

Sum-rules analysis of next-to-leading-order QCD perturbative contributions to a $J^{PC} = 0^{+-}$, $du\bar{d}\bar{u}$ tetraquark correlator

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We calculated next-to-leading-order (NLO) QCD perturbative contributions to a $J^{PC} = 0^{+-}$, $du\bar{d}\bar{u}$ tetraquark (diquark-antidiquark) correlator in the chiral limit of massless u and d quarks. At NLO, there are four quark self-energy diagrams and six gluon-exchange diagrams. Nonlocal divergences were canceled using diagrammatic renormalization. Dimensionally regularized integrals were numerically computed using pySecDec. The combination of pySecDec with diagrammatic renormalization establishes a valuable new methodology for NLO calculations of QCD correlation functions. Compared to leading-order (LO) perturbation theory, we found that NLO perturbation theory is significant. To quantify the impact of NLO perturbation theory on physical predictions, we computed NLO perturbative contributions to QCD Laplace, Gaussian, and finite-energy sum rules. Using QCD sum rules, we determined upper and lower bounds on the 0^{+-} , $du\bar{d}\bar{u}$ tetraquark ground-state mass, M ; at NLO in perturbation theory, we found $2.2 \text{ GeV} \lesssim M \leq 4.2 \text{ GeV}$ whereas, at LO, we found $2.4 \text{ GeV} \lesssim M \leq 4.6 \text{ GeV}$. This mass range suggests the possibility of mixing between 0^{+-} , light-quark (i.e., u and d quarks) hybrid and $du\bar{d}\bar{u}$ tetraquark states. Taking into account uncertainties in QCD parameters, we found no evidence for a 0^{+-} , $du\bar{d}\bar{u}$ tetraquark under 1.9 GeV .

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I. INTRODUCTION

Color confinement allows for hadron families beyond the well-known two-quark mesons and three-quark baryons [1]. Four-quark states are one such family (see e.g., [2,3]). Experimental evidence for the existence of four-quark states is strong (see, for example, the reviews [4–10]). Several states containing (a minimum of) four quarks have been reported including the $Z_c(3900)^+$ with quark content $cu\bar{c}\bar{d}$ [11,12], the $X(5568)^+$ with quark content $sub\bar{d}$ [13,14], and the $X(6900)$ with quark content $cc\bar{c}\bar{c}$ [15].

While all known manifestly four-quark states contain at least one heavy quark, a four-quark framework (with an inverted mass hierarchy for the scalar mesons) is also expected in systems without heavy quarks [2,16]. However, clearly identifying four-quark states that do not contain heavy quarks is difficult due in part to overlapping hadron multiplets and, presumably, hadron mixing. Given these difficulties, a promising search strategy is to look for

bosonic hadrons with exotic quantum numbers, i.e., J^{PC} combinations such as 0^{--} , 0^{+-} , and 1^{-+} that are forbidden for two-quark mesons. While not guaranteed to be four-quark states (e.g., hybrid mesons can have exotic J^{PC} [5,7,10]), hadrons with exotic quantum numbers are at least guaranteed to not be two-quark mesons.

One possible picture for the structure of a four-quark state is that of a diquark-antidiquark bound state, i.e., a tetraquark [2,3,16]. Tetraquarks with exotic quantum numbers that do not contain heavy quarks have been studied using QCD sum rules [17–22].¹ QCD sum rules are transformed dispersion relations that relate hadron properties to QCD correlation functions of interpolating currents [26–31]. Variants include Laplace, Gaussian, and finite-energy sum rules. In [17], an analysis of light-quark (i.e., u and d quarks) and hidden-strange, isovector tetraquarks with $J^{PC} = 1^{-+}$ yielded mass predictions of about 1.6 GeV for $qq\bar{q}\bar{q}$ states and 2.0 GeV for $qs\bar{q}\bar{s}$ states where q represents u or d . However, for many of the currents considered, the QCD spectral functions (i.e., imaginary parts) of corresponding correlators were negative (i.e., unphysical) in the squared-energy scale, t , range $1 \text{ GeV}^2 \lesssim t \lesssim 4 \text{ GeV}^2$. For $t \gtrsim 4 \text{ GeV}^2$, the QCD spectral functions were positive (and hence physical), but the

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¹They have also been studied with the MIT bag model [23] and a Coloumb gauge QCD Hamiltonian model [24,25].

