Zero-modes in magnetized $T^6/\mathbb{Z}_N$ orbifold models through $Sp(6,\mathbb{Z})$ modular symmetry

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We study fermion zero-modes in magnetized $T^6/\mathbb{Z}_N$ orbifold models. In particular, we focus on nonfactorizable orbifolds, i.e., $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$ corresponding to $SU(7)$ and E_6 Lie lattices, respectively. The number of degenerated zero-modes corresponds to the generation number of low-energy effective theory in four-dimensional (4D) spacetime. We find that three-generation models preserving 4D $\mathcal{N}=1$ supersymmetry can be realized by magnetized $T^6/\mathbb{Z}_{12}$, but not by $T^6/\mathbb{Z}_7$. We use $Sp(6,\mathbb{Z})$ modular transformation for the analyses.

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I. INTRODUCTION

Higher dimensional theory such as superstring theory is interesting as a candidate for unified theory of particle physics. When we start with higher dimensional theory, we need compactification of extra dimensions. In particular, compactifications leading to four-dimensional (4D) chiral theory are important, because the standard model is a chiral theory.

Inspired by superstring theory, we start with six-dimensional (6D) compact space. One of the simplest compactifications is the toroidal compactification T^6. However, that leads to 4D nonchiral theory. One way to derive a 4D chiral theory is orbifolding $T^6/\mathbb{Z}_N$ [1,2]. The 4D supersymmetry (SUSY) must be broken to $\mathcal{N}=1$ or 0 to realize a 4D chiral theory. The $\mathbb{Z}_N$ twists to preserve 4D $\mathcal{N}=1$ SUSY were classified [1,2]. In addition, six-dimensional lattices with those $\mathbb{Z}_N$ twist symmetries were studied in Refs. [3–8].

Another way to lead to a 4D chiral theory is the introduction of magnetic fluxes in compact space [9–12]. The degeneracy number of zero-modes, which corresponds to the generation number of 4D massless chiral fermions, is determined by the size of magnetic fluxes. Yukawa couplings in 4D low-energy effective field theory are computed by overlap integrals of zero-mode wave functions [13,14]. They can lead to suppressed Yukawa couplings as well as $\mathcal{O}(1)$ of couplings depending on moduli values.

One can combine the above geometrical background and gauge background and study the orbifold compactification with a magnetic flux background [15,16]. Adjoint matter fields can be projected out in magnetized orbifold models, and that corresponds to stabilization of Wilson line moduli, i.e., open string moduli in intersecting D-brane models on orbifolds [17], which are T-dual to magnetized D-brane models on orbifolds. Magnetized orbifold models have richer flavor structure. Three-generation models can be derived by various setups on the $T^2/\mathbb{Z}_N$ orbifold with magnetic flux [18,19]. Furthermore, realization of quark and lepton mass matrices were studied [20–26].

So far, the six-dimensional space, which can be factorizable to three two-dimensional spaces, was mainly studied, although some nonfactorizable $T^4/\mathbb{Z}_N$ orbifolds were studied [27,28]. Our purpose is to study nonfactorizable cases. Here, we study $T^6/\mathbb{Z}_N$ orbifold models with magnetic fluxes, whose T^4 or T^6 parts are nonfactorizable. We examine their zero-mode numbers. In particular, we show three-generation models. Such studies were done in magnetized $T^2/\mathbb{Z}_N$ orbifold models by several methods [15,16,29–31]. Among them, one way to analyze zero-mode numbers in magnetized $T^2/\mathbb{Z}_N$ orbifold models is to use the $SL(2,\mathbb{Z})$ modular symmetry of wave functions on T^2 [32]. (See also Ref. [28].) We extend such analysis to $T^6/\mathbb{Z}_N$ orbifolds as well as $T^4/\mathbb{Z}_N$ orbifolds. Higher dimensional compact spaces such as T^6 have several moduli and have larger $Sp(2g,\mathbb{Z})$ symplectic modular symmetries. (See for mathematical reviews, e.g., Refs. [33,34].) These large $Sp(2g,\mathbb{Z})$ symplectic modular symmetries appear in string compacitification. (See, e.g., Refs. [35–40].) Also, they were used in flavor model building [41,42]. Here, we construct the orbifold twists as elements of $Sp(2g,\mathbb{Z})$ and modular transformation behavior of wave functions.

Then, we study zero-modes in $T^6/\mathbb{Z}_N$ orbifolds as well as $T^4/\mathbb{Z}_N$ orbifolds.

The rest of our paper is organized as follows. In Sec. II, we review massless spinor modes on T^6 and bosonic spectra. In Sec. III, we give a brief review of 6D lattices leading to $T^6/\mathbb{Z}_N$ with 4D $\mathcal{N}=1$ SUSY. We study magnetized $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$ models in Sec. IV. In Sec. V, we discuss the tachyon-free condition in each orbifold model. Section VI is our conclusion. In Appendix A, we present results in magnetized $T^4/\mathbb{Z}_N$ orbifold models with $SO(8)$ Lie lattice. In Appendix B, we derive the transformations of zero-mode wave functions under $Sp(6,\mathbb{Z})$.

II. MAGNETIZED T^6 MODEL

First we consider magnetized D-brane models with T^6 compactification. We review the Dirac operator and the Dirac equation to introduce fermion zero-modes on magnetized T^6 [13,14].

A. The Dirac operator on magnetized T^6

To find wave functions on magnetized T^6, we construct the Dirac operator on the six-dimensional torus $T^6 \simeq \mathbb{C}^3/\Lambda$, where Λ is a lattice spanned by six basis vectors e'_i ($i=1$, 2, 3, 4, 5, 6) defined by

$$e'_1 = 2\pi R\vec{e}_1 = 2\pi R\begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad e'_2 = 2\pi R\vec{e}_2 = 2\pi R\begin{bmatrix}0\\1\\0\end{bmatrix},$$
$$e'_3 = 2\pi R\vec{e}_3 = 2\pi R\begin{bmatrix}0\\0\\1\end{bmatrix},$$
$$e'_4 = 2\pi R\vec{e}_4 = 2\pi R\begin{bmatrix}\omega_1\\\omega_4\\\omega_6\end{bmatrix}, \quad e'_5 = 2\pi R\vec{e}_5 = 2\pi R\begin{bmatrix}\omega_4\\\omega_2\\\omega_5\end{bmatrix},$$
$$e'_6 = 2\pi R\vec{e}_6 = 2\pi R\begin{bmatrix}\omega_6\\\omega_5\\\omega_3\end{bmatrix}. \tag{1}$$

Here, $R(>0)$ denotes the scale factor and $\omega_i \in \mathbb{C}$ characterize the shape of Λ. By factoring out R, we defined vectors $\vec{e}_i$. Here we focus on the six basis vectors corresponding to simply laced root lattices.

Also, we define real coordinates x^i, y^i ($i=1,2,3$) along the lattice vectors on T^6. They are related to complex coordinates $\vec{Z}=(Z^1,Z^2,Z^3)$ of $\mathbb{C}^3$ by

$$\vec{Z} = \begin{bmatrix}Z^1\\Z^2\\Z^3\end{bmatrix} = 2\pi R\left(\begin{bmatrix}x^1\\x^2\\x^3\end{bmatrix} + \begin{bmatrix}\omega_1 & \omega_4 & \omega_6\\ \omega_4 & \omega_2 & \omega_5\\ \omega_6 & \omega_5 & \omega_3\end{bmatrix}\begin{bmatrix}y^1\\y^2\\y^3\end{bmatrix}\right)$$
$$= 2\pi R(\vec{x}+\Omega\vec{y}) = 2\pi R\vec{z}, \tag{2}$$

where we identify $\vec{z}=\vec{x}+\Omega\vec{y}$ as complex coordinates on T^6 and

$$\Omega = \begin{bmatrix}\omega_1 & \omega_4 & \omega_6\\ \omega_4 & \omega_2 & \omega_5\\ \omega_6 & \omega_5 & \omega_3\end{bmatrix} \tag{3}$$

is complex structure moduli. We are interested in symmetric moduli $\Omega^T=\Omega$; thus the actions of the $Sp(6,\mathbb{Z})$ modular group can be consistently seen. Then we will discuss how to realize some models with generation structure in magnetized $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$ orbifold models.

Here, Ω is not necessarily an element of the Siegel upper-half plane $\mathcal{H}^3$ defined as [33]

$$\mathcal{H}_3 = \{\Omega \in GL(3,\mathbb{C}) | \Omega^T = \Omega, \mathrm{Im}\Omega > 0\}. \tag{4}$$

We will see that zero-modes of all positive chirality $(+,+,+)$ are well-defined if $N\Omega \in \mathcal{H}^3$, where N is a 3×3 integer matrix called flux and we will define later.

Then, we define the Dirac operator to write down the Dirac equation on magnetized T^6. The Kähler metric on $\mathbb{C}^3$ is defined as

$$ds^2 = 2H_{i\bar{j}}dZ^i d\bar{Z}^{\bar{j}}, \tag{5}$$

where $H_{i\bar{j}} = \frac{1}{2}\delta_{i,\bar{j}}$ and i, $j=1$, 2, 3.

The Gamma matrices on $\mathbb{C}^3$ are defined as

$$\begin{aligned}
\Gamma^{Z^1} &= \sigma^Z \otimes \sigma^3 \otimes \sigma^3, & \Gamma^{\bar{Z}^{\bar{1}}} &= \sigma^{\bar{Z}} \otimes \sigma^3 \otimes \sigma^3,\\
\Gamma^{Z^2} &= \mathbf{1}_2 \otimes \sigma^Z \otimes \sigma^3, & \Gamma^{\bar{Z}^{\bar{2}}} &= \mathbf{1}_2 \otimes \sigma^{\bar{Z}} \otimes \sigma^3,\\
\Gamma^{Z^3} &= \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^Z, & \Gamma^{\bar{Z}^{\bar{3}}} &= \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^{\bar{Z}},
\end{aligned} \tag{6}$$

where $\mathbf{1}_2$ is the 2×2 unit matrix and σ^i are Pauli matrices,

$$\mathbf{1}_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \qquad \sigma^1 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix},$$
$$\sigma^2 = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \tag{7}$$

$$\sigma^Z = \sigma^1 + i\sigma^2 = \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix}, \qquad \sigma^{\bar{Z}} = \sigma^1 - i\sigma^2 = \begin{bmatrix}0 & 0\\2 & 0\end{bmatrix}. \tag{8}$$

Then one can find the Kähler metric on T^6

$$ds^2 = 2h_{i\bar{j}}dz^i d\bar{z}^{\bar{j}}, \qquad h_{i\bar{j}} = (2\pi R)^2 H_{i\bar{j}}. \tag{9}$$

On the other hand, the Gamma matrices $\Gamma^{z^i}, \Gamma^{\bar{z}^{\bar{i}}}$ on complex coordinates of T^6 are as follows:

$$\Gamma^{z^i} = \frac{1}{2\pi R}\Gamma^{Z^i}, \qquad \Gamma^{\bar{z}^{\bar{i}}} = \frac{1}{2\pi R}\Gamma^{\bar{Z}^{\bar{i}}}, \tag{10}$$

where $i = 1, 2, 3$. Then we obtain the following anti-commutative relations called the Dirac algebra (or Clifford algebra):

$$\begin{aligned}\{\Gamma^{z^i}, \Gamma^{z^j}\} &= \{\Gamma^{\bar{z}^{\bar{i}}}, \Gamma^{\bar{z}^{\bar{j}}}\} = 0,\\ \{\Gamma^{z^i}, \Gamma^{\bar{z}^{\bar{j}}}\} &= 2h^{i\bar{j}}.\end{aligned} \tag{11}$$

We define the chirality operator Γ^5 by

$$\Gamma^5 = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 = \mathrm{diag}[+,-,-,+,-,+,+,-]. \tag{12}$$

We can write the Dirac operator on T^6 by the Gamma matrices

$$\begin{aligned} i\not{D} &\equiv i(\Gamma^{z^j}D_{z^j} + \Gamma^{\bar{z}^{\bar{j}}}\bar{D}_{\bar{z}^{\bar{j}}})\\ &= \frac{i}{\pi R}\begin{bmatrix} D_{2,3} & D_1 \\ \bar{D}_{\bar{1}} & D_{2,3}\end{bmatrix},\end{aligned} \tag{13}$$

where D_{z^j} and $\bar{D}_{\bar{z}^{\bar{j}}}$ are covariant derivatives and fermions are coupled to the $U(1)$ gauge field with unit charge ($q = 1$),

$$\begin{aligned} D_{z^j} &= \partial_{z^j} - iA_{z^j},\\ \bar{D}_{\bar{z}^{\bar{j}}} &= \partial_{\bar{z}^{\bar{j}}} - iA_{\bar{z}^{\bar{j}}}.\end{aligned} \tag{14}$$

