Coupled-channel influence on the $a_0(1700/1800)$ line shape

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(Received 7 June 2023; accepted 7 August 2023; published 24 August 2023)

The current situation with the recently discovered $a_0(1700/1800)$ resonance is very paradoxical: it is believed that $a_0(1700/1800)$ must be strongly coupled with two vector mesons, but there is no direct experimental confirmation of this yet. Based on the assumption that the $a_0(1700/1800)$ is a state similar to the four-quark state from the MIT bag, belonging to either the $\underline{9}^*$ or the $\underline{36}^*$ $q^2\bar{q}^2$ multiplet, we analyze the influence of the strong $a_0(1700/1800)$ coupling to the vector channels $K^*\bar{K}^*$, $\rho\phi$, and $\rho\omega$ on its line shape in the decay channels into pseudoscalar mesons $K\bar{K}$, $\pi\eta$, and $\pi\eta'$. This effect depends on the location of the resonance mass m_{a_0} relative to the nominal thresholds of vector channels. For example, if $m_{a_0} \approx 1700$ MeV, then the influence turns out to be hidden in a fairly wide range of coupling constants. On the whole, our analysis shows that, to confirm the presence of the strong $a_0(1700/1800)$ coupling to the vector channels, utterly required is the direct detection of the decays $a_0(1700/1800) \to K^*\bar{K}^*, \rho\phi, \rho\omega$. The appearance of even certain hints at the existence of these decays would make it possible to fundamentally advance in understanding the nature of the new a_0 state.

DOI: 10.1103/PhysRevD.108.036018

I. INTRODUCTION

Investigations of the $K\bar{K}$ and $\eta\pi$ mass spectra performed recently by the BESIII, *BABAR*, and LHCb Collaborations in $D_s^+ \to K^+K^-\pi^+$ [1], $\gamma\gamma \to \eta_c \to \eta\pi^+\pi^-$ [2], $D_s^+ \to K_S^0 K_S^0 \pi^+$ [3], $D_s^+ \to K_S^0 K^+ \pi^0$ [4], and $B^+ \to [\eta_c, \eta_c(2S), \chi_{c1}]K^+ \to (K_S^0 K^\pm \pi^\mp)K^+$ [5] indicate the existence of a new scalar isovector resonance a_0 with a mass in the region of 1700–1800 MeV and a width of about 100 MeV. Below, we will denote it conventionally as $a_0(1710)$ (or simply a_0). This state can be a partner of the known isoscalar $f_0(1710)$ [6]. The BESIII [1,3,4] and *BABAR* [2] data and the earlier theoretical study [7] stimulated discussion of the nature of the $a_0(1710)$ state and its possible manifestations in other reactions as well as construction of the models for description of the experimentally observed two-body mass spectra taking into account the $a_0(1710)$ contribution [8–17]. In most of these works, the $a_0(1710)$ is considered as a state dynamically generated by interactions between the vector mesons, including their coupling with the channels of the pseudoscalar mesons in the framework of the coupled-channel approach $K^*\bar{K}^*$, $\rho\phi$, $\rho\omega$, $K\bar{K}$, and $\pi\eta$ [7–9,14]. For the present, the $a_0(1710)$ state was observed only in the decay channels into $K\bar{K}$ and $\pi\eta$ that are not suppressed by the phase space. Note that the experimental data on the mass and total width of the $a_0(1710)$ in these channels [1–5] were obtained within the framework of the isobar model in which the usual relativistic Breit-Wigner formulas were used to describe the resonant contributions. Of course, it would be interesting to find direct evidence confirming the strong $a_0(1710)$ coupling to the decay channels into two vector mesons.

Recall that the activity in the sector of scalar mesons in the region of 1800 MeV in the channels $K^*\bar{K}^*$, $\rho\phi$, $\rho\omega$, $K\bar{K}$, $\pi\eta$, and $\pi\eta'$ with isospin $I=1$ and in similar channels with $I=0$ was predicted 46 years ago by Jaffe [18] within the MIT bag model which phenomenologically takes into account quark confinement. This activity owes to the four-quark scalar states $C_\pi^s(\underline{9}^*)$ and $C_\pi(\underline{36}^*)$ with $I=1$ (analogs of a_0) and also the $C^s(\underline{9}^*)$ and $C^0(\underline{36}^*)$ with $I=0$ (analogs of f_0) belonging to the $\underline{9}^*$ and $\underline{36}^*$ four-quark multiplets [18]. The Jaffe model also predicts the expansion of the $q^2\bar{q}^2$ state wave functions in terms of the $(q\bar{q})(q\bar{q})$ states PP, VV, $\underline{P}\cdot\underline{P}$, and $\underline{V}\cdot\underline{V}$, where symbols P ($\underline{P}$) and V ($\underline{V}$) denote colorless (color) pseudoscalar and colorless (color) vector $q\bar{q}$ mesons, respectively. The correct $q^2\bar{q}^2 \to (q\bar{q})(q\bar{q})$ recoupling coefficients for the $J=0$ four-quark bag states were obtained in Ref. [19] (see also Refs. [20,21]). These coefficients make it possible to form the rough idea of the relative strength of the coupling between the four-quark states and decay channels into pairs of pseudoscalar and vector mesons.

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In the present work, we consider two scenarios in which the $a_0(1710)$ resonance is strongly coupled to the decay channels into vector mesons (VV) and analyze the influence of this coupling on the line shape of the $a_0(1710)$ in its decay channels into pseudoscalar mesons (PP). In the first scenario, the $a_0(1710)$ is treated as a four-quark state containing a hidden $s\bar{s}$ pair and having the Okubo-Zweig-Iizuka (OZI)-superallowed coupling [18] to $K^*\bar{K}^*$, $\rho\phi$, $K\bar{K}$, and $\pi\eta_s$ (η_s is shorthand for $s\bar{s}$). Such a state can be associated with the state $C^s_\pi(\underline{9}^*)$ in the Jaffe model [18]. According to the second scenario, the $a_0(1710)$ is a four-quark state without strange quarks having the OZI-superallowed coupling to $\rho\omega$ and $\pi\eta_0$ [η_0 denotes $(u\bar{u}+d\bar{d})/\sqrt{2}$]. Its analog in the model [18] is the state $C_\pi(\underline{36}^*)$. The paper is organized as follows. Section II contains the necessary formulas for describing the solitary $a_0(1710)$ resonance. Section III analyzes in detail the influence of coupled channels on the shape of the PP and VV mass spectra in the $a_0(1710)$ decays depending on the location of the resonance mass relative to the nominal thresholds of the vector channels and on the values of the coupling constants. In many cases, the influence of the strong coupling of the $a_0(1710)$ to the vector channels on its line shape in the decay channels into pseudoscalar mesons turns out to be hidden or difficult to distinguish. Therefore, the decisive confirmation of the existence of a strong $a_0(1710)$ coupling to the vector channels would be the direct detection of the decays $a_0(1710)\to K^*\bar{K}^*, \rho\phi, \rho\omega$. Section IV summarizes our conclusions.