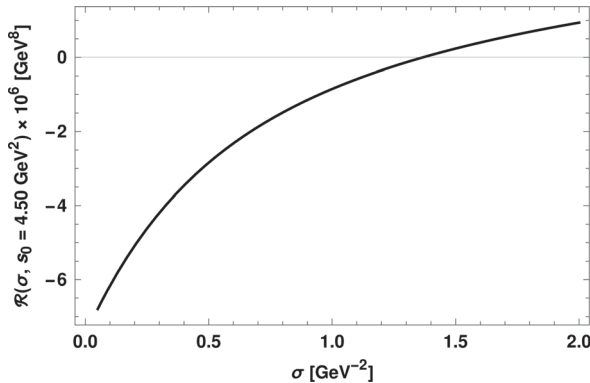
hadron masses obtained were greater than 2.5 GeV. As such, Ref. [17] concluded that these currents did not provide evidence for the existence of 1^{-+} tetraquarks under 2 GeV. In [18], an analysis of light-quark and hidden-strange, isoscalar tetraquarks with $J^{PC} = 1^{-+}$ yielded a $qs\bar{q}\bar{s}$ state mass prediction of 1.8 GeV–2.1 GeV. However, similar to what was seen in [17], many currents led to correlators with negative QCD spectral functions for $2 \text{ GeV}^2 \lesssim t \lesssim 4 \text{ GeV}^2$. All of the $qq\bar{q}\bar{q}$ currents showed this unphysical behavior, and, consequently, in Ref. [18], no mass predictions were obtained for 1^{-+} , light-quark, isoscalar tetraquarks. In [19], light-quark and hidden-strange, isovector and isoscalar tetraquarks with $J^{PC} = 0^{--}$ were studied using a set of scalar currents, but none of the sum-rules analyses stabilized. In [21], 0^{--} , light-quark tetraquark states were studied using a set of vector currents which yielded isospin-degenerate mass predictions of $(1.66 \pm 0.14) \text{ GeV}$. In [20], light-quark tetraquarks (as well as hidden-charm and hidden-bottom tetraquarks) with $J^{PC} = 0^{+-}$ were studied using scalar currents (that contained covariant derivative operators). For the $qq\bar{q}\bar{q}$ states, no sum-rules analyses were successful. In [22], light-quark and hidden-strange tetraquarks with $J^{PC} = 0^{+-}$ were studied using a set of vector currents. For $qq\bar{q}\bar{q}$ tetraquarks, a mass of $(1.43 \pm 0.09) \text{ GeV}$ was reported, and, for $qs\bar{q}\bar{s}$ tetraquarks, a mass of $(1.54 \pm 0.12) \text{ GeV}$ was reported.

Our focus in this paper is 0^{+-} , $dud\bar{u}$ tetraquarks, denoted as $T_{dud\bar{u}}^{0^{+-}}$ following the classification scheme of Ref. [32]. In [22], correlation functions of eight interpolating currents were considered at leading order (LO) in perturbation theory. In the chiral limit, correlation functions of these eight currents are pairwise degenerate, corresponding to four independent currents. Of these four, only two were identified as leading to Laplace sum rules (LSRs) with converging operator product expansion (OPE) series, and, of these two, only one led to an LSRs analysis stable

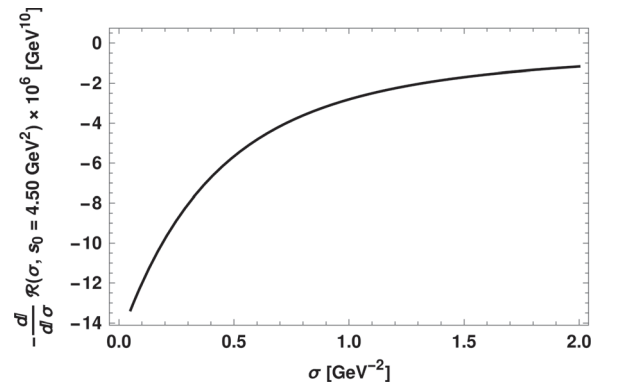
against deviations from vacuum saturation. This current, J_3 (or, equivalently, J_7) in the notation of [22], was analyzed using LSRs resulting in a mass prediction of 1.39 GeV at optimized continuum threshold parameter $s_0 = 4.50 \text{ GeV}^2$ and Borel parameter $\sigma = 0.265 \text{ GeV}^{-2}$ [22]. Because sum rules for two-point functions relate an integrated QCD-predicted $\rho^{(\text{OPE})}(t)$ to an integrated positive-valued hadronic spectral function $\rho(t)$ [see (46)–(47) below], QCD sum rules must be positive to be physically consistent with an integrated hadronic spectral function (see e.g., Refs. [33–36] that use the physical positivity constraint to obtain QCD sum-rule mass bounds on light quarks). However, as can be seen in Fig. 1, the LSRs of J_3 (J_7) from [22] are negative and, therefore, unphysical at the optimized s_0 and σ values; hence, the corresponding mass prediction is not reliable.

Focusing on the current J_7 from [22] (see (1) below), we extended the QCD sum-rules analysis of $T_{dud\bar{u}}^{0^{+-}}$ tetraquarks to include next-to-leading-order (NLO) QCD contributions to perturbation theory. For several 0^{++} , light-quark tetraquark currents, it has been shown that NLO contributions to perturbation theory are surprisingly large [37]. The effects of NLO perturbation theory on a QCD sum-rules analysis of a 0^{++} , light-quark tetraquark current are explored in [38]. It is therefore interesting to study whether light-quark, exotic- J^{PC} tetraquarks have similarly large NLO effects. Furthermore, as the NLO perturbative contributions are necessarily positive, they could potentially fix the negative, unphysical LSRs of Fig. 1.

The NLO diagrams that contribute to the $T_{dud\bar{u}}^{0^{+-}}$ diagonal correlator defined in (2)–(3) below are shown in Fig. 2. Each diagram has four loops and contains nonlocal divergences. Integrals were regulated using dimensional regularization, and nonlocal divergences were eliminated through diagrammatic renormalization [39–42] using the methodology for sum rules developed in Ref. [43].