Operators $D_{2,3}$ and D_1 are written by D_{z^i} and $\bar{D}_{\bar{z}^{\bar{i}}}$,

$$D_{2,3} = \begin{bmatrix} 0 & D_{z^3} & D_{z^2} & 0\\ \bar{D}_{\bar{z}^{\bar{3}}} & 0 & 0 & -D_{z^2}\\ \bar{D}_{\bar{z}^{\bar{2}}} & 0 & 0 & D_{z^3}\\ 0 & -\bar{D}_{\bar{z}^{\bar{2}}} & \bar{D}_{\bar{z}^{\bar{3}}} & 0\end{bmatrix},$$
$$D_1 = \begin{bmatrix} D_{z^1} & 0 & 0 & 0\\ 0 & -D_{z^1} & 0 & 0\\ 0 & 0 & -D_{z^1} & 0\\ 0 & 0 & 0 & D_{z^1}\end{bmatrix}. \tag{15}$$

B. Background magnetic flux and the F-term condition

In this subsection, we introduce background magnetic flux F on T^6 [14],

$$\begin{aligned} F = &\frac{1}{2}(p_{xx})_{ij}dx^i \wedge dx^j + \frac{1}{2}(p_{yy})_{ij}dy^i \wedge dy^j\\ &+ (p_{xy})_{ij}dx^i \wedge dy^j.\end{aligned} \tag{16}$$

In terms of complex coordinates z^i we get

$$\begin{aligned} F = &\frac{1}{2}(F_{zz})_{ij}dz^i \wedge dz^j + \frac{1}{2}(F_{\bar{z}\bar{z}})_{ij}d\bar{z}^i \wedge d\bar{z}^j\\ &+ (F_{z\bar{z}})_{ij}(idz^i \wedge d\bar{z}^j),\end{aligned} \tag{17}$$

where

$$\begin{aligned} (F_{zz})_{ij} &= (\bar{\Omega} - \Omega)^{-1}(\bar{\Omega}p_{xx}\bar{\Omega} + p_{yy} + p_{xy}^T\bar{\Omega} - \bar{\Omega}p_{xy})\\ &\quad\times(\bar{\Omega} - \Omega)^{-1},\\ (F_{\bar{z}\bar{z}})_{ij} &= (\bar{\Omega} - \Omega)^{-1}(\Omega p_{xx}\Omega + p_{yy} + p_{xy}^T\Omega - \Omega p_{xy})\\ &\quad\times(\bar{\Omega} - \Omega)^{-1},\\ (F_{z\bar{z}})_{ij} &= i(\bar{\Omega} - \Omega)^{-1}(\bar{\Omega}p_{xx}\Omega + p_{yy} + p_{xy}^T\Omega - \bar{\Omega}p_{xy})\\ &\quad\times(\bar{\Omega} - \Omega)^{-1}.\end{aligned} \tag{18}$$

We consider 10D $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. The Hermitian Yang-Mills equation [13] to conserve SUSY imposes an F-flat condition. That is, the magnetic flux has to be (1,1)-form, and thus $F_{zz} = F_{\bar{z}\bar{z}} = 0$. We will call it the F-term condition, and it is equivalent to

$$\Omega p_{xx}\Omega + p_{yy} + p_{xy}^T\Omega - \Omega p_{xy} = 0. \tag{19}$$

We can rewrite the magnetic flux as follows:

$$\begin{aligned} F &= (F_{z\bar{z}})_{ij}(idz^i \wedge d\bar{z}^j)\\ &= i(p_{xx}\Omega - p_{xy})(\bar{\Omega} - \Omega)^{-1}(idz^i \wedge d\bar{z}^j).\end{aligned} \tag{20}$$

For simplicity, we assume $p_{xx} = p_{yy} = 0$. Then we find

$$(F_{z\bar{z}})_{ij} = -i(p_{xy}(\bar{\Omega} - \Omega)^{-1})_{ij}. \tag{21}$$

From the F-term condition and the symmetry ($\Omega^T = \Omega$), one obtains

$$p_{xy}^T\Omega = \Omega p_{xy} = (p_{xy}^T\Omega)^T. \tag{22}$$

From the Dirac quantization condition, the flux is written by an integer matrix N as

$$p_{xy} = 2\pi N^T. \tag{23}$$

We just call N as flux. In summary, background magnetic flux F is given by

$$F = \pi(N^T(\mathrm{Im}\Omega)^{-1})_{ij}(idz^i \wedge d\bar{z}^j), \tag{24}$$

and the F-term condition is given by $(N\Omega)^T = N\Omega$.

1. Gauge potential

We find the gauge potential that corresponds to F in Eq. (24) as

$$\begin{aligned} A(\vec{z},\bar{\vec{z}}) &= \pi \mathrm{Im}\left(N(\bar{\vec{z}}+\bar{\vec{\zeta}})(\mathrm{Im}\Omega)^{-1})d\vec{z}\right) \\ &= -\frac{i\pi}{2}\left(N(\bar{\vec{z}}+\bar{\vec{\zeta}})(\mathrm{Im}\Omega)^{-1}\right)_i dz^i \\ &\quad + \frac{i\pi}{2}\left(N(\vec{z}+\vec{\zeta})(\mathrm{Im}\Omega)^{-1}\right)_i d\bar{z}^i \\ &\equiv A_{z^i}dz^i + A_{\bar{z}^{\bar{i}}}d\bar{z}^i, \end{aligned} \tag{25}$$

where $\vec{\zeta}$ is the Wilson line. The boundary conditions of the gauge potential on T^6 are

$$\begin{aligned} A(\vec{z}+\vec{e}_k) &= A(\vec{z}) + d\xi_{\vec{e}_k}(\vec{z}), \\ A(\vec{z}+\Omega\vec{e}_k) &= A(\vec{z}) + d\xi_{\Omega\vec{e}_k}(\vec{z}), \end{aligned} \tag{26}$$

where $\vec{e}_k$ $(k = 1, 2, 3)$ are three-dimensional standard Euclidean unit vectors and

$$\begin{aligned} \xi_{\vec{e}_k}(\vec{z}) &= \pi(N^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\vec{z}+\vec{\zeta}))_k, \\ \xi_{\Omega\vec{e}_k}(\vec{z}) &= \pi \mathrm{Im}(N\bar{\Omega}(\mathrm{Im}\Omega)^{-1}(\vec{z}+\vec{\zeta}))_k. \end{aligned} \tag{27}$$

In this paper, we assume that the Wilson line is vanishing, $\vec{\zeta} = \vec{0}$.

C. The Dirac equation

We introduce fermion massless modes (zero-modes) on magnetized T^6 which satisfy the following Dirac equation:

$$i\not{D}\Psi(\vec{z},\bar{\vec{z}}) = 0, \tag{28}$$

where $\Psi(\vec{z},\bar{\vec{z}})$ is an eight components spinor,

$$\Psi(\vec{z},\bar{\vec{z}}) = [\psi'_{1+},\psi_{3-},\psi_{2-},\psi_{1+},\psi_{1-},\psi_{2+},\psi_{3+},\psi'_{1-}]^T. \tag{29}$$

ψ_{j+} and ψ_{j-} denote the positive and negative chirality components, respectively. In particular, ψ'_{1+} denotes that all chiralities on 2D spinors are positive. That is, when we define a Majorana-Weyl spinor on each complex plane as $(+,-)$, $\Psi(\vec{z},\bar{\vec{z}})$ is given by

$$\Psi(\vec{z},\bar{\vec{z}}) \equiv \begin{bmatrix} + \\ - \end{bmatrix} \otimes \begin{bmatrix} + \\ - \end{bmatrix} \otimes \begin{bmatrix} + \\ - \end{bmatrix}, \tag{30}$$

where $(+,+,+)$, $(+,-,-)$, $(-,+,-)$, $(-,-,+)$ correspond to ψ'_{1+}, ψ_{1+}, ψ_{2+}, and ψ_{3+}.

From the definition of the Dirac operator $\not{D}$, we obtain the Dirac equation on each ψ_i as follows:

$$\begin{aligned} D_{z^3}\psi_{3-} + D_{z^2}\psi_{2-} + D_{z^1}\psi_{1-} &= 0, \\ \bar{D}_{\bar{z}^{\bar{3}}}\psi'_{1+} - D_{z^2}\psi_{1+} - D_{z^1}\psi_{2+} &= 0, \\ \bar{D}_{\bar{z}^{\bar{2}}}\psi'_{1+} + D_{z^3}\psi_{1+} - D_{z^1}\psi_{3+} &= 0, \\ -\bar{D}_{\bar{z}^{\bar{2}}}\psi_{3-} + \bar{D}_{\bar{z}^{\bar{3}}}\psi_{2-} + D_{z^1}\psi'_{1-} &= 0, \\ \bar{D}_{\bar{z}^{\bar{1}}}\psi'_{1+} + D_{z^3}\psi_{2+} + D_{z^2}\psi_{3+} &= 0, \\ -\bar{D}_{\bar{z}^{\bar{1}}}\psi_{3-} + \bar{D}_{\bar{z}^{\bar{3}}}\psi_{1-} - D_{z^2}\psi'_{1-} &= 0, \\ -\bar{D}_{\bar{z}^{\bar{1}}}\psi_{2-} + \bar{D}_{\bar{z}^{\bar{2}}}\psi_{1-} + D_{z^3}\psi'_{1-} &= 0, \\ \bar{D}_{\bar{z}^{\bar{1}}}\psi_{1+} - \bar{D}_{\bar{z}^{\bar{2}}}\psi_{2+} + \bar{D}_{\bar{z}^{\bar{3}}}\psi_{3+} &= 0, \end{aligned} \tag{31}$$

where

$$\begin{aligned} D_{z^j} &= \partial_{z^j} - \frac{\pi}{2}(N\bar{\vec{z}}(\mathrm{Im}\Omega)^{-1})_j, \\ \bar{D}_{\bar{z}^{\bar{j}}} &= \bar{\partial}_{\bar{z}^{\bar{j}}} + \frac{\pi}{2}(N\vec{z}(\mathrm{Im}\Omega)^{-1})_{\bar{j}}. \end{aligned} \tag{32}$$

Boundary conditions of Ψ consistent with Eq. (26) are

$$\begin{aligned} \Psi(\vec{z}+\vec{e}_k) &= e^{i\xi_{\vec{e}_k}(\vec{z})}\Psi(\vec{z}), \\ \Psi(\vec{z}+\Omega\vec{e}_k) &= e^{i\xi_{\Omega\vec{e}_k}(\vec{z})}\Psi(\vec{z}), \end{aligned} \tag{33}$$

where $k = 1, 2, 3$.

D. Zero-modes on magnetized T^6

We concentrate on zero-modes when the chirality $(+,+,+)$, satisfying $\bar{D}_{\bar{z}^{\bar{i}}}\psi'_{1+} = 0$, and hence Eqs. (31) are solved when other spinor components are vanishing. We also require the boundary conditions Eq. (33). The solution is given by [13]

$$\begin{aligned} \psi_N^{\vec{J}}(\vec{z},\Omega) &= \mathcal{N} \cdot e^{i\pi(N\vec{z})^T(N\mathrm{Im}\Omega)^{-1}\cdot \mathrm{Im}(N\vec{z})} \cdot \theta\begin{bmatrix} \vec{J}N^{-1} \\ 0 \end{bmatrix}(N\vec{z},N\Omega) \\ &= \mathcal{N} \cdot e^{i\pi(N\vec{z})^T(\mathrm{Im}\Omega)^{-1}\cdot \mathrm{Im}(\vec{z})} \cdot \theta\begin{bmatrix} \vec{J}N^{-1} \\ 0 \end{bmatrix}(N\vec{z},N\Omega), \end{aligned} \tag{34}$$

where the Riemann-theta function with characteristics $\theta\left[\begin{smallmatrix} \vec{J}N^{-1} \\ 0 \end{smallmatrix}\right](N\vec{z},N\Omega)$ is defined by

$$\begin{aligned} \theta\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(\vec{z},\Omega') &= \sum_{\vec{m}\in\mathbb{Z}^3} e^{\pi i(\vec{m}+\vec{a})^T\Omega'(\vec{m}+\vec{a})} e^{2\pi i(\vec{m}+\vec{a})^T(\vec{z}+\vec{b})}, \\ \Omega' \in \mathcal{H}_3, &\qquad \vec{a},\vec{b} \in \mathbb{R}^3. \end{aligned} \tag{35}$$

The indices of three components $\vec{J} \in \mathbb{Z}^3$ label the degeneracy of zero-modes. One can check the periodicity in the indices $\vec{J}$,

$$\psi_N^{\vec{J}+N^T\vec{e}_n} = \psi_N^{\vec{J}}, \tag{36}$$

where $\vec{e}_n$ ($n = 1, 2, 3$) are three-dimensional standard unit vectors. Thus, there are only $|\det N|$ of independent indices $\vec{J}$, and they lie inside the lattice Λ_N spanned by

$$N^T\vec{e}_n \quad (n = 1, 2, 3). \tag{37}$$

Here, note that zero-modes in Eq. (34) are well-defined if $\Omega' = N\Omega$ is an element of Siegel upper-half plane $\mathcal{H}_3$ defined in Eq. (4). We stress here that Ω is not necessarily an element of $\mathcal{H}_3$. We take the following normalization condition of wave functions:

$$\int_{T^6} d^3z d^3\bar{z} \psi_N^{\vec{J}} (\psi_N^{\vec{K}})^* = (2^3 \det(\mathrm{Im}\Omega))^{-1/2} \delta_{\vec{J},\vec{K}}. \tag{38}$$

Then the constant $\mathcal{N}$ is given by

$$\mathcal{N} = [\mathrm{Vol}(T^6)]^{-1/2} (\det N)^{1/4}, \\ \mathrm{Vol}(T^6) \propto \det(\mathrm{Im}\Omega), \tag{39}$$

where $\mathrm{Vol}(T^6)$ represents the volume of T^6 and is proportional to $\det(\mathrm{Im}\Omega)$.