II. SOLITARY $a_0(1710)$ RESONANCE

Consider the solitary $a_0^+(1710)$ resonance coupled to the decay channels $ab = K^{*+}\bar{K}^{*0}$, $\rho^+\phi$, $\rho^+\omega$, $K^+\bar{K}^0$, $\pi^+\eta$, and $\pi^+\eta'$ [in the following, we will indicate the charge of the $a_0(1710)$ only if necessary]. The $a_0(1710)$ propagator taking into account the finite width corrections has the form [22–26]

$$\frac{1}{D_{a_0}(s)} = \frac{1}{m^2_{a_0} - s + \sum_{ab}[\mathrm{Re}\Pi^{ab}_{a_0}(m^2_{a_0}) - \Pi^{ab}_{a_0}(s)]}, \quad (1)$$

where s is the square of the invariant mass of the virtual $a_0(1710)$ state, m_{a_0} is a mass of the $a_0(1710)$, and $\Pi^{ab}_{a_0}(s)$ is the matrix element of the $a_0(1710)$ polarization operator corresponding to the contribution of the ab intermediate state. The energy-dependent total width of the $a_0(1710)$ is given by

$$\Gamma^{\mathrm{tot}}_{a_0}(s) = -\mathrm{Im}D_{a_0}(s)/\sqrt{s} = \sum_{ab}\mathrm{Im}\Pi^{ab}_{a_0}(s)/\sqrt{s}. \quad (2)$$

The masses of particles in π, K, and K^* isotopic multiplets are putted to be equal to the masses of π^+, K^+, and K^{*+} mesons, respectively. In the case of the $a_0(1710)$ decay into pairs of pseudoscalar mesons, the imaginary part of $\Pi^{ab}_{a_0}(s)$, which is nonzero for $s > (m_a + m_b)^2$, has the form

$$\mathrm{Im}\Pi^{ab}_{a_0}(s) = \sqrt{s}\Gamma_{a_0\to ab}(s) = \frac{g^2_{a_0ab}}{16\pi}\rho_{ab}(s), \quad (3)$$

where g_{a_0ab} is the coupling constant of the $a_0(1710)$ to the ab channel, $\rho_{ab}(s) = \sqrt{s - m^{(+)2}_{ab}}\sqrt{s - m^{(-)2}_{ab}}/s$, and $m^{(\pm)}_{ab} = m_a \pm m_b$. In so doing, $\Pi^{ab}_{a_0}(s)$ is given by the once subtracted dispersion integral corresponding to the one-loop S-wave Feynman diagram with particles ab ($K\bar{K}$, $\pi\eta$, $\pi\eta'$) in the intermediate state:

$$\Pi^{ab}_{a_0}(s) = \frac{s}{\pi}\int_{m^{(+)2}_{ab}}^{\infty}\frac{\sqrt{s'}\Gamma_{a_0\to ab}(s')ds'}{s'(s'-s-i\varepsilon)}. \quad (4)$$

For $s > m^{(+)2}_{ab}$,

$$\Pi^{ab}_{a_0}(s) = \frac{g^2_{a_0ab}}{16\pi}\left[L_{ab}(s) + \rho_{ab}(s)\left(i - \frac{1}{\pi}\ln\frac{\sqrt{s-m^{(-)2}_{ab}}+\sqrt{s-m^{(+)2}_{ab}}}{\sqrt{s-m^{(-)2}_{ab}}-\sqrt{s-m^{(+)2}_{ab}}}\right)\right], \quad (5)$$

$$L_{ab}(s) = \frac{1}{\pi}\left[1 + \left(\frac{m^{(+)2}_{ab}+m^{(-)2}_{ab}}{2m^{(+)}_{ab}m^{(-)}_{ab}} - \frac{m^{(+)}_{ab}m^{(-)}_{ab}}{s}\right)\ln\frac{m_a}{m_b}\right]. \quad (6)$$

For $m^{(-)2}_{ab} < s < m^{(+)2}_{ab}$,

$$\Pi^{ab}_{a_0}(s) = \frac{g^2_{a_0ab}}{16\pi}\left[L_{ab}(s) - \rho_{ab}(s)\left(1 - \frac{2}{\pi}\arctan\frac{\sqrt{m^{(+)2}_{ab}-s}}{\sqrt{s-m^{(-)2}_{ab}}}\right)\right], \quad (7)$$

where $\rho_{ab}(s) = \sqrt{m^{(+)2}_{ab}-s}\sqrt{s-m^{(-)2}_{ab}}/s$. The region $s < m^{(-)2}_{ab}$ will not be considered.

If the vector mesons were stable, we can use Eqs. (3)–(7) for an estimate of the contributions of vector intermediate states. Let us write the S-wave amplitude of the $a_0(1710)$ decay into a pair of vector mesons as

$$\mathcal{A}^{S\text{-wave}}_{a_0\to ab} = \frac{g_{a_0ab}}{\sqrt{3}}\epsilon^*_{a\mu}\epsilon^{*\mu}_b, \quad (8)$$

TABLE I. Isotopic weights of the $a_0^+ \to K^{*+}\bar{K}^{*0} \to (K\pi)^+(\bar{K}\pi)^0$ decay modes.

Decay mode of $a_0^+ \to K^{*+}\bar{K}^{*0}$	(a) $K^0\pi^+K^-\pi^+$	(b) $K^0\pi^+\bar{K}^0\pi^0$	(c) $K^+\pi^0K^-\pi^+$	(d) $K^+\pi^0\bar{K}^0\pi^0$
Isotopic weight	4/9	2/9	2/9	1/9

where ϵ_a (ϵ_b) is the polarization four-vector of the a (b) vector meson and g_{a_0ab} is the corresponding coupling constant. Then the calculation of the width of the $a_0 \to VV$ decay near its threshold leads in the nonrelativistic approximation exactly to Eq. (3). The next approximation is to use Eq. (3) for all s above the ab threshold in order to calculate $\Pi_{a_0}^{ab}(s)$ according to Eq. (4) and as a result to have the expressions (5)–(7) for contributions of the vector channels. Below, we illustrate the specific differences between this hypothetical variant and the variant that takes into account the finite widths of the vector mesons.

