explicit dependence of \mathcal{V} on x^A and $x^{A'}$: $\mathcal{V}(\gamma) = \mathcal{V}(x^A, x^{A'}, \gamma)$.] Thus, all the discontinuities of G_R arise from the terms in $G_R^{\ell \geq 1}|_{k=0}$, given by (4.74), and in $G_R^{\ell \geq 1}|_{k=1}$, given by (4.76), which are discontinuous at values $\eta > 0$. As mentioned in Sec. III C, it is easy to check that the sum of these discontinuities in G_R agrees with (3.19).

We note that the $n = 0$ contributions in (4.74) and (4.76) correspond to the discontinuity terms of the Hadamard form of the GF in the normal neighborhood of the base point x —the familiar “ $\delta + \theta$ ” terms of Eq. (1.1). For $n \geq 1$, we note the presence of the four-fold singularity structure mentioned in the introduction [see Eqs. (1.2) and (1.3)]. Equations (4.74) and (4.76) together provide the full singularity structure of the GF at any points in Schwarzschild spacetime (subject to the convergence condition mentioned at the beginning of this subsection). This

excludes caustic points (where $\gamma = 0, \pi$); we obtain the singularity structure at caustics in Sec. V.

D. The sum of Hadamard forms

To obtain the complete ‘sum of Hadamard forms’ representation of the nonsmooth contribution to the retarded Green function, we now focus our attention on the $k \geq 2$ terms in (4.39). So we define

$$G_{R,NS}^{\ell \geq 1}|_{k \geq 2} \equiv G_{R,NS}^{\ell \geq 1} - (G_R^{\ell \geq 1}|_{k=0} + G_R^{\ell \geq 1}|_{k=1}). \quad (4.78)$$

We can then immediately write down

$$G_{R,NS}^{\ell \geq 1}|_{k \geq 2} = \frac{1}{\pi r \cdot r'} \frac{1}{\sqrt{\eta \sin \gamma}} \sum_{n \in \mathbb{Z}} G_{R,NS}^{(n)}(x^A, x^{A'}, \gamma), \quad (4.79)$$

where

$$\begin{aligned} G_{R,NS}^{(n)} \equiv & \sum_{k=1}^{\infty} \left[(X_{2k}^{(+)} \check{\mathcal{B}}_{2k}(\gamma + \eta - 2n\pi) + Y_{2k+1}^{(+)} \hat{\mathcal{B}}_{2k+1}(\gamma + \eta - 2n\pi)) \theta(\gamma + \eta - 2n\pi) \theta(2(n+1)\pi - \gamma - \eta) \right. \\ & + (X_{2k}^{(-)} \check{\mathcal{B}}_{2k}(\gamma - \eta - 2n\pi) + Y_{2k+1}^{(-)} \hat{\mathcal{B}}_{2k+1}(\gamma - \eta - 2n\pi)) \theta(\gamma - \eta - 2n\pi) \theta(2(n+1)\pi - \gamma + \eta) \\ & + Y_{2k}^{(+)} \left({}_n q_{2k}(\gamma + \eta) + a_{2k}(\gamma + \eta - 2n\pi)^{2k-1} \log \left| 1 - \frac{\gamma + \eta}{2n\pi} \right| \right) \\ & + X_{2k+1}^{(+)} \left({}_n q_{2k+1}(\gamma + \eta) + a_{2k+1}(\gamma + \eta - 2n\pi)^{2k} \log \left| 1 - \frac{\gamma + \eta}{2n\pi} \right| \right) \\ & + Y_{2k}^{(-)} \left({}_n q_{2k}(\gamma - \eta) + a_{2k}(\gamma - \eta - 2n\pi)^{2k-1} \log \left| 1 - \frac{\gamma - \eta}{2n\pi} \right| \right) \\ & \left. + X_{2k+1}^{(-)} \left({}_n q_{2k+1}(\gamma - \eta) + a_{2k+1}(\gamma - \eta - 2n\pi)^{2k} \log \left| 1 - \frac{\gamma - \eta}{2n\pi} \right| \right) \right], \quad n \in \mathbb{Z}_0, \end{aligned} \quad (4.80)$$

where

$$X_k^{(\pm)} = X_k^{(\pm)}(x^A, x^{A'}, \gamma) \equiv V_k^{(\pm)} + \bar{V}_k^{(\pm)}, \quad Y_k^{(\pm)} = Y_k^{(\pm)}(x^A, x^{A'}, \gamma) \equiv i(V_k^{(\pm)} - \bar{V}_k^{(\pm)}), \quad (4.81)$$

and

$${}_n q_k \equiv \begin{cases} -i {}_n Q_k, & k \text{ odd;} \\ {}_n Q_k, & k \text{ even;} \end{cases} \quad a_k \equiv \begin{cases} -i A_k, & k \text{ odd;} \\ A_k, & k \text{ even.} \end{cases} \quad (4.82)$$

(The latter definitions are required to remove the imaginary unit in the definition (4.59) of \mathcal{Z}_k .) The $n = 0$ contribution, $G_{R,NS}^{(0)}$, has the same form as (4.80) but with the replacements

$$\log \left| 1 - \frac{\gamma \pm \eta}{2n\pi} \right| \rightarrow \log |\gamma \pm \eta|. \quad (4.83)$$

It is a straightforward algebraic manipulation to put together Eqs. (4.32), (4.74), (4.76), and (4.79) and regroup them in the following form:

$$G_{R,NS} = \frac{\theta(-\sigma)\theta(\Delta t)}{\pi r \cdot r' \sqrt{\eta \sin \gamma}} \left\{ \sum_{n \in \mathbb{Z}} \left[U_0 \left(\pi(-1)^n \delta(x_{-,n}) + \cos\left(\frac{x_{+,0}}{2}\right) \text{PV}\left(\frac{1}{x_{+,n}}\right) + \sin\left(\frac{x_{-,0}}{2}\right) \text{PV}\left(\frac{1}{x_{-,n}}\right) \right) \right. \right. \\ \left. \left. + \mathcal{V}_{+,n} \theta(x_{+,n}) \theta(-x_{+,n+1}) + \mathcal{V}_{-,n} \theta(x_{-,n}) \theta(-x_{-,n+1}) + \tilde{\mathcal{V}}_{+,n} \ln |\tilde{x}_{+,n}| + \tilde{\mathcal{V}}_{-,n} \ln |\tilde{x}_{-,n}| \right] + \mathcal{W}_n \right\}, \quad (4.84)$$

where

$$x_{\pm,n} \equiv \gamma \pm \eta - 2n\pi \quad (4.85)$$

and

$$\tilde{x}_{\pm,n} \equiv \begin{cases} \frac{x_{\pm,n}}{2n\pi}, & n \in \mathbb{Z}_0, \\ x_{\pm,0}, & n = 0. \end{cases} \quad (4.86)$$

We note that the $\hat{\sigma}_k$ in Eq. (2.16), which is the ‘globalization’ of the world function $\hat{\sigma}$ on the 4D conformal Schwarzschild spacetime, can be expressed as

$$\hat{\sigma}_k = \frac{1}{2} x_{+,-k} \cdot x_{-,-k}, \quad k \in \mathbb{Z}. \quad (4.87)$$

This means that $x_{\pm,n} = 0$, $n \in \mathbb{Z}$, corresponds to null geodesics in the 4D conformal Schwarzschild spacetime and, therefore, also in Schwarzschild spacetime. The coefficients in Eq. (4.84) are obtained as follows. The coefficients of the Heaviside distributions are

$$\mathcal{V}_{\pm,n} \equiv \sum_{k=1}^{\infty} (X_{2k}^{(\pm)} \check{\mathcal{B}}_{2k}(x_{\pm,n}) + Y_{2k+1}^{(\pm)} \hat{\mathcal{B}}_{2k+1}(x_{\pm,n})). \quad (4.88)$$

The coefficients of the logarithms are

$$\begin{aligned} \tilde{\mathcal{V}}_{+,n} &\equiv \frac{\mathcal{V}(\gamma)}{8} \cos\left(\frac{x_{+,0}}{2}\right) + \sum_{k=1}^{\infty} (X_{2k+1}^{(+)} a_{2k+1} x_{+,n}^{2k} + Y_{2k}^{(+)} a_{2k} x_{+,n}^{2k-1}), \\ \tilde{\mathcal{V}}_{-,n} &\equiv \frac{\mathcal{V}(-\gamma)}{8} \sin\left(\frac{x_{-,0}}{2}\right) + \sum_{k=1}^{\infty} (X_{2k+1}^{(-)} a_{2k+1} x_{-,n}^{2k} + Y_{2k}^{(-)} a_{2k} x_{-,n}^{2k-1}). \end{aligned} \quad (4.89)$$

Finally, the coefficients in Eq. (4.84) which are not multiplying any distribution are

$$\begin{aligned} \mathcal{W}_n &\equiv \hat{\mathcal{W}}_n, \quad \forall n \in \mathbb{Z}_0, \\ \mathcal{W}_0 &\equiv \hat{\mathcal{W}}_0 + \pi \sqrt{\eta \sin \gamma} \mathcal{U}_0 - \frac{1}{2} \left[\left(U_0 + x_{+,0} \frac{V(\gamma)}{8} \right) \sin\left(\frac{x_{+,0}}{2}\right) + \left(U_0 + x_{-,0} \frac{V(-\gamma)}{8} \right) \cos\left(\frac{x_{-,0}}{2}\right) \right], \end{aligned} \quad (4.90)$$

where

$$\begin{aligned} \hat{\mathcal{W}}_n &\equiv \sum_{k=1}^{\infty} (Y_{2k}^{(+)} q_{2k}(\gamma + \eta) + X_{2k+1}^{(+)} q_{2k+1}(\gamma + \eta) + Y_{2k}^{(-)} q_{2k}(\gamma - \eta) + X_{2k+1}^{(-)} q_{2k+1}(\gamma - \eta)) \\ &\quad + \frac{\pi}{8} \mathcal{V}(\gamma) \sin\left(\frac{x_{+,0}}{2}\right) \left(\theta(x_{+,n}) - \frac{1}{2} \right) + \frac{\pi}{8} \mathcal{V}(-\gamma) \cos\left(\frac{x_{-,0}}{2}\right) \left(\theta(x_{-,n}) - \frac{1}{2} \right), \quad \forall n \in \mathbb{Z}. \end{aligned} \quad (4.91)$$

We remind the reader that:

- (i) $\check{B}_{2k}(x_{\pm,n})$ and $\check{B}_{2k+1}(x_{\pm,n})$, as given by Eqs. (4.51) and (4.52), are polynomials (of order $2k$ and $2k+1$ respectively) in $x_{\pm,n}$;
 - (ii) ${}_nq_k(\gamma \pm \eta)$, given by Eqs. (4.67) and (4.82), can be written as a polynomial (of order $k-1$) in $x_{\pm,n}$;
 - (iii) $Y_{2k}^{(\pm)}$ and $X_{2k+1}^{(\pm)}$, given by Eqs. (4.81), (4.41), (4.42), (4.36), and (4.37), depend on the coordinates γ , x^A and $x^{A'}$;
 - (iv) the a_k , given by Eqs. (4.82) and (4.68) are numbers.
- It then follows that:
- (i) the $\mathcal{V}_{\pm,n}$ can be formally expressed as an expansion in powers of $x_{\pm,n}$;
 - (ii) the k -sums in Eq. (4.89) for $\tilde{\mathcal{V}}_{\pm,n}$ can be formally expressed as an expansion in powers of $x_{\pm,n}$;
 - (iii) the k -sum of the terms containing $Y_{2k}^{(\pm)}$ and $X_{2k+1}^{(\pm)}$ in Eq. (4.90) for \mathcal{W}_n can be formally expressed as an expansion in powers of $x_{+,n}$ plus another expansion in powers of $x_{-,n}$.