Since $\mathrm{Im}\Omega'$ is positive-definite, we have

$$\det(N\mathrm{Im}\Omega) = \det N \cdot \det(\mathrm{Im}\Omega) > 0. \tag{40}$$

In the following, we will consider the case when $\det N > 0$ and $\det(\mathrm{Im}\Omega) > 0$ are satisfied.

1. Laplace operator

Here, we confirm that the zero-mode wave functions in Eq. (34) are eigenfunctions of the Laplace operator on magnetized T^6. The Laplace operator is defined as follows:

$$\Delta = -\frac{2}{(2\pi R)^2} \sum_{j=1,2,3} \{D_{z^j}, \bar{D}_{\bar{z}^{\bar{j}}}\}. \tag{41}$$

We focus on the spinor components that have positive chirality in the entire 6D compact space. When we use the Laplace operator Δ on ψ'_{1+}, we find the following eigenvalue equation:

$$\Delta\psi'_{1+} = 2(F_{z^1\bar{z}^{\bar{1}}} + F_{z^2\bar{z}^{\bar{2}}} + F_{z^3\bar{z}^{\bar{3}}})\psi'_{1+}, \tag{42}$$

where we used the following commutation relations that are valid under the F-term condition:

$$[D_{z^i}, D_{z^j}] = F_{z^i z^j} = 0, \\ [\bar{D}_{\bar{z}^{\bar{i}}}, \bar{D}_{\bar{z}^{\bar{j}}}] = F_{\bar{z}^{\bar{i}}\bar{z}^{\bar{j}}} = 0, \\ [D_{z^i}, \bar{D}_{\bar{z}^{\bar{j}}}] = F_{z^i\bar{z}^{\bar{j}}}. \tag{43}$$

Equation (42) shows that the eigenvalue is proportional to the trace of F. One can check that the Laplacian Δ on a compact manifold is positive semidefinite; that is, ψ'_{1+} is nonzero only if $F_{z^1\bar{z}^{\bar{1}}} + F_{z^2\bar{z}^{\bar{2}}} + F_{z^3\bar{z}^{\bar{3}}} \geq 0$.

E. Spectrum in the bosonic sector

In this subsection, we show the mass spectrum of the 4D scalar fields which comes from dimensional reduction of the 10D gauge boson. We will later use the obtained mass formula to discuss the stability and D-term SUSY condition of our magnetized orbifold models.

1. Dimensional reduction of 10D SYM

Here, we briefly review the dimensional reduction of 10D supersymmetric Yang-Mills theory (SYM). For simplicity, we consider the $U(2)$ gauge group. The extension to $U(N)$ is straightforward. Our discussion is based on Ref. [43] and the appendix of Ref. [13]. We assume compact space with no curvature such as T^6.

The bosonic part of the action is

$$S_{\mathrm{YM}} = -\frac{1}{4g^2} \int d^{10}w \mathrm{Tr}\{F^{MN}F_{MN}\}, \tag{44}$$

where M and N are the indices of ten-dimensional spacetime, that is, $M, N \in \{0, 1, \ldots, 9\}$. We take the real orthogonal coordinate system w^M with the following metric:

$$\eta_{MN} = \mathrm{diag}(-, +, +, \cdots, +, +). \tag{45}$$

F_{MN} is written as

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]. \tag{46}$$

Gauge boson A_M is written as

$$A_M = B_M + W_M = B_M^a U_a + W_M^{ab} e_{ab}, \tag{47}$$

where the elements of the Lie algebra of $U(2)$ are taken as $(U_a)_{ij} = \delta_{ai}\delta_{aj}$ and $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$ $(a \neq b)$. By noting $A_M^\dagger = A_M$, we see that B_M^a is real and $(W_M^{ab})^* = W_M^{ba}$. After the expansion, we obtain

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2g^2} \mathrm{Tr}(D_M W_N D^M W^N - D_M W_N D^N W^M \\ - iG_{MN}[W^M, W^N]) + \cdots, \tag{48}$$

where we have only shown quadratic terms of W_M^{ab} explicitly, because we focus on mass terms. Three- and four-point interactions are not relevant. Here, we denote the field strength of the Abelian direction by

$$G_{MN} = \partial_M B_N - \partial_N B_M. \tag{49}$$

We also defined the covariant derivative by

$$D_M W_N = \partial_M W_N - i[B_M, W_N]. \tag{50}$$

Then we consider vacuum expectation values of Abelian constant magnetic fluxes,

$$\begin{aligned} B_i^a(w) &= \langle B_i^a \rangle(\eta) + C_i^a(w), \\ W_i^{ab}(w) &= 0 + \Phi_i^{ab}(w). \end{aligned} \tag{51}$$

We consider $\langle B_i^1 \rangle \neq \langle B_i^2 \rangle$, and then $U(2)$ gauge symmetry is broken to $U(1) \times U(1)$. We take $w = (\xi, \eta)$ where ξ denotes the real orthogonal coordinates of 4D spacetime and η denotes that of the compact space. Spacetime indices are also decomposed as $M = (\mu, i)$ where $\mu = 0, \ldots, 3$ and $i = 4, \ldots, 9$.

By substituting Eq. (51) into Eq. (48), one obtains

$$\begin{aligned} \mathcal{L}_{\text{YM}} = {} & \frac{i}{4g^2}(G_{ij}^a - G_{ij}^b)(\Phi^{i,ab}\Phi^{j,ba} - \Phi^{j,ab}\Phi^{i,ba}) \\ & - \frac{1}{2g^2}[(D_\mu \Phi_i^{ba} D^\mu \Phi^{i,ab}) + (\tilde{D}_i \Phi_j^{ba} \tilde{D}^i \Phi^{j,ab}) \\ & - 2(\tilde{D}_i W_\mu^{ba})(D^\mu \Phi^{i,ab}) - (\tilde{D}_i \Phi_j^{ba})(\tilde{D}^j \Phi^{i,ab})] + \cdots, \end{aligned} \tag{52}$$

where

$$\tilde{D}_i W_j^{ab} = \partial_i W_j^{ab} - i(B_i^a - B_j^b) W_j^{ab}. \tag{53}$$

If one takes the gauge fixing condition $\tilde{D}^i \Phi_i^{ab} = 0$, Eq. (52) can be rewritten as

$$\begin{aligned} \mathcal{L}_{\text{YM}} = {} & \frac{2i}{2g^2}\Phi^{j,ba}\langle G \rangle_{ij}^{ab}\Phi^{i,ab} + \frac{1}{2g^2}[\Phi_i^{ba}(D_\mu D^\mu \Phi^{i,ab}) \\ & + \Phi_j^{ba}(\tilde{D}_i \tilde{D}^i \Phi^{j,ab})] + \cdots, \end{aligned} \tag{54}$$

where $\langle G \rangle_{ij}^{ab} = \langle G \rangle_{ij}^a - \langle G \rangle_{ij}^b$.

2. 4D scalar mass

Now, we consider the Kaluza-Klein decomposition,

$$\Phi_i^{ab}(w) = \sum_n \varphi_{n,i}^{ab}(\xi) \otimes \phi_{n,i}^{ab}(\eta), \tag{55}$$

where the wave functions in the compact space satisfy

$$\Delta \phi_{n,i}^{ab}(\eta) = \kappa_n^2 \phi_{n,i}^{ab}(\eta). \tag{56}$$

Here, $\kappa_n^2 \geq 0$ denotes the eigenvalue of the Laplace operator $\Delta = -\tilde{D}_i \tilde{D}^i$. The index $n(= 0, 1, \ldots)$ denotes the excitation number, and we set $\kappa_{n+1}^2 > \kappa_n^2$. Substituting Eq. (55) into Eq. (54) and integrating with respect to the internal coordinates η yield

$$S_{4D} = S_{\text{kinetic}} + S_{\text{mass}} + \cdots, \tag{57}$$

where

$$S_{\text{kinetic}} = \sum_n \int d^4\xi \delta_{ij} \varphi_n^{i,ba}(D_\mu D^\mu)\varphi_n^{j,ab} \quad (a > b), \tag{58}$$

$$S_{\text{mass}} = \sum_n \int d^4\xi \varphi_n^{j,ba}(2i\langle G \rangle_{ij}^{ab} - \kappa_n^2 \delta_{ij})\varphi_n^{i,ab} \quad (a > b). \tag{59}$$

Note that we have used the normalization condition

$$g^{-2} \int d^6\eta \phi_m^{i,ab}(\eta)^* \phi_n^{i,cd}(\eta) = \delta_{mn}\delta_{ac}\delta_{bd}. \tag{60}$$

Next, we consider the coordinate transformation to the complex basis. We define complex coordinates $\vec{z}$ as $z^j = (2\pi R)^{-1}(\eta^{2j+2} + i\eta^{2j+3})$ where $j = 1,\ 2,\ 3$. Note that complex coordinates defined here are identified as those defined on the right-hand side of Eq. (2). Then we have

$$\begin{aligned} \varphi^{z^j,ab}(\vec{z}) &= \frac{1}{2\pi R}(\varphi^{2j+2,ab}(\eta) + i\varphi^{2j+3,ab}(\eta)), \\ \varphi^{\bar{z}^{\bar{j}},ab}(\vec{z}) &= \frac{1}{2\pi R}(\varphi^{2j+2,ab}(\eta) - i\varphi^{2j+3,ab}(\eta)). \end{aligned} \tag{61}$$

Rewriting Eq. (58) in the new basis, we obtain

$$\begin{aligned} S_{\text{kinetic}} &= \sum_n \int d^4\xi [h_{i\bar{j}} \varphi_n^{z^i,ba}(D_\mu D^\mu)\varphi_n^{\bar{z}^{\bar{j}},ab} \\ &\quad + h_{\bar{i}j}\varphi_n^{\bar{z}^{\bar{i}},ba}(D_\mu D^\mu)\varphi_n^{z^j,ab}] \\ &= \frac{(2\pi R)^2}{2}\sum_n \int d^4\xi [\varphi_n^{z^j,ba}(D_\mu D^\mu)\varphi_n^{\bar{z}^{\bar{j}},ab} \\ &\quad + \varphi_n^{\bar{z}^{\bar{j}},ba}(D_\mu D^\mu)\varphi_n^{z^j,ab}], \end{aligned} \tag{62}$$

where we used a metric of the form Eq. (9). To get the canonical kinetic term, we redefine the 4D scalar field as

$$\hat{\varphi}^{z^j,ab} = \frac{2\pi R}{\sqrt{2}}\varphi^{z^j,ab}, \qquad \hat{\varphi}^{\bar{z}^{\bar{j}},ab} = \frac{2\pi R}{\sqrt{2}}\varphi^{\bar{z}^{\bar{j}},ab}. \tag{63}$$

Then we obtain the 4D scalar mass term as

$$S_{\rm mass} = \sum_n \int d^4\xi \left[\hat{\varphi}_n^{z^j,ba}\left(\frac{4i}{(2\pi R)^2}\langle G\rangle^{ab}_{\bar{z}^{\bar{j}}z^j} - \kappa_n^2\right)\hat{\varphi}_n^{\bar{z}^{\bar{j}},ab} + \hat{\varphi}_n^{\bar{z}^{\bar{j}},ba}\left(\frac{4i}{(2\pi R)^2}\langle G\rangle^{ab}_{z^j\bar{z}^{\bar{j}}} - \kappa_n^2\right)\hat{\varphi}_n^{z^j,ab}\right]. \quad (64)$$

If we assume $\langle G^{b=2}\rangle = 0$, scalar modes have only a $U(1)_{a=1}$ charge and we obtain a 4D scalar mass formula in the $U(1)$ gauge theory on which we have been focusing. Equation (64) realizes the results presented in Ref. [13] in the case of T^2 compactification.

