Penrose limits of inhomogeneous spacetimes, their diagonalizability, and twistors

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Penrose limits are considered in spacetimes admitting two Abelian, spacelike Killing vectors in general nonvacuum as well as in particular in the presence of an electromagnetic field. This class of spacetimes includes inhomogeneous cosmologies as well as colliding plane gravitational and the corresponding field waves or matter. Following the work of Tod [Classical Quantum Gravity **37**, 075021 (2020)] the conditions for diagonal Penrose limits are investigated in these backgrounds. The twistor equation is considered in these spacetimes, and solutions are given in the radial Penrose limit.

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I. INTRODUCTION

Plane wave solutions play an important role in classical as well as quantum field theories. In string theory they are also known to be examples of exact classical string vacua [1,2]. Penrose [3] showed that along a segment of a null geodesic without conjugate points any spacetime has a plane wave limit. The Penrose limit and its generalization to D dimensional spacetimes [4] is of particular interest in string/M theory (cf., e.g., [5,6]).

Recently Tod [7] demonstrated the condition on the spacetime so that *all* Penrose limits are diagonal. Moreover, an elegant and computational efficient way of calculating the Penrose limit has been presented there using two-spinor calculus [7]. Different from the original formulation of Penrose [3] it does not involve an explicit coordinate transformation adapted to the null geodesic along which the Penrose limit is taken.

Here the formulation of Tod [7] is applied to determine the Penrose limits of inhomogeneous spacetimes admitting two spacelike commuting Killing vectors generating an Abelian group G_2. This type of metrics allows one to describe different types of backgrounds such as cosmological backgrounds and colliding plane wave spacetimes. Moreover, some of the spatially homogeneous backgrounds with three spacelike Killing vectors, namely, models of Bianchi type I-VII as well as the locally rotationally symmetric (LRS) VIII and LRS IX admit two-dimensional Abelian subgroups (cf. [8]). Radial Penrose limits and Abelian as well as non-Abelian T-duality transformations of low energy string backgrounds have been considered in this type of background in [9].

An interesting aspect of the resulting plane wave spacetimes in the Penrose limit is that these backgrounds permit the existence of twistors [7,10,11]. Twistors were introduced to provide a fundamental description of spacetime structure and physical concepts. Penrose proposed considering the two spinors as more fundamental than spacetime points (e.g., [10]). Therefore it is natural to expect that twistors play an important role in the quantization of gravity. The relation of the twistor equation to massless fields and representation of solutions in terms of contour integrals of holomorphic functions has led to an important advancement in the mathematical aspects of solutions of differential equations by Penrose transforms (e.g., [12]). Global solutions of the twistor equation in a curved background are severely restricted by a consistency condition. This has led to additional concepts of local and asymptotic twistors in asymptotically flat spacetimes. However, there do exist global twistors in plane wave backgrounds. In this sense the importance of the resulting plane wave spacetimes in the Penrose limit is similar to the case of exact solutions of string theory in plane wave backgrounds which in part motivated the strong interest in the Penrose limit over recent years.

The plan of the paper is as follows. In Sec. II the Penrose limit procedure and all the relevant quantities in the Newman-Penrose formalism as well as in the two-spinor formulation are presented for G_2 metrics. In Sec. III the electromagnetic field two-spinor formulation for the G_2 metrics is given together with Einstein's equations to provide a particular example of a nonvacuum spacetime. In Sec. IV the question of diagonalizability will be discussed. Radial Penrose limits are considered as an example in Sec. V. The twistor equation is discussed in Sec. VI, and explicit solutions are given in the radial Penrose limit. Finally, in Sec. VII conclusions are presented.

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II. PENROSE LIMITS OF G_2 SPACETIMES

In [7] the calculation of the Penrose limit has been formulated in terms of two spinors. A normalized spin frame o^A, ι^A, with $A = 0, 1$ satisfies [13]

$$o_A \iota^A = 1 \qquad \iota_A o^A = -1 \qquad o_A o^A = 0 = \iota_A \iota^A. \tag{2.1}$$

Following [13] a four-dimensional spacetime is considered with metric $g_{\mu\nu}$, and a null tetrad of ("world") vectors l^μ, n^μ, m^μ and $\bar{m}^\mu$ is defined by

$$l^\alpha = o^A o^{A'} \quad n^\alpha = \iota^A \iota^{A'} \quad m^\alpha = o^A \iota^{A'} \quad \bar{m}^\alpha = \iota^A o^{A'}. \tag{2.2}$$

Then with respect to the spacetime metric $g_{\mu\nu}$ these vectors are defining an orthonormal null tetrad with two real null vectors l^μ and n^μ, respectively, and two complex null vectors which are complex conjugates of each other m^μ and $\bar{m}^\mu$ such that

$$l_\mu n^\mu = 1 \qquad m_\mu \bar{m}^\mu = -1, \tag{2.3}$$

and all other combinations are zero. Moreover, the metric is given by $g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu$. In the Newman-Penrose formalism the components of curvature and energy momentum tensor as well as covariant derivatives and Einstein's equations can be calculated efficiently with respect to this orthonormal null tetrad [8,13,14].