Here, an “expansion in powers of $x_{\pm,n}$ ” means a series expansion in non-negative integer powers of $x_{\pm,n}$, with coefficients that may depend on $(x^A, x^{A'}, \gamma)$, but that are independent of n . We note that the coefficients in these expansions depend crucially on the 2D Hadamard coefficients U_k of Eq. (2.26). Because of these various properties, together with the fact that Eq. (4.84) makes explicit the full singularity structure of the GF globally, we refer to the expression in Eq. (4.84) as a ‘sum of Hadamard forms’. We note that Eqs. (4.74), (4.76) and (4.84) are not valid when $\eta = 0$, $\gamma = 0$ or $\gamma \in [\gamma_0, \pi]$ (where $\gamma_0 = 2(\sqrt{2}-1)\pi \simeq 0.828\pi$)—we deal with the points $\gamma = 0$ and π in the next section.

V. SINGULARITY AT CAUSTICS

In this section we investigate the singularity of the GF at caustic points of Schwarzschild spacetime, which are points where null geodesics focus. Because of this property and the fact that the GF diverges when the two points in its argument are connected via a null geodesic, it is not surprising that the type of the singularity of the GF at caustics is different from that we have derived above [Eqs. (4.74) and (4.76)], which is valid away from $\gamma = 0$ and π , i.e., the values of γ for caustics in Schwarzschild spacetime.

Let us first deal with the caustics with $\gamma = 0$. In this case, we insert $P_\ell(1) = 1$, $\forall \ell$, in Eq. (4.31) with (2.32). We then just proceed as in Sec. IV and use Eq. (4.1) in order to first gather like powers of L [where we are now of course spared the use of Eq. (4.23)] and obtain

$$G_R^{\ell \geq 1}|_{\gamma=0} = \frac{1}{rr'\sqrt{2\pi\eta}} \sum_{m=0}^{\infty} \left\{ e^{-i\pi/4} \tilde{\mathcal{V}}_m \sum_{\ell=1}^{\infty} \frac{e^{iL\eta}}{L^{m-1/2}} + \text{c.c.} \right\}, \quad (5.1)$$

where ‘c.c.’ denotes the complex conjugation of the term that is preceding it and where

$$\tilde{\mathcal{V}}_m \equiv \sum_{j=0}^m e^{i\pi(j+m/2)} (2\eta)^{2j-m} U_j a_{j,m-j}. \quad (5.2)$$

We note that in this case we have half-integer powers of L , as opposed to Sec. IV where we had integer powers. The highest power of L here is $1/2$, as opposed to 0 in Sec. IV, anticipating a ‘stronger’ divergence of the GF at caustic points. As we are interested here only in the divergent terms, we introduce the notation $A \doteq B$ to mean that $A - B$ is a continuous function. We then have

$$G_R^{\ell \geq 1}|_{\gamma=0} \doteq \frac{1}{rr'\sqrt{2\pi\eta}} [\tilde{\mathcal{G}}_R^0|_{\gamma=0} + \tilde{\mathcal{G}}_R^1|_{\gamma=0}], \quad (5.3)$$

where

$$\begin{aligned} \tilde{\mathcal{G}}_R^0|_{\gamma=0} &\equiv e^{-i\pi/4} e^{i\eta/2} \mathcal{A}_{-1/2}(\eta) \tilde{\mathcal{V}}_0 + \text{c.c.}, \\ \tilde{\mathcal{G}}_R^1|_{\gamma=0} &\equiv e^{-i\pi/4} e^{i\eta/2} \mathcal{A}_{1/2}(\eta) \left(\frac{1}{4} \tilde{\mathcal{V}}_0 + \tilde{\mathcal{V}}_1 \right) + \text{c.c.}, \end{aligned} \quad (5.4)$$

and we have

$$\tilde{\mathcal{V}}_0 = U_0, \quad \tilde{\mathcal{V}}_1 = -\frac{i}{4} \frac{(U_0 + 16\eta^2 U_1)}{2\eta} \equiv -\frac{i}{4} \tilde{U}_1. \quad (5.5)$$

We already dealt with the distributions \mathcal{A}_k for $k \in \mathbb{Z}$ in Sec. IV B and, in the cases \mathcal{A}_0 and \mathcal{A}_1 , originally in [14]. Here we deal similarly with the distributions $\mathcal{A}_{\pm 1/2}$. We have [30],

$$\begin{aligned} \mathcal{A}_{3/2}(\eta) &= \Gamma\left(-\frac{1}{2}\right) (-i\eta)^{1/2} + \sum_{j=0}^{\infty} \zeta\left(\frac{3}{2}-j\right) \frac{(i\eta)^j}{j!}, \\ |\eta| &< 2\pi, \end{aligned} \quad (5.6)$$

where $s \mapsto \zeta(s)$ is the Riemann zeta function. It is straightforward to show that the infinite series in (5.6) is uniformly convergent and hence analytic for $|\eta| < 2\pi$. It follows that the series and its derivatives are smooth on $|\eta| < 2\pi$. Also, the first expression on the right-hand side of Eq. (5.6) involves the principal branch of the square root. Then we can write

$$\mathcal{A}_{3/2}(\eta) = s_{3/2}(\eta) + r_{3/2}(\eta), \quad |\eta| < 2\pi, \quad (5.7)$$

where

$$s_{3/2}(\eta) \equiv -2\sqrt{\pi}(-i\eta)^{1/2} = -2\sqrt{\pi}|\eta|^{1/2} e^{i\pi(1-2\theta(\eta))/4} \quad (5.8)$$

is the singular (nonsmooth) part of $\mathcal{A}_{3/2}$ on $|\eta| < 2\pi$, and $r_{3/2}$ is the regular part. By analyticity of $r_{3/2}$, it follows that

the singular and regular parts of $\mathcal{A}_{1/2}$ and $\mathcal{A}_{-1/2}$ are obtained by applying Eqs. (4.46) to the singular and regular parts of $\mathcal{A}_{3/2}$ separately. Thus, we define the singular parts of $\mathcal{A}_{\pm 1/2}$ (for $|\eta| < 2\pi$) by

$$s_{1/2} = -is'_{3/2}, \quad (5.9)$$

$$s_{-1/2} = -is'_{1/2}, \quad (5.10)$$

where the derivative is taken in the sense of distributions. Calculating the first of these derivatives yields

$$s_{1/2}(\eta) = \delta_1(\eta) + i\delta_2(\eta), \quad (5.11)$$

where

$$\delta_1(\eta) \equiv \sqrt{\frac{\pi}{2}}|\eta|^{-1/2}, \quad (5.12)$$

$$\delta_2(\eta) \equiv \sqrt{\frac{\pi}{2}}|\eta|^{-1/2}(2\theta(\eta) - 1). \quad (5.13)$$

Then

$$s_{-1/2}(\eta) = \delta'_2(\eta) - i\delta'_1(\eta). \quad (5.14)$$

We note that $s_{1/2}$ has the form of a locally integrable function, but the distribution $s_{-1/2}$ does not. The derivatives δ'_i are calculated (and applied) by using the distributional rule

$$\langle \delta'_i, u \rangle = -\langle \delta_i, u' \rangle, \quad i = 1, 2, \quad (5.15)$$

for all test functions u and where $\langle f, g \rangle = \int_{\mathbb{R}} fg dx$.

Collecting terms yields

$$\tilde{G}_R^0|_{\gamma=0} \doteq \sqrt{\pi} \left(\cos \frac{\eta}{2} (\delta'_2 - \delta'_1) + \sin \frac{\eta}{2} (\delta'_2 + \delta'_1) \right) U_0, \quad |\eta| < 2\pi, \quad (5.16)$$

and

$$\begin{aligned} \tilde{G}_R^1|_{\gamma=0} \doteq & \frac{1}{2\sqrt{2}} \left(\left(\cos \frac{\eta}{2} + \sin \frac{\eta}{2} \right) U_0 \delta_1 + \left(\cos \frac{\eta}{2} - \sin \frac{\eta}{2} \right) U_0 \delta_2 \right. \\ & \left. + \left(\cos \frac{\eta}{2} + \sin \frac{\eta}{2} \right) \tilde{U}_1 \delta_2 - \left(\cos \frac{\eta}{2} - \sin \frac{\eta}{2} \right) \tilde{U}_1 \delta_1 \right), \quad |\eta| < 2\pi. \end{aligned} \quad (5.17)$$

It is straightforward to establish that

$$\eta^p \delta_i(\eta) \doteq 0, \quad i = 1, 2, \quad p \geq 1, \quad (5.18)$$

$$\eta^q \delta'_i(\eta) \doteq 0, \quad i = 1, 2, \quad q \geq 2. \quad (5.19)$$

Then in a Taylor expansion of the trigonometric coefficients in (5.16) and (5.17), only $O(\eta^p)$, $p \leq 1$ and $O(1)$ terms contribute to the right-hand sides of (5.16) and (5.17), respectively. Thus, we can write

$$\tilde{G}_R^0|_{\gamma=0} \doteq \sqrt{\pi} \left(-\left(1 - \frac{\eta}{2}\right) U_0 \delta'_1(\eta) + \left(1 + \frac{\eta}{2}\right) U_0 \delta'_2(\eta) \right), \quad |\eta| < 2\pi, \quad (5.20)$$

and

$$\tilde{G}_R^1|_{\gamma=0} \doteq \frac{1}{2\sqrt{2}} ((U_0 - \tilde{U}_1) \delta_1(\eta) + (U_0 + \tilde{U}_1) \delta_2(\eta)), \quad |\eta| < 2\pi. \quad (5.21)$$

To determine the corresponding globally valid expressions, we note that the distributions $\mathcal{A}_k(\eta)$ are periodic with period 2π . This implies that there will also be discontinuous contributions to $\tilde{G}_R^0|_{\gamma=0}$ and $\tilde{G}_R^1|_{\gamma=0}$ when $\eta = 2n\pi$, $n \in \mathbb{Z}$. These contributions are determined by repeating the calculation above, using

$$\mathcal{A}_{3/2}(\eta) = \mathcal{A}_{3/2}(\eta_{e,n}) = \Gamma\left(-\frac{1}{2}\right)(-i\eta_{e,n})^{1/2} + \sum_{j=0}^{\infty} \zeta\left(\frac{3}{2}-j\right) \frac{(i\eta_{e,n})^j}{j!}, \quad |\eta_{e,n}| < 2\pi, \quad (5.22)$$

where

$$\eta_{e,n} = \eta + 2n\pi, \quad n \in \mathbb{Z}. \quad (5.23)$$

Note that

$$e^{i\eta_{e,n}/2} = (-1)^n e^{i\eta/2}. \quad (5.24)$$

This results in

$$\tilde{G}_R^0 \Big|_{\gamma=0} \doteq (-1)^n \sqrt{\pi} \left(-\left(1 - \frac{\eta_{e,n}}{2}\right) U_0 \delta'_1(\eta_{e,n}) + \left(1 + \frac{\eta_{e,n}}{2}\right) U_0 \delta'_2(\eta_{e,n}) \right), \quad |\eta_{e,n}| < 2\pi, \quad (5.25)$$

and

$$\tilde{G}_R^1 \Big|_{\gamma=0} \doteq \frac{(-1)^n}{2\sqrt{2}} ((U_0 - \tilde{U}_1) \delta_1(\eta_{e,n}) + (U_0 + \tilde{U}_1) \delta_2(\eta_{e,n})), \quad |\eta_{e,n}| < 2\pi. \quad (5.26)$$