III. $T^6/\mathbb{Z}_N$ ORBIFOLDS

A. 6D lattices

Here, we give a brief review on the $T^6/\mathbb{Z}_N$ orbifolds. The $\mathbb{Z}_N$ twists preserving 4D $\mathcal{N}=1$ SUSY were studied in Refs. [1,2], and they are also shown in the second column of Table I, where eigenvalues of the orbifold twist are written by $e^{2\pi i k_i/N}$ $(i = 1,\ 2,\ 3)$. The SUSY condition requires

$$k_1 + k_2 + k_3 = 0 \pmod{N}. \quad (65)$$

We divide the 6D flat space by a 6D lattice Λ_6 to construct the torus T^6. We divide T^6 by the $\mathbb{Z}_N$ twist so as to obtain the $T^6/\mathbb{Z}_N$ orbifold. Hence, the 6D lattice must have the $\mathbb{Z}_N$ symmetry. We use Lie lattices with dimensions $D \leq 6$ and combine them to construct the 6D lattice Λ_6. The $\mathbb{Z}_N$ twist corresponds to the Coxeter element C of the Lie root lattice [3–8]. For example, the Coxeter element C of the $SU(N)$ root lattice has the $\mathbb{Z}_N$ symmetry, i.e., $C^N = 1$. In particular, we use Lie root lattices with even dimensions, where we define complex coordinates and introduce magnetic fluxes. We also use the $SU(2)^2$ lattice, but they are not always orthogonal to each other. To represent $T^2/\mathbb{Z}_2$, we denote its lattice by $SU(2)^2$, because the product of their Coxeter elements is the $\mathbb{Z}_2$ twist in two dimensions. These root lattices are shown in Table I. In the table, $SU(3)^{[2]}$ and $SO(8)^{[2]}$ denote the use of generalized Coxeter elements including $\mathbb{Z}_2$ outer automorphisms of $SU(3)$ and $SO(8)$ Lie algebras, respectively. In addition, $SO(8)^{[3]}$ denotes the use of the generalized Coxeter element of $SO(8)$ including $\mathbb{Z}_3$ outer automorphism.

The flavor structure originated from $T^2/\mathbb{Z}_N$ has been studied already in Refs. [18,19]. Thus, here we study nonfactorizable $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$ orbifolds in Table I, in particular the numbers of zero-modes on these orbifolds with magnetic fluxes.

Nonfactorizable $T^6/\mathbb{Z}_N$ orbifolds include the $T^6/\mathbb{Z}_7$ orbifold with the $SU(7)$ root lattice and the $T^6/\mathbb{Z}_{12-I}$ orbifold with the E_6 root lattice. The zero-modes on such magnetized orbifolds are studied in the next sections. We reconstruct the orbifold twists as elements of $Sp(6,\mathbb{Z})$. We study zero-modes by analyzing $Sp(6,\mathbb{Z})$ modular transformation behaviors of wave functions. Similar analysis was carried out in magnetized $T^2/\mathbb{Z}_N$ orbifold models [32]. Our analysis extends to $T^6/\mathbb{Z}_N$ orbifolds as well as $T^4/\mathbb{Z}_N$ orbifolds. (See also Ref. [28].)

Nonfactorizable $T^4/\mathbb{Z}_N$ orbifolds include $T^4/\mathbb{Z}_6$ in $T^6/\mathbb{Z}_{6-II}$, $T^4/\mathbb{Z}_8$ in $T^6/\mathbb{Z}_{8-I}$ and $T^6/\mathbb{Z}_{8-II}$, and $T^4/\mathbb{Z}_{12}$ in $T^6/\mathbb{Z}_{12-II}$. All of them use the $SO(8)$ root lattice. However, we cannot introduce magnetic fluxes on $T^4/\mathbb{Z}_6$ and $T^4/\mathbb{Z}_{12}$. Its reason is explained in Appendix A. Also Appendix A shows zero-modes on a magnetized $T^4/\mathbb{Z}_8$ orbifold model.

B. $\mathbb{Z}_N$ twists for $T^6/\mathbb{Z}_N$

The 6D lattices have the modular symmetry, that is, the basis transformation of the basis vectors. We find some of the aforementioned Coxeter and generalized Coxeter elements can be expressed as $Sp(6,\mathbb{Z})$ modular transformation.

The symplectic modular group $Sp(6,\mathbb{Z})$ is given by the set of 6×6 integer matrices,

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (66)$$

satisfying

TABLE I. $T^6/\mathbb{Z}_N$ orbifolds and torus lattices.

Orbifold	Twists $(k_1,k_2,k_3)/N$	Lattice
$T^6/\mathbb{Z}_3$	$(1,1,-2)/3$	$SU(3)^3$
$T^6/\mathbb{Z}_4$	$(1,1,-2)/4$	$(SO(4)^{[2]})^2\times SU(2)^2$
$T^6/\mathbb{Z}_{6-I}$	$(1,1,-2)/6$	$(SU(3)^{[2]})^2\times SU(3)$
$T_6/\mathbb{Z}_{6-II}$	$(1,2,-3)/6$	$SU(3)^{[2]}\times SU(3)\times SU(2)^2$, $SO(8)\times SU(3)$
$T^6/\mathbb{Z}_7$	$(1,2,-3)/7$	$SU(7)$
$T^6/\mathbb{Z}_{8-I}$	$(1,2,-3)/8$	$(SO(8)^{[2]})\times SO(4)^{[2]}$
$T^6/\mathbb{Z}_{8-II}$	$(1,3,-4)/8$	$(SO(8)^{[2]})\times SU(2)^2$
$T^6/\mathbb{Z}_{12-I}$	$(1,4,-5)/12$	E_6
$T^6/\mathbb{Z}_{12-II}$	$(1,5-6)/12$	$(SO(8)^{[3]})\times SU(2)^2$

$$\gamma J \gamma^T = J, \qquad J = \begin{bmatrix} 0 & \mathbf{1}_3 \\ -\mathbf{1}_3 & 0 \end{bmatrix}. \tag{67}$$

A, B, C, and D are 3×3 integer matrices. The modular transformation of the complex coordinates $\vec{z}$ and the complex structure moduli Ω under γ are given by

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}, \tag{68}$$

$$\vec{z} \to (C\Omega + D)^{-1T}\vec{z}. \tag{69}$$

Generators S, T_i $(i = 1, 2, \ldots, 5, 6)$ are given by

$$S = \begin{bmatrix} O & \mathbf{1}_3 \\ -\mathbf{1}_3 & O \end{bmatrix}, \qquad T_i = \begin{bmatrix} \mathbf{1}_3 & B_i \\ O & \mathbf{1}_3 \end{bmatrix}, \tag{70}$$

where B_i are symmetric matrices given by

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{71}$$

$$B_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{72}$$

We will consider modular S and T_i transformations of zero-modes on magnetized T^6.

As we will see in the next section, the $\mathbb{Z}_7$ twist on the $SU(7)$ lattice can be written by $ST_3T_4T_5$ satisfying $(ST_3T_4T_5)^7 = \mathbf{1}_6$. The $\mathbb{Z}_{12}$ twist on the E_6 lattice can be written by $ST_1T_2T_3^{-1}T_5T_6$ satisfying $(ST_1T_2T_3^{-1}T_5T_6)^{12} = \mathbf{1}_6$.

In the $Sp(6, \mathbb{Z})$ modular group, we suppose symmetric moduli Ω, and one can see that the lattice of $T^6/\mathbb{Z}_N$ can be found by Ω.

In the following section, we represent lattice vectors $\vec{e}_i$ as following Euclidean basis representation

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_4 = \begin{bmatrix} \mathrm{Re}\omega_1 \\ \mathrm{Re}\omega_4 \\ \mathrm{Re}\omega_6 \\ \mathrm{Im}\omega_1 \\ \mathrm{Im}\omega_4 \\ \mathrm{Im}\omega_6 \end{bmatrix}, \quad \vec{e}_5 = \begin{bmatrix} \mathrm{Re}\omega_4 \\ \mathrm{Re}\omega_2 \\ \mathrm{Re}\omega_5 \\ \mathrm{Im}\omega_4 \\ \mathrm{Im}\omega_2 \\ \mathrm{Im}\omega_5 \end{bmatrix}, \quad \vec{e}_6 = \begin{bmatrix} \mathrm{Re}\omega_6 \\ \mathrm{Re}\omega_5 \\ \mathrm{Re}\omega_3 \\ \mathrm{Im}\omega_6 \\ \mathrm{Im}\omega_5 \\ \mathrm{Im}\omega_3 \end{bmatrix}. \tag{73}$$

Then we can find what lattices of $T^6/\mathbb{Z}_N$ correspond to root lattices of Lie algebra.

IV. NONFACTORIZABLE ORBIFOLDS

Here we perform counting of the zero-modes in magnetized $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$ orbifold models.

A. Magnetized $T^6/\mathbb{Z}_7$ orbifold

First, we study zero-modes on the magnetized $T^6/\mathbb{Z}_7$ orbifold. To realize the orbifold, we focus on the following algebraic relation:

$$(ST_3T_4T_5)^7 = \mathbf{1}_6. \tag{74}$$

This shows that $ST_3T_4T_5$ transformation can be identified as the $\mathbb{Z}_7$ twist. Thus, it is useful for constructing the $T^6/\mathbb{Z}_7$ orbifold. Under the transformation $ST_3T_4T_5$, the complex structure moduli Ω and complex coordinates $\vec{z}$ transform as

$$\begin{aligned} &\Omega \to -(\Omega + B_3 + B_4 + B_5)^{-1}, \\ &\vec{z} \to -(\Omega + B_3 + B_4 + B_5)^{-1}\vec{z}. \end{aligned} \tag{75}$$

Then one can verify that $ST_3T_4T_5$ invariant moduli Ω_7 are given by

$$\Omega_7 = \begin{bmatrix} \omega_1 & \omega_4 & \omega_6 \\ \omega_4 & \omega_2 & \omega_5 \\ \omega_6 & \omega_5 & \omega_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{7}}i & -\frac{1}{2}+\frac{\sqrt{7}}{14}i & \frac{i}{\sqrt{7}} \\ -\frac{1}{2}+\frac{\sqrt{7}}{14}i & -\frac{i}{\sqrt{7}} & -\frac{1}{2}+\frac{3\sqrt{7}}{14}i \\ \frac{i}{\sqrt{7}} & -\frac{1}{2}+\frac{3\sqrt{7}}{14}i & -\frac{1}{2}-\frac{\sqrt{7}}{14}i \end{bmatrix}, \tag{76}$$

where we take the case when $\det(\mathrm{Im}\Omega_7) > 0$. This corresponds to the $SU(7)$ root lattice as shown in Fig. 1.

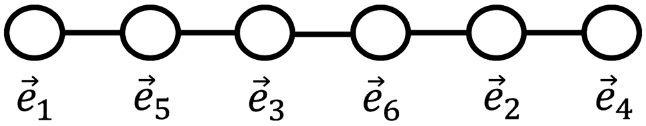

FIG. 1. The lattice of $T^6/\mathbb{Z}_7$.

We note that the shape of flux N is constrained as follows. First, from the F-term condition $(N\Omega_7)^T = N\Omega_7$, N is symmetric and parametrized as

$$N = \begin{bmatrix} n_{11} & n_{33}-n_{22} & n_{22}-n_{11} \\ n_{33}-n_{22} & n_{22} & n_{33}-n_{11} \\ n_{22}-n_{11} & n_{33}-n_{11} & n_{33} \end{bmatrix}, \qquad (77)$$

where n_{11}, n_{22}, and n_{33} are integers and we can see them as independent parameters.

Second, we consider the consistency with the T transformation. As we study in Appendix B 2, the matrix NB must be symmetric and all of its diagonal components are even. From

$$N(B_3+B_4+B_5) = \begin{bmatrix} n_{33}-n_{22} & n_{22} & n_{33}-n_{11} \\ n_{22} & 2n_{33}-n_{11}-n_{22} & n_{33}+n_{22}-n_{11} \\ n_{33}-n_{11} & n_{33}+n_{22}-n_{11} & 2n_{33}-n_{11} \end{bmatrix}, \qquad (78)$$

it is immediate that

$$n_{11} \equiv n_{22} \equiv n_{33} \equiv 0 \pmod 2 \qquad (79)$$

must be satisfied. As a result, we find that the background magnetic flux F is invariant under the $\mathbb{Z}_7$ twist, and $\det N$ is always a multiple of eight.

We will analyze how many zero-modes with $\mathbb{Z}_7$ charges exist under the flux N.

1. The number of zero-modes

Here we analyze the number of zero-modes on magnetized $T^6/\mathbb{Z}_7$ with modular transformation. Noting that Ω_7 satisfies $\Omega_7 = -(\Omega_7 + B_3 + B_4 + B_5)^{-1}$, zero-modes behave under the $ST_3T_4T_5$ transformation as

$$\psi_N^{\vec{J}}(\Omega_7\vec{z}, \Omega_7) = \frac{\sqrt{\det[-i(\Omega_7+B_3+B_4+B_5)]}}{\sqrt{\det N}} \times \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{J}^T N^{-1}\vec{K}} e^{\pi i \vec{K}^T N^{-1}(B_3+B_4+B_5)\vec{K}} \psi_N^{\vec{K}}(\vec{z}, \Omega_7), \qquad (80)$$

where Λ_N is defined in Eq. (37). We take a branch $\sqrt{t} > 0$ for $t > 0$. Then, it is found that

$$\sqrt{\det[-i(\Omega_7+B_3+B_4+B_5)]} = e^{-\pi i/4}. \qquad (81)$$

Then we find the trace of $\rho(ST_3T_4T_5)$ as follows:

$$\mathrm{tr}\rho(ST_3T_4T_5) = \frac{e^{-\pi i/4}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{K}^T N^{-1}(\mathbf{1}_3 + \frac{1}{2}(B_3+B_4+B_5))\vec{K}}. \qquad (82)$$

From numerical calculation in the region $|n_{ii}| \le 400$, we obtained only three values of $\mathrm{tr}\rho$,

$$\mathrm{tr}\rho = -1, +1, -\sqrt{7}i. \qquad (83)$$

On the other hand, $\mathrm{tr}\rho$ can be expressed as

$$\mathrm{tr}\rho = \sum_{k=0}^{6} n_k \gamma^k, \qquad \gamma = e^{2\pi i/7}, \qquad (84)$$

where n_k denotes the number of zero-modes which corresponds to the $\mathbb{Z}_7$ eigenvalue γ^k. Note that we also have $\sum_{k=0}^{6} n_k = \det N = 8\ell$. In the simple case, the modes with γ^0 correspond to the $\mathbb{Z}_N$ invariant states. However, we may embed the geometrical twist into the gauge sector. Then, which state with γ^k can survive through the $\mathbb{Z}_N$ projection depends on such gauge embedding.

