Off-shell hydrodynamic expansion

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Relativistic hydrodynamics is surprisingly predictive, even for systems that violate the fundamental assumptions of small gradients and small deviations from local thermal equilibrium. The method of moments can be used to extract second-order viscous hydrodynamics from a kinetic background, without the explicit request of small gradients and local equilibrium. The relativistic Boltzmann equation itself, however, is not general enough to justify hydrodynamics in all cases of interest. A possible explanation, often invoked but not yet formalized, focuses on the similarities between the relativistic Boltzmann equation and its quantum precursor, the evolution of the Wigner distribution. Indeed, it is possible to recover a systematically improvable hydrodynamic expansion. It is necessary, however, to work with a set of regularized, parametric moments. Their integrals reduce, in the classical limit, to the momentum integrals appearing in the moments expansion; most of the integrals appearing in the classical method of moments being undefined off shell. The regularized moments are, on the other hand, always well defined. Just like its classical counterpart, the evolution of the Wigner distribution can be solved exactly for a $(1+1)$-dimensional expansion with the relaxation time approximation. The convergence of the regularized expansion is checked numerically against the exact solutions. Second-order viscous hydrodynamics can be predictive even far from the kinetic limit, and far from local equilibrium.

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I. INTRODUCTION

Relativistic hydrodynamics has been used in a wide range of physical systems, from astrophysical plasmas to heavy-ion collisions [1–3]. It is often thought to be inescapably an expansion in gradients, or an approximation of the relativistic Boltzmann equation. Both instances are not general enough to explain the hydrodynamic behavior, namely, of the quark-gluon plasma in the crossover region. The Chapman-Enskog expansion [4] involves a systematic power counting of the gradients of the hydrodynamic variables. It can be performed in the context of relativistic kinetic theory [5] (weak coupling) as well as strongly coupled relativistic systems [6,7]. The leading order corresponds to ideal hydrodynamics, and the first order to the relativistic Navier-Stokes equations [8]. The latter equations, in general, violate causality [9] and are unstable [10–13], but it has been proven [14] that it is possible to obtain causal and stable theory *at first order in gradients* if different definitions for the hydrodynamic fields are used. The most common way to preserve causality and stability, however, is to add second-order corrections [6,7]. Third-order terms can be included [15,16], though precise statements regarding causality and stability are not available at higher orders. Recent works pointed out that the gradient series has zero radius of convergence [17–20]. The lowest orders of an asymptotic series can be numerically accurate, as it seems the case for second-order relativistic hydrodynamics. However, in the absence of a fast convergence, one cannot look at the next orders to guess the accuracy of the approximation. Second-order relativistic hydrodynamics can be obtained independently as an approximation of relativistic kinetic theory [21]. The method of moments can be used to systematically improve the hydrodynamic expansion to higher orders. In this approach the Boltzmann equation is expressed as a set of coupled equations for the tensor moments of the distribution function, the lower ranking moments being the hydrodynamic variables. One can truncate this set of equations at some order, approximating the leftover moments still present in the equations, and systematically improve the approximation including more moments as dynamical variables [21]. Under flow conditions of extreme symmetry, in which the relativistic Boltzmann equation can be solved exactly [22–26], this procedure has been shown to converge rapidly to the exact results of relativistic kinetic theory [19,27,28]. The relativistic Boltzmann equation itself, however, is expected to be a valid approximation in the small $\hbar$ limit, and only for weakly interacting asymptotic particle states [29]. It is questionable at best, in heavy-ion collisions to assume both. Close to the crossover region it is not clear what are (if any) the (quasi)particle asymptotic degrees of freedom. In general, given the temperature of the order of hundreds of MeV, but

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rapid changes over a fraction of 1 fm of distance, the use of natural units is not surprising. The Plank constant $\hbar c \simeq 200\ \mathrm{MeV} \cdot \mathrm{fm}$ is close to the typical action scales of the system.

An often invoked, and yet not properly formalized, justification of hydrodynamics relies on the similarities between the relativistic Boltzmann equation and the properties of the quantum precursor of the distribution function: the Wigner distribution.

The Wigner distribution is an off-shell generalization of the distribution function and it is linked to the (expectation values of) the stress energy tensor, much like the distribution function is in relativistic kinetic theory. The fields in quantum field theory fulfill a version of the Klein-Gordon equation with sources. This allows to prove that the Wigner distribution follows an off-shell version of the relativistic Boltzmann equation, among the other equations it fulfills in the general case.

The method of moments relies heavily on the kinetic part of the Boltzmann equation. From a similar structure of the equations it can be expected that similar results can be obtained. Since the Wigner distribution fulfills at least one equation which is the generalization of the Boltzmann equation, it can be expected that the method of moments can be generalized, thus providing an off-shell version of the hydrodynamic expansion. In turn this can be used to justify hydrodynamics also in the off-shell case, like it has been done for the classical case, and it provides higher order approximations.

Recently there has been a raising interest regarding the evolution of the Wigner distribution, mostly due to spin dynamics [30–33]. The method of moments has been applied successfully in the case of spin-1/2 fields, but only at first order in an $\hbar$ expansion [34–38]. The use the semiclassical expansion of the Wigner distribution and the equation it fulfills is more important than it looks, in the framework of the method of moments: its first order still on shell. This "on-shellness" requirement can be easily understood, if one notices that the higher ranking tensors appearing in the moments expansion are not well defined off shell. They diverge even if the Wigner distribution has very small, but finite, values off shell. In this approach, restricting to the first order of a semiclassical expansion, or in any case an on-shell approximation, is not just a simplification of the kinetic equations.

The purpose of this work is to show that, in fact, one can generalize the method of moments to the off-shell case, and recover a systematically improvable hydrodynamic expansion. It is necessary, though, to use a set of regularized moments. In the kinetic (on-shell) limit, they reproduce the classical tensor moments, and one recovers the classical expansion.

In Sec. II there is a brief introduction of the classical method of moments. Section III discusses the problem of a naive extension to the off-shell case, and the regularized expansion is introduced. In Sec. IV regularized expansion is tested against the exact solutions, for initial conditions which are very far from the kinetic limit. The conclusions are in Sec. V. The technical details and the generalizations are discussed in the Appendices. In Appendix A there is a mathematical discussion over free fields, proving that they can have a very large contribution from the off-shell parts, and that they already need regularized moments for the expansion. In Appendix B there is a discussion about the limits in which one can expect a single scalar Wigner distribution to describe an interacting system, and it discusses the more general case. In Appendix C there are details about the proper way to treat the higher order of the regularized expansion. In Appendix D there are some general details about the relaxation time approximation used for the numerical comparisons.

In this work the natural units are used $\hbar = c = k_B = 1$, as well as the Einstein convention of automatically summing over repeated upper and lower indices. The contraction between four-vectors is represented with a dot: $v \cdot w = v^\mu w^\nu g_{\mu\nu}$. The "mostly minus" convention for the Minkowski metric is used, i.e. $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The conventions $\Delta^{\mu\nu} = (g^{\mu\nu} - u^\mu u^\nu)$ for the projection orthogonal to the four-velocity u^μ and round parentheses for a symmetrization of indices are also used, e.g. $A^{(\mu} B^{\nu)} = \frac{1}{2!}(A^\mu B^\nu + A^\nu B^\mu)$.

II. THE CLASSICAL MOMENT EXPANSION

It is possible to extract the exact comoving derivative of the distribution function $u \cdot \partial f = \dot{f}$, directly from the relativistic Boltzmann equation,

$$p \cdot \partial f(x,p) = -\mathcal{C}[f] \Rightarrow (p \cdot u) u \cdot \partial f = -p \cdot \nabla \partial f - \mathcal{C}$$
$$\dot{f} = -\frac{1}{(p \cdot u)} \Big(p \cdot \nabla f + \mathcal{C} \Big), \tag{1}$$

independently of the particular definition of the four-velocity u^μ and the collisional kernel $\mathcal{C}$. The gradient orthogonal to u^μ is

$$\nabla_\mu = \Delta^\nu_\mu \partial_\nu = (g^{\mu\nu} - u^\mu u^\nu)\partial_\mu. \tag{2}$$

It is useful to introduce the generic tensor moments

$$\mathcal{F}^{\mu_1 \cdots \mu_s}_r = \int_{\mathbf{p}} (p \cdot u)^r p^{\mu_1} \cdots p^{\mu_s} f, \tag{3}$$

with $\int_{\mathbf{p}}$ the covariant (on-shell) momentum integral

$$\int_{\mathbf{p}} = \frac{N_{\mathrm{dof}}}{(2\pi)^3} \int \frac{d^3 p}{E_{\mathbf{p}}} = \frac{N_{\mathrm{dof}}}{(2\pi)^3} \int d^4 p 2\theta(E) \delta(p^2 - m^2), \tag{4}$$

and N_{dof} the eventual degeneracy factor.

Making use of (1), after some straightforward algebra, one can find the exact comoving derivatives of the moments (3),

$$\dot{\mathcal{F}}_r^{\mu_1\cdots\mu_s} + C_{r-1}^{\mu_1\cdots\mu_s} = r\dot{u}_\alpha \mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha \mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} + (r-1)\nabla_\alpha u_\beta \mathcal{F}_{r-2}^{\alpha\beta\mu_1\cdots\mu_s}, \quad (5)$$

in which $C_{r-1}^{\mu_1\cdots\mu_s}$ is a shorthand notation for

$$C_{r-1}^{\mu_1\cdots\mu_s} = \int_{\mathbf{p}} (p\cdot u)^{r-1} p^{\mu_1}\cdots p^{\mu_s} \mathcal{C}[f]. \quad (6)$$

In particular, the stress-energy tensor reads

$$\mathcal{F}_0^{\mu\nu} = T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + q^\mu u^\nu + u^\mu q^\nu - (\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \quad (7)$$

in which the general decomposition of the hydrodynamic degrees of freedom has been used. That is, starting from the geometric decomposition along and orthogonal to u^μ, the proper energy density is $\mathcal{E}$; that is, the energy in the local comoving frame. The energy flux is the spatial vector q^μ. The fully spatial and traceless part is the shear pressure correction $\pi^{\mu\nu}$. Then the isotropic pressure, that is, the only contribution to $\Delta_{\mu\nu}T^{\mu\nu}$, which is further decomposed in the hydrostatic pressure $\mathcal{P}$, is given by the equation of state (or simply the local equilibrium part of the distribution function in relativistic kinematics). The remaining part Π is the bulk pressure correction.