We then collect terms and use the distributional identities

$$\eta \delta'_i(\eta) = -\frac{1}{2} \delta_i(\eta), \quad i = 1, 2 \quad (5.27)$$

to obtain [reintroducing the θ 's as per (4.32)]

$$G_R \Big|_{\gamma=0} \doteq \theta(\Delta t) \theta(-\sigma) \frac{\sqrt{2}}{rr' \sqrt{\eta}} \sum_{n \in \mathbb{Z}} (-1)^n \left(\frac{U_0 + 16\eta^2 U_1}{8\eta} H_{-\frac{1}{2}}(-\eta_{e,n}) + U_0(\eta) H'_{-\frac{1}{2}}(-\eta_{e,n}) \right), \quad (5.28)$$

where

$$H_{-\frac{1}{2}}(x) \equiv |x|^{-\frac{1}{2}} \theta(x), \quad x \in \mathbb{R}. \quad (5.29)$$

This equation shows the singular behavior of the GF at caustics with $\gamma = 0$.⁷

In order to deal with the caustics with $\gamma = \pi$, we note that $P_\ell(-1) = (-1)^\ell = e^{\pm i\pi\ell}$. In this case, Eq. (5.3) and (5.4) become respectively

$$G_R^{\ell \geq 1} \Big|_{\gamma=\pi} \doteq \frac{1}{rr' \sqrt{2\pi\eta}} [\tilde{G}_R^0 \Big|_{\gamma=\pi} + \tilde{G}_R^1 \Big|_{\gamma=\pi}], \quad (5.30)$$

and

$$\tilde{G}_R^0 \Big|_{\gamma=\pi} \equiv e^{-i\pi/4} e^{i\eta/2} \mathcal{A}_{-1/2}(\eta + \pi) \tilde{V}_0 + \text{c.c.}, \quad \tilde{G}_R^1 \Big|_{\gamma=\pi} \equiv e^{-i\pi/4} e^{i\eta/2} \mathcal{A}_{1/2}(\eta + \pi) \left(\frac{1}{4} \tilde{V}_0 + \tilde{V}_1 \right) + \text{c.c.} \quad (5.31)$$

Otherwise, the calculation proceeds as above, and we find

$$G_R \Big|_{\gamma=\pi} \doteq \theta(\Delta t) \theta(-\sigma) \frac{\sqrt{2}}{rr' \sqrt{\eta}} \sum_{n \in \mathbb{Z}} (-1)^{n+1} \left(\frac{U_0 + 16\eta^2 U_1}{8\eta} H_{-\frac{1}{2}}(\eta_{o,n}) + U_0(\eta) H'_{-\frac{1}{2}}(\eta_{o,n}) \right), \quad (5.32)$$

where

$$\eta_{o,n} = \eta + (2n + 1)\pi, \quad n \in \mathbb{Z}. \quad (5.33)$$

This equation shows the singular behavior of the GF at caustics with $\gamma = \pi$.

⁷As mentioned in Sec. IV A, the coincidence limit (i.e., $\eta, \gamma \rightarrow 0$) is excluded from the region of validity of this result.

These results show that the singularity of the GF at caustic points is ‘stronger’ than at points which are not caustics and that its structure is twofold instead of the observed four-fold structure at points which are not caustics.

As stated in Sec. III A, our results in this paper for the GF in Schwarzschild spacetime should reproduce those in $\mathbb{M}_2 \times \mathbb{S}_2$, times the conformal factor $1/(rr')$, when setting $U_0 = 1$ and $U_k = 0$, $\forall k > 0$. In particular that should be true for Eq. (5.32) above when compared with Eq. (132) of Ref. [14] and it is easy to check that is indeed the case. Furthermore, in Schwarzschild spacetime, Fig. 13 [15] shows a numerical approximation to the GF near a caustic point both for the cases $\gamma = 0$ and $\gamma = \pi$. This figure seems to suggest that if the form, at least to leading order, of the GF about $\eta_e = 0$ when $\gamma = 0$ is, say, ‘ $F(\eta_e)$ ’, then its form about $\eta_o = 0$ when $\gamma = \pi$ is ‘ $-F(-\eta_o)$ ’. This symmetry is manifest to leading order in Eqs. (5.28) and (5.32) for G_R (note, however, that this symmetry is not satisfied by the terms \tilde{G}_R^0 and \tilde{G}_R^1 separately), thus serving as further support for our results at caustic points in Schwarzschild spacetime. In the next section, we also present numerical evidence for our results away from caustics.

VI. NUMERICAL EVIDENCE

In this section we provide numerical evidence for the analytical leading-order divergences that we have obtained: (i) in the Bessel expansion Eq. (2.32); and (ii) in the sum of Hadamard forms in Eq. (4.84). We provide the evidence for (i) and (ii) in, respectively, the first subsection and the second (and third) subsection(s). By “leading-order divergences” here we mean the divergences whose coefficients involve U_0 but no U_k with $k > 0$. We compare both of our analytical results to a semianalytic/numerical calculation of the full GF. In particular, we shall see that the comparison is good for points which are quite far apart, thus indicating that the region Ω_p of convergence of the series in Eq. (2.32) is either equal to the whole of \mathcal{M}_2 or, at least, that it covers a large enough region of physical interest. We further note that [3,20] also numerically corroborated the singularity structure that we have here obtained beyond the maximal normal neighborhood, and applied it to the setting of quantum field theory. In the third subsection, which is the last one, we do not present a new comparison but we briefly discuss the use of the “direct” divergence in Eq. (4.84) which has been made in [3,24,25] for a practical calculation of the GF, thus corroborating the validity of this expression of ours for this direct divergence.

The comparison in $G_R(x, x')$ will be made specifically for points x and x' on a timelike circular geodesic at $r = 6M$ in Schwarzschild spacetime. One can think of the field point x as being fixed and the base point x' varying to the past along the timelike geodesic. Then x and all points x' are connected via this timelike geodesic, and some of the points x' might also be connected to x via a null geodesic that has orbited around the black hole a certain number of times (a different

number for each of these such points x'). We refer to these points x' which are connected to x via a null geodesic (as well as via the timelike geodesic) as *light crossings*; the first light crossing signals the end of the normal neighborhood of x . As we know, the GF diverges at the light crossings. For an illustration of the light crossings and the corresponding null geodesics in the case of the timelike circular geodesic at $r = 6M$, see Fig. 1(a) in [18]. For this specific timelike geodesic, it is easy to check that the light crossings occur at the following times: $\Delta t/M \approx 27.62, 51.84, 58.05, 75.96, 100.09, 108.55, 124.21, \dots$

Before we proceed to the comparison, we give a brief description of the semianalytic/numerical solution. This solution is also obtained by doing an ℓ -mode decomposition but, in this method, each ℓ -mode is calculated as the semianalytic sum of its quasinormal mode and branch cut contributions—see [18] for details. This solution is very accurate but it is not exact, at least for three reasons: (i) its ℓ -modes have been calculated up to a certain numerical accuracy; (ii) the infinite sum has been truncated at the large but finite value of $\ell = 100$; (iii) a smoothing factor $e^{-\ell^2/(2\ell_{\text{cut}}^2)}$ with $\ell_{\text{cut}} = 25$ has been included in the ℓ -sum in order to avert spurious oscillations (see [18,37]). This means that the fine features of the exact GF close to the singularities, as well as its diverging behavior at the singularities themselves, are not captured exactly by the semianalytic/numerical solution (in particular, the singularities are smeared out). Finally, we note that, in a region “close enough” to x , the ℓ -sum as just described does not offer a practical way of calculating the GF. That region is the so-called ‘quasilocal’ region, which is within a normal neighborhood of x and so, if the points are timelike-separated, the GF is just equal to the 4D Hadamard biscalar $V(x, x')$ of Eq. (1.1). Therefore, for calculating the GF very accurately in a quasilocal region, instead of using the mentioned ℓ -sum, we directly calculate $V(x, x')$ using the techniques described in [18,33].

We next proceed to the comparison between the semianalytic/numerical solution and our mentioned leading-order analytical results.

A. Leading term in the Bessel expansion

In this subsection we use the values of σ and U_0 that we numerically calculated in [21]⁸ in order to evaluate the leading ($k = 0$) term in the Bessel expansion Eq. (2.32).

⁸In Fig. 6(a) in [21] we said that we plotted the final values of σ (which on the vertical axis there is labelled σ_2), i.e., the values upon return to the original radius $r = 6M$ after bouncing off at the turning point. However, we actually plotted the values of σ only when reaching the turning point. Equivalently, Fig. 6(a) in [21] is really a plot of $\sigma/4$ (not of σ as wrongly indicated there) as a function of the total coordinate time intervals $\Delta t/M$. [All the other plots in [21] are correct as they are, including Fig. 6(b), which is indeed a plot of the final values of $\Delta^{1/2}$ as a function of $\Delta t/M$, as indicated there.]

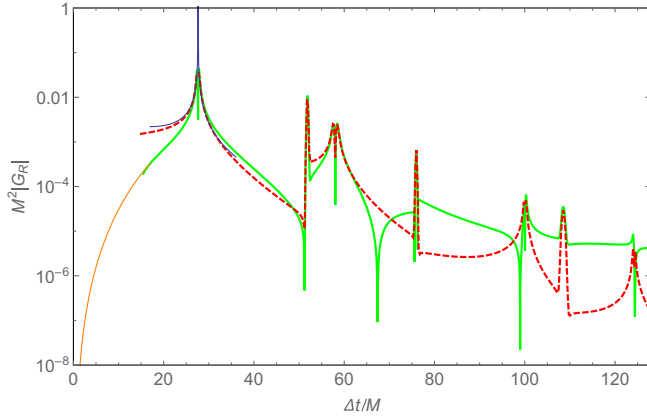


FIG. 1. Log-plots of the retarded Green function and approximations to it on Schwarzschild spacetime as functions of coordinate time interval Δt for points on a timelike circular geodesic at $r = 6M$. The ‘exact’ GF is plotted in the solid orange curve [up to $\Delta t \approx 17M$, which is within a ‘quasilocal’ region, where it is equal to $V(x, x')$ of Eq. (1.1)] matched on to the solid green curve (from $\Delta t \approx 17M$, i.e., outside a ‘quasilocal’ region, where it is calculated using a sum of quasinormal-mode and branch-cut contributions). The dashed red curve is the Bessel expansion Eq. (2.30) with (2.32), where we only use the leading $k = 0$ term and we sum up to a finite value of $\ell = 200$ including a smoothing factor $e^{-\ell^2/(2\ell_{\text{cut}}^2)}$ with $\ell_{\text{cut}} = 25$ (see [18]). The solid blue curve, which is concentrated around the first light crossing only, is the large- ℓ asymptotics of the quasinormal mode contribution to the GF as given by Eq. (55) in Ref. [12].