(a) The order-0 (subtracted) LSR.



(b) The order-1 (subtracted) LSR.

FIG. 1. The order-0 and order-1 (subtracted) LSRs of J_3 (J_7) from [22] at continuum threshold parameter $s_0 = 4.50 \text{ GeV}^2$, the optimized value of s_0 determined in [22]. Both LSRs are negative at Borel parameter $\sigma = 0.265 \text{ GeV}^{-2}$, the optimized value of σ determined in [22]. Note that, in [22], Laplace sum rules are denoted M and the Borel parameter is denoted τ .

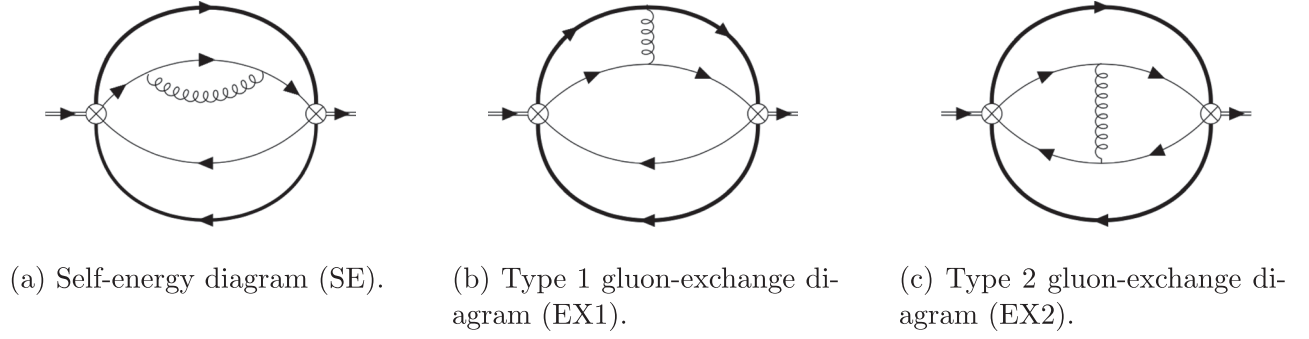


FIG. 2. The NLO perturbative diagrams of $\Pi(q^2)$. The \otimes denotes the Feynman rule for the current (1). Thin lines are u quarks. Thick lines are d quarks.

As discussed in [43], diagrammatic renormalization is particularly convenient for radiative corrections to tetraquark correlation functions as it circumvents the problem of composite-operator mixing under renormalization. Also, it provides helpful consistency checks as nonlocal divergences are eliminated diagram-by-diagram.

Rather than evaluating integrals analytically, we evaluated them using pySecDec, a program that numerically calculates dimensionally regularized integrals [44]. pySecDec makes use of FORM [45–47], GSL [48], and the CUBA library [49,50]. It has been demonstrated that pySecDec can be successfully incorporated into the QCD sum-rules methodology at LO using a 0^{+-} charmonium hybrid current as an example [51]. In this paper, we demonstrate that pySecDec numerical loop-integration methods combined with diagrammatic renormalization techniques can be successfully implemented at NLO, establishing new calculational methods for higher-loop corrections in QCD sum rules.

For the $T_{dud\bar{u}}^{0^{+-}}$ diagonal correlator (2)–(3) below, we found that NLO contributions are large relative to LO perturbation theory. This is similar to what was found for 0^{++} , light-quark tetraquarks [37,38]. To assess the importance of the NLO corrections to physical predictions, we computed NLO perturbative contributions to Laplace, Gaussian, and finite-energy sum rules. Using Gaussian sum rules (GSRs), we motivated a lower bound on the $T_{dud\bar{u}}^{0^{+-}}$ tetraquark ground state mass, M , and using LSRs, we determined an upper bound on M . Omitting NLO perturbation theory, we found that $2.4 \text{ GeV} \lesssim M \leq 4.6 \text{ GeV}$ contrary to the predictions of [22]. The discrepancy is due to our analysis being restricted to positive, physical sum rules. Including both LO and NLO perturbation theory, we found that $2.2 \text{ GeV} \lesssim M \leq 4.2 \text{ GeV}$. With NLO perturbation theory, the resulting mass scale was lowered somewhat, but we still found no evidence for a $T_{dud\bar{u}}^{0^{+-}}$ state lighter than 1.9 GeV , even when taking into account uncertainties in QCD parameters. Furthermore, it is worth noting that this mass range is comparable to the QCD sum-rule mass prediction for 0^{+-} , light-quark hybrids [52], suggesting the possibility of hybrid-tetraquark mixing.