Because of the limited phase spaces of the VV states near their nominal thresholds, i.e., at $\sqrt{s} \approx m_\rho + m_\phi \approx$ 1.795 GeV, $\sqrt{s} \approx 2m_{K^*} \approx 1.791$ GeV, and $\sqrt{s} \approx m_\rho + m_\omega \approx$ 1.558 GeV, the finite widths of the V mesons must be taken into account. We will do this for the K^* and ρ mesons, while the ϕ and ω mesons will be considered in the zero-width approximation [6]. Let us start with the $a_0^+ \to K^{*+}\bar{K}^{*0} \to (K\pi)^+(\bar{K}\pi)^0$ decay. Isotopic weights of its four charged modes are listed in Table I.

The a_0 decay into channel (b) is described by one amplitude $a_0^+ \to K^{*+}\bar{K}^{*0} \to (K^0\pi^+)(\bar{K}^0\pi^0)$. The same applies to the a_0 decay into channel (c). The a_0 decay into channel (a) is described by two amplitudes differing by the permutation of two identical π^+ mesons: $a_0^+ \to K^{*+}\bar{K}^{*0} \to [(K^0\pi_1^+)(\bar{K}^-\pi_2^+) + (K^0\pi_2^+)(\bar{K}^-\pi_1^+)]$. Their contribution enters into the a_0 decay width with a factor of 1/2!. The same applies to the a_0 decay into channel (d). The modulus squared of each charged amplitude gives (without taking its isotopic weight into account) an equal contribution to the a_0 width. As is seen from Table I, the sum of isotopic weights of all charged a_0 decay modes is normalized to 1 and the weight of the interference contribution originating from channels (a) and (d) is equal to 5/9. We note that calculations of the widths of the resonances decaying into VV channels may be found, for example, in Refs. [21,27–29] (see also references therein).

We denote by s_1 and s_2 the squares of the invariant masses of the virtual K^{*+} and $\bar{K}^{*0}$ mesons, respectively, and weigh the S-wave two-body invariant phase space

$$\rho(s,s_1,s_2) = \sqrt{s^2 - 2s(s_1+s_2) + (s_1-s_2)^2}/s \tag{9}$$

for the $K^{*+}\bar{K}^{*0}$ pair with the resonant K^{*+} and $\bar{K}^{*0}$ Breit-Wigner distributions (which we assume to be the same)

$$\bar{\rho}_{K^{*+}\bar{K}^{*0}}(s) = \frac{1+\frac{5}{9}C(s)}{\pi^2} \int_{(m_K+m_\pi)^2}^{(\sqrt{s}-m_K-m_\pi)^2} \frac{\sqrt{s_1}\Gamma_{K^*}(s_1)}{|D_{K^*}(s_1)|^2} ds_1 \times \int_{(m_K+m_\pi)^2}^{(\sqrt{s}-\sqrt{s_1})^2} \frac{\sqrt{s_2}\Gamma_{K^*}(s_2)}{|D_{K^*}(s_2)|^2} \rho(s,s_1,s_2) ds_2. \tag{10}$$

Here,

$$D_{K^*}(s_j) = m_{K^*}^2 - s_j - i\sqrt{s_j}\Gamma_{K^*}(s_j), \tag{11}$$

$$\sqrt{s_j}\Gamma_{K^*}(s_j) = m_{K^*}\Gamma_{K^*}(m_{K^*}^2)\frac{m_{K^*}}{\sqrt{s_j}}\left(\frac{q_{K\pi}(s_j)}{q_{K\pi}(m_{K^*}^2)}\right)^3 \times \frac{1+q_{K\pi}^2(m^2)r_{K^*}^2}{1+q_{K\pi}^2(s_j)r_{K^*}^2}, \tag{12}$$

$q_{K\pi}(s_j) = \sqrt{s_j}\rho(s_j, m_K^2, m_\pi^2)/2$, $j = 1, 2$; $m_{K^*} = 0.8955$ GeV, $\Gamma_{K^*}(m_{K^*}^2) = 0.05$ GeV, and $r_{K^*} = 3$ GeV^{-1} [6]. The function $C(s)$ in Eq. (10) describes the relative contribution of interference between diagrams differing by the permutation of two identical π mesons in the decay modes (a) and (d) in Table I. $C(s)$ is a smooth function of s, $0 < C(s) < 1$; it tends to zero as s increases and also for $\Gamma_{K^*}(m_{K^*}^2) \to 0$. The available estimate of the interference contribution in the reaction $\gamma\gamma \to \rho^0\rho^0 \to \pi^+\pi^-\pi^+\pi^-$ [29] shows that the quantity $\frac{5}{9}C(s)$ is certainly less than 0.1, and we neglect this contribution. So, taking into account the finite width of the K^*, we get

$$\mathrm{Im}\Pi_{a_0}^{K^{*+}\bar{K}^{*0}}(s) = \sqrt{s}\Gamma_{a_0\to K^{*+}\bar{K}^{*0}}(s) = \frac{g_{a_0K^{*+}\bar{K}^{*0}}^2}{16\pi}\bar{\rho}_{K^{*+}\bar{K}^{*0}}(s), \tag{13}$$

$$\Pi_{a_0}^{K^{*+}\bar{K}^{*0}}(s) = \frac{s g_{a_0K^{*+}\bar{K}^{*0}}^2}{16\pi^2} \int_{(2m_K+2m_\pi)^2}^{\infty} \frac{\bar{\rho}_{K^{*+}\bar{K}^{*0}}(s')ds'}{s'(s'-s-i\varepsilon)}. \tag{14}$$

Similarly, taking into account the finite width of the ρ meson in the decays $a_0^+ \to \rho^+(\phi/\omega) \to \pi^+\pi^0(\phi/\omega)$, we have

$$\bar{\rho}_{\rho^+(\phi/\omega)}(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{(\sqrt{s}-m_{\phi/\omega})^2} \frac{\sqrt{s_1}\Gamma_\rho(s_1)}{|D_\rho(s_1)|^2} \rho(s, s_1, m_{\phi/\omega}^2) ds_1, \tag{15}$$

$$\mathrm{Im}\Pi_{a_0}^{\rho^+(\phi/\omega)}(s) = \sqrt{s}\Gamma_{a_0\to\rho^+(\phi/\omega)}(s) = \frac{g_{a_0\rho^+(\phi/\omega)}^2}{16\pi}\bar{\rho}_{\rho^+(\phi/\omega)}(s), \tag{16}$$

TABLE II. Recoupling coefficients for $J^P = 0^+$ $q^2\bar{q}^2$ mesons into two $q\bar{q}$ mesons [19].