This leading term provides an approximation for \mathcal{U}_ℓ , which we then insert into Eq. (2.30), in order to obtain an approximation for the GF in Schwarzschild. As we demonstrated in Sec. IV, the leading $k = 0$ term is sufficient to determine the ‘leading order’ of the singularity of the GF along null geodesics; for the next-to-leading order singularity and the regular part of the GF, other $k > 0$ terms contribute as well as the $k = 0$ term. We truncate the ℓ -sum in Eq. (2.30) at $\ell = 200$ and, as in the semianalytic/numerical GF, we include a smoothing factor $e^{-\ell^2/(2\ell_{\text{cut}}^2)}$ with $\ell_{\text{cut}} = 25$ —as a consequence, the singularities are smeared out.

We plot the mentioned leading $k = 0$ approximation as a function of time in Fig. 1, which is to be compared with Figs. 1(b), 10(a), and 11(a) in [18]. In the figure, we compare our approximation to the semianalytic/numerical calculation of the GF mentioned above. We also compare it to another approximation near the first light crossing as given by Eq. (55) [12]. We note a couple of features about the plot. Firstly, as expected, the leading $k = 0$ approximation captures well the light-crossing times given above as well as the behavior of the (smeared-out) GF near the singularities; this is true up to the fourth, if not even seventh, light crossing. Secondly, the leading $k = 0$ term does not approximate well the GF *before* very near the first

light crossing.⁹ This is not surprising, since the time regimes in between singularities essentially correspond to a backscattering (due to the gravitational potential) ‘tail’ whereas we are only including the first 2D Hadamard coefficient U_0 (the square root of the van Vleck determinant Δ_{2d} in the 2D spacetime \mathcal{M}_2). Considering this, the agreement between the leading $k = 0$ term and the semianalytic/numerical GF is rather remarkable in the time regimes between the first, second, third, and fourth light crossings.

B. Leading term in the global Hadamard form

In this subsection we provide numerical evidence for our result in Eq. (4.84) by comparing the semianalytic/numerical calculation of the GF with the result of subtracting from it (a smeared version of) some of the divergences in that equation. Let us next describe exactly what we subtracted from the semianalytic/numerical calculation. We first define the following quantities, which contain the Dirac- δ singularities in Eq. (4.84):

$$G_{\delta,n} \equiv (-1)^n \frac{U_0}{r \cdot r' \sqrt{\eta \sin \gamma}} \Big|_{x_{-,n}=0} \delta(x_{-,n}). \quad (6.1)$$

We now define part of the PV and logarithmic singularities in Eq. (4.84) with the coefficients evaluated at the singularities as

$$G_{PVL,n} \equiv \frac{(-1)^n \theta(-\sigma) \theta(\Delta t)}{\pi} \frac{U_0}{r \cdot r' \sqrt{\eta \sin \gamma}} \Big|_{x_{+,n}=0} \times \left(\text{PV} \left(\frac{1}{x_{+,n}} \right) + \frac{1}{8} \left(\frac{1}{\eta} + \cot \gamma \right) \ln |x_{+,n}| \right). \quad (6.2)$$

We note that, here, we have only included the U_0 term¹⁰ in \mathcal{V} in Eq. (4.77) and we have evaluated $\cos(\frac{x_{+,0}}{2})$ at $x = x_{+,n}$, and so we replaced it by $(-1)^n$.

⁹We note that the behavior of the semianalytic/numerical calculation (solid green curve) around the first light crossing manifests a Dawsonian shape [see Eq. (6.4)], corresponding to the ℓ smoothing that has been applied. Although similar smoothing has also been applied to the calculation using the Bessel expansion Eq. (2.30) with (2.32) (dashed red curve), it manifests no similar Dawsonian shape around the first light crossing (although the peak at the light crossing is manifest) just due to the large step size in time used in the calculation (so it is just a ‘numerical artifact’); the Dawsonian shape in this curve, however, is indeed manifest around some later light crossings. In its turn, the calculation using the large- ℓ asymptotics of the quasinormal mode contribution (solid blue curve) exhibits a peak at the first light crossing and no Dawsonian shape because, in this case, no ℓ smoothing was actually applied.

¹⁰Including the U_1 term would involve an extra, more difficult numerical calculation.

We would use the quantities $G_{\delta,n}$ and $G_{PVL,n}$ if we were to subtract quantities from the exact GF. However, as mentioned, our semianalytic/numerical GF includes a finite upper limit in the ℓ -sum and a smoothing factor $e^{-\ell^2/(2\ell_{\text{cut}}^2)}$, thus smearing out the divergences. Therefore, instead of using $G_{\delta,n}$ and $G_{PVL,n}$, we shall use versions where their divergences have been smeared-out in a similar manner. Remembering that the leading-order divergences come from $\mathcal{A}_0(x_{\pm,n}) = \sum_{\ell=1}^{\infty} e^{i\ell x_{\pm,n}}$ [Eq. (4.40)] and taking cognizance of Eq. (50) of [12] but also including in the integrand the smoothing factor $e^{-\ell^2/(2\ell_{\text{cut}}^2)}$, the Dirac- δ singularity is smeared out as the following Gaussian-like distribution:

$$\begin{aligned} G_{\text{Gaus}}(x_{-,n}) &\equiv \frac{1}{\pi} \text{Re} \left(\int_0^{\infty} d\ell e^{i(\ell+1/2)x_{-,n} - \ell^2/(2\ell_{\text{cut}}^2)} \right) \\ &= \frac{\ell_{\text{cut}}}{\sqrt{2\pi}} e^{-\ell_{\text{cut}}^2 x_{-,n}^2 / 2} \left(-\sin\left(\frac{x_{-,n}}{2}\right) \text{Erfi}\left(\frac{\ell_{\text{cut}} x_{-,n}}{\sqrt{2}}\right) \right. \\ &\quad \left. + \cos\left(\frac{x_{-,n}}{2}\right) \right), \end{aligned} \quad (6.3)$$

where Erfi is the imaginary error function. Similarly, but taking the imaginary part instead of the real part, the PV singularity is smeared out as the following Dawson-like distribution (“Dawsonian”) (see Eq. 10 [15]),

$$\begin{aligned} G_{\text{Daws}}(x_{+,n}) &\equiv \text{Im} \left(\int_0^{\infty} d\ell e^{i(\ell+1/2)x_{+,n} - \ell^2/(2\ell_{\text{cut}}^2)} \right) \\ &= \ell_{\text{cut}} \sqrt{\frac{\pi}{2}} e^{-\ell_{\text{cut}}^2 x_{+,n}^2 / 2} \left(\cos\left(\frac{x_{+,n}}{2}\right) \right. \\ &\quad \left. \times \text{Erfi}\left(\frac{\ell_{\text{cut}} x_{+,n}}{\sqrt{2}}\right) + \sin\left(\frac{x_{+,n}}{2}\right) \right). \end{aligned} \quad (6.4)$$

We now denote by $G_{G,n}$ the result of replacing $\delta(x_{-,n})$ by $G_{\text{Gaus}}(x_{-,n})$ in Eq. (6.1) for $G_{\delta,n}$, and we denote by $G_{DL,n}$ the result of replacing $\text{PV}(\frac{1}{x_{+,n}})$ by $G_{\text{Daws}}(x_{+,n})$ in Eq. (6.2) for $G_{PVL,n}$. The quantities that we subtracted from our semianalytic/numerical GF were, separately in different plots: $G_{PVL,+1}$, $G_{PVL,+2}$ and $G_{\delta,-1}$, i.e., the PV- and logarithmic-singularities both at the first light crossing (i.e., at $\hat{\sigma}_1 = 0$, $\eta = 2\pi - \gamma$, $x_{+,+1} = \gamma + \eta - 2\pi = 0$) and at the third light crossing (i.e., at $\hat{\sigma}_3 = 0$, $\eta = 4\pi - \gamma$, $x_{+,+2} = \gamma + \eta - 4\pi = 0$), and the Dirac- δ singularity at the second light crossing (i.e., at $\hat{\sigma}_2 = 0$, $\eta = 2\pi + \gamma$, $x_{-,-1} = \gamma - \eta + 2\pi = 0$).

In Fig. 2 we show three plots which contain both the full semianalytic/numerical GF (solid green curve; same curve as in Fig. 1) as well as the result of subtracting (a) $G_{PVL,+1}$, (b) $G_{\delta,-1}$ and (c) $G_{PVL,+2}$ from it (dashed red curves). We do it for the same setting as in Fig. 1, i.e., as a function of Δt for points on a timelike circular geodesic at $r = 6M$. In Fig. 2 we can see that the singularities in Eq. (4.84)

smeared as per $G_{G,n}$ and $G_{DL,n}$ capture well the behavior of the semianalytic/numerical GF. This is true up to at least the third light crossing (after that, the overall magnitude has decreased too much to allow for such a precise check). The removal of $G_{PVL,n}$ and $G_{\delta,n}$, which only contain the U_0 -dependent coefficients in Eq. (4.84), “flattens” out the GF in a striking way, as can be seen in Fig. 2; these terms thus seem to form the “backbone” of the GF. The “flattening” of the divergences is not perfect since, after all, we are just carrying out approximations to the GF; there is an ℓ -mode truncation and a smoothing-factor in the semianalytic/numerical solution, which we somewhat mirror in $G_{PVL,n}$ and $G_{\delta,n}$; in Eq. (6.2) we only include the term that contains U_0 in the coefficient of the logarithmic divergence, not the full coefficient and the semianalytic/numerical solution has some numerical error; etc.

C. Direct term in the global Hadamard form

A separate numerical corroboration of, specifically, our expression for the “direct” divergence, namely, the $\delta(x_{-,n=0})$ term in Eq. (4.84), i.e., $G_{\delta,n=0}$ in Eq. (6.1), is provided in [3,24,25]. In these works, our expression for such a term proves to be very useful for a practical calculation of the GF. As mentioned above, in a quasilocal region, the semianalytic/numerical ℓ -sum is not practical for calculating the GF and so one typically needs to supplement it with a calculation of $V(x, x')$ inside the quasilocal region. The reason is essentially due to the fact that truncation of the ℓ -sum and inclusion of a smoothing factor cause the direct $\delta(\sigma_{4d})$ singularity in Eq. (1.1) to smear out in a way that it “contaminates” the calculation at field points x' near the base point x . In [25] it is shown that by obtaining the ℓ -modes of $G_{\delta,n=0}$ and subtracting them from the ℓ modes of the numerical/semianalytic ℓ -sum, the resulting sum converges at points x' much closer to x than without such subtraction. This means that a calculation of $V(x, x')$ might potentially not be needed or that, even if it is needed, it will be at a smaller region (and so if it is calculated via, e.g., an expansion near x , then many less terms are needed in the expansion). The application of this subtraction trick in [3], enabled a practical calculation of the GF in a much larger region of pairs of points in Schwarzschild spacetime, which is there applied to obtaining the signal strength between two quantum particle detectors in this spacetime.

We finish by noting a connection for the leading term in the exact Bessel expansion Eq. (2.32), i.e., the term considered in Sec. VI A. By using Eq. (4.23), it can be seen¹¹ that Eq. (20) in [25] for G_{ℓ}^d , which are the ℓ modes of $\theta(\Delta t)G_{\delta,n=0}$, reduces, for large- ℓ , to the $k = 0$ term in Eq. (2.32). That is, it reduces to the term $\theta(-\sigma)\theta(\Delta t)U_0 \cdot J_0((\ell + 1/2)\eta)$, except

¹¹The statements in this sentence and the one that follows it are true modulo a factor of 4π , since the definitions of G_{ℓ} here and of G_{ℓ}^{ret} in [25] differ by such a factor.