II. NEXT-TO-LEADING-ORDER PERTURBATION THEORY

We investigate $T_{dud\bar{u}}^{0^{+-}}$ tetraquarks using the current

$$J_\mu = u_a^T C d_b (\bar{u}_a \gamma_\mu C \bar{d}_b^T - \bar{u}_b \gamma_\mu C \bar{d}_a^T) - u_a^T C \gamma_\mu d_b (\bar{u}_a C \bar{d}_b^T - \bar{u}_b C \bar{d}_a^T), \quad (1)$$

denoted J_7 in [22], with charge-conjugation operator C , quark color indices a and b , and massless u and d quarks. As discussed in Ref. [22], this current couples to different isospin multiplets, but because our calculations do not include isospin-breaking effects, our conclusions concerning $T_{dud\bar{u}}^{0^{+-}}$ masses are isospin degenerate. The diagonal correlator of (1) is

$$\begin{aligned} \Pi_{\mu\nu}(q) &= i \int d^4x e^{iq \cdot x} \langle \Omega | T j_\mu(x) j_\nu^\dagger(0) | \Omega \rangle \\ &= q_\mu q_\nu \Pi^{(S)}(q^2) + (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi^{(V)}(q^2), \end{aligned} \quad (2)$$

where $\Pi^{(S)}(q^2)$ and $\Pi^{(V)}(q^2)$ probe 0^{+-} and 1^{--} states respectively. We focus on $\Pi^{(S)}(q^2)$ where

$$\Pi^{(S)}(q^2) = \frac{q_\mu q_\nu}{q^4} \Pi_{\mu\nu}(q^2). \quad (3)$$

We omit the superscript “(S)” from $\Pi^{(S)}(q^2)$ from here on. As discussed above, at LO, the LSRs based on this current have good OPE convergence properties and stability under variations in QCD parameter inputs [22]. For the QCD sum-rules analyses of Sec. III, we actually only need the imaginary part of Π ; thus, for convenience, we define

$$\rho(t) = \lim_{\delta \rightarrow 0^+} \frac{\Pi(t + i\delta) - \Pi(t - i\delta)}{2\pi i} = \frac{1}{\pi} \text{Im} \Pi(t).$$

We calculate $\rho(t)$ within the OPE [42,53] in which perturbation theory, $\rho^{(\text{pert})}(t)$, is supplemented by non-perturbative condensate terms, $\rho^{(\text{cond})}(t)$,

$\Rightarrow \Gamma_\mu^{(\text{EX1})} = 2$

(11)

where

$$\Gamma_\mu^{(\text{EX1})} = \frac{g_s^2}{24\pi^2\epsilon} (\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}) \times (4C_{ij}(C\gamma_\mu)_{\ell k} - (\gamma_\mu C)_{ij}C_{\ell k}). \quad (12)$$

The diagram of Fig. 2(c) contains two divergent subdiagrams, again, each comprising the gluon line, a current insertion, and the two quark lines that connect them. We find

$\Rightarrow \Gamma_\mu^{(\text{EX2})} = 3$

(13)

where

$$\Gamma_\mu^{(\text{EX2})} = \frac{g_s^2}{384\pi^2\epsilon} (5\delta^{ad}\delta^{bc} + \delta^{ac}\delta^{bd}) ((\gamma^\rho\gamma^\sigma C)_{ij}(C\gamma_\mu\gamma_\rho\gamma_\sigma)_{\ell k} - (\gamma_\mu\gamma^\rho\gamma^\sigma C)_{ij}(C\gamma_\rho\gamma_\sigma)_{\ell k}). \quad (14)$$

The counterterm diagrams needed to eliminate nonlocal divergences from the diagrams of Fig. 2 are shown in Fig. 3. The self-energy counterterm diagram (SEC) of Fig. 3(a) has a multiplicity of four. The Type 1 gluon-exchange counterterm diagram (EXC1) of Fig. 3(b) has a multiplicity of four, the multiplicity of the diagram of Fig. 2(b) multiplied by two as the counterterm vertex can replace either current insertion. The Type 2 gluon-exchange counterterm diagram (EXC2) of Fig. 3(c) has a multiplicity of eight, the multiplicity of the diagram of Fig. 2(c) multiplied by two.

We denote a particular NLO diagram from Fig. 2 or Fig. 3 with a superscript (A) where $A \in \{\text{SE}, \text{EX1}, \text{EX2}, \text{SEC}, \text{EXC1}, \text{EXC2}\}$. Then, including multiplicities,

$$\rho^{(\text{NLO})}(t) = 4\rho^{(\text{SE})}(t) + 4\rho^{(\text{SEC})}(t) + 2\rho^{(\text{EX1})}(t) + 4\rho^{(\text{EXC1})}(t) + 4\rho^{(\text{EX2})}(t) + 8\rho^{(\text{EXC2})}(t). \quad (15)$$

As the u and d quarks are massless, each $\rho^{(A)}(t)$ takes the form

$$\rho^{(A)}(t) = g_s^2 t^3 \left(\frac{a^{(A)}}{\epsilon} + b^{(A)} + c^{(A)}L \right), \quad (16)$$

where $a^{(A)}$, $b^{(A)}$, and $c^{(A)}$ are constants and where

$$L = \log\left(\frac{t}{\mu^2}\right). \quad (17)$$

Using pySecDec, we numerically evaluated the imaginary parts of all six NLO diagrams (excluding the g_s^2 factors) over a range of values of t at $\mu = 1$ GeV. The values of $a^{(A)}$ were easily identified as the coefficients of ϵ^{-1} in the resulting data. We extracted values of $b^{(A)}$ and $c^{(A)}$ by fitting the finite parts of (16), i.e., the terms free of ϵ , to the coefficients of ϵ^0 in the data. As a benchmark of our methodology, we first successfully reproduced (6) using this pySecDec method before calculating NLO corrections. In Fig. 4, for each NLO diagram, we plot the fitted finite part of (16) along with the coefficient of ϵ^0 in the