Flavor	PP	VV	$\underline{P}\cdot\underline{P}$	$\underline{V}\cdot\underline{V}$
$\underline{9}^*$	−0.177	0.644	0.623	0.407
$\underline{36}^*$	0.041	0.743	−0.646	−0.169

$$\Pi_{a_0}^{\rho^+(\phi/\omega)}(s) = \frac{s g^2_{a_0\rho^+(\phi/\omega)}}{16\pi^2} \int_{(m_{\phi/\omega}+2m_\pi)^2}^{\infty} \frac{\bar{\rho}_{\rho^+(\phi/\omega)}(s')ds'}{s'(s'-s-i\varepsilon)}. \quad (17)$$

The functions $D_\rho(s_1)$ and $\sqrt{s_1}\Gamma_\rho(s_1)$ are obtained from Eqs. (11) and (12), respectively, by using the obvious changing of indexes. Here, we put $m_\phi = 1.01961$ GeV, $m_\omega = 0.78265$ GeV, $m_\rho = 0.77526$ GeV, $\Gamma_\rho(m_\rho^2) = 0.1491$ GeV [6], and $r_\rho = 1.5\ \mathrm{GeV}^{-1}$ [30].

Let us now discuss the relations between the coupling constants g_{a_0ab}. According to these relations, one can judge to some extent about the nature of the decaying state. In the MIT bag model, the flavor structure of the wave functions of the four-quark scalars $C^s_\pi(\underline{9}^*)$ and $C_\pi(\underline{36}^*)$ with masses about 1800 MeV has the form [18,19]

$$C^s_\pi(\underline{9}^*) = -0.177\left(-\frac{1}{\sqrt{2}}K\bar{K} - \frac{1}{\sqrt{2}}\eta_s\pi\right) + 0.644\left(-\frac{1}{\sqrt{2}}K^*\bar{K}^* - \frac{1}{\sqrt{2}}\phi\rho\right) + \cdots, \quad (18)$$

$$C_\pi(\underline{36}^*) = 0.041(\pi\eta_0) + 0.743(\rho\omega) + \cdots. \quad (19)$$

Here, $\eta_0 = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\eta_s = s\bar{s}$ are the linear combinations of the physical states η and η': $\eta_0 = \eta\sin(\theta_i - \theta_p) + \eta'\cos(\theta_i - \theta_p)$ and $\eta_s = s\bar{s} = \eta'\sin(\theta_i - \theta_p) - \eta\cos(\theta_i - \theta_p)$, where $\theta_i = 35.3°$ is the so-called "ideal" mixing angle and $\theta_p = -11.3°$ is the mixing angle in the nonet of the light pseudoscalar mesons [6].

The coefficients in front of the expressions in parentheses in the right-hand sides of Eqs. (18) and (19) are taken from Table II; the dots imply that the wave functions of the $C^s_\pi(\underline{9}^*)$ and $C_\pi(\underline{36}^*)$ states involve the contributions from $q\bar{q}$ pairs with hidden color $\underline{P}\cdot\underline{P}$ and $\underline{V}\cdot\underline{V}$ (see Table II). Thus, if we identify $a_0(1710)$, for example, with the state $C^s_\pi(\underline{9}^*)$, then we will have the following coupling constants $a_0(1710)$ with PP and VV channels: $g_{a_0^+K^+\bar{K}^0} = 0.177\frac{1}{\sqrt{2}}g_0$, $g_{a_0^+\eta\pi^+} = -0.177\frac{1}{\sqrt{2}}g_0\cos(\theta_i - \theta_p)$, $g_{a_0^+\eta'\pi^+} = 0.177\frac{1}{\sqrt{2}}g_0\sin(\theta_i - \theta_p)$, and $g_{a_0^+K^{*+}\bar{K}^{*0}} = g_{a_0^+\rho^+\phi} = -0.644\frac{1}{\sqrt{2}}g_0$, where the universal coupling constant g_0 describes the OZI-superallowed decays of the $q^2\bar{q}^2$ mesons into two $q\bar{q}$ mesons [18]. We will not strictly adhere to the relations between the PP and VV components of the wave functions that follow from Table II.

As for the masses of $q^2\bar{q}^2$ states in the MIT bag model, they were quoted in Ref. [18] to the nearest 50 MeV. The modern values of masses of the light scalar mesons $f_0(500)$, $K_0^*(700)$, $a_0(980)$, and $f_0(980)$ [6], which are candidates for the four-quark states [18,31–33], indicate a shift about 100 MeV toward lower masses relative to the predictions of the $q^2\bar{q}^2$ model [18]. In this regard, any value for the mass of the $a_0(1710)$ in the region 1700–1800 MeV seems quite probable within the $q^2\bar{q}^2$ model.

Let us note two points. First, the $q^2\bar{q}^2$ MIT bag model predicts a strong coupling of the $a_0(1710)$ to two vector mesons, as is seen from Eqs. (18) and (19). For example, the ratios $g^2_{a_0^+K^{*+}\bar{K}^{*0}}/g^2_{a_0^+K^+\bar{K}^0}$ and $g^2_{a_0^+\rho^+\phi}/g^2_{a_0^+K^+\bar{K}^0}$ for the $a_0(1710) = C^s_\pi(\underline{9}^*)$ are approximately 13.2. If these ratios are of the order of 1, then the searches for the $a_0(1710) \to VV$ decays are hopeless. Second, the $q^2\bar{q}^2$ model predicts the existence of the isoscalar partner of the $a_0(1710)$ with a close or even degenerate mass. The known $f_0(1710)$ state [6] is one of the possible candidates to this role. An intriguing fact is its recent observation in the $\omega\phi$ decay channel [34] (see also references herein). Complementary studies of the $a_0(1710)$ and $f_0(1710)$ resonances are very promising.

Figure 1 shows the imaginary and real parts of the polarization operators $\Pi_{a_0}^{K^{*+}\bar{K}^{*0}}(s)$ and $\Pi_{a_0}^{\rho^+\phi}(s)$. In the hypothetical case of stable vector mesons, these functions change sharply near the nominal $K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$ thresholds. Accounting for the finiteness of the widths of the K^* and ρ mesons smooths out these sharp changes in $\Pi_{a_0}^{K^{*+}\bar{K}^{*0}}(s)$ and $\Pi_{a_0}^{\rho^+\phi}(s)$. Physically important here is the appearance of the quite noticeable energy-dependent widths for the $a_0^+ \to K^{*+}\bar{K}^{*0} \to (K\pi)^+(\bar{K}\pi)^0$ and $a_0^+ \to \rho^+\phi \to \pi^+\pi^0\phi$ decays for $\sqrt{s} < 2m_{K^*} \approx 1.791$ GeV and $\sqrt{s} < m_\rho + m_\phi \approx 1.795$ GeV, respectively. A similar picture takes place also for $\Pi_{a_0}^{\rho^+\omega}(s)$.