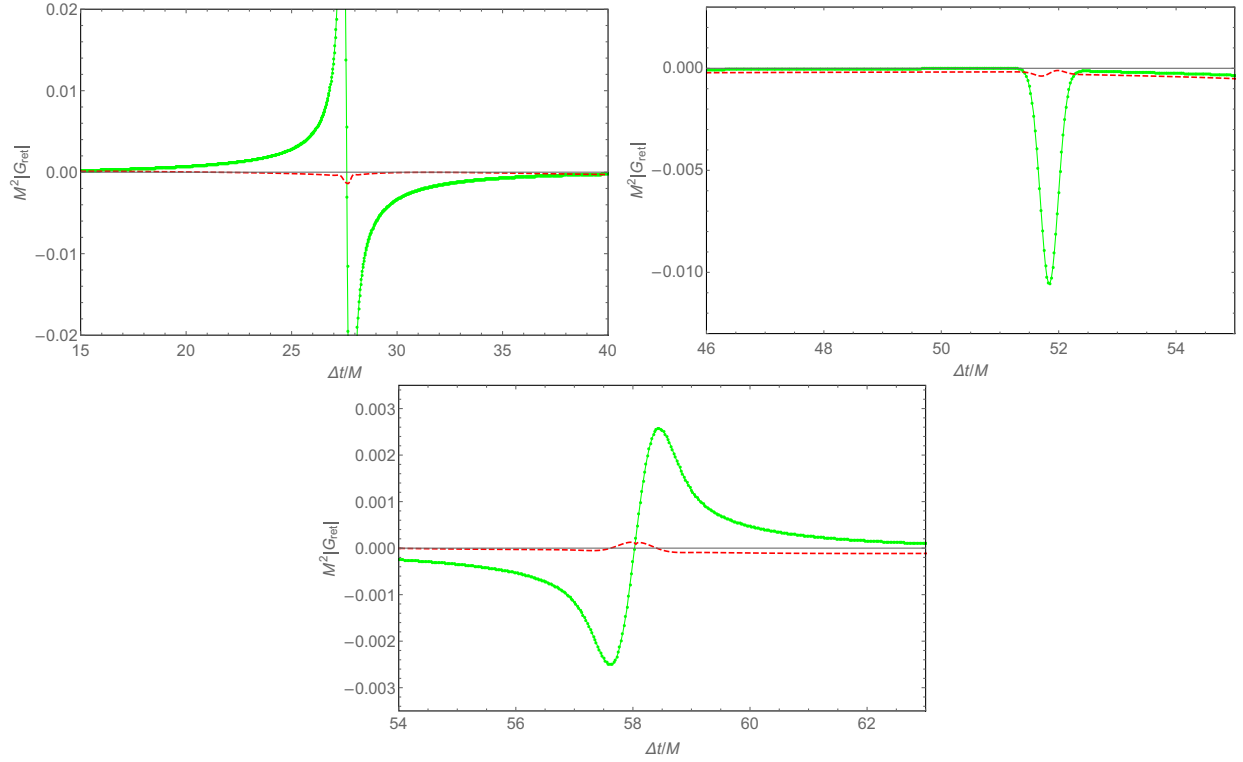


FIG. 2. Comparison of the semianalytic/numerical GF and its divergent behavior derived in Eq. (4.84) as a function of Δt for points on a timelike circular geodesic at $r = 6M$. The solid green curve is the semianalytic/numerical GF and the dashed red curve is the semianalytic/numerical GF minus one of the following diverging terms, depending on the plot: (a) (top left) $G_{PVL,+1}$, plotted near the first light crossing; (b) (top right) $G_{\delta,-1}$, plotted near the second light crossing; (c) (bottom) $G_{PVL,+2}$, plotted near the third light crossing. That is, the solid green curve is the GF minus a smeared version of the leading-PV and subleading-log divergences at the first and third light crossings, and minus the leading-Dirac- δ divergence at the second light crossing; see Eqs. (6.3) and (6.4).

for a factor $\theta(\pi - \eta)$ instead of $\theta(-\sigma)$, which is acceptable since, while G_ℓ^d in [25] is a clean-cut expression for the ℓ modes of the direct part, the $k = 0$ term in Eq. (2.32) does not correspond *just* to the direct part.

VII. CONCLUSIONS

In this paper we have derived representations for the retarded Green function of the wave equation for scalar field perturbations of Schwarzschild spacetime. Our starting point is the *globally valid* expression

$$G_R(x, x') = \frac{1}{r \cdot r'} \theta(-\sigma) \theta(\Delta t) \times \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \mathcal{U}_\ell(x^A, x^{A'}) P_\ell(\cos \theta). \quad (7.1)$$

See (2.30) and (2.32). This expression arises naturally by re-expressing the Schwarzschild metric as conformal to the direct product of the two-sphere and a 2-dimensional spacetime, \mathcal{M}_2 . Global validity stems from the global existence (and uniqueness) of the 2-dimensional Riemann functions \mathcal{U}_ℓ , defined throughout \mathcal{M}_2 . A necessary ingredient of this statement is the fact that \mathcal{M}_2 is a causal

domain; the normal neighborhood of any event of \mathcal{M}_2 is the whole spacetime. This was established in [21]. We then express \mathcal{U}_ℓ as a sum of Bessel functions with coefficients given by the Hadamard coefficients of the Riemann function U . This sum converges whenever the Hadamard series of U converges; numerical evidence (see Sec. VI as well as [3,20]) indicates that this holds on regions Ω_p of \mathcal{M}_2 that yield results extending well beyond the normal neighborhood of (the 4-dimensional) Schwarzschild spacetime.

We then took two different approaches to obtain complementary details of G_R on Schwarzschild spacetime. The first (see Sec. III) exploited a connection with the retarded Green function of Plebański-Hacyan spacetime, whereas the second (see Sec. IV) employed asymptotic expansions of Bessel functions. In each case, we obtained results describing the singularity structure of G_R on Schwarzschild and its representation as, using the former approach, integrals of the Plebański-Hacyan Green function and, using latter approach, a ‘sum of Hadamard forms’. These representations make explicit the *complete* singularity structure of the Schwarzschild Green function and is given in terms of the Hadamard coefficients of the ‘background’ (1+1)-wave equation, Eq. (2.22). Thus, Eq. (4.84) is effectively an extension of the Hadamard form for the Schwarzschild

Green function, which is only valid in normal neighborhoods. Specifically, our representation shows that, away from caustics, the singularity structure is fourfold; the ‘direct’ part changes as $\delta(\sigma_{4d})$, $\text{PV}(1/\sigma_{4d})$, $-\delta(\sigma_{4d})$, $-\text{PV}(1/\sigma_{4d})$, $\delta(\sigma_{4d})$, ..., whereas the ‘tail’ part changes as $\theta(-\sigma_{4d})$, $-\ln|\sigma_{4d}|$, $-\theta(-\sigma_{4d})$, $\ln|\sigma_{4d}|$, $\theta(-\sigma_{4d})$, Here, σ_{4d} represents a globally well-defined generalization of the world function in Schwarzschild spacetime once a geodesic is specified [see Eqs. (2.14) and (4.85)]. The changes in the character of the singularity take place as the null wave front passes through caustics of the spacetime. We have separately analyzed the singularity of the Green function along the line of caustics in Schwarzschild spacetime. We have shown that, in this case, the singularity structure is instead twofold and is ‘stronger’ than away from caustics—the result is a little more involved than in the noncaustic case and is given in Eq. (5.28), for $\gamma = 0$ and in (5.32) for $\gamma = \pi$. The type of singular behavior encountered here is strongly tied to the fact

that we are at a caustic where, in the present case, a 2-parameter family of null geodesics from x' reconverges at x . This is distinct to the situation addressed elsewhere in the paper, where singular behavior is encountered at light crossing, whereat a single causal geodesic from x' intersects the future light cone of x' . This behavior that we found at caustics occurs in Schwarzschild spacetime and in (some) other spherically symmetric spacetimes (such as in Plebański-Hacyan spacetime, see [14]), but it is not clear how general this behavior is.

In [11,14] the following heuristic explanation was put forward for the fourfold singularity structure of the ‘direct’ part of the Green function. The retarded Green function can be obtained from the Feynman Green function as $G_R(x, x') = 2\text{Re}(G_F(x, x'))\theta_+(x, x')$ [27]. The Hadamard form for the Feynman Green function in $(3+1)$ -dimensions is:

$$G_F(x, x') = \lim_{\epsilon \rightarrow 0^+} \frac{i}{2\pi} \left[\frac{U_{4d}(x, x')}{\sigma_{4d} + i\epsilon} - V_{4d}(x, x') \ln(\sigma_{4d} + i\epsilon) + W(x, x') \right], \quad (7.2)$$

where $W(x, x')$, like $U_{4d}(x, x')$ and $V_{4d}(x, x')$, is a regular and real-valued biscalar in a normal neighborhood of x' . Since $\lim_{\epsilon \rightarrow 0^+} 1/(\sigma_{4d} + i\epsilon) = \text{P.V.}(\frac{1}{\sigma_{4d}}) - i\pi\delta(\sigma_{4d})$ and $\lim_{\epsilon \rightarrow 0^+} \ln(\sigma_{4d} + i\epsilon) = \ln|\sigma_{4d}| + i\pi\theta(-\sigma_{4d})$, the Hadamard form Eq. (1.1) for the retarded Green function readily follows from that of the Feynman Green function, Eq. (7.2).

Now, in $(3+1)$ -dimensions, the biscalar $U_{4d}(x, x')$ is related to the van Vleck determinant $\Delta(x, x')$ in that $(3+1)$ -dimensional spacetime as $U_{4d}(x, x') = \Delta(x, x')^{1/2}$. It can be argued [11] that, in a spherically symmetric spacetime, the van Vleck determinant picks up a phase of ‘ $-\pi$ ’ as the geodesic along which it is evaluated crosses a caustic point. That is, $\Delta^{1/2} = e^{-i\pi/2}|\Delta|^{1/2}$ after the geodesic has crossed a first caustic. If one then tentatively evaluated the retarded Green function from the form Eq. (7.2) for the Feynman Green function after the geodesic has crossed a first caustic point (this is, of course, not rigorously justified since the Hadamard form is only valid within a normal neighborhood, which cannot contain caustic points; however, [38] provides numerical evidence that this works) the singularity $\text{PV}(1/\sigma_{4d})$ —instead of the $\delta(\sigma_{4d})$ —would be obtained for the ‘direct’ part. As the geodesic crosses later caustics, $\Delta^{1/2}$ picks up a phase ‘ $-\pi/2$ ’ every time and the fourfold structure (considering only the distributions, not their coefficients) for the leading singularity of the retarded Green function [that we have derived in this paper—see Eq. (1.2)] would ensue. As suggested in [14], if the same phase ‘ $-\pi/2$ ’ were picked up by the biscalar $V_{4d}(x, x')$ at each caustic crossing (this is known to be true—at least for ν_0 , the first term in the Hadamard series for $V_{4d}(x, x')$, see Eq. (3.13)—in the case of PH spacetime), then the fourfold structure for the subleading

discontinuity of the retarded Green function that we have derived [see Eq. (1.3)] would follow similarly. The corresponding leading and subleading singularity structures for the imaginary part of the Feynman Green function would then be, respectively, $\text{PV}(1/\sigma_{4d}) \rightarrow -\delta(\sigma_{4d}) \rightarrow -\text{PV}(1/\sigma_{4d}) \rightarrow \delta(\sigma_{4d}) \rightarrow \text{PV}(1/\sigma_{4d}) \dots$ (corroborated numerically in [20,38]) and $-\ln(\sigma_{4d}) \rightarrow -\theta(-\sigma_{4d}) \rightarrow \ln(\sigma_{4d}) \rightarrow \theta(-\sigma_{4d}) \rightarrow -\ln(\sigma_{4d})$.