First, we discuss the case when $\mathrm{tr}\rho = -\sqrt{7}i$. One can find that $[n_0, n_1, \ldots, n_5, n_6] = [1, 0, 0, 2, 0, 2, 2]$ reproduces $\mathrm{tr}\rho = -\sqrt{7}i$. Other possibilities are given by increasing each n_k by $m \in \mathbb{Z}^+$ because $\sum_{k=0}^{6}\gamma^k = 0$ holds. That is, $[n_0, n_1, \ldots, n_5, n_6] = [1+m, m, m, 2+m, m, 2+m, 2+m]$. As a result, $\det N$ is increased by $7m$ and we also observe that the minimal degeneracy number is given by n_1, n_2, and n_4 which are equal to m. Now, recall the fact that $\det N$ is a multiple of eight. We have the following equation:

$$8\ell = 7 + 7m = 7(m+1). \qquad (85)$$

Since 7 and 8 are coprime, $m+1$ must be a multiple of eight. Thus we obtain possible values of m as

$$m = 7, 15, \ldots. \qquad (86)$$

One can see that there are at least seven generations when $\mathrm{tr}\rho = -\sqrt{7}i$.

Next we discuss the case $\mathrm{tr}\,\rho = +1$. The first candidate is clearly $[n_0, n_1, \ldots, n_5, n_6] = [1, 0, 0, 0, 0, 0, 0]$ and the next one is $[n_0, n_1, \ldots, n_5, n_6] = [2, 1, 1, 1, 1, 1, 1]$. By a similar discussion as for the case $\mathrm{tr}\rho = -\sqrt{7}i$, we find

$$\det N = 8\ell = 1 + 7m = 7(m-1) + 8. \qquad (87)$$

Then $m-1$ must be a multiple of eight,

$$m = 1, 9, \ldots. \tag{88}$$

We can see that the minimal generation numbers are 1, 9, and so on. Therefore, we cannot obtain three-generation models in the case of $\mathrm{tr}\rho = +1$.

Similarly, we cannot find three-generation models when $\mathrm{tr}\rho = -1$.

In conclusion, there is no three-generation model on magnetized $T^6/\mathbb{Z}_7$ in the absence of Wilson lines and Scherk-Schwarz phases.

B. Magnetized $T^6/\mathbb{Z}_{12}$ orbifold

In this subsection, we focus on magnetized $T^6/\mathbb{Z}_{12}$ orbifold, whose twist is constructed by the following algebraic relation:

$$(ST_1T_2T_3^{-1}T_5T_6)^{12} = \mathbf{1}_6. \tag{89}$$

In the following, we denote $ST_1T_2T_3^{-1}T_5T_6$ as G. We adopt the following complex structure moduli $\Omega = \Omega_{12}$ which are invariant under the G transformation

$$\Omega_{12} = \begin{bmatrix} -\frac{1}{2}-\frac{\sqrt{3}}{6}i & \frac{\sqrt{3}}{3}i & -\frac{1}{2}+\frac{\sqrt{3}}{6}i \\ \frac{\sqrt{3}}{3}i & -\frac{1}{2}-\frac{\sqrt{3}}{6}i & -\frac{1}{2}+\frac{\sqrt{3}}{6}i \\ -\frac{1}{2}+\frac{\sqrt{3}}{6}i & -\frac{1}{2}+\frac{\sqrt{3}}{6}i & \frac{1}{2}-\frac{\sqrt{3}}{6}i \end{bmatrix}. \tag{90}$$

One can verify that this $T^6/\mathbb{Z}_{12}$ lattice corresponds to the E_6 root lattice as shown in Fig. 2.

The shape of the flux N is constrained. First, the F-term condition imposes a constraint, $(N\Omega_{12})^T = N\Omega_{12}$. We also require the invariance of the flux under G since the flux should be invariant for the $\mathbb{Z}_{12}$ twist. Then the flux is symmetric. Also this includes the consistency with $T = T_1T_2T_3^{-1}T_5T_6$ which demands that $N(B_1 + B_2 - B_3 + B_5 + B_6)$ is symmetric and all of its diagonal elements are even. One can find that fluxes of the form

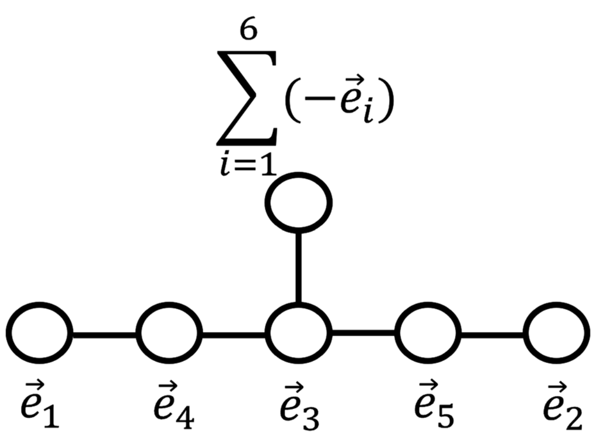

FIG. 2. The lattice of $T^6/\mathbb{Z}_{12}$.

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{11} & n_{13} \\ n_{13} & n_{13} & n_{11}+n_{12}-2n_{13} \end{bmatrix} \tag{91}$$

satisfy all requirements provided $n_{11} \equiv n_{12} \equiv n_{13} \pmod 2$.

It is obvious that G^2 can be regarded as a $\mathbb{Z}_6$ twist. Also we have

$$G^{12} = (G^2)^6 = (G^3)^4 = (G^4)^3 = (G^6)^2 = \mathbf{1}_6. \tag{92}$$

We will use this fact to count the number of zero-modes.

1. Representation of modular transformation of zero-modes

Modular transformation of the wave function on magnetized $T^6/\mathbb{Z}_{12}$ is written as

$$\psi_N^{\vec{J}}(\Omega_{12}\vec{z}, \Omega_{12}) = \frac{\sqrt{\det[-i(\Omega_{12}+B)]}}{\sqrt{\det N}} \times \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{J}^T N^{-1}\vec{K}} e^{\pi i \vec{K}^T N^{-1} B \vec{K}} \psi_N^{\vec{K}}(\vec{z}, \Omega_{12}), \tag{93}$$

where $B = B_1 + B_2 - B_3 + B_5 + B_6$. Here the representation itself is written by indices $\vec{K}_1$ and $\vec{K}_2$,

$$\rho_{\vec{K}_1\vec{K}_2}(G) = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{\det N}} e^{2\pi i \vec{K}_1^T N^{-1}\vec{K}_2} e^{i\pi(\vec{K}_1^T N^{-1} B \vec{K}_1)}. \tag{94}$$

Thus, a trace of the representation $\rho(G)$ is given by

$$\mathrm{tr}\rho(G) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{K}^T N^{-1}(\mathbf{1}_3 + B/2)\vec{K}}. \tag{95}$$

Then we immediately see the following relation of tr $\rho(G^n)$ from the property of modular transformation:

$$\mathrm{tr}\rho(G^n) = \frac{e^{-\frac{\pi i}{4}n}}{(\det N)^{n/2}} \times \sum_{\vec{K}_1,\vec{K}_2,\ldots\vec{K}_n\in\Lambda_N} e^{2\pi i(\vec{K}_1^T N^{-1}\vec{K}_2 + \vec{K}_2^T N^{-1}\vec{K}_3 + \cdots + \vec{K}_n^T N^{-1}\vec{K}_1)} \cdot e^{\pi i(\vec{K}_1^T N^{-1} B \vec{K}_1 + \cdots + \vec{K}_n^T N^{-1} B \vec{K}_n)}. \tag{96}$$

2. Zero-modes on magnetized $T^6/\mathbb{Z}_{12}$

We show how to count zero-modes with $\mathbb{Z}_{12}$ charges. We denote the number of degenerated zero-modes corresponding to the $\mathbb{Z}_{12}$ eigenvalues $e^{\frac{k\pi}{6}i}$ ($k = 0, 1, \ldots, 11$ by n_k. Note that the summation of all the degeneracy numbers is equal to det N,

TABLE II. The number of zero-modes on magnetized $T^6/\mathbb{Z}_{12}$.

det N	4	8	12	16	20	24	24	28	32	32	36	36	40	⋯
tr$\rho(G)$	−1	1	$-\sqrt{3}i$	−1	1	$-\sqrt{3}i$	−1	−1	1	1	$-\sqrt{3}i$	−1	−1	
tr$\rho(G^2)$	1	−1	$-\sqrt{3}i$	1	−1	$-\sqrt{3}i$	1	1	−1	−1	$-\sqrt{3}i$	1	1	
tr$\rho(G^3)$	2	4	6	8	10	12	2	14	16	4	18	2	20	
tr$\rho(G^4)$	1	−1	$\sqrt{3}i$	1	−1	$\sqrt{3}i$	−3	1	−1	−1	$\sqrt{3}i$	−3	1	
tr$\rho(G^6)$	4	8	12	16	20	24	4	28	32	8	36	4	40	
n_0	1	2	$\boxed{3}$	4	5	6	2	7	8	4	9	$\boxed{3}$	10	
n_1	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	
n_2	0	1	0	1	2	1	2	2	$\boxed{3}$	$\boxed{3}$	2	$\boxed{3}$	$\boxed{3}$	
n_3	0	0	0	0	0	0	1	0	0	2	0	2	0	
n_4	1	2	$\boxed{3}$	4	5	6	$\boxed{3}$	7	8	4	9	4	10	
n_5	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	
n_6	1	0	1	2	1	2	2	$\boxed{3}$	2	2	$\boxed{3}$	$\boxed{3}$	4	
n_7	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	
n_8	1	2	$\boxed{3}$	4	5	6	$\boxed{3}$	7	8	4	9	4	10	
n_9	0	0	0	0	0	0	1	0	0	2	0	2	0	
n_{10}	0	1	2	1	2	$\boxed{3}$	2	2	$\boxed{3}$	$\boxed{3}$	4	$\boxed{3}$	$\boxed{3}$	
n_{11}	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	

$$\det N = \sum_{k=0}^{11} n_k. \tag{97}$$

Next we consider relations between tr$\rho(G^n)$ and coefficients n_k. Since $G^{12} = \mathbf{1}_6$ and tr$\rho(G)$ is the summation of $\rho(G)$'s eigenvalues, we find

$$\mathrm{tr}\rho(G) = \sum_{k=0}^{11} n_k e^{\frac{k\pi}{6}i}. \tag{98}$$

We can represent tr$\rho(G^n)$ as linear combinations of n_k as

$$\mathrm{tr}\rho(G^2) = \sum_{k\equiv 0(\mathrm{mod}\ 6)} n_k + e^{\frac{\pi}{3}i} \sum_{k\equiv 1(\mathrm{mod}\ 6)} n_k + e^{\frac{2\pi}{3}i} \sum_{k\equiv 2(\mathrm{mod}\ 6)} n_k - \sum_{k\equiv 3(\mathrm{mod}\ 6)} n_k + e^{\frac{4\pi}{3}i} \sum_{k\equiv 4(\mathrm{mod}\ 6)} n_k + e^{\frac{5\pi}{3}i} \sum_{k\equiv 5(\mathrm{mod}\ 6)} n_k, \tag{99}$$

$$\mathrm{tr}\rho(G^3) = \sum_{k\equiv 0(\mathrm{mod}\ 4)} n_k + i \sum_{k\equiv 1(\mathrm{mod}\ 4)} n_k - \sum_{k\equiv 2(\mathrm{mod}\ 4)} n_k - i \sum_{k\equiv 3(\mathrm{mod}\ 4)} n_k, \tag{100}$$

TABLE III. Continuation of Table II.

det N	44	48	52	52	56	60	64	64	68	72	72	72	⋯
tr$\rho(G)$	1	$-\sqrt{3}i$	−1	−1	1	$-\sqrt{3}i$	−1	−1	1	$-\sqrt{3}i$	1	$-\sqrt{3}i$	
tr$\rho(G^2)$	−1	$-\sqrt{3}i$	1	1	−1	$-\sqrt{3}i$	1	1	−1	$-\sqrt{3}i$	−1	$-\sqrt{3}i$	
tr$\rho(G^3)$	22	24	26	2	28	30	32	8	34	36	4	6	
tr$\rho(G^4)$	−1	$\sqrt{3}i$	1	1	−1	$\sqrt{3}i$	1	1	−1	$\sqrt{3}i$	3	$-3\sqrt{3}i$	
tr$\rho(G^6)$	44	48	52	4	56	60	64	16	68	72	8	12	
n_0	11	12	13	5	14	15	16	8	17	18	8	8	
n_1	0	0	0	4	0	0	0	4	0	0	5	4	
n_2	4	$\boxed{3}$	4	4	5	4	5	5	6	5	6	6	
n_3	0	0	0	4	0	0	0	4	0	0	6	5	
n_4	11	12	13	5	14	15	16	8	17	18	7	7	
n_5	0	0	0	4	0	0	0	4	0	0	5	6	
n_6	$\boxed{3}$	4	5	5	4	5	6	6	5	6	6	6	
n_7	0	0	0	4	0	0	0	4	0	0	5	4	
n_8	11	12	13	5	14	15	16	8	17	18	7	9	
n_9	0	0	0	4	0	0	0	4	0	0	6	5	
n_{10}	4	5	4	4	5	6	5	5	6	7	6	6	
n_{11}	0	0	0	4	0	0	0	4	0	0	5	6	

$$\mathrm{tr}\rho(G^4) = \sum_{k\equiv 0 (\mathrm{mod}\, 3)} n_k + e^{\frac{2\pi}{3}i} \sum_{k\equiv 1 (\mathrm{mod}\ 3)} n_k + e^{\frac{4\pi}{3}i} \sum_{k\equiv 2 (\mathrm{mod}\ 3)} n_k, \tag{101}$$

$$\mathrm{tr}\rho(G^6) = \sum_{k\equiv 0 (\mathrm{mod}\ 2)} n_k - \sum_{k\equiv 1 (\mathrm{mod}\ 2)} n_k. \tag{102}$$

Then one can obtain the number of zero-modes on magnetized $T^6/\mathbb{Z}_{12}$. Here we conduct numerical calculations in the region $|n_{1i}| \le 400$, and Tables II and III show the results with the F-term condition.