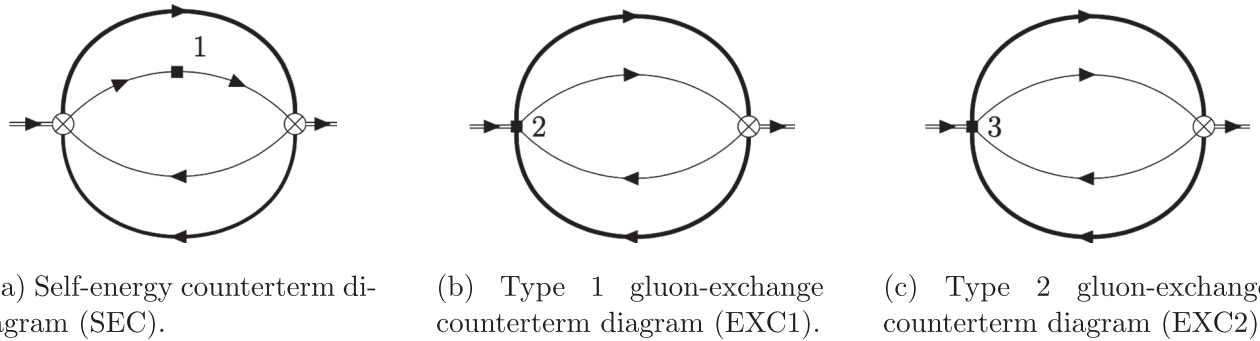


FIG. 3. The counterterm diagrams needed to eliminate nonlocal divergences from the diagrams of Fig. 2. The \otimes denotes the Feynman rule for the current (1). The \blacksquare denotes a counterterm vertex. Thin lines are u quarks. Thick lines are d quarks.

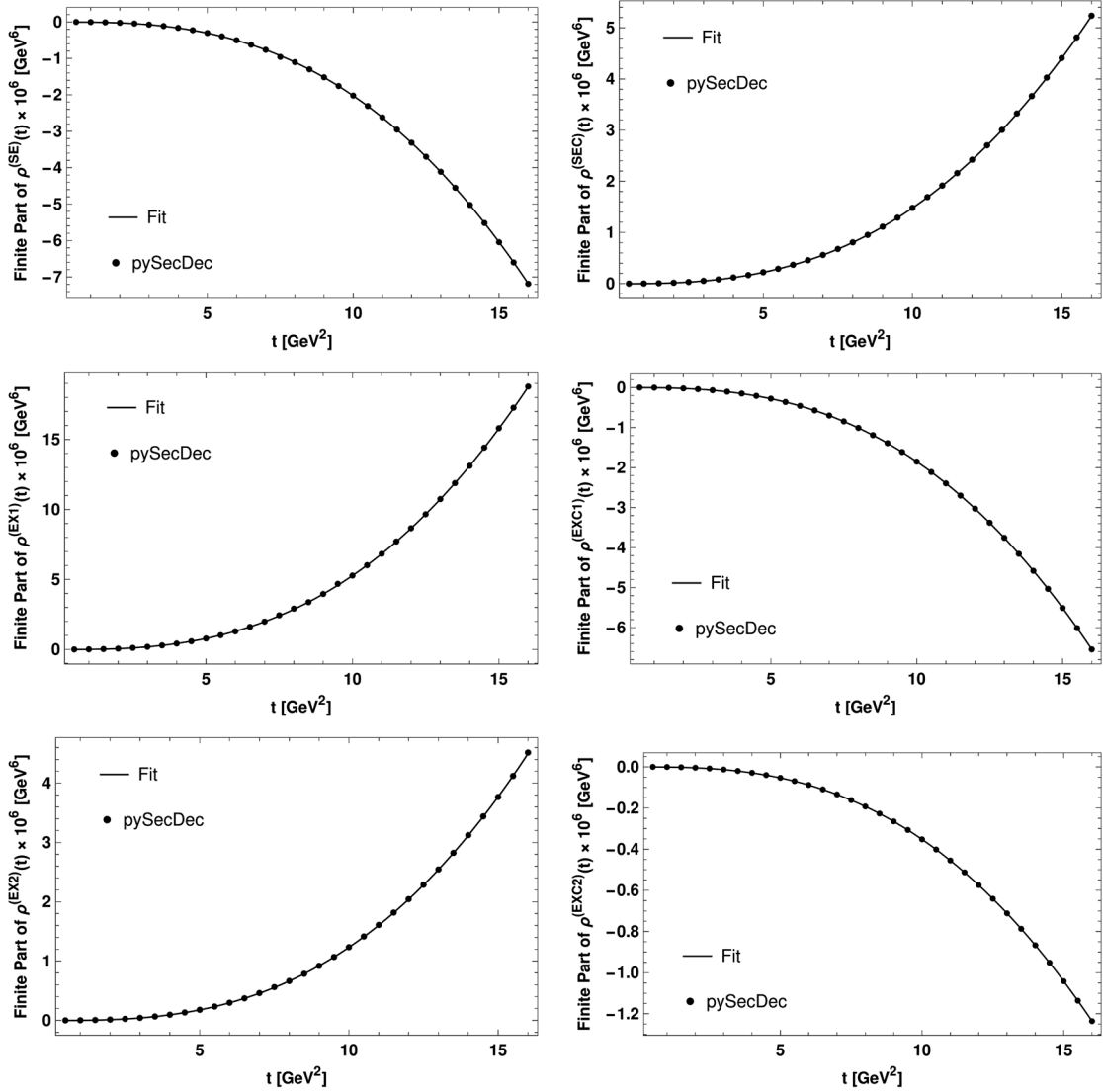


FIG. 4. Fits (solid lines) of the finite parts of (16) to pySecDec-generated data (dots). The error bars due to pySecDec numerical uncertainties are much smaller than the dots.