Although we have proved these singularity structures for the retarded Green function specifically for the case of a scalar field, we expect that the leading singularity structures $[\delta(\sigma_{4d}) \rightarrow \text{PV}(1/\sigma_{4d}) \dots]$ carry over to higher-spin field perturbations of Schwarzschild spacetime. The reason is that the leading singularity structure of the Green function is dictated by the coefficients of the second derivatives in the wave equation—which may be any of the generalized Regge-Wheeler or Teukolsky equations. The spin does not appear in the second-order terms of these equations.

Apart from the intrinsic theoretical interest in having a ‘global Hadamard form’ for the retarded Green function, such an expression may have various applications. For example, in the calculation of the self-force acting on a particle moving in a curved background spacetime. In [11,18,19] the self-force was calculated via a time integration of the GF along the past worldline of the particle. These calculations illustrated how the self-force may be seen as arising from backscattering of the field perturbation and from trapping of null geodesics, both of which are perfectly encapsulated within our ‘global Hadamard form’. In its turn, in the context of quantum field theory, the global singularity structure/Hadamard form has found

applications within quantum communication [3] as well as entanglement harvesting [20]. In terms of specific practical use, it has been found in [3,24,25] that subtracting, mode-by-mode, an ℓ -mode decomposition of an approximation of the direct term $U_{4d}(x, x')\delta(\sigma_{4d})$ that we have found here [specifically, the term containing δ in Eq. (4.74) for $n = 0$] from a numerical calculation (such as the one carried out in [19]) of the ℓ -modes of the retarded Green function significantly increases the accuracy of the resulting, numerical retarded Green function.

One other possible application of our results to the self-force would be to combine one of the methods used in [18,19] (these methods are valid outside the quasilocal region; it can either be a numerical calculation of the retarded Green function or else a semianalytic sum of its quasinormal-mode and branch-cut contributions) together with our Eq. (4.84) in different time regimes. We hope to investigate these possible applications in the future.

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APPENDIX A: EXAMPLES OF MULTIPOLAR MODES IN OTHER SPACETIMES

For illustration purposes, in this appendix we present the multipolar modes G_ℓ of the retarded Green function Eq. (2.30) in two simple 4-dimensional cases¹²: flat spacetime and a static region of Nariai spacetime (in this case we only do it for the direct part of the Hadamard form). Note that the modes G_ℓ of PH spacetime are readily inferable from Eq. (3.3).

1. Flat spacetime

Consider 4-dimensional flat spacetime. The line element in spherical coordinates can be obtained from that in Schwarzschild in (2.2) by setting the mass to zero ($M = 0$),

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 = r^2(ds_2^2 + d\Omega_2^2), \quad (\text{A1})$$

where

$$ds_2^2 = \frac{1}{r^2}(-dt^2 + dr^2) \quad (\text{A2})$$

is the line element of the 2-D conformal space \mathcal{M}_2 corresponding to this case, whose Synge world function we denote in the current section by σ .

¹²The multipolar modes of the direct part of the retarded Green function in Schwarzschild spacetime have been obtained in [25].

The Hadamard form for the GF of the wave equation for a massless scalar field in flat spacetime is valid everywhere in the spacetime. In spherical coordinates it is given by

$$G_R = \theta(\Delta t)\delta(\sigma_{4d}) = \frac{\theta(\Delta t)}{r \cdot r'}\delta(z - u), \quad (\text{A3})$$

where γ is the angular separation between the two points,

$$z \equiv \cos \gamma, \quad u \equiv \frac{-\Delta t^2 + r^2 + r'^2}{2r \cdot r'}, \quad (\text{A4})$$

and $\sigma_{4d} = r \cdot r'(u - z)$ is Synge's world function in flat spacetime (after a trivial use of the law of cosines). From Eq. (2.30), and making use of the orthogonality of the Legendre polynomials, the multipolar modes G_ℓ can readily be evaluated as

$$\begin{aligned} G_\ell &= 2\pi r \cdot r' \int_{-1}^{+1} dz P_\ell(z) G_R \\ &= 2\pi \theta(\Delta t) \theta(-\sigma_{4d}(\gamma = 0)) \theta(\sigma_{4d}(\gamma = \pi)) P_\ell(u). \end{aligned} \quad (\text{A5})$$

The presence of the θ -step distributions comes from ensuring that the argument of the second δ -distribution in Eq. (A3) is within the limits of integration in the integral in Eq. (A5). We note that $\theta(\Delta t)\theta(-\sigma_{4d}(\gamma = 0)) = \theta(\Delta t - |r - r'|)$ corresponds to the shortest possible spatial distance between two points in flat spacetime given Δt positive, r and r' , whereas $\theta(\Delta t)\theta(\sigma_{4d}(\gamma = \pi)) = \theta(\Delta t)\theta(r + r' - \Delta t)$ corresponds to the largest possible one—see Fig. 3 for an illustration. This combination of θ 's is, therefore, as constraining as possible if their arguments are to be ℓ -independent; there are no two points in flat spacetime with $\Delta t < |r - r'|$ or with $r + r' < \Delta t$ which may be joined by a null geodesic (and a massless field in flat spacetime propagates purely along null geodesics).

We also note that the Legendre function $P_\ell(u)$ in Eq. (A5) is precisely the Riemann function \mathcal{U}_ℓ of the wave equation Eq. (2.21) on the 2D conformal space with $M = 0$ (see also Appendix B). This is because (1) it is a homogeneous solution of this equation,

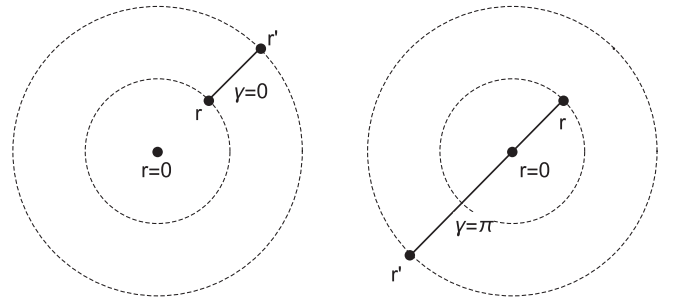


FIG. 3. Illustration of two spatial points in flat spacetime at the shortest (a) and largest (b) possible spatial distances given fixed radii r and r' .

$$\left(-\partial_t^2 + \partial_r^2 - \frac{\ell(\ell+1)}{r^2}\right)P_\ell\left(\frac{-\Delta t^2 + r^2 + r'^2}{2r \cdot r'}\right) = 0, \quad (\text{A6})$$

and (2) it satisfies the boundary condition: $P_\ell(u=1) = 1$, as can be readily checked. We note that $\theta(-\sigma_{4d}(\gamma=0)) = \theta(-\sigma)$ and that, in normal neighborhoods in \mathcal{M}_2 [as explained below Eq. (B1)], \mathcal{M}_2 is not a convex domain when $M=0$, so that, in this case, the maximal normal neighborhood of a point is not the whole of \mathcal{M}_2 , $\theta(\sigma_{4d}(\gamma=\pi)) = 1$. Therefore, in normal neighborhoods in \mathcal{M}_2 , the expression (A5) for G_ℓ reduces to the expression (2.32) for the ℓ -modes of the GF, as expected. Thus, for $M=0$, the exact (A5) extends globally the expression (2.32) which is only valid in normal neighborhoods in \mathcal{M}_2 , rendering the ℓ -modes exactly zero beyond these neighborhoods due to the distributional factor $\theta(\sigma_{4d}(\gamma=\pi))$ (such a Heaviside distribution cannot appear in an expression for G_ℓ in a general spherically symmetric spacetime because, as opposed to the flat spacetime case here, the Hadamard tail is generally nonzero).

2. Nariai spacetime

In a static region of Nariai spacetime ($dS_2 \times \mathbb{S}_2$) the line-element can be written as [11]

$$ds^2 = -(1-\rho^2)dt^2 + (1-\rho^2)^{-1}d\rho^2 + d\Omega_2^2, \quad (\text{A7})$$

where $\rho \in (-1, +1)$, $t \in (-\infty, +\infty)$ and we have set the cosmological constant to be equal to unity. Synge's world function is given by $\sigma_{4d} = \sigma_2 + \gamma^2/2$ where σ_2 is the world function in 2-dimensional de Sitter spacetime, dS_2 , and is given by

$$\cosh \sqrt{-2\sigma_2} = \rho \cdot \rho' + (1-\rho^2)^{1/2}(1-\rho'^2)^{1/2} \cosh \Delta t. \quad (\text{A8})$$

In this case, we were not able to find the multipolar decomposition G_ℓ of the full G_R . However, it is straightforward to find the multipolar decomposition of the direct part of the Hadamard form for G_R . This direct part is given by $G^{dir}(x, x') \equiv U_{4d}(x, x')\delta(\sigma_{4d})\theta(\Delta t)$ where [11,14]

$$U_{4d}(x, x') = \left(\frac{\sqrt{-2\sigma_2}}{\sinh \sqrt{-2\sigma_2}}\right)^{1/2} \left(\frac{\gamma}{\sin \gamma}\right)^{1/2} \quad (\text{A9})$$

and

$$\delta(\sigma_{4d}) = \frac{\delta(\gamma + \sqrt{-2\sigma_2}) + \delta(\gamma - \sqrt{-2\sigma_2})}{|\sqrt{-2\sigma_2}|}. \quad (\text{A10})$$

We can now readily calculate the ℓ -modes of the direct part,

$$\begin{aligned} & \int_{-1}^{+1} d(\cos \gamma) P_\ell(\cos \gamma) G^{dir} \\ &= \theta(\Delta t) \theta(-\sigma_{4d}(\gamma=0)) \theta(\sigma_{4d}(\gamma=\pi)) \\ & \times P_\ell(\cos \sqrt{-2\sigma_2}) \left(\frac{\sin \sqrt{-2\sigma_2}}{\sinh \sqrt{-2\sigma_2}}\right)^{1/2}. \end{aligned} \quad (\text{A11})$$

The causality structure is equivalent to that in flat spacetime, Eq. (A5).

APPENDIX B: CASE OF MASS $M=0$

In this appendix, we consider the case of vanishing mass, $M=0$. This allows us to investigate the convergence of the Hadamard series (2.26) in this limiting case, and to make connections between Green functions on different maximally symmetric manifolds, and direct products of such manifolds. As a by-product, we obtain an improved approach to the calculation of Hadamard coefficients in certain cases, via the result of Szego mentioned above [32].