The Penrose limit yields a plane wave spacetime. This admits five Killing vectors. The only nonvanishing Weyl scalar is $\Psi_4 = \Psi$, and $\Phi_{22} = \Phi$ is the only nonvanishing scalar determining the components of the Ricci tensor (cf., e.g., [7,8,15]). Equivalently the Weyl spinor and Ricci spinor are given by, respectively, [7]

$$\psi_{ABCD} = \Psi o_A o_B o_C o_D \quad \phi_{ABA'B'} = \Phi o_A o_B o_{A'} o_{B'}. \tag{2.4}$$

Following [7] the Penrose limit of a spacetime M is obtained by choosing any null geodesic Γ in M and considering a spinor field α^A parallelly propagated tangent to Γ and defining an affine parameter ξ (up to an additive constant) by $\alpha^A \bar{\alpha}^{A'} \nabla_{AA'} \xi = 1$. The plane wave in the Penrose limit is then determined by

$$\Psi(\xi) = \psi_{ABCD} \alpha^A \alpha^B \alpha^C \alpha^D \qquad \Phi(\xi) = \phi_{ABA'B'} \alpha^A \alpha^B \bar{\alpha}^{A'} \bar{\alpha}^{B'}. \tag{2.5}$$

Assuming the spinor field α^A to be of the form

$$\alpha^A = A(\xi) o^A + B(\xi) \iota^A \tag{2.6}$$

the condition for its parallel transport along the null geodesic Γ,

$$\alpha^A \bar{\alpha}^{A'} \nabla_{AA'} \alpha_B = 0, \tag{2.7}$$

yields the evolution equations for the two complex functions $A(\xi)$ and $B(\xi)$:

$$\frac{dA}{d\xi} = -|A|^2 (A\epsilon + B\alpha - B\tau') + |B|^2 (A\rho' + B\kappa' - A\gamma) - A^2 \bar{B}\beta + B^2 \bar{A}\sigma' \tag{2.8}$$

$$\frac{dB}{d\xi} = |A|^2 (A\kappa + B\epsilon + B\rho) - |B|^2 (-A\beta - A\tau - B\gamma) + A^2 \bar{B}\sigma + B^2 \bar{A}\alpha \tag{2.9}$$

using that $\{o^A, \iota^A\}$ is a normalized spinor basis.

In the following the Penrose limit construction following [7] will be considered for metrics admitting two Abelian spacelike Killing vectors. These are described by the line element (e.g., [15])

$$ds^2 = 2e^{-M} du dv - \frac{2e^{-U}}{Z + \bar{Z}} (dx + iZdy)(dx - i\bar{Z}dy), \tag{2.10}$$

where $M = M(u, v)$ and $U = U(u, v)$ are real functions of the null coordinates u and v. $Z = Z(u, v)$ is a complex function of u and v. Colliding plane wave spacetimes can be separated in four different regions. Two of these describe the two incoming plane waves for which all metric functions have as argument either the null coordinate u or the null coordinate v, respectively (e.g., [15]). The interaction region of these plane waves constitutes the third region in which all metric functions depend on both null coordinates in general. The fourth region describes the background spacetime on which the waves propagate, most commonly taken to be flat. The Penrose limits calculated of the interaction region might have interesting relations to the incoming, initial plane waves.

The spin coefficients and curvature components for the G_2 metric (2.10) are given in the Appendix. In particular the spin coefficients τ, τ', α, β, κ, and κ' vanish [cf. Eq. (A4)] simplifying Eqs. (2.8) and (2.9).

When considering cosmological spacetimes it is useful to introduce a timelike coordinate t and a spacelike coordinate z by defining $t = u - v$ and $z = u + v$. The explicit forms of the metric functions M, U, and Z for the case of the spatially homogenous models of Bianchi type which have three Killing vectors but admit two-dimensional Abelian subgroups G_2 have been found explicitly in [9].

The two Killing vectors ∂_x and ∂_y admitted by the metric (2.10) expressed in terms of the null tetrad are given by

$$^{(1)}K^\mu = \frac{e^{-\frac{U}{2}}}{(Z + \bar{Z})^{\frac{1}{2}}} m^\mu + \frac{e^{-\frac{U}{2}}}{(Z + \bar{Z})^{\frac{1}{2}}} \bar{m}^\mu \tag{2.11}$$

$$^{(2)}K^\mu = i \frac{e^{-\frac{U}{2}}}{(Z + \bar{Z})^{\frac{1}{2}}} Z m^\mu - i \frac{e^{-\frac{U}{2}}}{(Z + \bar{Z})^{\frac{1}{2}}} \bar{Z} \bar{m}^\mu. \tag{2.12}$$

Therefore there are two constants of motion $^{(i)}E = {}^{(i)}K_\mu V^\mu$ where the tangent world vector of the null geodesic is determined by $V^\mu = \alpha^A \bar{\alpha}^{A'}$; thus

$$V^\mu = A\bar{A}l^\mu + B\bar{B}n^\mu + A\bar{B}m^\mu + \bar{A}B\bar{m}^\mu. \tag{2.13}$$

This yields to

$$^{(1)}E = -\frac{e^{-\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}(\bar{A}B + A\bar{B}) \tag{2.14}$$

$$^{(2)}E = -i\frac{e^{-\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}(\bar{A}BZ - A\bar{B}\,\bar{Z}), \tag{2.15}$$

implying that

$$\bar{A}B = \frac{e^{\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}(-{}^{(1)}E\bar{Z} + i{}^{(2)}E). \tag{2.16}$$

This allows one to obtain expressions for the product of the modulus of A and B as well as the differences in their phases, namely,

$$|A||B| = \frac{e^{\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}[({}^{(1)}E)^2 Z\bar{Z} + i(\bar{Z}-Z){}^{(1)}E{}^{(2)}E + ({}^{(2)}E)^2]^{\frac{1}{2}} \tag{2.17}$$

$$e^{\pm 2i(\varphi_B - \varphi_A)} = \left(\frac{{}^{(1)}E\bar{Z} - i{}^{(2)}E}{{}^{(1)}EZ + i{}^{(2)}E}\right)^{\pm 1} \tag{2.18}$$

with $X = |X|e^{i\varphi_X}$, $X = A,\ B$. Finally, using the spin coefficients as given in Eq. (A4) this leads to