The exact evolution of the stress-energy tensor $T^{\mu\nu} = \mathcal{F}_0^{\mu\nu}$ reads, thanks to (5),

$$\dot{T}^{\mu\nu} + C_{-1}^{\mu\nu} = -\nabla_\alpha \mathcal{F}_{-1}^{\alpha\mu\nu} - \nabla_\alpha u_\beta \mathcal{F}_{-2}^{\alpha\beta\mu\nu}. \quad (8)$$

The contraction of the last equation with the four-velocity u_μ is the local conservation of four-momentum $\partial_\mu T^{\mu\nu} = 0$, which is included at all orders of the hydrodynamic expansion. The remaining equations provide the exact evolution of the pressure tensor,

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C_1^{\langle\mu\rangle\langle\nu\rangle} = -\mathcal{P}^{\mu\alpha}\nabla_\alpha u^\nu - \mathcal{P}^{\nu\alpha}\nabla_\alpha u^\mu - q^\mu \dot{u}^\nu - \dot{u}^\mu q^\nu - \theta\mathcal{P}^{\mu\nu} - \nabla_\alpha \mathfrak{f}_{-1}^{\alpha\langle\mu\rangle\langle\nu\rangle} - \nabla_\alpha u_\beta \mathfrak{f}_{-2}^{\alpha\beta\mu\nu}. \quad (9)$$

Both instances can be obtained without approximations from Eq. (8) making use of the exact relations

$$u_{\nu_1}\cdots u_{\nu_n}\mathcal{F}_r^{\nu_1\cdots\nu_n\mu_1\cdots\mu_s} = \mathcal{F}_{r+n}^{\mu_1\cdots\mu_s}, \quad (10)$$

and the similar ones for the collisional integrals. The brackets in Eq. (9) represent the projection $\mathcal{O}^{\langle\mu\rangle\cdots} = \Delta_\nu^\mu \mathcal{O}^{\nu\cdots}$, and the shorthand notation,

$$\mathfrak{f}_r^{\mu_1\cdots\mu_s} = \mathcal{F}_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}, \quad (11)$$

is in use. The energy flux is then $q^\mu = \mathfrak{f}_1^\mu$ (vanishing in the Landau frame), and $\mathfrak{f}_0^{\mu\nu}$ is the pressure tensor: the fully spatial part of $T^{\mu\nu}$. In other words, according to the general decomposition (7)

$$\mathfrak{f}_0^{\mu\nu} = \mathcal{P}^{\mu\nu} = -(\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}. \quad (12)$$

On the right-hand side of Eq. (9) all terms except the last two are components of $T^{\mu\nu}$. One can either approximate these tensors (second-order relativistic hydrodynamics or modified versions of it [39–41]) or treat them as independent degrees of freedom and use Eq. (5) for their evolution [39–41]. The same arguments hold for the collisional integral in the left-hand side of Eq. (9), and these steps can be repeated further to get the higher orders of the expansion; in other words, making use of the general Eq. (5) and the appropriate projections orthogonal to the four-velocity to obtain the exact evolution of all the $\mathfrak{f}_r^{\mu_1\cdots\mu_s}$ tensor moments (11) appearing, order by order, among the sources.

III. THE QUANTUM GENERALIZATION

In its most simple instance, the Wigner distribution is the Fourier transform of the two point expectation value [29]

$$W(x,k) = 2\int \frac{d^4v}{(2\pi)^4} e^{-ik\cdot v}\left\langle \phi^\dagger\left(x+\frac{1}{2}v\right)\phi\left(x-\frac{1}{2}v\right)\right\rangle, \quad (13)$$

with the expectation value for a generic state (pure or mixed) of the system. The stress-energy tensor reads

$$T^{\mu\nu} = \int d^4k k^\mu k^\nu W(x,k). \quad (14)$$

The Wigner distribution satisfies an equation very similar to Eq. (1), which can be derived from the field equations

$$k\cdot\partial W = -\mathcal{C}_W[W,\ldots] \Rightarrow (k\cdot u)\dot{W} = -k\cdot\nabla W - \mathcal{C}_W. \quad (15)$$

The quantum version of the collisional kernel $\mathcal{C}_W$ depends on the interaction, and it vanishes for free fields. In Ref. [29] it is shown that for the canonical equilibrium, the right-hand side of Eq. (13) is space-time independent, on shell, and it reduces to the Bose-Einstein distribution as expected; additionally, close to equilibrium and at the first order of an $\hbar$ expansion, $W(x,k) \propto \delta(k^2-m^2)f(x,k)$. Because of the similarities with relativistic kinematics, one would expect to recover the hydrodynamics expansion with the minimal substitution,

$$\int_{\mathbf{p}} \to \int d^4k, \qquad \frac{N_{\mathrm{dof}}}{(2\pi)^3} f(x,p) \to W(x,k). \quad (16)$$

Before discussing that, it should be pointed out that Eqs. (13)–(15) are not the most general. They are surely

correct for a free scalar field. For instance, the discussion in Appendix A to highlight the importance of the off-shell contributions and the need of a coherent off-shell treatment, is done in this case. There are however also matrix valued Wigner distributions (for spinors or vector bosons, one matrix index per field in the bilinear). In gauge theories Eq. (13) is modified to preserve gauge covariance, etc.

In the method of moments, on the other hand, the kinetic part, the left-hand side of Eq. (1), plays the most important role, allowing for the exact manipulations that lead to Eq. (9), which in turn highlights the many self-couplings of the hydrodynamic degrees of freedom in their evolution. The rest of this section will show the problem of ill defined moments in a naive off-shell generalization of the method of moments, and the way to treat it. Both "the problem and the cure" stem from the left-hand side of Eq. (15), which is the same in the general case (for each component of the matrix distributions, see Appendix B). For mathematical simplicity, the rest of this work considers only the simplest, scalar case of Eqs. (13)–(15); understanding that one has similar results, and must follow the same passages, in the more general case. More information about the interacting scalar case and the generalizations is in Appendix B.

Going back to the method of moments, following the steps outlined in the last section and making use of the minimal substitution (16), one would recover Eq. (9), however the last two integrals are ill defined in the quantum case, for instance

$$\mathfrak{f}^{\alpha\beta\mu\nu}_{-2}|_{\text{Wigner}} = \int d^4k \frac{k^{\langle\alpha\rangle}k^{\langle\beta\rangle}k^{\langle\mu\rangle}k^{\langle\nu\rangle}}{(k\cdot u)^2} W(x,k). \quad (17)$$

The Wigner distribution is not on shell in general, and Eq. (17) has a $1/x^2$ nonintegrable pole in the energy. Even in the case of free fields, the simplest imaginable case, the spacelike part of the Wigner distribution cannot be excluded [42,43], $W(x,k)|_{k^2<0} \neq 0$. In fact, it is not difficult to write a quantum state in which the spacelike part of W provides very large contributions to $T^{\mu\nu}$ for free scalar fields, as discussed in Appendix A. There is no particular reason, either, to expect only on-shell contribution in the interacting case. One is also left with a paradoxical outcome: it does not matter if the numerical value of W is very small for all spacelike values $k^2 < 0$ and, thus, it gives an irrelevant contribution to the total $T^{\mu\nu}$. The integrals like (17) remain divergent for any nonvanishing W in $k\cdot u = 0$, completely breaking the naive generalization. For instance, consider a Wigner distribution of the form

$$W(x,k) = \frac{N_{\text{dof}}}{(2\pi)^3} 2\theta(k^0)\delta_\varepsilon(k^2-m^2) f(x,\mathbf{k}), \quad (18)$$

with $\delta_\varepsilon(k^2-m^2)$ a delta family rather than the exact Dirac delta, e.g. a Gaussian $\exp\{-(k^2-m^2)^2/2\varepsilon^2\}/\sqrt{2\pi}\varepsilon$. In other words, an arbitrarily close situation to the actual kinetic case. The tensor moment (17) still diverges, not even the principal value is well defined. The off-shell distribution, however, is arbitrarily close to the on-shell limit, the same for the stress energy tensor. For all practical purposes, one can substitute $\delta_\varepsilon(k^2-m^2)$ in (18) with the actual Dirac delta, obtaining the classical counterpart (in which the method of moments works), but the usual moments expansion is ill defined for any $\varepsilon \neq 0$.

The classical method of moments has to be reformulated. In the situations like (18), the corrections are, by definition, arbitrarily small and one could use the regular expansion. One could also stick to the classical results even if some off-shell contributions are present. However, in this way it is simply not possible to guess if the results of the classical expansion are any good, a problem that can be important for significant off-shell contributions (like in the case presented in Appendix A). The similarities between the evolution of the off-shell Wigner distribution and the classical distribution function cannot be used to justify hydrodynamics in the off-shell case; not with the naive, ill-defined, generalization of the method of moments.

It is straightforward, at this point, to ask if there is a class of generalized moments, which are well defined off shell, but can be used to recover the classical tensor moments (11) in the kinetic limit, in which they are well defined, in order to build a consistent off-shell generalization of the hydrodynamic expansion.