V. THE D-TERM CONDITION

In this section, we study the D-term condition by computing 4D scalar mass spectrum. Three-generation models satisfying the F-term condition were discussed in Sec. IV. However, if tachyonic-modes appeared in the models, we would treat the unstable vacuum. If the model satisfies the D-term condition, it is stable and phenomenologically attractive.

We have introduced 3×3 integer flux N satisfying the F-term condition $(N\Omega)^T = N\Omega$. Thus, we have the following background flux F:

$$F = \pi[N^T(\mathrm{Im}\Omega)^{-1}]_{ij}(idz^i \wedge d\bar{z}^j) = F_{z^i\bar{z}^j}(idz^i \wedge d\bar{z}^j), \tag{103}$$

where $F_{z^i\bar{z}^j} = F_{z^j\bar{z}^i}$.

We analyze the mass spectrum of 4D scalar modes resulting from the magnetized T^6 compactification. When $N\mathrm{Im}\Omega > 0$, we have confirmed that the wave functions in Eq. (34) are eigenstates of the Laplacian on T^6. Their eigenvalue is equal to a trace of $\frac{2\pi}{(2\pi R)^2}[N^T(\mathrm{Im}\Omega)^{-1}]$ which corresponds to the lowest energy, i.e., κ_0^2 in Eq. (56). Hence, the lightest 4D scalar mode is given by their linear combination. We have shown the mass squared matrix of the 4D scalar modes in Eq. (64) where $\langle G\rangle^{ab}_{z^i\bar{z}^j} = iF^{ab}_{z^i\bar{z}^j}(= iF_{z^i\bar{z}^j})$. We find that the lightest scalar mode corresponds to the eigenvector of the following mass squared matrix $\mathcal{M}^2$ with the smallest eigenvalue,

$$\mathcal{M}^2 = \Delta + \frac{4\pi}{(2\pi R)^2}\begin{bmatrix} -A & O \\ O & A \end{bmatrix}. \tag{104}$$

Here, O is 3×3 zero matrix and A is 3×3 real-symmetric matrix defined as

$$A = \begin{bmatrix} [N^T(\mathrm{Im}\Omega)^{-1}]_{11} & [N^T(\mathrm{Im}\Omega)^{-1}]_{12} & [N^T(\mathrm{Im}\Omega)^{-1}]_{13} \\ [N^T(\mathrm{Im}\Omega)^{-1}]_{12} & [N^T(\mathrm{Im}\Omega)^{-1}]_{22} & [N^T(\mathrm{Im}\Omega)^{-1}]_{23} \\ [N^T(\mathrm{Im}\Omega)^{-1}]_{13} & [N^T(\mathrm{Im}\Omega)^{-1}]_{23} & [N^T(\mathrm{Im}\Omega)^{-1}]_{33} \end{bmatrix}, \tag{105}$$

and $\Delta = \frac{2\pi}{(2\pi R)^2}\sum_{j=1}^3 [N^T(\mathrm{Im}\Omega)^{-1}]_{jj}$.

We therefore obtain the D-term condition from eigenvalues of the above matrix. Since real symmetric matrices have real eigenvalues and are diagonalizable using orthogonal matrices, one can diagonalize mass matrix $\mathcal{M}^2$ as follows:

$$\mathcal{M}^2 = \frac{2\pi}{(2\pi R)^2}(\lambda_1+\lambda_2+\lambda_3) + \frac{4\pi}{(2\pi R)^2}\begin{bmatrix} -\lambda_1 & & & & & \\ & -\lambda_2 & & & & \\ & & -\lambda_3 & & & \\ & & & \lambda_1 & & \\ & & & & \lambda_2 & \\ & & & & & \lambda_3 \end{bmatrix}. \tag{106}$$

We note that $\lambda_i > 0$ $(i = 1, 2, 3)$ is satisfied because of the positive definite condition $N^T(\mathrm{Im}\Omega)^{-1} > 0$. The D-term condition is that the smallest eigenvalue of $\mathcal{M}^2$ is zero. That is, if the largest eigenvalue λ_i is equal to the summation of the rest eigenvalues λ_j, λ_k, or $\lambda_i = \lambda_j + \lambda_k$, the lightest mode is identified as the superpartner of the chiral fermion zero-modes. If the eigenvalue is negative corresponding to a tachyonic-mode, the system is unstable and there is no supersymmetry.

On the factorizable $T^6 = T_1^2 \times T_2^2 \times T_3^2$, the D-term condition can be satisfied by tuning ratios of areas of T_i^2 [20,44]. However, we have no such free parameter in nonfactorizable T^6. That leads to severe constraints in models.

In the next subsection, we discuss three-generation models in magnetized $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$ satisfying both the F- and D-term conditions, hence phenomenologically attractive.

A. 4D $\mathcal{N} = 1$ SUSY and magnetized $T^6/\mathbb{Z}_7$

We perform numerical calculations to check whether there exists a model such that the D-term condition is satisfied. In the region $|n_{ii}| \le 400$ and $|\det N| \le 1.0\times 10^{10}$, we find there are no such models except for the uninteresting case $N = O$, where O denotes the 3×3 zero matrix. Therefore, the magnetized $T^6/\mathbb{Z}_7$ model seems not suitable for the realization of realistic models not only because it cannot reproduce three generations, but we do not find any D-flat models.

B. 4D $\mathcal{N}=1$ SUSY and magnetized $T^6/\mathbb{Z}_{12}$

We obtain three-generation models satisfying both the F- and D-term conditions.

When flux N takes the following values:

$$N = \begin{bmatrix} -5 & 13 & 3 \\ 13 & -5 & 3 \\ 3 & 3 & 2 \end{bmatrix}, \tag{107}$$

the D-term condition is satisfied. We find that $\det N = 36$, $\mathrm{tr}\rho(G) = \mathrm{tr}\rho(G^2) = -\sqrt{3}i$, $\mathrm{tr}\rho(G^3) = 18$, $\mathrm{tr}\rho(G^4) = \sqrt{3}i$, $\mathrm{tr}\rho(G^6) = 36$, and the numbers of zero-modes are $[n_0, n_1, \ldots, n_{11}] = [9,0,2,0,9,0,3,0,9,0,4,0]$.

Also when the flux is

$$N = \begin{bmatrix} -3 & 3 & 1 \\ 3 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{108}$$

the D-term condition is satisfied. We find that $\det N = 12$, $\mathrm{tr}\rho(G) = \mathrm{tr}\rho(G^2) = -\sqrt{3}i$, $\mathrm{tr}\rho(G^3) = 6$, $\mathrm{tr}\rho(G^4) = \sqrt{3}i$, $\mathrm{tr}\rho(G^6) = 12$, and the numbers of zero-modes are $[n_0, n_1, \ldots, n_{11}] = [3,0,0,0,3,0,1,0,3,0,2,0]$.

VI. CONCLUSION

We have studied the zero-modes of nonfactorizable $T^6/\mathbb{Z}_N$ orbifold models with background magnetic flux. We have classified zero-modes with $\mathbb{Z}_N$ charges in magnetized $T^6/\mathbb{Z}_N$ models by $Sp(6,\mathbb{Z})$ modular transformation.

We have focused on degenerated fermion zero-modes with the chirality $(+,+,+)$. Corresponding zero-mode wave functions are normalizable, if $N\mathrm{Im}\Omega$ is positive-definite where N and Ω are symmetric fluxes and complex structure moduli, respectively.

Our results are important to check whether three-generation models in the effective field theory exist or not systematically. We have constructed magnetized $T^6/\mathbb{Z}_N$ twisted orbifolds by generators of $Sp(6,\mathbb{Z})$ and symmetric flux N. By modular transformation of zero-modes and its representation, we can classify how many zero-modes with $\mathbb{Z}_N$ charges exist on each sector.

For $T^6/\mathbb{Z}_7$, we have not found any three-generation models when $\mathrm{tr}\,\rho$ is equal to either one of $+1$, -1, or $-\sqrt{7}i$. This result can be proved by properties of number theory. Therefore, if $\mathrm{tr}\ \rho$ could take no other values than the above ones, it implies that the magnetized $T^6/\mathbb{Z}_7$ model is not suitable for realizing three-generation models in four-dimensional effective field theory. In addition, we have never found any solutions when the models have no tachyonic-modes.

For $T^6/\mathbb{Z}_{12}$, on the other hand, we have found some three-generation models. The F-term condition is satisfied in three-generation models when a determinant of flux N is from 12 to 48. However, the D-term SUSY condition is satisfied when $\det N$ is only 12 or 36 and N are particular values. It means that there are three-generation models without tachyonic-modes.

In conclusion, we have seen that a few three-generation models can be realized on magnetized nonfactorizable $T^6/\mathbb{Z}_N$ orbifolds. We have only used zero-modes with the chirality $(+,+,+)$, but there are zero-modes with other chiralities such as $(+,-,-)$. Zero-modes with other chiralities have different wave functions, and such analysis is nontrivial. We need those wave functions to write Yukawa couplings in 4D low-energy effective field theory. Such studies on other chiralities are beyond our scope, and we would study them elsewhere including realization of fermion masses and mixing angles.

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APPENDIX A: $T^4/\mathbb{Z}_N$

In this appendix, we study nonfactorizable $T^4/\mathbb{Z}_N$ orbifolds with background magnetic flux.

1. $\mathbb{Z}_N$ twists for $T^4/\mathbb{Z}_N$

The 4D lattices have the modular symmetry, $Sp(4,\mathbb{Z})$. Some of the Coxeter and generalized Coxeter elements can be realized by $Sp(4,\mathbb{Z})$ transformation.

Generators of $Sp(4,\mathbb{Z})$ are given by

$$S = \begin{bmatrix} O & I_2 \\ -I_2 & O \end{bmatrix}, \quad T_i = \begin{bmatrix} I_2 & B_i \\ O & I_2 \end{bmatrix} \quad (i = 1,2,3), \tag{A1}$$

where B_i are 2×2 symmetric matrices defined by

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{A2}$$

Referring to Table I, we may expect to realize the $\mathbb{Z}_6$, $\mathbb{Z}_8$, and $\mathbb{Z}_{12}$ orbifolds with the $SO(8)$ Lie root lattice. However, we succeed in describing only the $\mathbb{Z}_8$ in terms of $Sp(4,\mathbb{Z})$ as discussed in the following subsections.