$$\begin{aligned}\frac{d|A|^2}{d\xi} = {}&\frac{1}{2}e^{\frac{M}{2}}(\partial_v M)|A|^4 + \frac{1}{2}\frac{e^{\frac{M}{2}+U}}{Z+\bar{Z}}\Bigg[\partial_u(2U-M)[({}^{(1)}E)^2 Z\bar{Z} + i(\bar{Z}-Z){}^{(1)}E{}^{(2)}E + ({}^{(2)}E)^2] \\ &+ \frac{2}{Z+\bar{Z}}\partial_u\left[\frac{1}{3}({}^{(1)}E)^2(Z^3+\bar{Z}^3) - i{}^{(1)}E{}^{(2)}E(Z^2-\bar{Z}^2) - ({}^{(2)}E)^2(Z+\bar{Z})\right]\Bigg]\end{aligned} \tag{2.19}$$

$$\begin{aligned}\frac{d|B|^2}{d\xi} = {}&\frac{1}{2}e^{\frac{M}{2}}(\partial_u M)|B|^4 + \frac{1}{2}\frac{e^{\frac{M}{2}+U}}{Z+\bar{Z}}\Bigg[\partial_v(2U-M)[({}^{(1)}E)^2 Z\bar{Z} + i(\bar{Z}-Z){}^{(1)}E{}^{(2)}E + ({}^{(2)}E)^2] \\ &+ \frac{2}{Z+\bar{Z}}\partial_v\left[\frac{1}{3}({}^{(1)}E)^2(Z^3+\bar{Z}^3) + i{}^{(1)}E{}^{(2)}E(Z^2-\bar{Z}^2) - ({}^{(2)}E)^2(Z+\bar{Z})\right]\Bigg].\end{aligned} \tag{2.20}$$

Moreover, the condition ξ to be an affine parameter leads to

$$\begin{aligned}&\frac{du}{d\xi} = |B|^2 e^{\frac{M}{2}} &&\frac{dv}{d\xi} = |A|^2 e^{\frac{M}{2}} \\ &\frac{dx}{d\xi} = \frac{e^{\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}(A\bar{B}\,\bar{Z} + \bar{A}BZ) &&\frac{dy}{d\xi} = -i\frac{e^{\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}(A\bar{B} - \bar{A}B).\end{aligned} \tag{2.21}$$

Taking into account Eq. (2.17) the last two equations can be rewritten as

$$\frac{dx}{d\xi} = -\frac{2e^U}{Z+\bar{Z}}\left(Z\bar{Z}{}^{(1)}E - \frac{i}{2}(Z-\bar{Z}){}^{(2)}E\right) \tag{2.22}$$

$$\frac{dy}{d\xi} = -\frac{2e^U}{Z+\bar{Z}}\left(-\frac{i}{2}(Z-\bar{Z}){}^{(1)}E + {}^{(2)}E\right). \tag{2.23}$$

Thus null geodesics with $\dot{x} = 0$, $\dot{y} \neq 0$, a dot denoting the derivative with respect to ξ, are possible for ${}^{(1)}E = 0$, ${}^{(2)}E \neq 0$ in spacetimes with Z real. Interchanging the constants with the same condition on Z allows for the case $\dot{x} \neq 0$, $\dot{y} = 0$. In the case $\dot{x} = 0 = \dot{y}$ both constants have to vanish. Moreover, together with Eq. (2.16) this implies that one of the functions A or B has to vanish, making the affine parameter ξ a function only of either one of the null coordinates u or v of the background spacetime.

This describes radial Penrose limits in which the Penrose limit is considered along radial geodesics (e.g., [9]). Radial Penrose limits within the spinor formulation of Tod [7] are considered below in Sec. V.

The only nonvanishing Ricci spinor and Weyl scalar in the Penrose limit [cf. Eq. (2.5)] are given by

$$\begin{aligned}\Phi(\xi) = {}&|A|^4\Phi_{00} + (A\bar{B})^2\Phi_{02} + (\bar{A}B)^2\Phi_{20} \\ &- 4|A|^2|B|^2\Phi_{11} + |B|^4\Phi_{22}\end{aligned} \tag{2.24}$$

$$\Psi(\xi) = A^4\Psi_0 + 6A^2B^2\Psi_2 + B^4\Psi_4, \tag{2.25}$$

where Φ_{ab} and Ψ_i are the tetrad components of the Ricci tensor and the Weyl scalars for the G_2 metric (2.10) as given in Eqs. (A8)–(A12) and (A5)–(A7), respectively. The wave profiles of plane wave spacetimes are obtained from the Brinkmann form (cf., e.g., [7,8,15])

$$ds^2 = 2drd\xi + (h_{11}X^2 + 2h_{12}XY + h_{22}Y^2)d\xi^2 - dX^2 - dY^2, \tag{2.26}$$

where

$$h_{11} = \Phi(\xi) + \frac{1}{2}(\Psi(\xi) + \bar{\Psi}(\xi)) \tag{2.27}$$

$$h_{12} = -\frac{i}{2}(\Psi(\xi) - \bar{\Psi}(\xi)) \tag{2.28}$$

$$h_{22} = \Phi(\xi) - \frac{1}{2}(\Psi(\xi) + \bar{\Psi}(\xi)) \tag{2.29}$$

with the Ricci tensor component $\Phi(\xi)$ (2.24) and the Weyl scalar $\Psi(\xi)$ (2.25). If $\Psi(\xi)$ is real then $h_{12} \equiv 0$, and the corresponding G_2 metric is diagonal (cf. [15]).

III. THE ELECTROMAGNETIC FIELD IN G_2 SPACETIMES AND ITS PENROSE LIMIT

In this section the Penrose limit construction [7] is applied to a nonvacuum spacetime, namely including a nonvanishing electromagnetic field. The behavior of the electromagnetic spinor is determined in the Penrose limit.