It is possible to revisit a regularization introduced to deal with similar infrared divergencies appearing at higher orders in the hydrodynamic expansion of the Boltzmann-Vlasov equation [44,45]. The the main objects are the regularized tensors,

$$\Phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int d^4k (k\cdot u)^n e^{-\zeta(k\cdot u)^2} k^{\mu_1}\cdots k^{\mu_s} W, \quad (19)$$

with $\zeta \geq 0$ a parameter with the dimensions of a length squared in natural units. Integrating from $\zeta = 0$ to infinity one recovers the quantum version of the moments in (3), if they are well defined. In particular, $T^{\mu\nu} = \mathcal{F}_0^{\mu\nu}$ is the ζ integral of the $\Phi_2^{\mu\nu}$ moment. The dynamical equations obtained from (15) then read

$$\begin{aligned}\dot{\Phi}_n^{\mu_1\cdots\mu_s} + C_{n-1}^{\mu_1\cdots\mu_s} &= \dot{u}_\alpha[n\Phi_{n-1}^{\alpha\mu_1\cdots\mu_s} - 2\zeta\Phi_{n+1}^{\alpha\mu_1\cdots\mu_s}] \\ &\quad + \nabla_\alpha u_\beta[(n-1)\Phi_{n-2}^{\alpha\beta\mu_1\cdots\mu_s} - 2\zeta\Phi_n^{\alpha\beta\mu_1\cdots\mu_s}] \\ &\quad - \nabla_\alpha\Phi_{n-1}^{\alpha\mu_1\cdots\mu_s}. \end{aligned} \quad (20)$$

For $n \geq 1$ no diverging integral ever appears. It is immediate to check that in the kinetic limit $W \propto \delta(k^2-m^2)$, the ζ integrals can be performed exactly for each separate term of Eq. (20). Doing so, one recovers the classical counterparts (5). The exact relations (10) can be rewritten for the $\Phi_n^{\mu_1\cdots\mu_s}$ moments. Just like in the classical case, it is convenient to the switch to the moments orthogonal to u^μ,

$$\phi_n^{\mu_1\cdots\mu_s} = \Delta_{\nu_1}^{\mu_1}\cdots\Delta_{\nu_s}^{\mu_s}\Phi_n^{\nu_1\cdots\nu_s} = \int d^4k (k\cdot u)^n e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\cdots k^{\langle\mu_s\rangle} W, \quad (21)$$

that is, the quantum (and parametric in ζ) counterparts of the (11) moments. Making use again of the notation $O^{\langle\mu\rangle\cdots} = \Delta_\nu^\mu O^{\nu\cdots}$, from the exact evolution (20), one has

$$\begin{aligned}&\dot\phi_n^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \tilde C_{n-1}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}\\ &= -s\dot u^{(\mu_1}\phi_{n+1}^{\mu_2\cdots\mu_s)} - \theta\phi_n^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_n^{\mu_2\cdots\mu_s)\alpha}\\ &\quad - \nabla_\alpha\phi_{n-1}^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot u_\alpha[n\phi_{n-1}^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_{n+1}^{\alpha\mu_1\cdots\mu_s}]\\ &\quad + \nabla_\alpha u_\beta[(n-1)\phi_{n-2}^{\alpha\beta\mu_1\cdots\mu_s} - 2\zeta\phi_n^{\alpha\beta\mu_1\cdots\mu_s}],\end{aligned} \quad (22)$$

where $\theta = \nabla_\mu u^\mu$ is the scalar expansion, and $\tilde C_{n-1}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$ has the same prescription of the regularized integrals in (21) weighted with $\mathcal{C}_W$ instead of W. The exact evolution of the pressure tensor $\mathcal{P}^{\mu\nu} = \mathfrak{P}_0^{\mu\nu} = \int d\zeta\phi_2^{\mu\nu}$ then reads

$$\begin{aligned}\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} &= -q^\mu\dot u^\nu - \dot u^\mu q^\nu - \theta\mathcal{P}^{\mu\nu} - \mathcal{P}^{\mu\alpha}\nabla_\alpha u^\nu - \mathcal{P}^{\nu\alpha}\nabla_\alpha u^\mu\\ &\quad + \int_0^\infty d\zeta\{-\tilde C_1^{\langle\mu\rangle\langle\nu\rangle} - \nabla_\alpha\phi_1^{\alpha\langle\mu\rangle\langle\nu\rangle}\\ &\quad + \dot u_\alpha[2\phi_1^{\alpha\mu\nu} - 2\zeta\phi_3^{\alpha\mu\nu}] + \nabla_\alpha u_\beta[\phi_0^{\alpha\beta\mu\nu} - 2\zeta\phi_2^{\alpha\beta\mu\nu}]\}.\end{aligned} \quad (23)$$

The direct couplings with the elements of $T^{\mu\nu}$ are exactly the same as in the classical case (9). Different from the naive extension to the off-shell case mentioned at the beginning of this section, Eq. (22) is well defined. However, in the general case, one must be careful and avoid splitting the ζ integral in the right-hand side. It has to be convergent for any differentiable $T^{\mu\nu}$, but it does not have to converge *uniformly*. It can still be approximated to obtain second-order viscous hydrodynamics (e.g. substituting the higher ranking tensors with their equilibrium expectation values). Otherwise one can treat the new degrees of freedom as dynamical variables, using Eq. (22) for their evolution, and so on for the higher orders. From the very definition (21) the following relations hold, for any $n \geq 0$:

$$\int_\zeta^\infty d\zeta'\phi_{n+2}^{\mu_1\cdots\mu_s}(x,\zeta') = \phi_n^{\mu_1\cdots\mu_s}(x,\zeta), \qquad \partial_\zeta\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = -\phi_{n+2}^{\mu_1\cdots\mu_s}(x,\zeta), \quad (24)$$

in which one is the inverse of the other. As a consequence, one can use a limited number of generations with fixed n, as has been done in the classical case [44,45]; additional details in Appendix C. More interestingly, if the situation is tame enough, it is possible to perform exactly the integration in the right-hand side and obtain a set of well-defined, ζ-independent sources for the evolution of the pressure tensor, and for the higher order equations. In fact, this happens in the system used in this work for the numerical comparisons.

IV. AN EXACTLY SOLVABLE CASE

In order to check the prescription presented in the last chapter and, more interestingly, to check the validity of second-order hydrodynamics, it is interesting to look at the simplest solvable instance of a Wigner distribution out of the kinetic limit. The approach is a generalization to the off-shell case of [24]. The starting point is very similar. In the forward pointing light cone $t > |z|$, one assumes the so-called "Bjorken symmetry"; that is, homogeneity in the transverse plane, longitudinal boost invariance and reflection symmetry $z \to -z$. The generalized collisional kernel $\mathcal{C}_W$ is treated in the relaxation time approximation (RTA):

$$W(x,k) = W(\tau, k_T, w^2, v^2), \qquad k\cdot\partial W = -\frac{(k\cdot u)}{\tau_R}[W - W_{\rm eq}] \equiv -\frac{(k\cdot u)}{\tau_R}\delta W. \quad (25)$$

The same notation is in use as in [24]:

$$\tau = \sqrt{t^2 - z^2}, \qquad k_T = \sqrt{(k^x)^2 + (k^y)^2}, \qquad v = tk^0 - zk^z, \qquad w = zk^0 - tk^z. \quad (26)$$

The relaxation time approximation is a way to handle the generalized collisional kernel $\mathcal{C}_W$, which in general is notoriously very complicated, see for instance Refs. [29,31,37,38]. The physical meaning is to assume that the leading effect of the interaction is to pull back the system to a local equilibrium configuration $W_{\rm eq}$. The relaxation time τ_R represents the time scale for this drive toward equilibrium, and the energy in the numerator is necessary for dimensional reasons, and it grants, together with the minus sign, that the collisional kernel always drives the system toward equilibrium. Indeed, plugging the RTA in Eq. (15),

$$(k\cdot u)\dot W = -k\cdot\nabla W - \frac{(k\cdot u)}{\tau_R}\delta W. \quad (27)$$

For $k\cdot u = 0$ this is just an orthogonality condition for the spatial gradients. If $(k\cdot u) \neq 0$ instead

$$\dot W = -\frac{k\cdot\nabla W}{(k\cdot u)} - \frac{1}{\tau_R}\delta W. \quad (28)$$

This is the evolution in the proper time defined by the four-velocity field u^μ. The spatial gradient can either drive towards or out of equilibrium, but the collisional kernel is manifestly driving toward equilibrium.

According to the phenomenological assumption of a leading effect driving towards equilibrium, the RTA treatment can be expected to be more accurate for highly out-of-equilibrium systems. If the system were already close to equilibrium, one could expect the otherwise subleading effects to become dominant.

The RTA is however limited in accuracy, as it is in the classical case as well. It is a common approximation and it is used for the numerical comparison here because it is very simple, and it can be solved exactly. It has already been used extensively for the relativistic Boltzmann equation [16,19,20,24,26,41,44–48], and also for the Wigner distribution [34–36].

One of the few other simple approximations of the extended collisional kernel is the Vlasov interaction with external, classical fields. It does not drive towards equilibrium and, for a classical plasma, it is a very different physical situation than the ones that can be approximated with the RTA. Of course both of them can be used simultaneously, in an off-shell version of the Boltzmann-Vlasov equation. This is precisely the situation that lead to the classical version of the regularized moments [44]. In this work, the RTA will be used only for its simplicity, as a first, nontrivial step to check the validity of the generalized expansion in conditions similar to the ones expected in heavy ion-collisions.