2. $T^4/\mathbb{Z}_8$ orbifold

We consider the number of zero-modes on magnetized $T^4/\mathbb{Z}_8$ [28] by the following algebraic relation

$$(ST_1T_2^{-1}T_3^{-1})^8 = \mathbf{1}_4. \tag{A3}$$

The invariant moduli Ω_8 under the transformation $ST_1T_2^{-1}T_3^{-1}$ satisfy

$$-(\Omega_8 + B_1 - B_2 - B_3)^{-1} = \Omega_8. \tag{A4}$$

One of the solutions $\Omega_8 \in \mathcal{H}_2$ is given by

$$\Omega_8 = \begin{bmatrix} -\frac{1}{2} + \frac{i}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} + \frac{i}{\sqrt{2}} \end{bmatrix}. \tag{A5}$$

Flux N is constrained by the F-term condition to the form

$$N = \begin{bmatrix} n_1 & \frac{n_2 - n_1}{2} \\ \frac{n_2 - n_1}{2} & n_2 \end{bmatrix}, \qquad n_1 \equiv n_2 \pmod{2}. \tag{A6}$$

Therefore, N is $ST_1T_2^{-1}T_3^{-1}$ invariant. Furthermore, for the consistent transformation of zero-mode wave functions under $T_1T_2^{-1}T_3^{-1}$, additional constraints are imposed. That is, $N(B_1 - B_2 - B_3)$ is symmetric, and its diagonal elements are all even. This leads to

$$n_1 \equiv n_2 \equiv 0 \quad \text{or} \quad n_1 \equiv n_2 \equiv 2 \pmod{4}. \tag{A7}$$

Then, $\det N$ is always a multiple of 4. We obtain the following $ST_1T_2^{-1}T_3^{-1}$ transformation of zero-modes:

$$\psi_N^{\vec{J}}(\Omega_8\vec{z}, \Omega_8) = \frac{1}{\sqrt{\det N}} \sum_{\vec{K} \in \Lambda_N} e^{2\pi i \vec{J}^T N^{-1} \vec{K}} e^{\pi i \vec{K}^T N^{-1} B \vec{K}} \psi_N^{\vec{K}}(\vec{z}, \Omega_8), \tag{A8}$$

where $B_1 - B_2 - B_3$ is denoted by B. We define Λ_N as a lattice spanned by $N\vec{e}_i$. The representation of the algebraic structure $G = ST_1T_2^{-1}T_3^{-1}$ is given by

$$\rho_{\vec{K}_1 \vec{K}_2}(G) = \frac{1}{\sqrt{\det N}} e^{2\pi i \vec{K}_1^T N^{-1} \vec{K}_2} e^{\pi i \vec{K}_1^T N^{-1} B \vec{K}_1}. \tag{A9}$$

Trace of the representation ρ is

$$\mathrm{tr}\rho = \frac{1}{\sqrt{\det N}} \sum_{\vec{K} \in \Lambda_N} e^{2\pi i \vec{K}^T N^{-1} \vec{K}} e^{\pi i \vec{K}^T N^{-1} B \vec{K}}. \tag{A10}$$

Thus, we can obtain the zero-modes on magnetized $T^4/\mathbb{Z}_8$ that have $\mathbb{Z}_N$ charges from $\beta^n = \mathrm{tr}\rho(G^n)$, and $n = 1$, 2, 4. Table IV shows the zero-modes with $\det N$, $\mathrm{tr}\rho(G^n)$, the number of zero-modes in $\mathbb{Z}_N$ sector n_k. We see that there are three-generation models in the range of $16 \leq \det N \leq 32$.

TABLE IV. The number of zero-modes on magnetized $T^4/\mathbb{Z}_8$.

D	β^1	β^2	β^4	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
4	0	0	4	1	0	1	0	1	0	1	0
8	$\sqrt{2}i$	2	4	2	1	1	1	2	0	1	0
16	$\sqrt{2}i$	2	4	$\boxed{3}$	2	2	2	$\boxed{3}$	1	2	1
28	0	0	4	4	$\boxed{3}$	4	$\boxed{3}$	4	$\boxed{3}$	4	$\boxed{3}$
32	$\sqrt{2}i$	2	4	5	4	4	4	5	$\boxed{3}$	4	$\boxed{3}$
36	0	0	4	5	4	5	4	5	4	5	4

3. $T^4/\mathbb{Z}_{12}$ orbifold

The generalized Coxeter element $\mathbb{Z}_{12}$ on $SO(8)$ has the negative determinant. That is, such an element is not included in $Sp(4, \mathbb{Z})$. Any nonvanishing magnetic fluxes are not consistent with the $\mathbb{Z}_{12}$ twist. Thus, one cannot describe the $\mathbb{Z}_{12}$ twist by the $Sp(4, \mathbb{Z})$ transformation.

4. $T^4/\mathbb{Z}_6$ orbifold

Here we consider the $T^4/\mathbb{Z}_6$ orbifold defined by the Coxeter element $\mathbb{Z}_6$ on $SO(8)$. We have not succeeded in finding the corresponding $Sp(4, \mathbb{Z})$ generator. We discuss possible reasons behind this.

First, note that classifications of the fixed points of $Sp(4, \mathbb{Z})$ were studied [45]. There are six independent zero-dimensional fixed points $\Omega_f \in \mathcal{H}_2$ being invariant under the actions of certain subgroups (stabilizer) of $Sp(4, \mathbb{Z})$.

One of the fixed points is given by

$$\Omega_f = \begin{bmatrix} \eta & \frac{1}{2}(\eta - 1) \\ \frac{1}{2}(\eta - 1) & \eta \end{bmatrix} \in \mathcal{H}_2, \tag{A11}$$

where $\eta = \frac{1}{3}(1 + 2\sqrt{2}i)$. The corresponding stabilizer group is generated by [40,41]

$$h_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad h_2 = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}. \tag{A12}$$

They form a finite group of order 48 known as $GL(2, 3)$. One can find that Ω_f in Eq. (A11) is equivalent to Ω_8 in Eq. (A5) corresponding to the $SO(8)$ lattice. They are related by the following $Sp(4, \mathbb{Z})$ transformation:

$$\gamma\Omega_8 = \Omega_f, \qquad \gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in Sp(4, \mathbb{Z}). \tag{A13}$$

This shows that Ω_f also corresponds to the same $SO(8)$ lattice, but with a different basis choice related by $Sp(4,\mathbb{Z})$.

Then the stabilizer of Ω_8 is also $GL(2,3)$, and its representation matrices are given by the matrix conjugation $\gamma^{-1}h_i\gamma$ $(i=1,\ 2)$. The $\mathbb{Z}_6$ generator of the Coxeter element should be included in this stabilizer group if it is describable by $Sp(4,\mathbb{Z})$ generators. However, by the following argument we conclude that there is no such element.

One can compute the values of trace of all 48 elements in $GL(2,3)$ generated by h_i $(i=1,\ 2)$ in Eq. (A12). One obtains only even numbers.

On the other hand, the $\mathbb{Z}_6$ Coxeter element of $SO(8)$ is given by [6]

$$\theta_{\mathbb{Z}_6}=\begin{bmatrix}0&1&0&0\\1&1&1&1\\-1&-1&-1&0\\-1&-1&0&-1\end{bmatrix},\tag{A14}$$

and its trace is odd, $\mathrm{tr}\theta_{\mathbb{Z}_6}=-1$. Note that the above matrix representation Eq. (A14) assumes a different basis choice compared with Eq. (A11) although they span the same $SO(8)$ Lie lattice. If $\theta_{\mathbb{Z}_6}$ has an expression in terms of $Sp(4,\mathbb{Z})$, there exists $L\in GL(4,\mathbb{Z})$ such that

$$\theta_{\mathbb{Z}_6}\to\theta'_{\mathbb{Z}_6}=L^{-1}\theta_{\mathbb{Z}_6}L,\qquad L\in GL(4,\mathbb{Z}),\tag{A15}$$

which corresponds to a possible change of basis vectors. Since the trace is independent of the basis choice, we suspect that the $\mathbb{Z}_6$ Coxeter element of $SO(8)$ cannot be realized by the $Sp(4,\mathbb{Z})$ transformation.

APPENDIX B: MODULAR TRANSFORMATION

We consider modular transformation $\gamma\in Sp(6,\mathbb{Z})$. It is defined as the basis transformation of the lattice Λ defining $T^6\simeq\mathbb{C}^3/\Lambda$. The symplectic modular group $Sp(6,\mathbb{Z})$ is composed of 6×6 integer matrices γ,

$$\gamma=\begin{bmatrix}A&B\\C&D\end{bmatrix}\in Sp(6,\mathbb{Z}),\tag{B1}$$

satisfying

$$\gamma J\gamma^T=J,\qquad J=\begin{bmatrix}0&\mathbf{1}_3\\-\mathbf{1}_3&0\end{bmatrix}.\tag{B2}$$

We introduce the modular transformation γ for the complex coordinates $\vec{z}$ and the complex structure moduli Ω under γ as follows:

$$\begin{aligned}&\Omega\to(A\Omega+B)(C\Omega+D)^{-1},\\&\vec{z}\to(C\Omega+D)^{-1T}\vec{z},\end{aligned}\tag{B3}$$

where A, B, C, and D are 3×3 integer matrices and the generators S, T_i $(i=1,2,\ldots,5,6)$ are given by

$$S=\begin{bmatrix}O&\mathbf{1}_3\\-\mathbf{1}_3&O\end{bmatrix},\qquad T_i=\begin{bmatrix}\mathbf{1}_3&B_i\\O&\mathbf{1}_3\end{bmatrix}.\tag{B4}$$

The symmetric matrices B_i are given by

$$\begin{aligned}&B_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad B_2=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad B_3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\\&B_4=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\quad B_5=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad B_6=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}.\end{aligned}\tag{B5}$$

We will consider modular S and T_i transformations of zero-modes on magnetized T^6.

1. The S transformation

Under the S transformation, $\vec{z}=\vec{x}+\Omega\vec{y}$ and Ω behave as

$$S:\ (\vec{z},\Omega)\to(\vec{z}_S,\Omega_S)=(-\Omega^{-1}\vec{z},-\Omega^{-1}).\tag{B6}$$

We see that the complex coordinates, the moduli after the S transformation, are given by $\vec{z}_S=-\Omega^{-1}\vec{z}=\vec{x}_S-\Omega^{-1}\vec{y}_S$ and $\Omega_S=-\Omega^{-1}$. Also we obtain the transformation of real coordinates $\vec{x}_S$ and $\vec{y}_S$ as

$$\begin{aligned}&\vec{x}_S=-\vec{y},\\&\vec{y}_S=\vec{x}.\end{aligned}\tag{B7}$$

a. Magnetic flux and the F-term condition in the S transformation

Magnetic flux on T^6 is defined by

$$\begin{aligned}F&=\frac{1}{2}p_{IJ}dX^I\wedge dX^J\\&=\frac{1}{2}(p_{xx})_{ij}dx^i\wedge dx^j+\frac{1}{2}(p_{yy})_{ij}dy^i\wedge dy^j\\&\quad+(p_{xy})_{ij}dx^i\wedge dy^j,\end{aligned}\tag{B8}$$

where $X^I=(x^i,y^i)$, $i=1,2,3$ is the real coordinate along the lattice. Therefore the magnetic flux F after the S transformation is written by $\vec{x}_S=-\vec{y}$ and $\vec{y}_S=\vec{x}$:

$$\begin{aligned} F &= \frac{1}{2}(p^S_{xx})_{ij}dx^i_S \wedge dx^j_S + \frac{1}{2}(p^S_{yy})_{ij}dy^i_S \wedge dy^j_S \\ &\quad + (p^S_{xy})_{ij}dx^i_S \wedge dy^j_S \\ &= \frac{1}{2}(p^S_{xx})_{ij}dy^i \wedge dy^j + \frac{1}{2}(p^S_{yy})_{ij}dx^i \wedge dx^j \\ &\quad + (p^S_{xy})^T_{ij}dx^i \wedge dy^j. \end{aligned} \tag{B9}$$

We therefore find that

$$\begin{aligned} p^S_{xx} &= p_{yy}, \\ p^S_{yy} &= p_{xx}, \\ p^S_{xy} &= (p_{xy})^T. \end{aligned} \tag{B10}$$

When we impose the condition $p_{xx} = p_{yy} = 0$, it is clear that the magnetic flux is consistent with the S transformation. Thus the flux $N = \frac{p^T_{xy}}{2\pi}$ is transformed under S as

$$S\colon N \to N^T, \tag{B11}$$

and the F-term condition $(N\Omega)^T = N\Omega$ in the S transformation is given by

$$\begin{aligned} (N_S\Omega_S)^T &= (N^T(-\Omega^{-1}))^T \\ &= ((\Omega^T)^{-1}N\Omega(-\Omega^{-1}))^T \\ &= N_S\Omega_S, \end{aligned} \tag{B12}$$

where we use $(N\Omega)^T = N\Omega$ and the condition that Ω is symmetric. Therefore, we find that the F-term condition as well as the magnetic flux is consistent with the S transformation.

b. The S transformation of zero-modes

Zero-mode with the chirality $(+,+,+)$ is introduced by the Riemann-theta function with characteristics

$$\psi^{\vec{J}}_N(\vec{z},\Omega) = N \cdot e^{i\pi(N\vec{z})^T(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}(\vec{z})} \cdot \theta\begin{bmatrix} \vec{J}N^{-1} \\ 0 \end{bmatrix}(N\vec{z}, N\Omega). \tag{B13}$$

Since fluxes N in our magnetized orbifold models are constrained to be symmetric, we consider the S transformation under the condition $N = N^T$.

To come to the point, the S transformation of zero-modes $\psi^{\vec{J}}_N(\vec{z},\Omega)$ is written as

$$\begin{aligned} \psi^{\vec{J}}_N(-\Omega^{-1}\vec{z}, -\Omega^{-1}) &= \sqrt{\det(N^{-1}\Omega/i)} \\ &\quad \times \sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{J}^TN^{-1}\vec{K}}\psi^{\vec{K}}_N(\vec{z},\Omega) \\ &= \frac{\sqrt{\det(-i\Omega)}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{J}^TN^{-1}\vec{K}}\psi^{\vec{K}}_N(\vec{z},\Omega), \end{aligned} \tag{B14}$$

where Λ_N is a lattice spanned by $N\vec{e}_i$ and $\vec{e}_i$ are unit vectors.