III. $a_0(1710)$ MASS SPECTRA

Let us write the mass spectrum for the $a_0(1710) \to ab$ decay as

$$\frac{d\mathcal{B}(a_0 \to ab; s)}{d\sqrt{s}} = \frac{2\sqrt{s}}{\pi}\frac{\sqrt{s}\Gamma_{a_0\to ab}(s)}{|D_{a_0}(s)|^2} \quad (20)$$

[we note that in our model $\mathcal{B}(a_0 \to \text{all}) = 1$ in accordance with the unitarity requirement]. Following our first scenario, we assume that the $a_0(1710)$ state looks like the four-quark state $C^s_\pi(\underline{9}^*)$ and consider the influence of the coupled VV channels on the shape of its $K^+\bar{K}^0$ mass spectrum depending on three parameters m_{a_0}, $g_1 \equiv g_{a_0^+K^+\bar{K}^0}$, and $g_2 \equiv g_{a_0^+K^{*+}\bar{K}^{*0}}$. In so doing, $g_{a_0^+\eta\pi^+} = -g_1\cos(\theta_i - \theta_p)$, $g_{a_0^+\eta'\pi^+} = g_1\sin(\theta_i - \theta_p)$, and $g_{a_0^+\rho^+\phi} = g_2$. Table III presents a few sets of the values

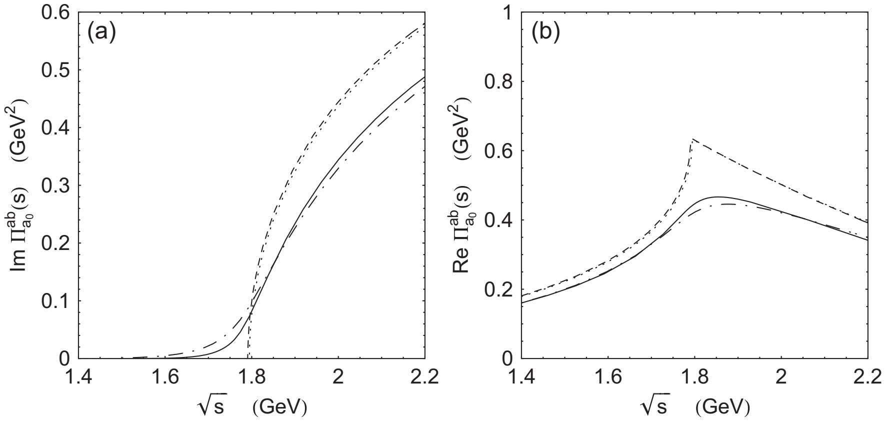

FIG. 1. (a) $\mathrm{Im}\Pi^{ab}_{a_0}(s)$ and (b) $\mathrm{Re}\Pi^{ab}_{a_0}(s)$ as functions of $\sqrt{s}$ for $ab = K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$. The dashed and dotted curves correspond to the $K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$ decay channels of the $a_0(1710)$, respectively, for the case of the stable vector mesons; see Eqs. (3)–(7). The solid and dot-dashed curves correspond to the $K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$ decay channels, respectively, with taking into account of the finite widths of the K^* and ρ mesons; see Eqs. (10) and (13)–(17). Here, we put $g^2_{a_0ab}/(16\pi) = 1$ GeV^2 to easier understand the scale of the functions $\mathrm{Im}\Pi^{ab}_{a_0}(s)$ and $\mathrm{Re}\Pi^{ab}_{a_0}(s)$.

of these parameters which will help us to understand the general situation. For m_{a_0}, it suffices to consider two extreme values: $m_{a_0} \approx 1700$ MeV and $m_{a_0} \approx 1800$ MeV. Sets 1, 4, and 6 we use as reference. They correspond to the a_0 resonance coupled only with pseudoscalar mesons. The values of the constant $g_1^2/(16\pi)$ have been estimated for these variants based on the assumption that the total a_0 decay width $\Gamma^{\mathrm{tot}}_{a_0}(m^2_{a_0}) = -\mathrm{Im}D_{a_0}(m^2_{a_0})/m_{a_0} = 100$ MeV [see Eqs. (2) and (3)]. The a_0 line shapes in the PP decay channels for variants 1, 4, and 6 correspond to the standard Breit-Wigner resonance curves with which it is convenient to compare the shapes of the PP mass spectra corresponding to the cases of the strong a_0 coupling to VV channels. It is obvious that for small values of the ratio g_2^2/g_1^2 the effect of the a_0 coupling to vector mesons on the PP mass spectra is to be small (as well as the a_0 manifestation in VV channels). Therefore, it is interesting to consider the situation when the ratio g_2^2/g_1^2 is much greater than one

TABLE III. Parameters of the a_0 resonance. Its mass and widths are in units of MeV and the coupling constants squared in units of GeV^2. $\Gamma_{a_0\to PP}(m^2_{a_0})$ is the sum of the $a_0^+ \to K^+\bar{K}^0$, $\pi^+\eta$, and $\pi^+\eta'$ decay widths. For all variants, the visible width of the a_0 peak (i.e., its full width at half maximum) is about 100 MeV.

No.	m_{a_0}	$g_1^2/(16\pi)$	$g_2^2/(16\pi)$	g_2^2/g_1^2	$\Gamma_{a_0\to PP}(m^2_{a_0})$	$\Gamma^{\mathrm{tot}}_{a_0}(m^2_{a_0})$
1	1710	0.1075	0	0	100	100
2	1717	0.15	0.84	5.6	139	159
3	1722	0.235	3	12.8	218	234
4	1817	0.1108	0	0	100	100
5	1840	0.035	0.785	22.4	31	158
6	1790	0.11	0	0	100	100
7	1830	0.11	1.25	11.4	99	281

(as in the $q^2\bar{q}^2$ model). For its illustration, we use variants 2, 3, 5, and 7 shown in Table III. Before proceeding to their analysis, we explain the notation of the curves in Figs. 2–4. For example, the solid curve labeled by 1 in Fig. 2 represents the mass spectrum $d\mathcal{B}(a_0^+ \to K^+\bar{K}^0; s)/d\sqrt{s}$ corresponding to the value set of the a_0 parameters from Table III heaving the same number 1. That is, the numbers of the curves in Figs. 2–4 are attached to the numbers of the value sets of the a_0 resonance parameters in Table III. Let us start with a discussion of the mass spectra $d\mathcal{B}(a_0 \to ab; s)/d\sqrt{s}$ shown in Fig. 2. The solid curves 1–3 show the mass spectra $d\mathcal{B}(a_0^+ \to K^+\bar{K}^0; s)/d\sqrt{s}$. The mass spectra in the decays $a_0 \to K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$ are shown by dashed and dotted curves 2 and 3, respectively. Naturally, we are talking about the mass spectra constructed taking into account the instability of the K^* and ρ mesons. If we multiply solid curves 2 and 3 by 1.79 and 3.785, respectively, and depict results by the dotted curves, then, as can be seen from the figure, they coincide with a good accuracy with curve 1 (for which $g_2 = 0$) in the range 1.62 GeV $< \sqrt{s} <$ 1.76 GeV. Thus, all three $K^+\bar{K}^0$ mass spectra have practically the same visible width of the resonance peaks approximately equal to 100 MeV. At the same time, Table III indicates that for parameter sets 2 and 3 $\Gamma_{a_0\to PP}(m^2_{a_0}) \approx 139$ and 218 MeV and $\Gamma^{\mathrm{tot}}_{a_0}(m^2_{a_0}) \approx$ 159 and 294 MeV [see Eq. (2)], respectively. The fact that the visible width of the a_0^+ peak in the $K^+\bar{K}^0$ channel turns out to be noticeably smaller than $\Gamma_{a_0\to PP}(m^2_{a_0})$ and $\Gamma^{\mathrm{tot}}_{a_0}(m^2_{a_0})$ is a direct consequence of the strong coupling of the a_0 with vector channels, which narrows the PP mass spectra. Deviations of the shape of solid curves 2 and 3 from curve 1 outside the interval 1.62 GeV $< \sqrt{s} <$ 1.76 GeV are not so large in order that they can be effectively used to detect the a_0 coupling to VV channels.