The main difference between the classical RTA and the off-shell one is that the Wigner distribution must depend on v. The frequency k^0 (hence v itself) is not dictated by the other variables for a generic, off-shell k^μ. The requirement in (30) to depend on w^2 is to make manifest the invariance for axis reflections, while the requirement to depend on v^2 rather than v does not come from the Bjorken symmetry, and it is a mathematical simplification. Among other things, it entails that the charge density in the comoving frame is vanishing (no net charge). Considering that the only timelike four vector consistent with the symmetry is $u^\mu = (t, 0, 0, z)/\tau$ and using the definitions (26), the evolution (25) simplifies:

$$\left[\partial_\tau + 2\frac{v^2 - w^2}{\tau}\partial_{v^2}\right] W = -\frac{1}{\tau_R}\delta W. \tag{29}$$

Just like in the classical case, the conservation of energy and momentum requires the effective temperature be the one from the Landau matching. Choosing for the equilibrium distribution and relaxation time

$$W_{\mathrm{eq}} = \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2 + \frac{w^2}{\tau^2}}}, \qquad \tau_R = \frac{5\bar{\eta}}{T(\tau)}, \tag{30}$$

one has

$$\mathcal{E}(\tau) = \mathcal{E}_{\mathrm{eq}}(T) = \frac{6}{\pi^2}T^4 \Rightarrow T(\tau) = \left(\frac{\pi^2\mathcal{E}(\tau)}{6}\right)^{\frac{1}{4}}. \tag{31}$$

This choice (30) is mainly for mathematical simplicity. It entails that at equilibrium the system has a conformal equation of state, and in the kinetic limit it has a constant ratio of shear viscosity over entropy $\bar{\eta}$; more details about it in Appendix D. Using the method of characteristics it is possible to write an implicit solution of (29),

$$W = D(\tau, \tau_0) W_{\mathrm{f.s.}} + \frac{2\delta(k^2)}{(2\pi)^3}\int_{\tau_0}^{\tau} ds \frac{D(\tau, s)}{\tau_R(s)} e^{-\frac{\sqrt{k_T^2 + \frac{w^2}{s^2}}}{T(s)}}, \tag{32}$$

with the damping function D and the free-streaming $W_{\mathrm{f.s.}}$,

$$\begin{aligned} D(\tau_1, \tau_2) &= e^{-\int_{\tau_1}^{\tau_2}\frac{ds}{\tau_R(s)}}, \\ W_{\mathrm{f.s.}}(\tau, k_T, w^2, v^2) &= W_0(k_T, w^2, v_0^2), \\ v_0^2 &= v^2\left(\frac{\tau_0}{\tau}\right)^2 + w^2\frac{\tau^2 - \tau_0^2}{\tau^2}, \end{aligned} \tag{33}$$

for a generic initial condition $W_0 = W(\tau = \tau_0)$. It is possible then to use the self-consistency method used in Refs. [24,26,48] to obtain the numerical values of (32) up to an arbitrary precision. Because of the strong symmetry of the system, the nontrivial components of the stress-energy tensor are the proper energy density, the longitudinal pressure $\mathcal{P}_L = T^{\mu\nu}z_\mu z_\nu$, and the transverse pressure $\mathcal{P}_T = T^{\mu\nu}x_\mu x_\nu$, with $x^\mu = (0, 1, 0, 0)$ and $z^\mu = (z, 0, 0, t)/\tau$. The only moments (21) related to them, directly or indirectly, are

$$\begin{aligned} L_n(\tau, \zeta) &= \phi_2^{\mu_1\cdots\mu_n} z_{\mu_1}\cdots z_{\mu_n}, \\ T_n(\tau, \zeta) &= \phi_2^{\alpha_1\alpha_2\mu_1\cdots\mu_n} x_{\alpha_1}x_{\alpha_2}z_{\mu_1}\cdots z_{\mu_n}, \end{aligned} \tag{34}$$

for instance, $\mathcal{E}(\tau) = L_0(\tau, \zeta = 0)$, $\mathcal{P}_T = \int d\zeta T_0$, $\mathcal{P}_L = \int d\zeta L_1$. Plugging (25) into (22), one has the evolution of the L_n and T_n moments,

$$\begin{aligned} \dot{L}_n + \frac{1}{\tau_R}\delta L_n &= -\frac{2n+1}{\tau}L_n + \frac{1}{\tau}\hat{\mathcal{L}}L_{n+1}, \\ \dot{T}_n + \frac{1}{\tau_R}\delta T_n &= -\frac{2n+1}{\tau}L_n + \frac{1}{\tau}\hat{\mathcal{L}}T_{n+1}, \end{aligned} \tag{35}$$

where δL_n and δT_n are the difference between the moments and their local equilibrium expectation value. It is possible to directly integrate in ζ the nonhydrodynamic sources on the right-hand side of (23), and obtain the ζ-independent set of equations. Introducing the linear operator,

$$\hat{\mathcal{L}}f(\zeta) = 2\zeta f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta'), \tag{36}$$

the dynamical equations for the components of $T^{\mu\nu}$ then read

$$\dot{\mathcal{E}} = -\frac{1}{\tau}\Big(\mathcal{E} + \mathcal{P}_L\Big),$$
$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\delta\mathcal{P}_L = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)},$$
$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\delta\mathcal{P}_T = -\frac{1}{\tau}\mathcal{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}, \tag{37}$$

the residual moments (an their equilibrium expectation value) being

$$\mathcal{R}_L^{(n)} = \int_0^\infty d\zeta\left[(\hat{\mathcal{L}})^n L_{n+1}\right] \xrightarrow{\text{eq}} \frac{(2n-1)!!}{2n+3}\mathcal{E},$$
$$\mathcal{R}_T^{(n)} = \int_0^\infty d\zeta\left[(\hat{\mathcal{L}})^n T_n\right] \xrightarrow{\text{eq}} \frac{(2n-1)!!}{(2n+3)(2n+1)}\mathcal{E}, \tag{38}$$

and their evolution, stemming directly from (35)

$$\dot{\mathcal{R}}_L^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_L^{(n)} = -\frac{2n+3}{\tau}\mathcal{R}_L^{(n)} + \frac{1}{\tau}\mathcal{R}_L^{(n+1)},$$
$$\dot{\mathcal{R}}_T^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+1}{\tau}\mathcal{R}_T^{(n)} + \frac{1}{\tau}\mathcal{R}_T^{(n+1)}. \tag{39}$$

Second-order viscous hydrodynamics corresponds to taking only (37) as the dynamical equations, and substituting the residual moments, e.g., with their equilibrium expectation values. For the higher orders, one considers the residual moments up to a maximum n as dynamical variables, evolving according to (39) and approximating the $(n+1)$-residual moments.

An interesting initial condition, which is far from the kinetic limit but still provides initial values of $T^{\mu\nu}$ close to the ones appearing in heavy ion collisions, is

$$W_0(k_T, w^2, v^2) = \frac{2}{(2\pi)^3}\frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{v^2}{2\tau_0^2\sigma^2}} e^{-\frac{\sigma}{T_{\text{in}}}}, \tag{40}$$

with the variance of the Gaussian factor being the on-shell energy $\sigma = \sqrt{k_T^2 + \frac{w^2}{\tau_0^2}}$. The initial energy and pressure correspond to a Boltzmann gas with temperature T_{in}, however the distribution is strongly off shell and during the whole evolution the weight corresponding to $(k\cdot u) = 0$ is never vanishing. The naive quantum generalization of (5) is ill defined at all orders. On the other hand, the regularized version (22), hence (37) and (39), are well defined and can be used. In particular,

$$\mathcal{R}_L^{(n)}(\tau_0) = \frac{(-1)^n}{2n+3}\mathcal{E}(\tau_0),$$
$$\mathcal{R}_T^{(n)}(\tau_0) = \frac{(-1)^n}{(2n+3)(2n+1)}\mathcal{E}(\tau_0), \tag{41}$$

therefore the initial error in the substitution $\mathcal{R}^{(1)} \to \mathcal{R}|_{\text{eq}}$ is an outstanding 200%, which would have been simply impossible for the classical Boltzmann equation, since the residual moments are defined to be positive for an on-shell non-negative distribution. However, the self-coupling of the longitudinal and transverse pressure is dominant and the total error $\Delta\dot{\mathcal{P}}_L/\dot{\mathcal{P}}_L = \Delta\dot{\mathcal{P}}_T/\dot{\mathcal{P}}_T = 1/3$. With a 30% error on the sources, one can expect some qualitative agreement between hydrodynamics and the exact results, however it is interesting to note that in a composite quantity like the trace anomaly $T^{\mu\nu}g_{\mu\nu}$ the self-coupling part of the hydrodynamic quantities compensates and one has a 100% error of its time derivative at the beginning, making it likely that, differently from the energy density and the anisotropy $\mathcal{P}_L/\mathcal{P}_T$, the trace anomaly will not be well reproduced by hydrodynamics.

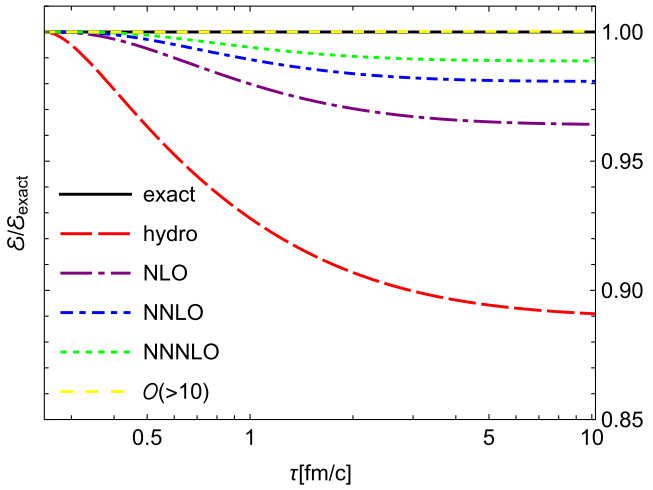

FIG. 1. Ratio of the energy density over the exact energy density for second-order hydrodynamics (red dashed line), and the next orders in the regularized expansion purple: dash-dotted line for the next-to-leading order (NLO), blue dash-dotted for next-to-next-to-leading-order (NNLO), green dotted for the next one, and the yellow dotted line for the tenth order or higher.

Selecting an initial temperature in (40) as $T_{\text{in}} = 600$ MeV, $4\pi\bar{\eta} = 3$, and $\tau_0 = 0.25$ fm/c, one has similar conditions to the ones in heavy-ion collisions in which hydrodynamics is applied.