pySecDec-generated data. In all cases, there is excellent agreement between the data and the fitted function, and the theoretical uncertainty in the coefficients arising from the fitting procedure is negligible.

The sum of a diagram and its counterterm diagrams must be free of nonlocal divergences implying, here, that the various divergent parts, i.e., the ϵ^{-1} terms, of (16) must cancel in pairs. Therefore,

$$a^{(\text{SE})} = -a^{(\text{SEC})}, \quad (18)$$

$$a^{(\text{EX1})} = -2a^{(\text{EXC1})}, \quad (19)$$

$$a^{(\text{EX2})} = -2a^{(\text{EXC2})}. \quad (20)$$

The factors of two in (19) and (20) are due to the two possible locations of the counterterm vertex in Fig. 3(b) and Fig. 3(c) respectively. In Fig. 5, we plot $a^{(\text{SE})}/a^{(\text{SEC})}$,

$a^{(\text{EX1})}/(2a^{(\text{EXC1})})$, and $a^{(\text{EX2})}/(2a^{(\text{EXC2})})$ using values of $a^{(\text{A})}$ obtained from fitting. Within numerical uncertainty, each ratio is consistent with a constant value of -1 in excellent agreement with (18)–(20). There are, however, a handful of outliers that violate (18)–(20) by a few percent. As the u and d quarks are massless, there is no special physical significance of any value of t , and so the outliers seem to be minor numerical anomalies. We speculate that the origin of these numerical anomalies is associated with our choice $\mu = 1$ GeV corresponding to a modified minimal subtraction ($\overline{\text{MS}}$) scale [see (21) below] of $\bar{\mu}^2 = 7.05$ GeV² in close proximity to the values of t at the anomalies. It seems plausible that pySecDec could encounter numerical challenges at this scale because of the natural combination $1/\epsilon + \log(t/\mu^2) - \log(4\pi) + \gamma_E$ occurring in dimensional regularization. From Fig. 4, it is clear that the finite parts do not contain any such numerical anomalies.

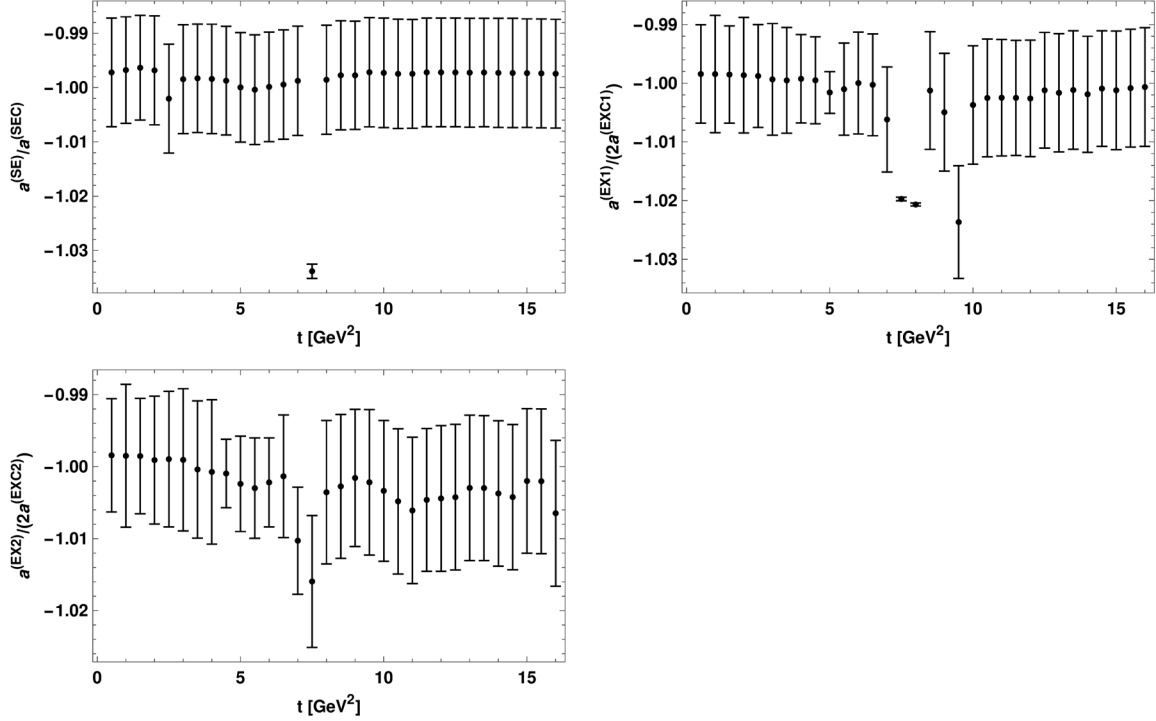


FIG. 5. Ratios of the divergent parts of diagrams and their corresponding counterterm diagrams. Error bars correspond to numerical uncertainties estimated by pySecDec.