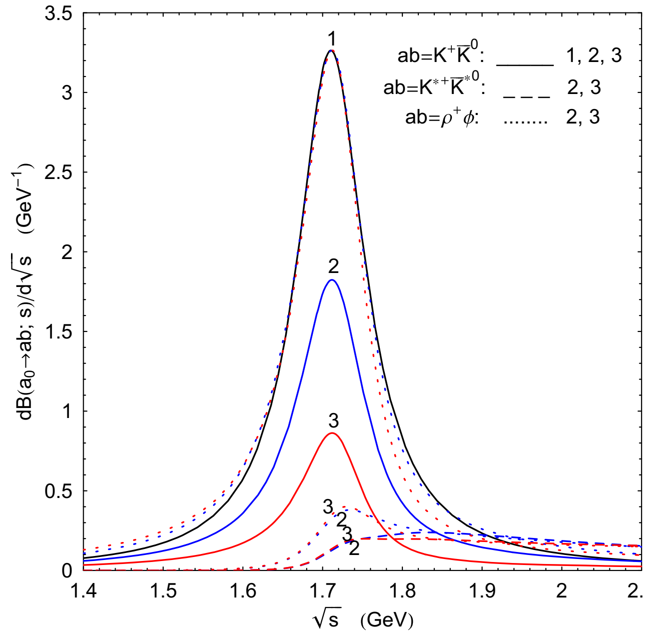

FIG. 2. The mass spectra $d\mathcal{B}(a_0 \to ab; s)/d\sqrt{s}$. Solid curves 1–3 correspond to the $a_0^+ \to K^+\bar{K}^0$ decay; dashed curves 2 and 3 to $a_0^+ \to K^{*+}\bar{K}^{*0}$; dotted curves 2 and 3 to $a_0^+ \to \rho^+\phi$. The curve numbers are attached to the numbers of the value sets of the a_0 resonance parameters in Table III. The dotted curves without numbers which practically coincide with solid curve 1 in the peak region are explained in the text.

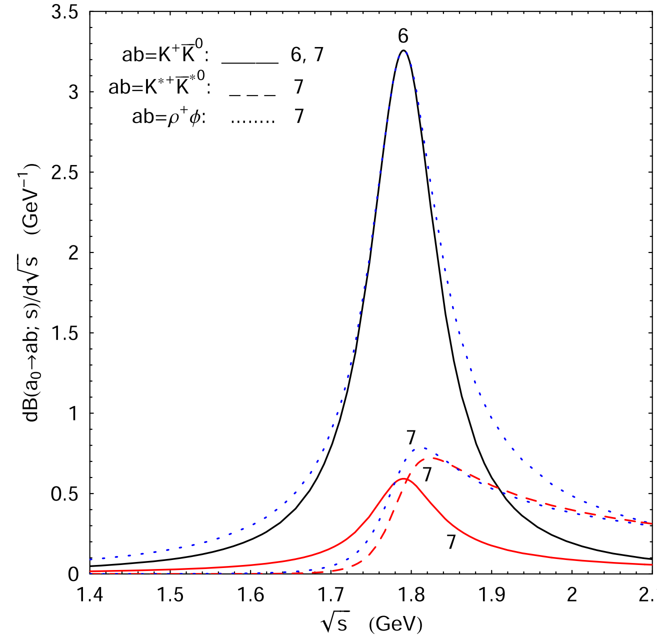

FIG. 4. The mass spectra $d\mathcal{B}(a_0 \to ab; s)/d\sqrt{s}$. Solid curves 6 and 7 correspond to the $a_0^+ \to K^+\bar{K}^0$ decay; dashed curve 7 to $a_0^+ \to K^{*+}\bar{K}^{*0}$; dotted curve 7 to $a_0^+ \to \rho^+\phi$. The curve numbers are attached to the numbers of the value sets of the a_0 resonance parameters in Table III. The dotted curve without number which practically coincides with solid curve 6 in the region of the peak maximum is explained in the text.

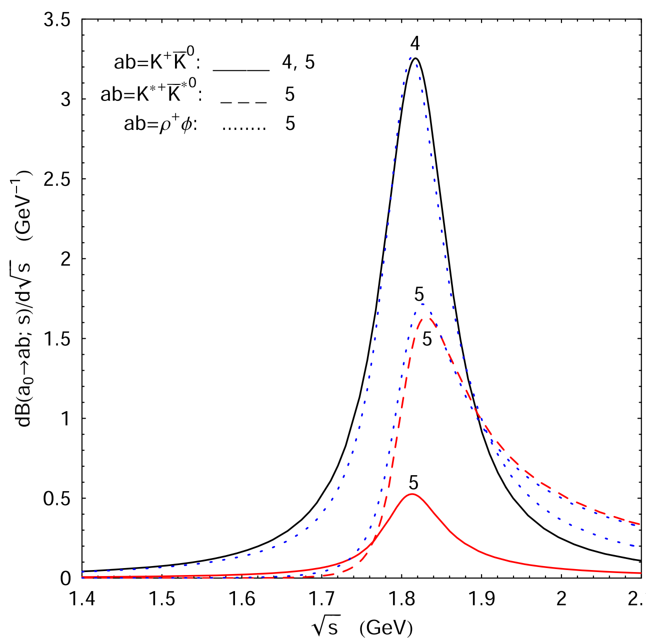

FIG. 3. The mass spectra $d\mathcal{B}(a_0 \to ab; s)/d\sqrt{s}$. Solid curves 4 and 5 correspond to the $a_0^+ \to K^+\bar{K}^0$ decay; dashed curve 5 to $a_0^+ \to K^{*+}\bar{K}^{*0}$; dotted curve 5 to $a_0^+ \to \rho^+\phi$. The curve numbers are attached to the numbers of the value sets of the a_0 resonance parameters in Table III. The dotted curve without number which practically coincides with solid curve 4 in the peak region is explained in the text.