As shown in Figs. 1 and 2, the temperature and the anisotropy are relatively well reproduced, despite large initial gradients $[\tau_R(\tau_0)\theta(\tau_0) \simeq 1.6]$ and large pressure corrections $(\mathcal{P}_L/\mathcal{P}_T \ll 1)$. This is at odds with the classical requirements of hydrodynamics but well within the expectations from the previous discussion. Indeed the trace anomaly $T^{\mu\nu}g_{\mu\nu} = \mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L$ is badly reproduced by hydrodynamics alone, as expected, missing both the correct sign and the magnitude of the absolute value (see Fig. 3). The next steps in the regularized expansion always provide a significant improvement (Figs. 1–3), recovering the fast convergence already seen for the classical moments expansion. Starting from the tenth order, the approximated values for the temperature,

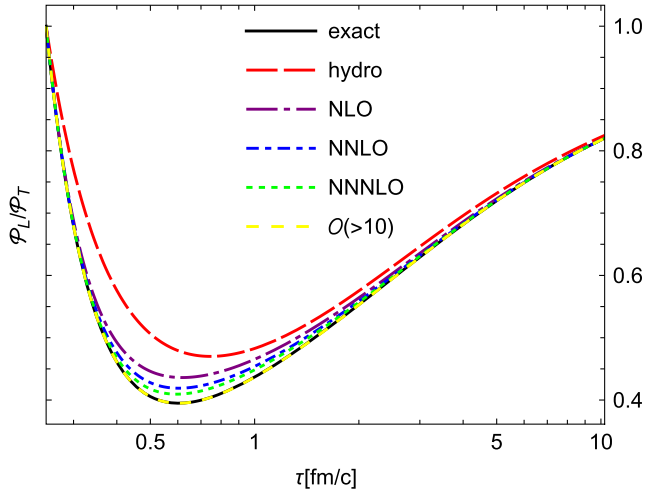

FIG. 2. Anisotropy evolution. The exact results in the black solid line, hydrodynamics (red dashed line), and the next orders: NLO (purple dash-dotted), NNLO (blue dash-dotted), NNNLO (green dotted) and tenth order or higher (yellow dotted).

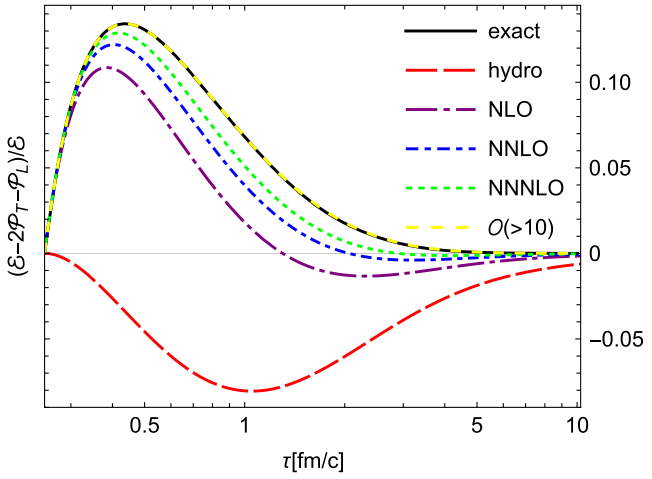

FIG. 3. Trace anomaly normalized to the energy density (color code as in Figs. 1 and 2).

anisotropy and trace anomaly reproduce substantially the exact results, and they are stable.[1]

Since hydrodynamics, and more in general the hydrodynamic expansion from the method of moments, was already known to be a good approximation of kinetic theory, one may wonder if the good approximation shown in the comparison is just an artifact of the initial conditions. One starts with a $T^{\mu\nu}$ close to the one achievable in kinetic theory, and the system does not have the time show significant difference in the evolution with respect to the on-shell case, despite being very far from the kinetic limit. In order to check if this is the case, there is an additional numerical comparison.

[1]I numerically checked up to the 100th order.

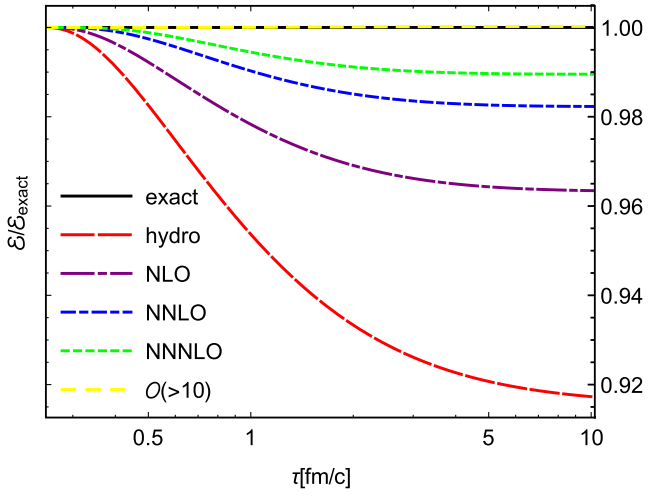

FIG. 4. Same as Fig. 1, but for the anisotropic initial condition (42). Ratio of the energy density over the exact energy density. The exact results in the black solid line, hydrodynamics (red dashed line), and the next orders: NLO (purple dash-dotted), NNLO (blue dash-dotted), and NNLO (green dotted), tenth order or higher (yellow dotted).

Another interesting initial condition is

$$W_0 = \frac{2}{(2\pi)^3}\frac{e^{-\frac{v^2}{2\tau_0^2\sigma^2}-\frac{\sigma}{T_{\rm in}}}}{\sqrt{2\pi}\sigma}\left[1-3P_2\left(\frac{w}{\sigma}\right)\right], \tag{42}$$

with the same values for σ, τ_0, and T_0 as in the previous case, and $P_2(x)$ being the second Laguerre polynomial. The initial values of the energy density and pressure then read

$$\mathcal{E}(\tau_0) = \frac{6}{\pi^2}T_{\rm in}^4, \qquad \mathcal{P}_L(\tau_0) = -\frac{1}{15}\mathcal{E}(\tau_0),$$
$$\mathcal{P}_T(\tau_0) = \frac{8}{15}\mathcal{E}(\tau_0), \tag{43}$$

while for the nonhydrodynamic moments

$$\mathcal{R}_L^{(n)}(\tau_0) = -\frac{4n+1}{2n+5}\frac{(-1)^n}{2n+3}\mathcal{E}(\tau_0),$$
$$\mathcal{R}_T^{(n)}(\tau_0) = -\frac{4n-8}{2n+5}\frac{(-1)^n}{(2n+3)(2n+1)}\mathcal{E}(\tau_0). \tag{44}$$

The negative longitudinal pressure is just impossible in the kinetic limit. It is particularly important to look at the comparisons with the exact solutions during the initial states, since any agreement cannot be an artifact for the "closeness to the kinetic results." The qualitative expectations due to the approximations in hydrodynamics are rather similar as in the previous case. At the beginning of the expansion there is a sizable error $\Delta\mathcal{R}_L^{(1)}/\mathcal{R}_L^{(1)} = 40\%$ and a very large $\Delta\mathcal{R}_T^{(1)}/\mathcal{R}_T^{(1)} = 275\%$. However, the self-coupling of the hydrodynamic degrees of freedom is dominant, and there is a moderate $\Delta\dot{\mathcal{P}}_L/\dot{\mathcal{P}}_L \simeq 22\%$ and

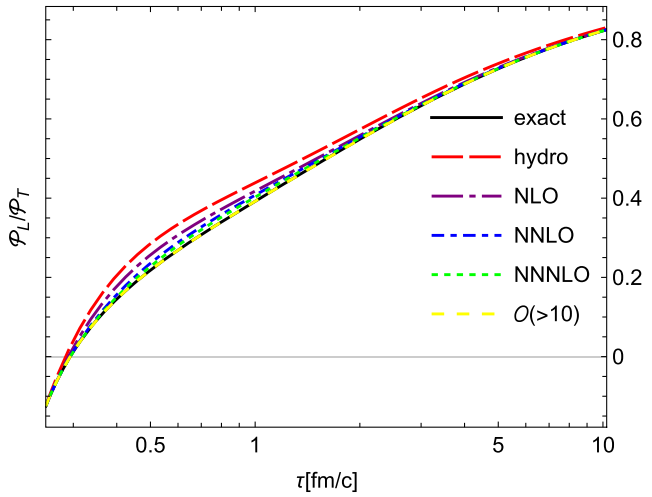

FIG. 5. Anisotropy evolution for the (42) initial condition (color code as in Fig. 4).

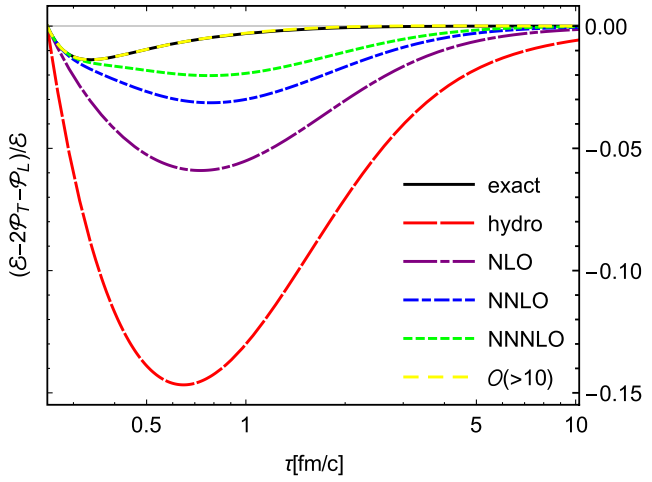

FIG. 6. Trace anomaly normalized to the energy density or the (42) initial condition (color code as in Fig. 4).

$\Delta\dot{\mathcal{P}}/\dot{\mathcal{P}}_T \simeq 15\%$. It is reasonable then to expect a qualitative agreement with the exact results. For the trace anomaly $T^{\mu\nu}g_{\mu\nu}$, the self-coupling part of the hydrodynamic quantities compensates again, and one has $\simeq$121% error in its time derivative at the beginning, making it likely that, also for this anisotropic initial condition, the trace anomaly will not be well reproduced by hydrodynamics.

Indeed, as shown in Figs. 4 and 5, the energy density and the anisotropy are relatively well reproduced; while the trace anomaly, Fig. 6, is largely overestimated. A definitive answer on the convergence of the regularized expansion is hard to get from a theoretical point of view, even for simple cases like the ones presented here. However, Figs. 4–6 clearly show that each additional step improves the accuracy also for the anisotropic initial condition (42). At the tenth order and higher, again, there is substantially no practical difference with the exact solutions.

V. CONCLUSIONS

In this work, the method of moments is generalized, from the relativistic Boltzmann equation, to the off-shell case of the Wigner distribution, the quantum counterpart to the classical distribution function. Despite following a very similar equation, it is not possible to make a naive generalization and recover the hydrodynamic expansion. Because of the off-shell nature of the Wigner distribution, the higher ranking tensor moments appearing at the very beginning of the expansion are ill defined.