The Maxwell tensor is determined by the electromagnetic spinor φ_{AB} by [13]

$$F_{\alpha\beta} = F_{AA'BB'} = \varphi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\bar{\varphi}_{A'B'} \tag{3.1}$$

with $\varphi_{AB} = \varphi_{(AB)} = \frac{1}{2}F_{ABC'}{}^{C'}$ with the correspondence between spacetime and spinor components given by $\varphi_0 = \varphi_{00}$, $\varphi_1 = \varphi_{01}$, $\varphi_2 = \varphi_{11}$ and its complex conjugate. The free space Maxwell's equations can be expressed in terms of a zero rest-mass field equation [13] which in the compacted spin-coefficient form [Geroch-Held-Penrose (GHP) formalism] is efficiently written as

$$þ\varphi_1 - ð'\varphi_0 = -\tau'\varphi_0 + 2\rho\varphi_1 - \kappa\varphi_2 \tag{3.2}$$

$$þ'\varphi_1 - ð\varphi_2 = -\tau\varphi_2 + 2\rho'\varphi_1 - \kappa'\varphi_0 \tag{3.3}$$

$$þ\varphi_2 - ð'\varphi_1 = \sigma'\varphi_0 - 2\tau'\varphi_1 + \rho\varphi_2 \tag{3.4}$$

$$þ'\varphi_0 - ð\varphi_1 = \sigma\varphi_2 - 2\tau\varphi_1 + \rho'\varphi_0. \tag{3.5}$$

In the following the coupled Einstein-Maxwell equations will be considered in G_2 spacetimes. Therefore it is assumed that the Maxwell field has the same symmetries as the background spacetime. Writing Eqs. (3.2)–(3.5) explicitly for the G_2 metric (2.10) and its spin coefficients (cf. Appendix) and taking into account that all metric functions only depend on the null variables u and v the first two equations yield to

$$\varphi_1(u, v) = c_{\varphi_1}e^U, \tag{3.6}$$

where c_{φ_1} is a constant. Equations (3.4) and (3.5) result in

$$\partial_u\left[\frac{e^{-\frac{1}{2}(M+U)}}{\sqrt{Z+\bar{Z}}}\varphi_0\right] = -\partial_v\left[\frac{e^{-\frac{1}{2}(M+U)}}{\sqrt{Z+\bar{Z}}}\varphi_2\right] \tag{3.7}$$

showing the existence of a potential function $H(u, v)$ (cf. [15]) such that the remaining components of the electromagnetic spinor are determined by

$$\varphi_0 = -e^{\frac{1}{2}(M+U)}\sqrt{Z+\bar{Z}}H_{,v} \tag{3.8}$$

$$\varphi_2 = e^{\frac{1}{2}(M+U)}\sqrt{Z+\bar{Z}}H_{,u}. \tag{3.9}$$

Indices $,u$ and $,v$ denote the corresponding partial derivatives. Moreover, using expressions (3.8) and (3.9) in Eqs. (3.4) and (3.5) yields to (cf. also [15])

$$H_{,uv} + \frac{Z_{,u}}{Z+\bar{Z}}H_{,v} + \frac{\bar{Z}_{,v}}{Z+\bar{Z}}H_{,u} = 0, \tag{3.10}$$

where $X_{,mn} \equiv \frac{\partial^2 X}{\partial m \partial n}$ with m, n denoting the null variables u and v.

Einstein's equations imply that the components of the Ricci tensor in terms of the energy-momentum tensor T_{ab} are given by [13]

$$\Phi_{\alpha\beta} = 4\pi G_N\left(T_{\alpha\beta} - \frac{1}{4}T^{\mu}_{\mu}g_{\alpha\beta}\right), \tag{3.11}$$

with G_N Newton's constant of gravitation. Together with the energy momentum tensor of the electromagnetic field $T_{\alpha\beta} = \frac{1}{2\pi}\varphi_{AB}\bar{\varphi}_{A'B'}$. Using Eqs. (A8)–(A12) together with Eq. (3.11) yields to

$$(Z+\bar{Z})(2Z_{,uv} - Z_{,u}U_{,v} - Z_{,v}U_{,u}) - 4Z_{,u}Z_{,v} - 4G_N e^U(Z+\bar{Z})^3 H_{,u}\bar{H}_{,v} = 0 \tag{3.12}$$

$$e^{-2U}(U_{,uv} - U_{,u}U_{,v}) + 16G_N|c_{\varphi_1}|^2 e^{-M} = 0 \tag{3.13}$$

$$2U_{,uu} - U_{,u}^2 + 2M_{,u}U_{,u} - 4\frac{Z_{,u}\bar{Z}_{,u}}{(Z+\bar{Z})^2} - 8G_N e^U(Z+\bar{Z})H_{,u}\bar{H}_{,u} = 0 \tag{3.14}$$

$$2U_{,vv} - U_{,v}^2 + 2M_{,v}U_{,v} - 4\frac{Z_{,v}\bar{Z}_{,v}}{(Z+\bar{Z})^2} - 8G_N e^U(Z+\bar{Z})H_{,v}\bar{H}_{,v} = 0, \tag{3.15}$$

which forms a complete set of equations together with Eq. (3.10).