For $M=0$, the line element (2.5) of conformal Schwarzschild spacetime becomes

$$d\hat{s}^2 = -\frac{1}{r^2}(dt^2 - dr^2) + d\Omega_2^2. \quad (\text{B1})$$

This can readily be recognized as the line element of Bertotti-Robinson spacetime, $\text{AdS}_2 \times \mathbb{S}_2$ [39,40], i.e., \mathcal{M}_2 is (a patch of) 2-dimensional anti-de Sitter spacetime AdS_2 . In this case we have an 'enhanced' symmetry between the two factors of conformal Schwarzschild spacetime $\hat{\mathcal{M}}$, namely between the factor $M_2 = \text{AdS}_2$ and the factor \mathbb{S}_2 ; both are maximally symmetric manifolds and the corresponding Riemann functions for certain differential operators are formally the same (as functions of their corresponding geodesic distances), as we shall next see. Before that, though, we point out that the global structures of AdS_2 and the 2-dimensional conformal Schwarzschild spacetime for $M \neq 0$ are very different. The former is not geodesically convex; there are events x in the causal future of any event x' which are not linked by any future-directed causal geodesic from x' to x . (We note that the proof of the geodesic convexity of the 2-dimensional conformal Schwarzschild spacetime of [21] is not valid when $M=0$.) We can link this to the fact that the 'soft' potential barrier of \mathcal{M}_2 at $r=3M$ (see [21]) is replaced by a repulsive barrier at $r=0$ which forms part of the boundary of the patch of AdS_2 under consideration; $\{r=0\}$ is accessible only to null geodesics of infinite affine length. However, this difference does not impact the analysis below which applies on normal neighborhoods of AdS_2 . In particular, the analysis is valid for regions of the form $\{x^A: \eta(x^A, x^{A'}) \in [0, \pi)\}$ where $\eta(x^A, x^{A'})$ is the geodesic distance along geodesics emanating from $x^A \in \text{AdS}_2$; geodesics remain unique for (at least) this amount of geodesic distance. So, in the particular case that

x^A and $x^{A'}$ are connected by a timelike geodesic, η is the proper time along that geodesic.

As seen in Sec. II above, the 4-dimensional retarded Green function G_R can be constructed from a sequence of 2-dimensional retarded Green functions G_ℓ , $\ell \geq 0$ which in the present case we write as G_{ℓ, AdS_2} . This takes the form [see (2.32)]

$$G_{\ell, \text{AdS}_2} = 2\pi \mathcal{U}_\ell \theta(-\sigma_{\text{AdS}_2}) \theta(\Delta t), \quad (\text{B2})$$

where¹³ $\sigma_{\text{AdS}_2} = \epsilon \eta^2/2$ and $\epsilon = -1, 0, +1$ for, respectively, timelike, null, and spacelike geodesics. The two-point function \mathcal{U}_ℓ is the Riemann function of the operator $P = \square_2 + \lambda$ in the case $\lambda = -\ell(\ell + 1)$ (i.e., the operator in (2.24) with $M = 0$). It is convenient to consider the Riemann function U of P more generally, which satisfies

$$(\square_2 + \lambda)U = 0, \quad U|_{\eta=0} = 1. \quad (\text{B3})$$

In this maximally symmetric spacetime, U depends only on the geodesic distance η (see e.g., [41]). So $U = U(\eta)$, and we can then show that

$$U''' + \cot \eta U' - \lambda U = 0, \quad U(0) = 1, \quad (\text{B4})$$

where the prime represents derivative with respect to argument. This problem has a unique solution given by the Legendre function $U = P_\nu(\cos \eta)$ with $\nu \equiv (-1 + \sqrt{1 - 4\lambda})/2$. We note that the second linearly independent solution of the ODE (B4) is singular at $\eta = 0$. Specializing to the case $\lambda = -\ell(\ell + 1)$, we see that (as already noted in Appendix A 1)

$$\mathcal{U}_\ell(x^A, x^{A'}) = P_\ell(\cos \eta), \quad \eta \in [0, \pi), \quad \ell \geq 0. \quad (\text{B5})$$

This is the Riemann function of the operator $\square_2 - \ell(\ell + 1) = \square_2 + \frac{1}{4} - L^2$ where $L = \ell + 1/2$. Thus in this case, the ‘background’ wave operator is $\square_2 + \frac{1}{4}$ [consistent with (2.23) in the case $M = 0$], and the background Riemann function U [cf. Eq. (2.26)] is given in this case by setting $\lambda = 1/4$, which yields $\nu = -1/2$,

$$U(x^A, x^{A'}) = P_{-1/2}(\cos \eta), \quad \eta \in [0, \pi). \quad (\text{B6})$$

Alternatively, we can use the fact that, for $M = 0$, Schwarzschild spacetime is just Minkowski spacetime and so its GF is just [see (A3)]¹⁴ but also written here in terms of van Vleck determinants as per, e.g., [25]

¹³In the generic-mass case throughout the main body of the paper, σ_{AdS_2} is denoted by σ .

¹⁴In normal neighborhoods of $\mathcal{M}_2 = \text{AdS}_2$ and for causal separations, u in (A3) is equal to $\cos \eta$ here.

$$\begin{aligned} G_R(x, x') &= \theta(\Delta t) \delta(\sigma_{4d}) = \frac{\theta(\Delta t)}{r \cdot r'} \Delta_{2d}^{1/2} \Delta_{\mathbb{S}_2}^{1/2} \delta(\hat{\sigma}) \\ &= \frac{1}{r \cdot r'} \frac{\delta(\eta - \gamma)}{\sin \eta}, \end{aligned} \quad (\text{B7})$$

where

$$\Delta_{2d} = \frac{\eta}{\sin \eta}, \quad \Delta_{\mathbb{S}_2} = \frac{\gamma}{\sin \gamma}, \quad (\text{B8})$$

are the van Vleck determinants in $\mathcal{M}_2 = \text{AdS}_2$ [already introduced in (2.28)] and in \mathbb{S}_2 , respectively. Using the standard distributional representation of the Dirac delta function in terms of Legendre polynomials [e.g., Eq. (1.17.22) in Ref. [30]],

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{\delta(\eta + 2\pi + \gamma) + \delta(\eta + 2\pi - \gamma)}{|\sin \eta|} \\ = \delta(\cos \eta - \cos \gamma) = \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) P_\ell(\cos \eta) P_\ell(\cos \gamma). \end{aligned} \quad (\text{B9})$$

Using this representation in (B7), yields

$$\begin{aligned} G_R(x, x') &= \frac{1}{r \cdot r'} \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) P_\ell(\cos \eta) P_\ell(\cos \gamma), \\ &\text{for } \eta \in (0, \pi). \end{aligned} \quad (\text{B10})$$

Comparing this expression with Eqs. (2.30) and (2.32), we reproduce (B5).

A third quite distinct approach is available to us. This is to use a Hadamard series representation for the Riemann functions U and \mathcal{U}_ℓ . As seen above, Zauderer’s theorem allows us to relate the latter to the former. That is, writing [see (2.26)]

$$U = \sum_{k=0}^{\infty} U_k \frac{\eta^{2k}}{k!} \quad (\text{B11})$$

yields [see (2.32)]

$$\mathcal{U}_\ell = \sum_{k=0}^{\infty} U_k \left(\frac{2\eta}{L} \right)^k J_k(L\eta), \quad L \geq 0. \quad (\text{B12})$$

The Hadamard coefficients $U_k = U_k(\eta)$, $k \geq 0$, obey a sequence of transport equations, and can, in theory, be calculated recursively.

On the other hand, since $\mathcal{U}_\ell = P_\ell(\cos \eta)$, Szego’s result expressing $P_\ell(\cos \eta)$ as a series of Bessel functions provides a very similar representation and we have

$$P_\ell(\cos \eta) = \Delta_{2d}^{1/2} \sum_{k=0}^{\infty} V_k(\eta) \frac{J_k(L\eta)}{L^k}. \quad (\text{B13})$$

The coefficients V_k may be obtained in the following manner as given in [32]:

$$V_0(\gamma) = 1, V_j(\gamma) = (2j-1)!! \gamma^j \psi_j(\gamma), \quad \forall j > 0, \quad (\text{B14})$$

where the ψ_k are given via

$$\left[1 + \sum_{k=1}^{\infty} (\gamma^2 - v^2)^k \phi_{k+1}(\gamma) \right]^{-1/2} = \sum_{k=0}^{\infty} (\gamma^2 - v^2)^k \psi_k(\gamma), \quad (\text{B15})$$

with $v \in [0, \gamma]$ and

$$\phi_k(\gamma) \equiv \frac{(2\gamma)^{1-k} J_{k-1/2}(\gamma)}{k! J_{1/2}(\gamma)}, \quad (\text{B16})$$

Crucially for the current discussion, Szego provides a convergence result for the series (B13); this is uniformly convergent on any interval $[0, \gamma_0 - \epsilon]$ with $0 < \epsilon < \gamma_0$ where $\gamma_0 = 2(\sqrt{2} - 1)\pi \simeq 0.828\pi$.

Thus we have two representations for the Riemann function \mathcal{U}_ℓ in (B5) in terms of Bessel functions: comparing (B5), (B12) and (B13), we see that

$$\begin{aligned} \sum_{k=0}^{\infty} (2\eta)^k U_k(\eta) \frac{J_k(L\eta)}{L^k} \\ = P_\ell(\cos \eta) = \Delta_{2d}^{1/2} \sum_{k=0}^{\infty} V_k(\eta) \frac{J_k(L\eta)}{L^k}, \end{aligned} \quad (\text{B17})$$

and so

$$0 = \sum_{k=0}^{\infty} C_k \frac{J_k(L\eta)}{L^k}, \quad C_k \equiv (2\eta)^k U_k(\eta) - \Delta_{2d}^{1/2} V_k(\eta). \quad (\text{B18})$$

Furthermore, we know from (2.28) that $U_0 = \Delta_{2d}^{1/2}$ and from (B14) that $V_0 = 1$, so that $C_0 = 0$. We now prove that $C_k = 0$ for any $k > 0$.

Crucial to the proof are the following ‘‘orthogonality’’ properties of the Bessel functions with respect to their argument [Eqs. (10.22.62) and (10.22.63) in Ref. [30]],

$$\begin{aligned} \int_0^\infty dL L^{m-k+1} J_m(L\eta) J_k(L\eta) \\ = \begin{cases} 0, & \text{if } k > m > -1 \text{ and } m \neq k \text{ and } m \neq k-1, \\ \frac{1}{2\eta}, & \text{if } m = k-1. \end{cases} \end{aligned} \quad (\text{B19})$$

So, multiplying (B18) by $LJ_0(L\eta)$, integrating over $L: 0 \rightarrow \infty$ [the relevant expansions above being valid for all $L \geq 0$: see Eq. (15) on page 58 of [35], volume 2 and Theorem 6.4.2 of [6]] and using (B19) together with $C_0 = 0$, it readily follows that $C_1 = 0$ (interchange of the integral and the sum is permitted by uniform convergence of the series). Next, we multiply (B18) by $L^2 J_1(L\eta)$, integrate over $L: 0 \rightarrow \infty$ and use (B19) together with $C_0 = C_1 = 0$, from which it follows that $C_2 = 0$. Proceeding similarly for $k = 3, 4, \dots$, it follows that $C_k = 0$, and so

$$U_k = (2\eta)^{-k} \Delta_{2d}^{1/2} V_k(\eta), \quad \forall k \geq 0. \quad (\text{B20})$$

We thus have that, in this zero-mass case, the terms in the Bessel series (2.32) are the same as the terms in Szego’s expansion (4.23), and therefore the former series converges where we already know that the latter does: namely, in the case $M = 0$, (2.32) converges uniformly in (at least) $\eta \in [0, \gamma_0]$, with $\gamma_0 = 2(\sqrt{2} - 1)\pi$.