The above examples corresponding to m_{a_0} at about 1700 MeV show that the strong coupling of the a_0 resonance with the VV channels is undoubtedly possible. At the same time, the shape of the $K\bar{K}$ mass spectrum in the region of the a_0 peak can be quite satisfactorily described without taking into account the a_0VV coupling. Certainly, a similar situation takes place for the decay channels of the a_0 into $\pi\eta$ and $\pi\eta'$. That is, the strong a_0 coupling to VV turns out to be hidden in the $a_0 \to PP$ decay channels, and, therefore, one can speak about it only presumably, at least until direct detection of the decays $a_0 \to VV$.

With m_{a_0} increasing from 1700 to 1800 MeV (and further), the $a_0 \to VV$ decay channels become more and more open and the contribution from the width $\Gamma_{a_0\to VV}(m^2_{a_0})$ becomes dominated in the total width $\Gamma^{\rm tot}_{a_0}(m^2_{a_0})$. Therefore, if we want to retain the visible width of the a_0 peak in the PP channels at a level of about 100 MeV, it is necessary to reduce the contribution from $\Gamma_{a_0\to PP}(m^2_{a_0})$ to $\Gamma^{\rm tot}_{a_0}(m^2_{a_0})$ [i.e., decrease $g_1^2/(16\pi)$]. Let us illustrate the above by using variants 4 and 5 in Table III in which m_{a_0} takes the values slightly above the nominal thresholds of the $K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$ decay channels. The corresponding mass spectra are shown in Fig. 3. The $K^+\bar{K}^0$ mass spectra are shown by solid curves 4 and 5. Dashed curve 5 and dotted curve 5 show the mass spectra in the a_0 decay channels into $K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$, respectively. It is worth to pay attention to the difference between the mass distributions of $K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$ from the case of $K^+\bar{K}^0$. If the solid curve 5 is enlarged 6.2 times and depicted as a dotted one, then, as can be seen from the figure, its shape repeats the shape of curve 1 [for which $g_2 = 0$ and $\Gamma^{\rm tot}_{a_0}(m^2_{a_0}) = 100$ MeV] with a good accuracy in the interval 1.76 GeV $< \sqrt{s} <$ 1.9 GeV. Note that solid curve 5 corresponds to $\Gamma^{\rm tot}_{a_0}(m^2_{a_0}) \approx 158$ MeV and $\Gamma_{a_0\to PP}(m^2_{a_0}) \approx$ 31 MeV (see Table III). Thus, for $m_{a_0} \approx 1800$ MeV, the strong a_0 coupling to VV pairs remains hidden as before, i.e., has very little effect on the visible width and shape of the mass distributions in the PP channels. On the other hand, the values of $m_{a_0} \approx 1800$ MeV favor the bright manifestation of the a_0 resonance in the $K^{*+}\bar{K}^{*0}$ and $\rho^+\phi$ mass spectra, as can be seen from the comparison of the corresponding curves in Figs. 2 and 3.

Let now the constant $g_1^2/(16\pi)$ takes the values greater than in variant 5 indicated in Table III. The corresponding mass spectra are shown in Fig. 4. They correspond to variants 6 and 7 in Table III. If the solid curve 7 is magnified by a factor of 5.5 and depicted as a dotted one, then, as can be seen from the figure, its right wing deviates noticeably from the reference curve 6 [for which $g_2 = 0$ and $\Gamma^{\rm tot}_{a_0}(m^2_{a_0}) = 100$ MeV]. Such a shape asymmetry of the $K^+\bar{K}^0$ mass spectrum can be discovered, in principle, provided that the background contributions accompanying the a_0 resonance are small and the region of high $\sqrt{s}$ is not limited by the phase space of the reaction.

Finally, consider the scenario where the a_0 state is similar to the $q^2\bar{q}^2$ state $C_\pi(\underline{36}^*)$; see Eq. (19). The coupled channels in this case are $\pi\eta$, $\pi\eta'$, and $\rho\omega$ channels; the a_0 coupling to the latter is dominant; see Table II. The $\rho\omega$ channel is the open one, since its the nominal threshold is approximately equal to 1558 MeV, and the a_0 resonance decaying into $\rho\omega$ is in the region of 1700–1800 MeV. Adhering to the $q^2\bar{q}^2$ model, we can express the coupling constant a_0 to $\rho\omega$ in terms of the constant g_2 introduced above: $g_3 \equiv g_{a_0\rho\omega} = 0.743 g_0 = -\sqrt{2}(0.743/0.644)g_2 \approx -1.63 g_2$ [see Eqs. (18) and (19) and Table II]. Since $g_3^2 \approx 2.66 g_2^2$, the a_0 resonance can be very broad. As an illustration, we set $g_2^2/(16\pi) = 0.75$ GeV^2 [for comparison, see the values of $g_2^2/(16\pi)$ indicated in Table III]. We also neglect a tiny a_0 coupling to $\pi\eta$ and $\pi\eta'$ channels [see Eq. (19)]. The corresponding examples of the mass spectrum $d\mathcal{B}(a_0 \to \rho\omega; s)/d\sqrt{s}$ are shown in Fig. 5. The curves (a) and (b) correspond to $d\mathcal{B}(a_0 \to \rho\omega; s)/d\sqrt{s}$ calculated at $m_{a_0} = 1710$ and 1817 MeV, respectively. Note that the total widths of distributions (a) and (b) calculated by Eq. (2) at $\sqrt{s} = m_{a_0}$ are of about 600 and 800 MeV, respectively. At the same time, their visible widths turn out to be approximately 300 and 500 MeV, respectively, owing to the energy-dependent finite width corrections in the a_0 propagator; see Eq. (1).

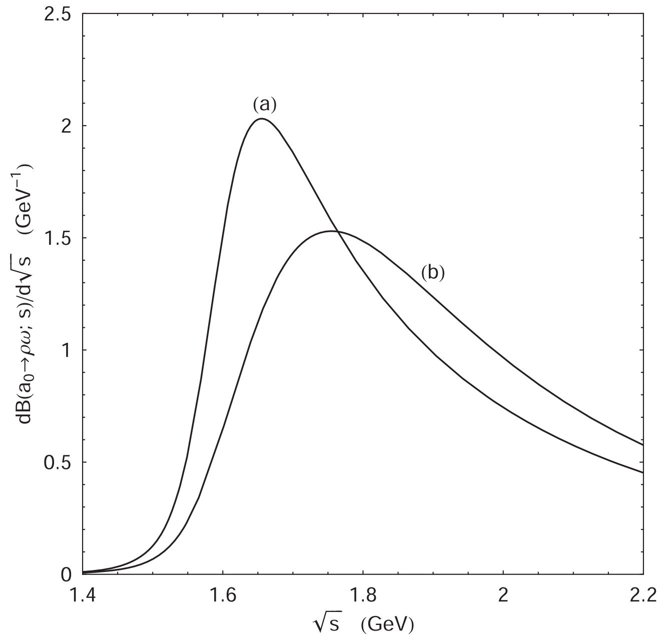

FIG. 5. Curves (a) and (b) show the mass spectrum $d\mathcal{B}(a_0 \to \rho\omega; s)/d\sqrt{s}$ calculated at $m_{a_0} = 1710$ and 1817 MeV, respectively. See the text for details.