A. The electromagnetic field in the Penrose limit

In the pp wave spacetime of the Penrose limit in the direction of the flagpole of the spinor α^A the only nonvanishing component of the electromagnetic spinor is given by $\varphi_2(\xi) = \varphi_{11} = \varphi_{AB}\alpha^A\alpha^B$ and its complex conjugate. In terms of the components of the electromagnetic spinor of the original G_2 spacetime this is found to be

$$\varphi(\xi) = A^2(\xi)\varphi_{00} + 2A(\xi)B(\xi)\varphi_{01} + B^2(\xi)\varphi_{11}. \quad (3.16)$$

IV. DIAGONAL PENROSE LIMITS

Tod [7] showed that the plane wave spacetime in the Penrose limit is diagonizable if

$$\Sigma(\xi) \equiv i(\bar{\Psi}(\xi)\dot{\Psi}(\xi) - \Psi(\xi)\dot{\bar{\Psi}}(\xi)) = 0, \quad (4.1)$$

where a dot denotes the derivative with respect to the affine parameter ξ. For the G_2 metric under consideration [cf. Eq. (2.25)]

$$\Psi(\xi) = \sum_{N=0,2,4} \mu_N \Psi_N, \quad \text{with} \quad \mu_0 = A^4(\xi),$$
$$\mu_2 = 6A^2B^2, \qquad \mu_4 = B^4(\xi). \quad (4.2)$$

This yields to

$$\begin{aligned} -i\Sigma(\xi) = \sum_{N,M=0,2,4} &[T_2(\mu_N,\mu_M)T_3(\Psi_N,\Psi_M) \\ &+ T_1(\Psi_N,\Psi_M)T_4(\mu_N,\mu_M) \\ &+ T_1(\mu_N,\mu_M)T_4(\bar{\Psi}_N,\bar{\Psi}_M) \\ &+ T_2(\bar{\Psi}_N,\bar{\Psi}_M)T_3(\mu_N,\mu_M)], \end{aligned} \quad (4.3)$$

where with $X_N = |X_N|e^{i\varphi_{X_N}}$,

$$\begin{aligned} T_1(X_N,X_M) &= |X_N||X_M|(e^{-i(\varphi_{X_N}-\varphi_{X_M})} - e^{i(\varphi_{X_N}-\varphi_{X_M})}) \\ T_2(X_N,X_M) &= |X_N||\dot{X_M}|(e^{-i(\varphi_{X_N}-\varphi_{X_M})} - e^{i(\varphi_{X_N}-\varphi_{X_M})}) \\ &\quad + i\dot{\varphi}_{X_M}|X_N||X_M|(e^{-i(\varphi_{X_N}-\varphi_{X_M})} \\ &\quad + e^{i(\varphi_{X_N}-\varphi_{X_M})}) \\ T_3(X_N,X_M) &= |X_N||X_M|e^{-i(\varphi_{X_N}-\varphi_{X_M})} \\ T_4(X_N,X_M) &= |X_N||\dot{X_M}|e^{i(\varphi_{X_N}-\varphi_{X_M})} \\ &\quad - i\dot{\varphi}_{X_M}|X_N||X_M|e^{i(\varphi_{X_N}-\varphi_{X_M})}. \end{aligned} \quad (4.4)$$

It can be seen that the diagonalizability condition (4.1) is satisfied for $\varphi_{\mu_N} = \varphi_\mu = \text{const}$ and $\varphi_{\Psi_N} = \varphi_\psi = \text{const}$ From Eq. (2.18) it follows that in this case the imaginary part of Z is constant, namely, $\frac{i}{2}(Z - \bar{Z}) = \frac{{}^{(2)}E}{{}^{(1)}E}$. Thus the Weyl scalars Ψ_0, Ψ_2 and Ψ_4 are real functions and $\varphi_\psi \equiv 0$. This still leaves the possibility that the constant phase $\varphi_\mu \neq 0$ and thus the Weyl scalar $\Psi(\xi)$ of the plane wave spacetime in the Penrose limit is complex. Formally this induces a nondiagonal wave amplitude h_{12} [cf. Eq. (2.28)]. However, the constant phase $4\varphi_\mu$ of $\Psi(\xi)$ can be made to vanish by a rotation of the null tetrad or equivalently the spinor dyad. In the former case following [14] a rotation of class III leaves the directions of the null tetrad vectors **l** and **n** unchanged but rotates **m** and $\bar{\mathbf{m}}$ by an angle θ in the $(\mathbf{m}, \bar{\mathbf{m}})$ plane. Only considering this rotation leads to a transformation of the Weyl scalars $\Psi_j \to e^{i(2-j)\theta}\Psi_j$, $j = 0, \ldots, 4$. Thus $\Psi(\xi)$ will be transformed to $e^{-2i\theta}\Psi(\xi)$, and choosing $\theta = 2\varphi_A$ leads to a real Weyl scalar. Thus h_{12} (2.28) vanishes. The components of the Ricci tensor transform as $\Phi_{kj} \to e^{i(j-k)\theta}\Phi_{kj}$, $j, k = 0, 1, 2$. Therefore, $\Phi(\xi)$ stays invariant under the considered rotation of the null tetrad. Finally, the behavior of the spacetime components of the electromagnetic field is determined by $\varphi_j \to \varphi_j e^{i(1-j)\theta}$, $j = 0, 1, 2$, and correspondingly the spinor components [cf. Eq. (3.1)]. Thus under the specified rotations of the null tetrad vectors **m** and $\bar{\mathbf{m}}$ $\varphi(\xi) \to e^{-i\theta}\varphi(\xi)$.

In the Penrose limit the spin frame will be chosen to be determined by the normalized spinor dyad $\{\beta, \alpha\}$ with α given by Eq. (2.6); then β is found to be

$$\beta^A = Fo^A + G\iota^A, \quad (4.5)$$

with the normalization $\beta_A\alpha^A = 1$ implying $FB - GA = 1$ and choosing F and G such that in the Penrose limit the only nonvanishing component of the Weyl spinor is $\Psi(\xi)$. Note α^A determines the flagpole, as well as together with β^A the flagplane [13]. Equally, as the transformation of the null tetrad vectors renders the only nonvanishing Weyl scalar of the plane wave space time in the Penrose limit a real function [14] this can also be achieved by transforming the spinor dyad $\{\beta, \alpha\}$, namely, by $\beta^A \to e^{i\varphi_A}\beta^A$ and $\alpha^A \to e^{-i\varphi_A}\alpha^A$ [13].