The off-shell effects are usually neglected but they can be very large even for free fields, and a generalized method is needed to extend the results of the method of moments to the quantum case.

It is necessary to use a regularized set of parametric moments to recover a well-defined and systematically improvable expansion. In the kinetic (hence on-shell) limit the generalized expansion reduces the classical one.

This opens the possibility to generalize the recent progress of spin hydrodynamics that stem from a semiclassical approximation of the Wigner distribution to the full off-shell case, on top of giving the instruments to check the validity of hydrodynamics in the quantum case, and providing higher order approximations, if necessary.

The generalized expansion has been tested against the exact solutions of the simplest solvable instance of the Wigner distribution. The initial conditions have been chosen to be very far from the kinetic limit. In fact, the initial time derivative of the hydrodynamic degrees of freedom is impossible to obtain in the on-shell case. It shows the same fast convergence properties already seen for the traditional (nonquantum) moments expansion, also in the case of negative pressures. It is possible in this case to check the systematic error committed by approximating the nonhydrodynamic moments with a combination of the hydrodynamic degrees of freedom. It appears that it is the relative magnitude of these errors, compared to the contribution from the self-coupling of the hydrodynamic moments, that dictates if second-order hydrodynamics is a good approximation or not. The magnitude of the gradients or the size of the pressure corrections does not really affect the accuracy, at least in the tested cases. Different observables couple with different strength to the nonhydrodynamic moments; it is therefore possible to estimate which one is going to be well reproduced by hydrodynamics, like the pressure anisotropy, or badly reproduced, like the trace anomaly in the tested case. This can be done, mostly, looking at the relative error at the beginning of the evolution, and regardless of the Knudsen number and inverse Reynolds numbers, as one would have otherwise expected from considerations in the gradient expansion approach.

ACKNOWLEDGMENTS

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APPENDIX A: AN EXACT RESULT FROM THE FREE, UNCHARGED, SCALAR FIELD

Free scalar fields are simple enough that one can compute the exact Wigner distribution $W(x,k)$ for a given state. As it is shown, for instance, in Refs. [42,43], the Wigner distribution of free fields can be written in terms off the expectation values of the two ladder operators. Using the popular notation and normalization of the Peskin-Schroeder book [49],

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right], \tag{A1}$$

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}), \tag{A2}$$

the Wigner distribution of a free-uncharged field then reads

$$\begin{aligned} W(x,k) &= \int \frac{d^4v}{(2\pi)^4}e^{-ik\cdot v}\left\langle :\phi\left(x+\frac{1}{2}v\right)\phi\left(x-\frac{1}{2}v\right):\right\rangle \\ &= \int \frac{d^3pd^3q}{2(2\pi)^6\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left\{\delta^4\left(k-\frac{p+q}{2}\right)\langle a^\dagger_{\mathbf{p}}a_{\mathbf{q}}\rangle e^{i(p-q)\cdot x} + \delta^4\left(k+\frac{p+q}{2}\right)\langle a^\dagger_{\mathbf{p}}a_{\mathbf{q}}\rangle e^{-i(p-q)\cdot x}\right. \\ &\quad \left. + \delta^4\left(k-\frac{p-q}{2}\right)\langle a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}\rangle e^{i(p+q)\cdot x} + \delta^4\left(k-\frac{p-q}{2}\right)\langle a_{\mathbf{p}}a_{\mathbf{q}}\rangle e^{-i(p+q)\cdot x}\right\}. \end{aligned} \tag{A3}$$

The lack of a factor of 2 in Eq (A3) compared to Eq (13) is due to the $1/2$ in the Lagrangian of a free scalar field, compared to a charged one. It is a convention to preserve Eq. (14) for both charged and uncharged fields. The normal ordering is written explicitly because it is the only renormalization needed in the free theory.

Since both p and q are on-shell parameters of the original fields $p^2 = q^2 = m^2$, one can prove that $(\frac{p+q}{2})^2 \geq m^2$ and $(\frac{p-q}{2})^2 < 0$. Because of the Dirac deltas, the $\langle a^\dagger_{\mathbf{p}}a_{\mathbf{q}}\rangle$ terms contribute solely to the timelike part $W(x,k)|_{k^2\geq m^2}$, while the other two to the spacelike one $W(x,k)|_{k^2<0}$. It is interesting to note that both terms are not on shell in general. However the spacelike part can never be on shell. Any relevant contribution to the hydrodynamic variables from this part and one can exclude that the relativistic Boltzmann equation is a valid approximation, and a consistent treatment of the off-shell hydrodynamic expansion must be used.

It is rather easy at this point to specify a state of the system having contributions to $T^{\mu\nu}$ from the spacelike parts. Needless to say, it is also easy to find examples in which its contribution vanishes.

A very simple example, possibly the simplest, is a superposition of states having N and $N+2$ particles. Being the n-particles eigenstates of the four-momentum

$$|p_1,\ldots,p_n\rangle = \sqrt{\frac{2E_{\mathbf{p}_1}\cdots 2E_{\mathbf{p}_n}}{n!}}a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_n}|0\rangle, \tag{A4}$$

the state considered in this section is

$$\begin{aligned} |\Psi\rangle &= \frac{1}{\sqrt{2}}\left(\prod_{i=1}^{N}\int\frac{d^3p_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\psi(\mathbf{p}_i)\right)|p_1\ldots,p_N\rangle \\ &\quad + \frac{e^{i\Delta\phi}}{\sqrt{2}}\left(\prod_{i=1}^{N+2}\int\frac{d^3p_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\psi(\mathbf{p}_i)\right)|p_1\ldots,p_{N+2}\rangle. \end{aligned} \tag{A5}$$

Both partial waves have a wave function which is manifestly invariant in the exchange $p_i \leftrightarrow p_j$, since they are direct products of a single ψ. No additional symmetrization is needed. With the normalization

$$\int\frac{d^3p}{(2\pi)^2 2E_{\mathbf{p}}}\left|\psi(\mathbf{p})\right|^2 = 1, \tag{A6}$$

and the regular normalization of the $|p_1,\ldots p_n\rangle$ states, the total state $|\Psi\rangle$ is manifestly normalized to 1.

For mathematical simplicity it is assumed rotation invariance in the lab frame $\psi(\mathbf{p}) = \psi(p)$ and a real $\psi = \psi^*$. The only phase between the two partial waves is the $\Delta\phi$.

Making use of the commutation relations (A2), one can prove

$$\begin{aligned} \langle a^\dagger_{\mathbf{p}}a_{\mathbf{q}}\rangle &= \frac{N}{2}\frac{\psi(p)\psi(q)}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}} + \frac{N+2}{2}\frac{\psi(p)\psi(q)}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}} \\ &= (N+1)\frac{\psi(p)\psi(q)}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}, \end{aligned} \tag{A7}$$

which is equal to the reduced density matrix of a state constructed as in (A5) but without a superposition of state, just with a single wave with $N+1$ particles, and the wave function which is just the $N+1$ products of ψ.

The skeleton of the proof is noting that the $a^\dagger_{\mathbf{p}} a_{\mathbf{q}}$ in front of the free particle state defined as in (A4), can be "moved one step to the right" leaving each time a $(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p}_i)$ and substituting the p_i in the $|p_1,\dots p_n\rangle$ state with a p form $a^\dagger_{\mathbf{p}}$. The process is repeated a number of times equal to the number of particles in the state, then the last $a_{\mathbf{q}}|0\rangle$ vanishes and the process stops without leaving states with a different particles' number. The numerical factor stems from the normalization of the states (A4) and the Lorentz covariant momentum measure used in the definition of Ψ (A5).

With similar considerations, one can compute the other two expectation values, which are even simpler since $\langle a_{\mathbf{p}} a_{\mathbf{q}}\rangle = (\langle a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}}\rangle)^*$. The two creation operators just add two particles to the N particles state, and the only nonvanishing overlap is with the $N+2$ state. Summing up,

$$\langle a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}}\rangle = \frac{e^{-i\Delta\phi}}{2}\sqrt{\left(N+2\right)\left(N+1\right)}\frac{\psi(p)\psi(q)}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}. \tag{A8}$$

Because of the rotational system of the state, the only four-velocity consistent with the symmetry is $u=(1,0,0,0)$, as seen from the lab frame. The isotropic pressure in $x=0$ then reads

$$\frac{1}{3}\int d^4k W(0,k)\mathbf{k}\cdot\mathbf{k} = 2(N+1)\tilde{\mathcal{P}} + \cos(\Delta\phi) \times\sqrt{(N+2)(N+1)}\tilde{\mathcal{P}}, \tag{A9}$$

with $\tilde{\mathcal{P}}$ being the integral

$$\tilde{\mathcal{P}} = \int \frac{d^3p d^3q}{4(2\pi)^6 E_{\mathbf{p}}E_{\mathbf{q}}}\left(\frac{\mathbf{p}\pm\mathbf{q}}{2}\right)^2\psi(p)\psi(q). \tag{A10}$$

The integral does not depend on the sign $\pm$ because of the spherical symmetry. In Eq. (A9), the first term in the right-hand side is the one from the timelike part of W [the $+$ sign in the integral (A10)], the second term is the one due to the spacelike part (and originally with the $-$ sign). Modulating the relative phase $\Delta\phi$ it is possible to add or remove up to half of the other term. To be precise slightly more than that since $\sqrt{(N+2)(N+1)} > N+1$, if very close to it in the large N limit. This relative difference of the order of $\pm 50\%$ is definitely not a small correction. The isotropic pressure is a relevant variable in hydrodynamics, and one cannot assume to be in the kinematic (thus on-shell) limit for the hydrodynamic expansion regarding this state.