Thus, the a_0 resonance which is discovered in the $K\bar{K}$ and $\pi\eta$ decay channels in the region of 1700–1800 MeV can be a manifestation of the four-quark state $C^s_\pi(\underline{9}^*)$. This state can also manifest itself in the $K^*\bar{K}^*$ and $\rho\phi$ decay channels that have not yet been studied. In addition, a very broad $\rho\omega$ enhancement at about 1600–1800 MeV (if found) can be due to the $C_\pi(\underline{36}^*)$ state. In Ref. [35], for completeness, we provide a brief explanation of our choice of all above variants.

IV. CONCLUSION

The discovery of a new rather narrow, heavy, isovector scalar meson $a_0(1700/1800)$ decaying to $K\bar{K}$ and $\eta\pi$ was a sufficiently unexpected event [1–5]. From the theoretical considerations mentioned in the introduction, it follows that $a_0(1710)$ can be strongly coupled to decay channels into two vector mesons. Note that good probes for the search for the $a_0 \to VV$ decays can be reactions in which the a_0 state was observed in the PP decay channels: $D_s^+ \to K^+K^-\pi^+$ [1], $\gamma\gamma \to \eta_c \to \eta\pi^+\pi^-$ [2], $D_s^+ \to K^0_S K^0_S \pi^+$ [3], $D_s^+ \to K^0_S K^+\pi^0$ [4], and $B^+ \to [\eta_c, \eta_c(2S), \chi_{c1}]K^+ \to (K^0_S K^\pm \pi^\mp)K^+$ [5]. A discussion of the processes that have the potential for detecting $a_0(1710) \to PP, VV$ decays can also be found in Refs. [14,16,17]. Of course, the detection of the decays $a_0 \to K^*\bar{K}^*, \rho\phi, \rho\omega$ is not an easy task in all the a_0 production reactions. For example, in the processes mentioned above, it will be necessary to study the $K\bar{K}\pi\pi\pi$ or 6π final states (instead of $K\bar{K}\pi$ and $\eta\pi\pi$) in order to extract the contributions of the quasi-two-body components

$\rho\phi$, $K^*\bar{K}^*$, or $\rho\omega$ within the isobar model. In this paper, we did not set ourselves the goal of studying specific processes involving the production of the vector meson pairs in final states. Each such process is unique and requires special consideration. Nevertheless, we hope that our work contributes future investigations just in this direction. The mass spectra constructed by us for the decays $a_0 \to K^{*+}\bar{K}^{*0}$, $a_0 \to \rho^+\phi$, and $a_0 \to \rho^+\omega$ in Figs. 2–5 could be confronted with future data.

In this paper, we have analyzed the effect of the strong coupling of the $a_0(1710)$ resonance to vector-vector channels on its line shape in the decay channels into two pseudoscalar mesons. We have assumed that the $a_0(1710)$ state strongly coupled to the VV channels might be similar to the four-quark state belonging to either the $\underline{9}^*$ or $\underline{36}^*$ multiplet. Our goal was to find evidence in favor of the strong coupling of the $a_0(1710)$ to vector mesons in the decay channels into pseudoscalar mesons. The impetus to this was the well-known effect of the narrowing of the $\pi\eta$ mass spectrum in the $a_0(980) \to \pi\eta$ decay caused by the influence of the strong coupling $a_0(980)$ to the $K\bar{K}$ channel [24,25]. This effect helped, in particular, to eliminate the obvious contradiction between the observed narrowness of the $a_0(980)$ peak in the $\pi\eta$ channel and the assumption of the $q^2\bar{q}^2$ model about the superallowed coupling of the $a_0(980)$ to $\pi\eta$ and $K\bar{K}$ channels [25,26]. We have found out that in the case of the state $a_0(1710)$ its strong coupling to VV channels can work similarly to the coupling of the $a_0(980)$ to $K\bar{K}$, i.e., to narrow the $a_0(1710)$ peak in the PP mass spectra. If, in the presence of this coupling, the visible width of the $a_0(1710)$ in the $K\bar{K}$ and $\pi\eta$ channels turns out to be (as in experiment) about 100 MeV, then, in its absence, the width could be from 150 to 300 MeV. At present, the fundamental difference between the situation with the $a_0(1710)$ and the situation with the $a_0(980)$ is in the absence of the data on $a_0(1710) \to VV$ decays. In addition, we have shown that in many cases the shape of the $a_0(1710)$ mass spectra in PP channels can be satisfactorily described without taking into account the $a_0(1710)VV$ coupling at all. That is, the $a_0(1710)$ coupling to VV turns out to be hidden in the PP channels, and so far it is possible to speak about its existence only hypothetically. Additional investigations are needed in this direction, and first of all, of course, the direct detection of the $a_0(1710) \to VV$ decays is necessary. Combined studies of the $a_0(1710) \to VV$ and $f_0(1710) \to VV$ decays are also very promising.

ACKNOWLEDGMENTS

The work was carried out within the framework of the state contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0021.

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[35] At present, the position of m_{a_0} is not well defined. In Sec. II, we noted that in the $q^2\bar{q}^2$ model the values of m_{a_0} in the interval 1700–1800 MeV are quite admissible. In the literature, the values $m_{a_0} = 1710$ and 1817 MeV are the most frequently mentioned (tentative) values. Therefore, we used these values for m_{a_0} in reference sets 1 and 4 in Table III. Set 6 with $m_{a_0} = 1790$ MeV from the considered interval we also used as the reference one. Parameters a_0 for sets 2, 3, 5, and 7 were selected so as to best reproduce the mass spectra for the corresponding reference variants (for which $g_2 = 0$). In so doing, all the given examples had to correspond to large values of the ratio g_2^2/g_1^2 (as indicated by the $q^2\bar{q}^2$ model), as well as the visible width of the a_0 peak in the PP channels of about 100 MeV. It is clear that the selected examples for the $a_0 = C_\pi^s(\underline{9}^*)$ scenario are quite sufficient to understand the possible situation in the a_0 mass range from 1700 to 1800 MeV. The choice of the a_0 parameters to illustrate the $a_0 = C_\pi(\underline{36}^*)$ scenario is exhaustively described in the text.