V. EXAMPLE: RADIAL PENROSE LIMIT

The radial Penrose limit is an important particular case since the affine parameter ξ of the null geodesic becomes a function of just one of the null coordinates. In particular, the radial Penrose limit is taken along a null geodesic with tangent parallel to one of the real null tetrad vectors l^μ or n^μ, respectively. Here, the latter is chosen such that the null geodesic (2.13) reads as $V^\mu = |B|^2 n^\mu$. Equation (2.9) for $A \equiv 0$ and $B = |B|e^{i\varphi_B}$ with φ_B a real function implies

$$\begin{aligned} \frac{1}{|B|}\frac{d|B|}{d\xi} &= \frac{1}{4}\frac{dM}{d\xi} \\ \frac{d\varphi_B}{d\xi} &= \frac{i}{4}\frac{1}{Z + \bar{Z}}\frac{d(Z - \bar{Z})}{d\xi}, \end{aligned} \quad (5.1)$$

where Eq. (2.21)(1) has been used. Thus the Weyl scalar of the plane wave spacetime is found to be

$$\Psi(\xi) = B^4(\xi)\Psi_4(\xi) = |B|^4|\Psi_4(\xi)|e^{i(4\varphi_B(\xi)+\varphi_{\Psi_4}(\xi))} \tag{5.2}$$

with

$$|B| = C_0 e^{\frac{M}{4}} \tag{5.3}$$

$$e^{i\varphi_{\Psi_4}} = \left(\frac{1+i\mathcal{P}}{1-i\mathcal{P}}\right)^{\frac{1}{2}} \tag{5.4}$$

$$\mathcal{P} = -\frac{\frac{d^2\varphi_B}{d\xi^2} - \frac{1}{2}\frac{d}{d\xi}(2U+M+\ln(Z+\bar{Z}))\frac{d\varphi_B}{d\xi}}{\frac{d^2}{d\xi^2}\ln(Z+\bar{Z}) - \frac{1}{2}\frac{d}{d\xi}(2U+M)\frac{d}{d\xi}\ln(Z+\bar{Z}) - 8(\frac{d\varphi_B}{d\xi})^2}, \tag{5.5}$$

and the affine parameter is determined by $\xi = C_0^{-2}\int du e^{-M(u,v=v_0)}$ with C_0 and v_0 constants. The plane wave metric in the radial Penrose limit is diagonal or diagonizable, respectively, for $4\varphi_B + \varphi_{\Psi_4}$ equal to zero or constant, respectively. In the latter case the complex phase can be removed by a transformation of the spinor dyad as discussed in the previous section. As can be seen from the solution (5.5) a constant phase φ_B implies $\varphi_{\Psi_4} = 0$. Moreover, it imposes $Z - \bar{Z}$ at most a constant implying the G_2 metric is diagonal (or diagonizable). Thus the radial Penrose limit is diagonal for diagonal G_2 spacetimes and not diagonal for nondiagonal ones. In [9] an explicit example of this has been presented.

VI. TWISTOR EQUATION AND SOLUTIONS IN THE PENROSE LIMIT

The twistor equation is given by [10]

$$\nabla_{A'}^{(A}\omega^{B)} = 0. \tag{6.1}$$

It is also interesting to note that because of its conformal invariance solutions in Minkowski spacetime can be transformed to solutions in conformally flat, curved backgrounds. Moreover, in [11] all metrics of four-dimensional real spacetimes locally admitting a solution to the twistor equation have been found.

In general in curved spacetimes solutions are severely restricted by the consistency condition

$$\nabla^{A'(C}\nabla_{A'}^{A}\omega^{B)} = -\Box^{(CA}\omega^{B)} = -\Psi^{CA}{}_{D}{}^{B}\omega^D - ie\varphi^{(CA}\omega^{B)}, \tag{6.2}$$

permitting the presence of an electromagnetic field and a twistor ω^B with charge e. Equation (6.1) yields the condition

$$\Psi_{ABCD}\omega^D = -ie\varphi_{(AB}\omega_{C)}. \tag{6.3}$$

For uncharged twistors this implies that either the spacetime is conformally flat ($\Psi_{ABCD} = 0$) or the Weyl spinor is null implying that it has a fourfold principal spinor. The latter is the case of plane wave spacetimes. Metrics of the form (2.10) admit conformally flat as well as plane wave solutions. Writing the uncharged twistor $\omega^A = \omega^0 o^A + \omega^1 \iota^A$ its equation is given by [13] in the GHP formalism

$$\begin{gathered}\kappa\omega^0 = \text{þ}\omega^1, \qquad \sigma\omega^0 = \eth\omega^1, \qquad \eth'\omega^0 = \sigma'\omega^1, \qquad \text{þ}'\omega^0 = \kappa'\omega^1,\\ \text{þ}\omega^0 + \rho\omega^0 = \eth'\omega^1 + \tau'\omega^1, \qquad \eth\omega^0 + \tau\omega^0 = \text{þ}'\omega^1 + \rho'\omega^1.\end{gathered} \tag{6.4}$$

For the metric (2.10) Eq. (6.4) yields to

$$\partial_u\omega^0 = -\frac{1}{4}\left(\partial_u M - \frac{\partial_u(Z-\bar{Z})}{Z+\bar{Z}}\right)\omega^0 \tag{6.5}$$

$$\partial_v\omega^1 = -\frac{1}{4}\left(\partial_v M + \frac{\partial_v(Z-\bar{Z})}{Z+\bar{Z}}\right)\omega^1 \tag{6.6}$$

$$\partial_x\omega^0 = \frac{e^{\frac{1}{2}(M-U)}}{(Z+\bar{Z})^{\frac{1}{2}}}\left(\partial_u\omega^1 - \frac{1}{4}\left(\partial_u M - 2\partial_u U - \frac{\partial_u(Z-\bar{Z})}{Z+\bar{Z}} + 4\frac{\partial_u Z}{Z+\bar{Z}}\right)\omega^1\right) \tag{6.7}$$