In the following, we present a derivation of Eq. (B14). From the literature in mathematics [34], it is known that

$$\theta(-\Omega^{-1}\vec{z}, -\Omega^{-1}) = \sqrt{\det(\Omega/i)}e^{\pi i\vec{z}^T\Omega^{-1}\vec{z}} \cdot \theta(\vec{z},\Omega) \tag{B15}$$

holds where $\vec{z} \in \mathbb{C}^3, \Omega \in \mathcal{H}_3$. Note that we must take a branch of the square root which returns a positive number if Ω is purely imaginary.

When we replace $\vec{z}$ with $\vec{z} + N^{-1}\vec{J}$ in Eq. (B15), we obtain

$$\begin{aligned} &\theta\begin{bmatrix} \vec{J}^TN^{-1} \\ \vec{0} \end{bmatrix}(-\Omega^{-1}\vec{z}, -\Omega^{-1}) \\ &= \sqrt{\det(\Omega/i)} \cdot e^{i\pi\vec{z}^T\Omega^{-1}\vec{z}} \cdot \theta\begin{bmatrix} \vec{0} \\ \vec{J}^TN^{-1} \end{bmatrix}(\vec{z},\Omega). \end{aligned} \tag{B16}$$

The following relations were used:

$$\begin{aligned} &\theta\begin{bmatrix} \vec{a} \\ \vec{b}+\vec{b}' \end{bmatrix}(\vec{z},\Omega) = \theta\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(\vec{z}+\vec{b}',\Omega), \\ &\theta\begin{bmatrix} \vec{a}+\vec{a}' \\ \vec{b} \end{bmatrix}(\vec{z},\Omega) = e^{i\pi\vec{a}'^T\Omega\vec{a}'+2\pi i\vec{a}'^T(\vec{z}+\vec{b})}\theta\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}(\vec{z}+\Omega\vec{a}',\Omega), \end{aligned} \tag{B17}$$

where $\vec{a}', \vec{b}' \in \mathbb{R}^3$.

Now we replace $\Omega \in \mathcal{H}_3$ with $N^{-1}\Omega_I$ where N is to be identified as the flux in magnetized D-brane models. Even if Ω_I is not an element of Siegel upper-half plane $\mathcal{H}_3$, our replacement is consistent if $N^{-1}\Omega_I \in \mathcal{H}_3$. In the following, since we denote Ω_I by Ω, we have

$$\begin{aligned} &\theta\begin{bmatrix} \vec{J}^TN^{-1} \\ \vec{0} \end{bmatrix}(-\Omega^{-1}N\vec{z}, -\Omega^{-1}N) \\ &= \sqrt{\det(N^{-1}\Omega/i)} \cdot e^{i\pi\vec{z}^T(N^{-1}\Omega)^{-1}\vec{z}} \cdot \theta\begin{bmatrix} \vec{0} \\ \vec{J}^TN^{-1} \end{bmatrix}(\vec{z}, N^{-1}\Omega). \end{aligned} \tag{B18}$$

The right-hand side of Eq. (B18) can be rewritten as

$$\begin{aligned}
&\theta\begin{bmatrix}\vec{0}\\ \vec{J}^T N^{-1}\end{bmatrix}(\vec{z}, N^{-1}\Omega)\\
&= \sum_{\vec{\ell}\in\mathbb{Z}^3} e^{\pi i \vec{\ell}^T N^{-1}\Omega\vec{\ell}} e^{2\pi i \vec{\ell}^T(\vec{z}+N^{-1}\vec{J})}\\
&= \sum_{\vec{K}\in\Lambda_N}\sum_{\vec{a}\in\mathbb{Z}^3} e^{\pi i (N\vec{a}+\vec{K})^T N^{-1}\Omega(N\vec{a}+\vec{K})} e^{2\pi i (N\vec{a}+\vec{K})^T(\vec{z}+N^{-1}\vec{J})}\\
&= \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{K}^T N^{-1}\vec{J}} \sum_{\vec{a}\in\mathbb{Z}^3} e^{\pi i (N\vec{a}+\vec{K})^T N^{-1}\Omega(N\vec{a}+\vec{K})} e^{2\pi i (N\vec{a}+\vec{K})^T\vec{z}}\\
&= \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{J}^T N^{-1}\vec{K}} \theta\begin{bmatrix}\vec{K}^T N^{-1}\\ \vec{0}\end{bmatrix}(N\vec{z}, N\Omega),
\end{aligned} \tag{B19}$$

where the summation variable $\vec{l}$ is decomposed into two variables $\vec{a}\in\mathbb{Z}^3$ and $\vec{K}\in\mathbb{Z}^3$ as $\vec{l} = N\vec{a}+\vec{K}$. Note that $\vec{K}$ are integer points inside the lattice Λ_N.

Also since we know the F-term condition $(N\Omega)^T = N\Omega$ as well as the symmetries $N^T = N$, $\Omega^T = \Omega$, we find the relation $[N,\Omega] = 0$. From this relation, $\Omega^{-1}[N,\Omega]\Omega^{-1} = 0$, and thus N and Ω^{-1} are also commutative.

We therefore find that Eq. (B18) can be expressed as

$$\begin{aligned}
&\theta\begin{bmatrix}\vec{J}^T N^{-1}\\ \vec{0}\end{bmatrix}(N(-\Omega^{-1}\vec{z}), N(-\Omega^{-1}))\\
&= \sqrt{\det(N^{-1}\Omega/i)}\cdot e^{i\pi\vec{z}^T(N^{-1}\Omega)^{-1}\vec{z}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{J}^T N^{-1}\vec{K}}\\
&\quad\cdot\theta\begin{bmatrix}\vec{K}^T N^{-1}\\ \vec{0}\end{bmatrix}(N\vec{z}, N\Omega).
\end{aligned} \tag{B20}$$

Next, when we focus on the S transformation on the phase of zero-modes, we find

$$S\colon\ e^{i\pi(N\vec{z})^T\cdot(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\vec{z})} \to e^{\pi i(-N\Omega^{-1}\vec{z})^T(\mathrm{Im}(-\Omega^{-1}))^{-1}\mathrm{Im}(-\Omega^{-1}\vec{z})}. \tag{B21}$$

Then, when this is multiplied by the exponential factor $e^{i\pi\vec{z}^T(N^{-1}\Omega)^{-1}\vec{z}}$ in the right-hand side of Eq. (B20), we get

$$\begin{aligned}
&e^{\pi i(-N\Omega^{-1}\vec{z})^T(\mathrm{Im}(-\Omega^{-1}))^{-1}\mathrm{Im}(-\Omega^{-1}\vec{z})} e^{i\pi\vec{z}^T(N^{-1}\Omega)^{-1}\vec{z}}\\
&= e^{i\pi(N\vec{z})^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}.
\end{aligned} \tag{B22}$$

Finally, we find the S transformation of zero-modes as follows:

$$\psi_N^{\vec{J}}(-\Omega^{-1}\vec{z}, -\Omega^{-1}) = \frac{\sqrt{\det(-i\Omega)}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{J}^T N^{-1}\vec{K}} \psi_N^{\vec{K}}(\vec{z},\Omega). \tag{B23}$$

2. The T transformation of zero-modes

Under the T_i $(i = 1,2,\ldots,5,6)$ transformation, complex coordinates $\vec{z}$ and complex structure moduli Ω are transformed as

$$T\colon\ (\vec{z},\Omega) \to (\vec{z}_T, \Omega_T) = (\vec{z}, \Omega + B_i). \tag{B24}$$

Let $\vec{z}_T$ represent complex coordinates after the T transformation, and we omit the index i in this subsection. We obtain transformation of real coordinates,

$$\begin{aligned}
&\vec{z} = \vec{x} + \Omega\vec{y} \to \vec{z}_T = \vec{x}_T + (\Omega + B)\vec{y}_T = \vec{z} = \vec{x} + \Omega\vec{y}\\
&\leftrightarrow \vec{x}_T = \vec{x} - B\vec{y}, \qquad \vec{y}_T = \vec{y}.
\end{aligned} \tag{B25}$$

Therefore, the magnetic flux F after the T transformation is

$$\begin{aligned}
F &= \frac{1}{2}(p_{xx}^{(T)})_{ij} dx_T^i \wedge dx_T^j + \frac{1}{2}(p_{yy}^{(T)})_{ij} dy_T^i \wedge dy_T^j\\
&\quad + (p_{xy}^{(T)})_{ij} dx_T^i \wedge dy_T^j\\
&= \frac{1}{2}(p_{xx}^{(T)})_{ij} dx^i \wedge dx^j + \frac{1}{2}[p_{yy}^{(T)} + Bp_{xx}^{(T)}B - (Bp_{xy}^{(T)})\\
&\quad + (Bp_{xy}^{(T)})^T]_{ij} dy^i \wedge dy^j + [(p_{xy}^{(T)}) - (p_{xx}^{(T)}B)]_{ij} dx^i\\
&\quad \wedge dy_j.
\end{aligned} \tag{B26}$$

From Eq. (B26), we can see the following transformation of the components p_{xx}, p_{yy}, and p_{xy}:

$$\begin{aligned}
p_{xx}^{(T)} &= p_{xx},\\
p_{yy}^{(T)} &= p_{yy} + Bp_{xx}B + (Bp_{xy} - (Bp_{xy}^{(T)})^T),\\
p_{xy}^{(T)} &= p_{xy} - (p_{xx}B).
\end{aligned} \tag{B27}$$

Thus, we find that the following constraint is required for the condition $p_{xx} = p_{yy} = 0$ to be consistent with the T transformation,

$$(Bp_{xy})^T = Bp_{xy}. \tag{B28}$$

Noting Dirac's quantization condition $p_{xy} = 2\pi N^T$, and the fact that matrices B in $Sp(6,\mathbb{Z})$ are symmetric, we can write Eq. (B28) as

$$(NB)^T = NB. \tag{B29}$$

Then it follows that the F-term condition is consistent with the T transformation

$$\begin{aligned}(N_T\Omega_T)^T &= (\Omega+B)^T N^T \\ &= (N\Omega)^T + (NB)^T \\ &= N\Omega + NB \\ &= N_T\Omega_T. \end{aligned} \tag{B30}$$

Next we consider the T transformation of zero-modes. In the following, we require that all diagonal components of NB are even, and then we obtain

$$\begin{aligned}&\theta(N\vec{z}, N(\Omega+B)) \\ &= \sum_{\vec{m}\in\mathbb{Z}^3} \exp(\pi i\vec{m}^T N(\Omega+B)\vec{m} + 2\pi i\vec{m}^T N\vec{z}) \\ &= \sum_{\vec{m}\in\mathbb{Z}^3} e^{\pi i\vec{m}^T N\Omega\vec{m}+2\pi i\vec{m}^T N\vec{z}} e^{\pi i\vec{m}^T NB\vec{m}} \\ &= \sum_{\vec{m}\in\mathbb{Z}^3} e^{\pi i\vec{m}^T N\Omega\vec{m}+2\pi i\vec{m}^T N\vec{z}} \cdot 1 \\ &= \theta(N\vec{z}, N\Omega). \end{aligned} \tag{B31}$$

Here, when we replace complex coordinates $\vec{z}$ with $\vec{z} + (\Omega+B)N^{-1T}\vec{J}$, we obtain the following formula:

$$\begin{aligned}&\theta(N(\vec{z} + (\Omega+B)N^{-1T}\vec{J}), N(\Omega+B)) \\ &\quad = \theta(N(\vec{z} + (\Omega+B)N^{-1T}\vec{J}), N\Omega). \end{aligned} \tag{B32}$$

Then, by the use of Eq. (B17), we can express it as

$$\begin{aligned}&\theta\begin{bmatrix}\vec{J}^T N^{-1} \\ \vec{0}\end{bmatrix}(N\vec{z}, N(\Omega+B)) \\ &= e^{-\pi i\vec{J}^T N^{-1}B\vec{J}}\theta\begin{bmatrix}\vec{J}^T N^{-1} \\ \vec{0}\end{bmatrix}(N\vec{z}+B\vec{J}, N\Omega) \\ &= e^{-\pi i\vec{J}^T N^{-1}B\vec{J}}\theta\begin{bmatrix}\vec{J}^T N^{-1} \\ \vec{J}^T B\end{bmatrix}(N\vec{z}, N\Omega) \\ &= e^{\pi i\vec{J}^T N^{-1}B\vec{J}}\theta\begin{bmatrix}\vec{J}^T N^{-1} \\ \vec{0}\end{bmatrix}(N\vec{z}, N\Omega). \end{aligned} \tag{B33}$$

The phase factor $e^{\pi i(N\vec{z})^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}$ of zero-modes is clearly invariant under the T transformation. Therefore, we find the T transformation of zero-modes as follows:

$$\psi_N^{\vec{J}}(\vec{z}, \Omega+B) = e^{\pi i\vec{J}^T N^{-1}B\vec{J}}\psi_N^{\vec{J}}(\vec{z}, \Omega). \tag{B34}$$

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