Two further observations are worth making, the first in relation to the calculation of Szego’s coefficients V_k , or equivalently, the Hadamard coefficients U_k . As noted, the latter coefficients are determined by a sequence of transport equations. These take the form

$$\begin{aligned} 2\eta U'_k + (2k-1 + \eta \cot \eta) U_k \\ = -\frac{1}{2} \left(U''_{k-1} + \cot \eta U'_{k-1} - \frac{1}{4} U_{k-1} \right), \quad k \geq 1, \end{aligned} \quad (\text{B21})$$

and where $U_0 = (\eta / \sin \eta)^{1/2}$. The (exact) integration is nontrivial, but is amenable to calculation using a computer algebra system. Szego’s formulas (B14)–(B16) provide a purely algebraic approach to the calculation of the coefficients V_k . However, the calculations involved are again nontrivial—but are simplified via the following approach.

In (B15), let $\rho \equiv \gamma^2 - v^2$ and write $\sum_{k=1}^{\infty} \phi_{k+1} \rho^k =: S$. Then we have

$$\sum_{k=0}^{\infty} \psi_k \rho^k = (1 + S)^{-1/2} = \sum_{n=0}^{\infty} \left(\frac{-1}{2} \right)^n \frac{(2n-1)!!}{n!} S^n. \quad (\text{B22})$$

Let D be the differential operator $\frac{d}{d\rho}$. Then

$$\begin{aligned} \psi_k &= \frac{1}{k!} D^k \left(\sum_{n=0}^{\infty} \psi_n \rho^n \right) \Big|_{\rho=0} \\ &= \frac{1}{k!} \sum_{n=0}^{\infty} \left(\frac{-1}{2} \right)^n \frac{(2n-1)!!}{n!} D^k(S^n) \Big|_{\rho=0}. \end{aligned} \quad (\text{B23})$$

Applying Faà di Bruno’s theorem for higher order derivatives of compositions (the generalized chain rule; see e.g., [42]), we have, for $n \geq 1$,

$$D^k(S^n)|_{\rho=0} = \begin{cases} 0, & k < n \\ k! \hat{B}_{k,n}(\phi_2, \dots, \phi_{k-n+2}), & k \geq n, \end{cases} \quad (\text{B24})$$

where

$$\begin{aligned} & \hat{B}_{k,n}(x_1, x_2, \dots, x_{k-n+1}) \\ &= \sum \frac{n!}{m_1! m_2! \dots m_{k-n+1}!} x_1^{m_1} x_2^{m_2} \dots x_{k-n+1}^{m_{k-n+1}}, \end{aligned} \quad (\text{B25})$$

with the sum taken over all finite sequences of non-negative integers $(m_1, m_2, \dots, m_{k-n+1})$ such that

$$m_1 + m_2 + \dots + m_{k-n+1} = n, \quad (\text{B26})$$

and

$$m_1 + 2m_2 + \dots + (k-n+1)m_{k-n+1} = k. \quad (\text{B27})$$

The functions $\hat{B}_{k,n}$ are Bell polynomials, and they may be calculated either directly using (B25), or via recursion formulae [43]. It follows that $\psi_0 = 1$, and

$$\psi_k = \sum_{n=1}^k \left(\frac{-1}{2}\right)^n \frac{(2n-1)!!}{n!} \hat{B}_{k,n}(\phi_2, \dots, \phi_{k-n+2}), \quad (\text{B28})$$

which [along with (B14)] gives an explicit formula for Szego's coefficients V_k , and hence for the Hadamard coefficients U_k .

Finally, we note a connection between the Hadamard coefficients considered here, and the Hadamard coefficients of the 4-dimensional Plebański-Hacyan (PH) spacetime considered in Sec. III above. In PH spacetime, the direct product of 2-dimensional Minkowski spacetime \mathbb{M}_2 and the (unit) 2-sphere \mathbb{S}_2 , the world function decomposes as

$$\sigma_{\text{PH}} = \sigma_{\mathbb{M}_2} + \sigma_{\mathbb{S}_2}. \quad (\text{B29})$$

It is straightforward to show that

$$\square_{\text{PH}} \sigma_{\text{PH}} = 3 + \gamma \cos \gamma, \quad (\text{B30})$$

where γ is geodesic distance on \mathbb{S}_2 and \square_{PH} is the D'Alembertian in PH. Writing the tail term of the retarded Green function of the wave operator $P = \square_{\text{PH}} - \frac{1}{4}$ in the form of a Hadamard series [see Eq. (3.12)]

$$V_{\text{PH}} = \sum_{n=0}^{\infty} \nu_n \sigma_{\text{PH}}^n, \quad (\text{B31})$$

we can show that the Hadamard coefficients ν_n depend only on γ . (We note that this operator is the one considered in

Sec. III above: $P = \square_{\text{PH}} - (m^2 + 2\xi)$ with $m^2 + 2\xi = 1/4$.) Defining \bar{V}_n by

$$\nu_n(\gamma) = \frac{2^n}{n!} \bar{V}_n(\gamma), \quad (\text{B32})$$

we can show that the \bar{V}_n satisfy the sequence of transport equations

$$\begin{aligned} & 2\gamma \bar{V}'_k + (2k+1 + \gamma \cot \gamma) \bar{V}_k \\ &= -\frac{1}{2} \left(\bar{V}''_{k-1} + \cot \gamma \bar{V}'_{k-1} - \frac{1}{4} V_{k-1} \right), \quad k \geq 1. \end{aligned} \quad (\text{B33})$$

These are precisely the transport equations (B21) for the Hadamard coefficients of AdS_2 —but with a shift of index. We calculate that $\bar{V}_0(\gamma) = U_1(\gamma)$ (note the change of argument in U_1), and we can conclude that, in this case of $M = 0$,

$$\frac{k!}{2^k} \nu_k(\eta) = \bar{V}_k(\eta) = U_{k+1}(\eta), \quad k \geq 0. \quad (\text{B34})$$

Thus the Hadamard coefficients ν_k of 4-dimensional PH spacetime are essentially the Hadamard coefficients U_k of AdS_2 , and can therefore be calculated using Bell polynomials [see Eqs. (B14) and (B28)], via Szego's theorem.

APPENDIX C: REGULARIZED SELF-FIELD

We here show how our results could be used to calculate the regularized self-field Φ_R (that is, the regularized value of the scalar field created by a scalar-point charge, evaluated on the worldline of the charge itself) in Schwarzschild spacetime (see [25]). As noted earlier, the self-field is relevant to self-force calculations of radiation reaction; the self-force is the covariant derivative of the regularized self-field. A practical calculation of the regularized self-field can be achieved by integrating what we refer to as the *nondirect* GF $G_{\text{nd}}(x, x')$ over the worldline of the scalar charge,¹⁵

$$\Phi_R(\tau) = \int_{-\infty}^{\tau} d\tau' G_{\text{nd}}(z(\tau), z(\tau')) \quad (\text{C1})$$

where τ is the proper time along the worldline $z(\tau)$ of the charge and

¹⁵Another expression for the regularized self-field [see, e.g., [2]] is like Eq. (C1) but with the integrand being the full GF G_R instead of its nondirect part G_{nd} and the upper limit of integration being τ^- instead of τ ; in that case, the upper limit τ^- would effectively remove the sharp divergence of the integrand $G_R(z(\tau), z(\tau'))$ at coincidence which is alternatively removed in (C1) by having subtracted from the GF its divergence at coincidence [see (C2)].

$$G_{\text{nd}}(x, x') \equiv \begin{cases} G_R(x, x') - G_d(x, x') = V_{4d}(x, x')\theta(-\sigma_{4d})\theta(\Delta t), & x' \in \mathcal{N}(x), \\ G_R(x, x'), & x' \notin \mathcal{N}(x). \end{cases} \quad (\text{C2})$$

where

$$G_d(x, x') \equiv U_{4d}(x, x')\delta(\sigma_{4d})\theta(\Delta t) \quad (\text{C3})$$

is the direct part of the GF. For obvious reasons, the integral in (C1) is usually called the tail integral. One may calculate the nondirect part of the GF by carrying out an ℓ -mode decomposition as

$$G_{\text{nd}}(x, x') \equiv \frac{1}{4\pi r r'} \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(\cos \gamma) (G_{\ell}(r, r'; \Delta t) - G_{\ell}^d(r, r'; \Delta t)) \quad (\text{C4})$$

where

$$G_{\ell}^d(r, r'; \Delta t) = 2\pi\theta(\Delta t)\theta(\pi - \eta)U_0(x^A, x^{A'})P_{\ell}(\cos \eta) \left(\frac{\sin \eta}{\eta} \right)^{1/2} \quad (\text{C5})$$

are the modes of the direct part G_d as obtained in Eq. (20) [25] and U_0 is the first Hadamard series coefficient in (2.26). In Ref. [25], G_{ℓ} as well as U_0 , η and, with them, G_{ℓ}^d , were all evaluated numerically to obtain the regularized self-field Φ_R . In here, we instead use results in the main part of our paper in order to give an analytic expression for the integrand in the regularized self-field expression (C1). Using Eq. (4.23) for P_{ℓ} in the modes (C5) of the direct part and Eq. (2.32) for the modes G_{ℓ} of the full GF, we notice a remarkable similarity between the representations for the two sets of modes, which allows us to write,

$$G_{\ell}(r, r'; \Delta t) - G_{\ell}^d(r, r'; \Delta t) = 2\pi\theta(\Delta t) \sum_{k=0}^{\infty} (\theta(-\sigma)U_k(2\eta)^k - \theta(\pi - \eta)U_0V_k(\eta)) \frac{J_k(L\eta)}{L^k}. \quad (\text{C6})$$

We note that the expressions for G_R in Sec. IV *et seq.* would also apply to the nondirect part G_{nd} of the GF merely by applying the replacement $U_k \rightarrow U_k - U_0V_k(\eta)/(2\eta)^k$ in (4.22) for $\eta < \pi$, and with no replacement for $\eta \geq \pi$. The regularized self-field (C1) can now be written in terms of the nondirect GF as

$$\Phi_R(\tau) = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \int_{-\infty}^{\tau^-} d\tau' \frac{P_{\ell}(\cos \gamma)}{r'} \sum_{k=0}^{\infty} (U_k(2\eta)^k - \theta(\pi - \eta)U_0V_k(\eta)) \frac{J_k(L\eta)}{L^k}, \quad (\text{C7})$$

where we have used the fact that $\theta(-\sigma) = 1$ along the worldline. This integral can be split as

$$\Phi_R(\tau) = \int_{-\infty}^{\tau_0} d\tau' G_R(z(\tau), z(\tau')) + \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2} \right) \int_{\tau_0}^{\tau} d\tau' \frac{P_{\ell}(\cos \gamma)}{r'} \sum_{k=1}^{\infty} (U_k(2\eta)^k - U_0V_k(\eta)) \frac{J_k(L\eta)}{L^k}, \quad (\text{C8})$$

where $\tau_0 < \tau$ is the value of the proper time corresponding to $\eta = \pi$. The nice feature here is that we have subtracted the direct part from the GF in an exact analytical manner (note the starting value of the sum at $k = 1$, as there has been an exact cancellation of the $k = 0$ term). We note that, in the massless case $M = 0$ and using Eq. (B20), the k -sum inside the second partial tail integral in (C8) vanishes (in agreement with the fact that the massless scalar field in flat spacetime has no tail) *term-by-term*.

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