To finish, it is rather simple to prove that the naive extension of the method of moments to the off-shell case is not appropriate for this state. Reminding that $k\cdot u = k_0$ due to the spherical symmetry, for any $\|\mathbf{k}\|\neq 0$ it is possible to compute the Wigner distribution,

$$\begin{aligned} W(x;0,\mathbf{k}) &= \frac{\sqrt{(N+2)(N+1)}}{2}\int\frac{d^3pd^3q}{2(2\pi)^6}\frac{\psi(p)\psi(q)}{2E_{\mathbf{p}}E_{\mathbf{q}}}\delta\left(\frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{2}\right)\delta^3\left(\mathbf{k}-\frac{\mathbf{p}-\mathbf{q}}{2}\right)\left[e^{-i(p+q)\cdot x+i\Delta\phi}+e^{i(p+q)\cdot x-i\Delta\phi}\right] \\ &= \frac{4\sqrt{(N+2)(N+1)}}{(2\pi)^6}\int d^2p_T\frac{\left(\psi(\sqrt{p_T^2+k^2})\right)^2}{k\sqrt{m^2+p_T^2+k^2}}\cos\left[2\sqrt{m^2+p_T^2+k^2}t-\mathbf{p}_T\cdot\mathbf{x}+\Delta\phi\right], \end{aligned} \tag{A11}$$

being the variable of integration $\mathbf{p}_T$ an orthogonal vector to the $\mathbf{k}$, $\mathbf{p}_T\cdot\mathbf{k}=0$. The last formula is not vanishing in general, in particular for $x=0$ and $\Delta\phi$ not a multiple of $\pi/2$, making the moment of the naive expansion (17) unusable.

APPENDIX B: THE WIGNER DISTRIBUTION IN MORE GENERAL CASES

Equations (13)–(15) are not the most general. Just as in relativistic kinematics, there is usually more than one field (different particle species to consider [50]), and they all contribute to the stress-energy tensor. The fields are not necessarily scalars, and the Wigner distribution fulfills another equation, usually referred to as the constraint equation, on top of the kineticlike one (15). For weak couplings (or just noninteracting fields), the form of (14) remains the same, with more scalar contributions from the different fields [29,51], but in general the stress-energy tensor is more complicated. Because of the interaction terms or the spin structure, see for instance Ref. [29] for a general overview from the axiomatic field theory framework. After all of this, it must be mentioned that one has to introduce gauge links in gauge theories. The gauge-covariant Wigner distributions might look the same but it is not any longer two point functions of the fields. It includes a contribution from the gauge fields up to infinity orders [52,53]. It is not possible to take into account all of these cases in detail. The main result which is crucial for this work is that for all these cases one can apply the Klein-Gordon operator $\Box - m^2$ to one of the fields in the definition of the Wigner distribution (see Refs. [51,54] for additional details) and obtain the equation

$$\left[\frac{\hbar^2}{4}\Box - (k^2 - m^2c^2) + i\hbar k\cdot\partial\right] W(x,k) = \text{Source terms}. \tag{B1}$$

Separating the real from the purely imaginary components of this equation[2]

$$\frac{\hbar^2}{4}\Box W - (k^2 - m^2c^2)W(x,k) = \text{Source terms},$$
$$k\cdot\partial W(x,k) = \text{Source terms}. \tag{B2}$$

The first equation is an off-shell, interacting, generalization of the classical on-shell constraint $(k^2 - m^2)W \simeq 0 \Rightarrow W \propto \delta(k^2 - m^2)$. The second one is essentially (15), without any particular assumption about the generalized collisional kernel. The Wigner distribution always fulfills an overdetermined set of equations. The details over the constraint equation are beyond the scope of this work. Its aim is not so solve off-shell transport, but to generalize the hydrodynamic expansion. Equation (15) is the one needed for that, as illustrated in the main text.

Regarding the form of the stress-energy tensor in (14), either one has a free field, and it is correct, or it has to be considered a phenomenological approximation; in the sense that the contribution of the other fields does not appear, except in the interaction itself, responsible of the right-hand side in (15). In other words, the contribution of the other fields is assumed to give a trivial (constant, within the space-time scales of the problem) contribution to the total $T^{\mu\nu}$, the dynamical part is given just by the scalar field. This is not very different from the role of the Higgs field in the standard model. The energy scales to excite the Higgs bosons are usually considered too high for it to contribute in a dynamical way to the local $T^{\mu\nu}$, but it still affects the other fields with the masses (also in QCD), the shortening of the range of the weak interaction, and overall it contributes to the renormalized propagators.

More in general, for instance in the case of gauge theories (with no approximations), one has the form

$$T^{\mu\nu} = \int d^4k W^{\mu\nu}(x,k). \tag{B3}$$

In the generalized $W^{\mu\nu}$ one includes all the fields. The scalars will contribute through $W_s k^\mu k^\nu$; the spinors through traces of the γ^μ matrices, on top of the k^μ (for instance, see Ref. [53]). Substantially all the considerations bringing to (19) and their evolution (20) are the same, with the difference that two of the indices are not coming from $k^{\langle\mu\rangle}$ momenta in the momentum integration and have to be treated separately $\phi_n^{\mu\nu,\mu_1\cdots\mu_s}$. Consequently, there is a (finite) number of extra terms in their evolution equations. The formulas are even lengthier than the already lengthy ones appearing in the main text, and they do not add any physical insight to the discussion. The generalization of the method of moments for the regularized expansion follows exactly the same logic passages. Because of that they will not be reported here; it suffices to say that they are a straightforward, if tedious, extension. It is also very likely that in specific cases, because of symmetry or some phenomenological approximations similar to the ones already used in [32,36–38,51], one can further simplify them.

[2]The Wigner distribution of scalar fields is real by construction. For matrix valued distributions, like in the case of spin 1/2, one can repeat for the Hermitian and the anti-Hermitian part of the Wigner distribution.

APPENDIX C: THE REGULARIZED MOMENTS IN THE QUANTUM CASE

From a first look at Eqs. (22) and (23), it seems like the procedure highlighted at the end of Sec. III for the higher orders is just postponing the problem with the ill-defined moments to the higher orders. The rank three and four tensors in the right-hand side of Eq. (23) evolve according to Eq. (22), which couples also to moments having even lower index n than the starting $\phi_n^{\mu_1\cdots\mu_s}$. Going to the higher orders, one would see again some nonintegrable poles due to the $1/(k\cdot u)$ or $1/(k\cdot u)^2$ in the integrand.

The purpose of this appendix is to present the modified version of the procedure used in the classical case [44,45] to restrict everything to one single generation of moments with a fixed n; or, more precisely, two generations of moments with fixed n in the off-shell case.

It also highlights why the regularized moments $\phi_n^{\mu_1\cdots\mu_s}$ have been defined as in (21) with an exponential in the extra parameter ζ, instead of the classical analog with a Gaussian in the ξ [44,45].

In fact, the two instances are related. In the classical case, that is, on-shell and positive frequency, the dot product $(p\cdot u) \le 0$ is never negative. As it can be easily seen going to the locally comoving frame in which $u = (1,0,0,0)$ and reminding that $(p\cdot u)$ is a scalar and does not change under Lorentz boosts.

In other words, there is no difference, in the classical case, between the contraction of $(p\cdot u)$ and its absolute value. Therefore, as long as the integral in the parameter ξ can be performed before the momentum integrals

$$\frac{2}{\sqrt{\pi}}\int_0^\infty d\xi e^{-\xi^2(p\cdot u)^2} = \frac{1}{|p\cdot u|} = \frac{1}{(p\cdot u)}, \tag{C1}$$

thus removing the Gaussian in ξ and reducing by one the index of $(p\cdot u)^n$, as it has been used extensively in Ref. [44].

All of this does not work in the off-shell case. The sign of the frequency k_0 of spacelike k^μ vectors is a frame dependent quantity, or, if one prefers a different perspective: if the frequency is vanishing in the lab frame $k^0 = 0$, the sign of $k\cdot u = -\mathbf{k}\cdot\mathbf{u}$ depends on the projection of two three-vectors and it cannot be known in advance. In general $(k\cdot u) \neq |k\cdot u|$; hence, the choice of the definitions with an

exponential $e^{-\zeta(k\cdot u)^2}$ in ζ for (19) and (21). Differentiation and integration with respect to ζ will multiply or divide by $(k\cdot u)^2$, with no absolute values. One could have tried to use the older prescription, as in Ref. [44], but with an absolute value $|k\cdot u|$ in the definition of moments with $n=1$ rather than $(k\cdot u)$ in the off-shell case to preserve the relations between the ξ dependent moments and the ξ independent ones. Calling these classically inspired moments $\tilde{\phi}_1^{\mu_1\cdots\mu_s}(x,\xi)=\tilde{\Phi}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}(x,\xi)$, to avoid confusion with the ones in use in this work, the stress energy tensor would read $T^{\mu\nu}=\int d\xi\tilde{\Phi}_1^{\mu\nu}$ just like in the classical case [44,45]. On the other hand, it has the unfortunate consequence of having the evolution of the $\tilde{\phi}_1^{\mu_1\cdots\mu_s}$ depending on moments having a $\mathrm{sign}(k\cdot u)$ in their integral definition instead of $|k\cdot u|$ because of the general relation

$$\begin{aligned}\partial_\mu|k\cdot u| &= \partial_\mu[\mathrm{sign}(k\cdot u)(k\cdot u)]\\ &= [2\delta(k\cdot u)(k\cdot u)k^\nu\partial_\mu u_\nu]+\mathrm{sign}(k\cdot u)k^\nu\partial_\mu u_\nu\\ &= 0+\mathrm{sign}(k\cdot u)k^\nu\partial_\mu u_\nu. \end{aligned}\tag{C2}$$

This sign cannot be written in terms of integral or derivatives in ξ, hence the necessity of a new generation of moments. Using very similar calculations it is immediate to show that the exact evolution of the latter generation of moments couples to moments having a $\delta(k\cdot u)$ in the integral definition. Their evolution couples to moments having a δ', and hence, typically, integrals of derivative of the Wigner distribution with respect to the k^μ. All of it is an unnecessary complication, which can be avoided by using the definitions (19) and (21) instead of the classically inspired ones.

The only side effect is the need of two generations of regularized moments, also in the on-shell limit in which one could have used safely the previous prescription.