$$-i\partial_y\omega^0 = \frac{e^{\frac{1}{2}(M-U)}}{(Z+\bar{Z})^{\frac{1}{2}}}\left(Z\partial_u\omega^1 - \frac{Z}{4}\left(\partial_u M - 2\partial_u U - \frac{\partial_u(Z-\bar{Z})}{Z+\bar{Z}}\right)\omega^1 - \bar{Z}\frac{\partial_u Z}{Z+\bar{Z}}\omega^1\right) \tag{6.8}$$

$$\partial_x\omega^1 = \frac{e^{\frac{1}{2}(M-U)}}{(Z+\bar{Z})^{\frac{1}{2}}}\left(\partial_v\omega^0 - \frac{1}{4}\left(\partial_v M - 2\partial_v U + \frac{\partial_v(Z-\bar{Z})}{Z+\bar{Z}} + 4\frac{\partial_v\bar{Z}}{Z+\bar{Z}}\right)\omega^0\right) \tag{6.9}$$

$$i\partial_y\omega^1 = \frac{e^{\frac{1}{2}(M-U)}}{(Z+\bar{Z})^{\frac{1}{2}}}\left(\bar{Z}\partial_v\omega^0 - \frac{\bar{Z}}{4}\left(\partial_v M - 2\partial_v U + \frac{\partial_v(Z-\bar{Z})}{Z+\bar{Z}}\right)\omega^0 - Z\frac{\partial_v\bar{Z}}{Z+\bar{Z}}\omega^0\right). \tag{6.10}$$

In the Penrose limit (cf. Sec. II) the consistency condition for uncharged twistors ω^A in the G_2 background [cf. Eq. (6.3) with $e \equiv 0$] yields $\Psi_{ABCD}\omega^D\alpha^A\alpha^B\alpha^C = 0$ implying

$$A(\xi)\omega^0(A^2(\xi)\Psi_0 + B^2(\xi)\Psi_2) + B(\xi)\omega^1(A^2(\xi)\Psi_2 + B^2(\xi)\Psi_4) = 0. \tag{6.11}$$

Moreover, the only nonvanishing twistor component $\omega(\xi) = \omega^A\alpha_A$ is given by

$$\omega(\xi) = B(\xi)\omega^0 - A(\xi)\omega^1 \tag{6.12}$$

in terms of the twistor components in the G_2 spacetime.

Assuming that the spacetime is not conformally flat Eq. (6.11) and considering as examples radial Penrose limits yields solutions ω^A of the twistor equation for

(i) the Penrose limit along a null geodesic Γ tangent to the real null tetrad vector n^μ: $A(\xi) \equiv 0$, $\omega^1 \equiv 0$.

This radial Penrose limit has been considered in Sec. V. The twistor component along Γ is given by $\omega(\xi) = B(\xi)\omega^0$. The solution for $B(\xi)$ is determined by Eqs. (5.1) and (5.3). Equations (6.5)–(6.10) yield to

$$\omega^0(u) = \exp\left(-\frac{1}{4}M(u) + \frac{1}{4}\int^u d\tilde{u}\frac{\partial_{\tilde{u}}(Z-\bar{Z})}{Z+\bar{Z}}\right), \tag{6.13}$$

taking into account that all metric functions become effectively only functions of u.

Taking into account the source-free Maxwell equations the solution is of the same general form. However, the solutions for the metric functions are different in general because of the contribution from the electromagnetic spinor component $\varphi_2(u) = e^{\frac{1}{2}(M+U)}H_{,u}$ with $H = H(u)$ to Eqs. (3.12)–(3.15).

(ii) the Penrose limit along a null geodesic Γ tangent to the real null tetrad vector l^μ: $B(\xi) \equiv 0$, $\omega^0 \equiv 0$.

The twistor component along Γ is given by $\omega(\xi) = -A(\xi)\omega^1$. Using Eq. (2.8) it is found that the solution for $A(\xi) = |A|e^{i\varphi_A}$ has the same form for the modulus $|A|$ as $|B|$ [cf. Eq. (5.3)] in the anterior radial Penrose limit *i.)*, but the complex phase φ_A is determined by $-\varphi_B$ [cf. Eq. (5.1)]. Moreover, ω^1 is given by

$$\omega^1(v) = \exp\left(-\frac{1}{4}M(v) - \frac{1}{4}\int^v d\tilde{v}\frac{\partial_{\tilde{v}}(Z-\bar{Z})}{Z+\bar{Z}}\right) \tag{6.14}$$

using that all metric functions become effectively only functions of v.

Including a source-free electromagnetic field implies that the spinor component $\varphi_0(v) = -e^{\frac{1}{2}(M+U)}H_{,v}$ with $H = H(v)$ contributes to Eqs. (3.12)–(3.15).

VII. CONCLUSIONS

Penrose limits of G_2 spacetimes for general nonvacuum as well as for source-free electromagnetic fields have been considered using the formulation of Tod [7] in the spinor formalism. Moreover, the condition for the diagonalizabilty of the resulting plane wave spacetime has been considered for G_2 spacetimes. In terms of the Brinkmann form of the plane wave metric the nondiagonal wave profile is determined by the imaginary part of the only nonvanishing Weyl scalar. The Tod condition implies that the complex phase can be at most constant [7]. This could also be seen by arguing that a transformation of the null tetrad vectors renders the Weyl scalar to be a real function for a constant complex phase. As an example the radial Penrose limit has been considered in detail. Finally, the twistor equation in the Penrose limit has been considered. Explicit solutions including a Maxwell field for uncharged twistors have been found in the radial Penrose limit. This points toward an additional, interesting aspect of Penrose limits. In general the consistency condition of the twistor equation severely restricts solutions in arbitrary, curved spacetimes. However, it is possible to associate corresponding twistor solutions in a Penrose limit of a general curved spacetime.