For our final expression for $\rho^{(\text{NLO})}(t)$, we transform from an $\overline{\text{MS}}$ to an $\overline{\text{MS}}$ result in order to make use of $\overline{\text{MS}}$ QCD quantities provided in Ref. [60], for example. With

$$\mu^2 = \frac{e^{\gamma_E}}{4\pi} \bar{\mu}^2, \quad (21)$$

we have

$$L = \bar{L} - \gamma_E + \log(4\pi), \quad (22)$$

where

$$\bar{L} = \log\left(\frac{q^2}{\bar{\mu}^2}\right) \quad (23)$$

in terms of $\overline{\text{MS}}$ renormalization scale $\bar{\mu}$. Then, ignoring the divergent parts of (16) as the sum of all such contributions has been shown to cancel in pairs, we find

$$\rho^{(\text{SE})}(t) = g_s^2 t^3 (-2.23 \times 10^{-9} + 5.76 \times 10^{-10} \bar{L}), \quad (24)$$

$$\rho^{(\text{EX1})}(t) = g_s^2 t^3 (5.80 \times 10^{-9} - 1.49 \times 10^{-9} \bar{L}), \quad (25)$$

$$\rho^{(\text{EX2})}(t) = g_s^2 t^3 (1.33 \times 10^{-9} - 2.87 \times 10^{-10} \bar{L}), \quad (26)$$

$$\rho^{(\text{SEC})}(t) = g_s^2 t^3 (1.63 \times 10^{-9} - 4.29 \times 10^{-10} \bar{L}), \quad (27)$$

$$\rho^{(\text{EXC1})}(t) = g_s^2 t^3 (-2.04 \times 10^{-9} + 5.36 \times 10^{-10} \bar{L}), \quad (28)$$

$$\rho^{(\text{EXC2})}(t) = g_s^2 t^3 (-3.89 \times 10^{-10} + 1.07 \times 10^{-10} \bar{L}). \quad (29)$$

Substituting (24)–(29) into (15) gives

$$\rho^{(\text{NLO})}(t) = g_s^2 t^3 (3.30 \times 10^{-9} - 5.40 \times 10^{-10} \bar{L}). \quad (30)$$

Then, substituting (6) and (30) into (5) gives

$$\rho^{(\text{pert})}(t) = (1.69 \times 10^{-8}) t^3 (1 + \alpha_s (2.45 - 0.401 \bar{L})), \quad (31)$$

where

$$\alpha_s = \frac{g_s^2}{4\pi} \quad (32)$$

is the running strong coupling at the renormalization scale $\bar{\mu}$. For four active flavors (i.e., $n_f = 4$) at one-loop order,

$$\alpha_s(\bar{\mu}) = \frac{\alpha_s(M_\tau)}{1 + \frac{25}{12\pi} \alpha_s(M_\tau) \log\left(\frac{\bar{\mu}^2}{M_\tau^2}\right)}, \quad (33)$$

where M_τ , the τ mass, is 1.77 GeV and where $\alpha_s(M_\tau) = 0.330$ [60].

The perturbative results are consolidated by, once again, expressing $\rho^{(\text{pert})}(t)$ as [recall (5)]

$$\rho^{(\text{pert})}(t) = \rho^{(\text{LO})}(t) + \rho^{(\text{NLO})}(t), \quad (34)$$

where

$$\rho^{(\text{LO})}(t) = d_1 t^3, \rho^{(\text{NLO})}(t) = d_1 t^3 \frac{\alpha_s}{\pi} (d_2 + d_3 \bar{L}) \quad (35)$$

implies

$$\rho^{(\text{pert})}(t, \alpha_s, \bar{\mu}) = d_1 t^3 \left(1 + \frac{\alpha_s}{\pi} (d_2 + d_3 \bar{L}) \right) \quad (36)$$

with, from (31),

$$d_1 = 1.69 \times 10^{-8}, \quad d_2 = 7.70, \quad d_3 = -1.26. \quad (37)$$

The NLO perturbative terms in (36) imply that $\rho(t)$ satisfies a renormalization-group (RG) equation that contains an anomalous-dimension $\gamma_\rho(\alpha_s)$ contribution

$$\left(\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \beta(\alpha_s) \alpha_s \frac{\partial}{\partial \alpha_s} - 2\gamma_\rho(\alpha_s) \right) \rho(t) = 0, \quad (38)$$

where

$$\beta(\alpha_s) = \beta_1 \frac{\alpha_s}{\pi} + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2 \quad \text{with} \quad \beta_1 = -\frac{11}{2} + \frac{n_f}{3} \quad (39)$$

and

$$\gamma_\rho(\alpha_s) = \gamma_1 \frac{\alpha_s}{\pi} + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^2 \quad \text{with} \quad \gamma_1 = -d_3. \quad (40)$$

However, up to NLO, the quantity

$$\tilde{\rho}(t) = \alpha_s^{2\gamma_1/\beta_1} \rho(t) \quad (41)$$

satisfies an RG equation that does not contain an anomalous-dimension contribution, enabling standard RG approaches to QCD sum rules as discussed below.