Coming to the reduction of the exact equation to only two generations of moments, because of the exact relations (24), one can rewrite Eq. (22), as two sets of well defined equations for the $n=2$ and $n=1$ generations,

$$\begin{aligned}&\dot{\phi}_2^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}+\tilde{C}_1{}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}\\ &=s\dot{u}^{\langle\mu_1}\partial_\zeta\phi_1^{\mu_2\cdots\mu_s\rangle}-\theta\phi_2^{\mu_1\cdots\mu_s}-s\nabla_\alpha u^{\langle\mu_1}\phi_2^{\mu_2\cdots\mu_s\rangle\alpha}\\ &\quad-\nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle}+\dot{u}_\alpha[2\phi_1^{\alpha\mu_1\cdots\mu_s}+2\zeta\partial_\zeta\phi_1^{\alpha\mu_1\cdots\mu_s}]\\ &\quad+\nabla_\alpha u_\beta\left[\int_\zeta^\infty d\zeta\phi_2^{\alpha\beta\mu_1\cdots\mu_s}-2\zeta\phi_2^{\alpha\beta\mu_1\cdots\mu_s}\right],\end{aligned}\tag{C3}$$

$$\begin{aligned}&\dot{\phi}_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}+\tilde{C}_0{}^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}\\ &=-s\dot{u}^{\langle\mu_1}\phi_2^{\mu_2\cdots\mu_s\rangle}-\theta\phi_1^{\mu_1\cdots\mu_s}-s\nabla_\alpha u^{\langle\mu_1}\phi_1^{\mu_2\cdots\mu_s\rangle\alpha}-\nabla_\alpha\\ &\quad\times\int_\zeta^\infty d\zeta\phi_2^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle}+\dot{u}_\alpha\left[\int_\zeta^\infty d\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}-2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]\\ &\quad-2\zeta\nabla_\alpha u_\beta\phi_1^{\alpha\beta\mu_1\cdots\mu_s}.\end{aligned}\tag{C4}$$

The choice of $n=2$, instead of other even numbers, is rather intuitive, noting that the energy density is $\mathcal{E}=\lim_{\zeta\to0^+}\phi_2$, and the pressure tensor is $\mathcal{P}^{\mu\nu}=\int d\zeta\phi_2^{\mu\nu}$. The choice of $n=1$ for the odd number is because it enters in the exact equations of the $\phi_2^{\mu_1\cdots\mu_s}$ without integrals in ζ, and it might be more convenient numerically to limit the integral part of this set of integro-differential equations. There is no particular reason, however, to prefer this choice over any other couple of numbers, as long as one is odd and one is even, and both positive. The defining integrals of $\phi_n^{\mu_1\cdots\mu_s}$ are typically fast converging. In particular for ζ and n positive, they are uniformly converging for any Wigner distribution $W(x,k)$ which is absolutely integrable in d^4k. In particular, all of the ζ integrals and derivatives in Eqs. (C3) and (C4) are well defined (one can perform the integral and derivatives before the d^4k integration). The equations themselves are well defined at all orders. The same can be said of the other possible choices of fixed even and odd positive n. Having W integrable and absolutely integrable is rather weak as a requirement. For instance, at least W multiplied by a rank two polynomial in the k^μs should be integrable too. The components of $T^{\mu\nu}$ are in this category and they should be finite integrals, and differentiable in the x^μs.

The fact that Eqs. (C3) and (C4) are well defined equations with minimal requirements on the mathematical structure of the Wigner distribution grants that the equation of hydrodynamics (or modified versions of it) can be systematically extended to higher orders like in the on-shell case. However, it does not grant that any of the approximations are causal or stable. After all even second-order viscous hydrodynamics can violate causality, depending on the initial conditions [55]. General stability and causality statements are not ready at the moment and possibly one should extend the work done in Ref. [55]. It is also unclear, at the moment, if the choice of the generations with $n=2$ and $n=1$ is the most convenient from a phenomenological point of view, the same can be said of the choice of the $e^{-\zeta(p\cdot u)}$ in the regularized moments. All of this is left for future research.

APPENDIX D: RELAXATION TIME APPROXIMATION

In this appendix there are some clarifications about the relaxation time approximation (RTA), and the link between the relaxation time and shear viscosity. Many of the practical constraints in the use of the RTA in the off-shell case are the same as the ones in the classical one. For instance, the formulas in the main text are "agnostic" regarding the four-velocity, it can be the Landau definition (following the energy flux) or the Eckart definition (following one timelike conserved current, e.g. the Baryon or electric current) or other, less popular, prescriptions. Equation (23) is correct regardless of the definition in use of the four-velocity u^μ.

On the other hand, making use of the RTA with a k independent τ_R forces one to use the Landau prescription, since it is the only one compatible with the local conservation of four-momentum,

$$\begin{aligned} 0 = \partial_\mu T^{\mu\nu} &= \int d^4k k^\nu (k\cdot\partial) W(x,k) \\ &= -\frac{u_\mu}{\tau_R}\int d^4k k^\mu k^\nu [W(x,k) - W_{\mathrm{eq}}(x,k)] \\ &= -\frac{u_\mu}{\tau_R}[T^{\mu\nu} - T^{\mu\nu}_{\mathrm{eq}}]. \end{aligned} \tag{D1}$$

Taking into account the local isotropy of the equilibrium part, for instance $W_{\mathrm{eq}} \propto e^{-(k\cdot u)/T(x)}$, which has solely the four-velocity u^μ as a privileged vector

$$T^{\mu\nu}_{\mathrm{eq}} = \mathcal{E}_{\mathrm{eq}} u^\mu u^\nu - \Delta^{\mu\nu}\mathcal{P}_{\mathrm{eq}} \tag{D2}$$

hence

$$u_\mu T^{\mu\nu} = \mathcal{E} u^\nu + q^\nu = \mathcal{E}_{\mathrm{eq}} u^\nu \tag{D3}$$

using the general decomposition of the stress-energy tensor (7). The only way to fulfill the last equation is to assume that the four-velocity u^μ entering the RTA is the one in the Landau prescription, which is the timelike eigenvector of the physical $T^{\mu\nu}$. Therefore $q^\mu = 0$ by construction and $\mathcal{E} = \mathcal{E}_{\mathrm{eq}}$. In the particular case of W_{eq} depending exclusively on u^μ and T (e.g., no chemical potential associated to conserved charges), the last equation is usually used as the definition of the effective temperature field, the temperature is the inverse function of $\mathcal{E}$, which can be obtained exclusively from $T^{\mu\nu}$: $T = \mathcal{E}^{-1}_{\mathrm{eq}}(\mathcal{E})$, [clearly $\mathcal{E}^{-1}_{\mathrm{eq}}$ is the inverse of the function $\mathcal{E}_{\mathrm{eq}}(T)$]. For additional information about the matching procedure in different cases (chemical potential etc.) see for instance Refs. [14,39].

Clearly if $T^{\mu\nu}$ does not have, even in a small neighborhood, any timelike eigenvector, the RTA cannot be used at all, the system is too wild. For the purpose of this paper it is less relevant, since the Bjorken symmetry assumed for the numerical calculations has only one vector compatible with the symmetry, and it is timelike, therefore all the prescriptions for the four-velocity are equivalent to the Landau one. The more general part is not dependent on the choice and it does not matter. It is flexible with respect to the choice of the u^μ prescription.

It is relevant however that the $\bar{\eta}$ is, in general, not only for the Bjorken symmetry, the shear-viscosity to entropy ratio; more precisely the shear viscosity η divided by the temperature scaled equilibrium enthalpy density $(\mathcal{E}+\mathcal{P})/T$, which is of course the canonical equilibrium entropy density for a classical gas. Starting from the exact (23) and removing the trace $\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} - \Delta^{\mu\nu}\Delta_{\alpha\beta}\dot{\mathcal{P}}^{\langle\alpha\rangle\langle\beta\rangle}/3$, making use of the RTA one can remove from the ζ integral the collisional kernel contribution, obtaining just $-\pi^{\mu\nu}\tau_R$. The equation then reads

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{1}{\tau_R}\pi^{\mu\nu} = \cdots, \tag{D4}$$

with $O^{\langle\mu\nu\rangle}$ a shorthand notation for the traceless, spacelike part of a tensor. The function[3] $2\beta_\pi$ is the function of temperature[4] that multiplies the $\sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle}$ in the right-hand side. It is computed by splitting all the moments in the right-hand side in their local equilibrium expectation value and the remaining part, then adding all terms proportional to $\sigma^{\mu\nu}$ through an equilibrium expectation value integral. Most of these moments, like the components of $T^{\mu\nu}$, are already decomposed in a local equilibrium part and corrections according to (7), but the other tensors in the ζ integral are not. For an on-shell local equilibrium W_{eq} their local equilibrium part is classical, differently from the full tensors, their ζ integral can be performed exactly term by term. Therefore the β_π is the classical one, despite having off-shell nonequilibrium contributions in the equation that can be very different from the classical ones, the β_π is the same. The shear viscosity is defined in the method of moments with the RTA as $\eta = \beta_\pi\tau_R$ and it corresponds to the first order approximation of $\pi^{\mu\nu}$ for arbitrarily small perturbations from global equilibrium (hence, small deviations from the local equilibrium and small gradients and time derivatives): $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} +$ higher order corrections.[5] Since the off-shell β_π in RTA is by construction the same function as in the classical case, one has $\beta_\pi = (\mathcal{E}+\mathcal{P})/5$ and therefore $\eta = \tau_R(\mathcal{E}+\mathcal{P})/5$. For the specific relaxation time in (30) one has $\eta = \bar{\eta}(\mathcal{E}+\mathcal{P})/T$, therefore the ration of the shear viscosity η with $(\mathcal{E}+\mathcal{P})/T$ is $\bar{\eta}$, a constant by construction. This is why such a choice is called "fixed shear viscosity over entropy."

[3]The 2 is a naming convention to fit the classical definition in the nonrelativistic limit.

[4]In the more general case, it is also a function of the chemical potential, and possible other intensive variables.

[5]Among the higher order terms, there is the time derivative of $\pi^{\mu\nu}$ itself, which is considered doubly small, a deviation from equilibrium and a derivative in an almost-equilibrated, almost-static state.

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Correction: The author affiliation was incomplete and has been rectified.