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APPENDIX: QUANTITIES IN THE NEWMAN-PENROSE FORMALISM

The relevant quantities for the metric (2.10) are given here, some of which have been calculated using the MATHEMATICA package xAct [16]. The null tetrad metric is given by

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{A1}$$

Tetrad indices are latin indices running from 1 to 4, enclosed in brackets. The null tetrad vectors for the metric (2.10) are given by

$$l^\mu = (0 \quad e^{\frac{M}{2}} \quad 0 \quad 0) \qquad n^\mu = (e^{\frac{M}{2}} \quad 0 \quad 0 \quad 0)$$
$$m^\mu = \left(0 \quad 0 \quad \frac{e^{\frac{U}{2}}\bar{Z}}{(Z+\bar{Z})^{\frac{1}{2}}} \quad -i\frac{e^{\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}\right) \qquad \bar{m}^\mu = \left(0 \quad 0 \quad \frac{e^{\frac{U}{2}}Z}{(Z+\bar{Z})^{\frac{1}{2}}} \quad i\frac{e^{\frac{U}{2}}}{(Z+\bar{Z})^{\frac{1}{2}}}\right). \tag{A2}$$

Using the notation of [13] the directional derivatives are defined by

$$D = l^\nu\nabla_\nu \qquad D' = n^\nu\nabla_\nu \qquad \delta = m^\nu\nabla_\nu \qquad \delta' = \bar{m}^\nu\nabla_\nu. \tag{A3}$$

For the definitions of the spin-weighted directional derivatives as used in the compacted spin-coefficient (GHP) formalism, þ, þ′, ð and ð′, cf. [13] and references therein.

The only nonvanishing spin coefficients are

$$\epsilon = -\frac{1}{4}e^{\frac{1}{2}M(u,v)}\left[\partial_v M + \frac{\partial_v(Z-\bar{Z})}{Z+\bar{Z}}\right] \qquad \gamma = \frac{1}{4}e^{\frac{1}{2}M(u,v)}\left[\partial_u M - \frac{\partial_u(Z-\bar{Z})}{Z+\bar{Z}}\right]$$
$$\rho' = \frac{1}{2}e^{\frac{1}{2}M(u,v)}\partial_u U \qquad \sigma' = e^{\frac{1}{2}M(u,v)}\frac{\partial_u Z}{Z+\bar{Z}}$$
$$\rho = \frac{1}{2}e^{\frac{1}{2}M(u,v)}\partial_v U \qquad \sigma = e^{\frac{1}{2}M(u,v)}\frac{\partial_v \bar{Z}}{Z+\bar{Z}}. \tag{A4}$$

The components of the Weyl tensor are encoded in five complex Weyl scalars in the Newmann-Penrose formalism. The nonvanishing Weyl scalars for the metric G_2 metric (2.10) are given by

$$\Psi_0 = \frac{e^M}{(Z+\bar{Z})^2}[(Z+\bar{Z})[\bar{Z}_{,vv} + M_{,v}\bar{Z}_{,v} - U_{,v}\bar{Z}_{,v}] - 2(\bar{Z}_{,v})^2] \tag{A5}$$

$$\Psi_2 = -\frac{e^M}{4}\left[2U_{,uv} - U_{,u}U_{,v} - 4\frac{Z_{,u}\bar{Z}_{,v}}{(Z+\bar{Z})^2}\right] - 2\Pi \tag{A6}$$

$$\Psi_4 = \frac{e^M}{(Z+\bar{Z})^2}[(Z+\bar{Z})[Z_{,uu} + M_{,u}Z_{,u} - U_{,u}Z_{,u}] - 2(Z_{,u})^2], \tag{A7}$$

with the notation $X_{,m} \equiv \frac{\partial X}{\partial m}$ and $X_{,mn} \equiv \frac{\partial^2 X}{\partial m \partial n}$ for m, n denoting the null variables u and v. Moreover, in the expression for the Weyl scalar Ψ_2 (A6) $\Pi = \Lambda = \frac{1}{24}R$ with R the Ricci scalar, as given below.

The components of the Ricci tensor are encoded in the four real and three complex scalars given of which the following are nonvanishing for the G_2 metric under consideration:

$$\Phi_{00} = \frac{e^M}{4}\left[2U_{,vv} - U_{,v}^2 + 2M_{,v}U_{,v} - 4\frac{Z_{,v}\bar{Z}_{,v}}{(Z+\bar{Z})^2}\right] \tag{A8}$$

$$\Phi_{02} = -\frac{1}{2}\frac{e^M}{Z+\bar{Z}}\left[2\bar{Z}_{,uv} - \bar{Z}_{,u}U_{,v} - \bar{Z}_{,v}U_{,u} - 4\frac{\bar{Z}_{,u}\bar{Z}_{,v}}{Z+\bar{Z}}\right] \tag{A9}$$

$$\Phi_{20} = \bar{\Phi}_{02} \tag{A10}$$

$$\Phi_{11} = \frac{e^M}{8}\left[U_{,u}U_{,v} + 2M_{,uv} - 2\frac{Z_{,u}\bar{Z}_{,v} + \bar{Z}_{,u}Z_{,v}}{(Z+\bar{Z})^2}\right] \tag{A11}$$

$$\Phi_{22} = \frac{e^M}{4}\left[2U_{,uu} - U_{,u}^2 + 2M_{,u}U_{,u} - 4\frac{Z_{,u}\bar{Z}_{,u}}{(Z+\bar{Z})^2}\right] \tag{A12}$$

$$\Lambda = -\frac{e^M}{24}\left[4U_{,uv} - 3U_{,u}U_{,v} + 2M_{,uv} - 2\frac{Z_{,u}\bar{Z}_{,v} + \bar{Z}_{,u}Z_{,v}}{(Z+\bar{Z})^2}\right]. \tag{A13}$$

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