The relative size of the LO and NLO terms in (35) can be examined through the ratio

$$\frac{\rho^{(\text{NLO})}(t)}{\rho^{(\text{LO})}(t)} = \frac{\alpha_s}{\pi} (d_2 + d_3 \bar{L}). \quad (42)$$

Hence, with $d_2 \sim 10$ from (37), it is expected that NLO effects could be important. Recalling from (5) and (34) that $\rho^{(\text{pert})}(t) = \rho^{(\text{LO})}(t) + \rho^{(\text{NLO})}(t)$, we plot in Fig. 6 the ratio (42) for a characteristic renormalization scale $\bar{\mu} = M_\tau$. Over the range of values of t considered in the figure, $\rho^{(\text{NLO})}(t)$ is, on average, roughly 75% the size of $\rho^{(\text{LO})}(t)$. In Fig. 7, we plot $\rho^{(\text{OPE})}(t)$ [see (4), (7), and (31)] with and without $\rho^{(\text{NLO})}(t)$ at $\bar{\mu} = M_\tau$. For $t \lesssim 4 \text{ GeV}^2$, $\rho^{(\text{OPE})}(t) < 0$ due to the large-magnitude, negative contribution from the 6d quark condensate term. However, the NLO contributions do mitigate the effect of the 6d condensates by extending the $\rho^{(\text{OPE})} > 0$ region to lower values of t compared to LO. (Note that the zeroes of $\rho^{(\text{OPE})}(t)$ and

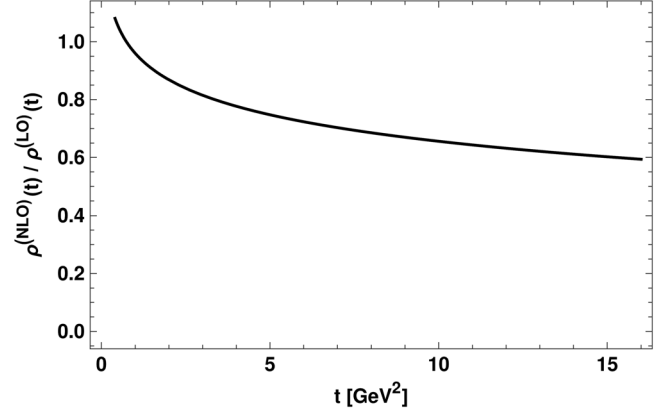


FIG. 6. The ratio of $\rho^{(\text{NLO})}(t)$ to $\rho^{(\text{LO})}(t)$ (see Eq. (42) for $\bar{\mu} = M_\tau$).

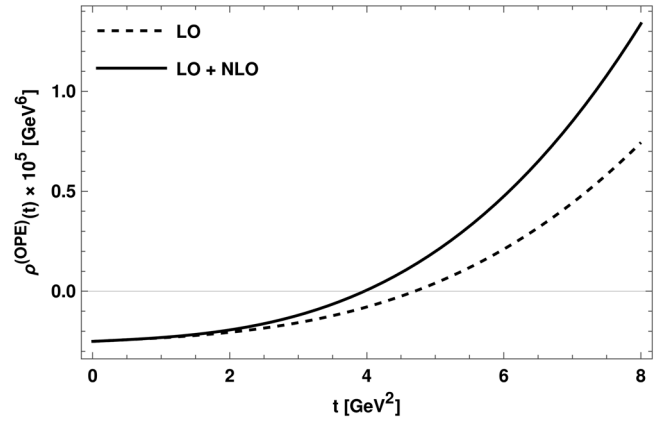


FIG. 7. $\rho^{(\text{OPE})}(t)$ with NLO perturbation theory (the solid line) and without (the dashed line) at $\bar{\mu} = M_\tau$.

$\tilde{\rho}^{(\text{OPE})}(t)$ are the same.) This behavior of $\rho^{(\text{OPE})}$ is similar to that seen for 1^{-+} , light-quark tetraquarks in [17,18] and disfavors the existence of $T_{du\bar{d}\bar{u}}^{0^{+-}}$ states lighter than $\approx 2 \text{ GeV}$. But, of course, hadronic predictions cannot be extracted directly from the QCD-calculated $\rho^{(\text{OPE})}(t)$, and so, for a more rigorous analysis, we relate $\rho^{(\text{OPE})}(t)$ to the channel's hadronic spectral function through QCD sum rules.

III. TETRAQUARK GROUND STATE MASS BOUNDS FROM QCD SUM RULES

At $Q^2 = -q^2 > 0$, the correlator Π defined in (2)–(3) satisfies the dispersion relation

$$\Pi(Q^2) = -Q^6 \int_{t_0}^{\infty} \frac{\rho(t)}{t^3(t+Q^2)} dt + \dots, \quad (43)$$

where $\rho(t)$ is the hadronic spectral function, $t_0 \approx 0$ is a threshold parameter corresponding to the squared energy needed to create real constituents, and \dots represents a

polynomial in Q^2 (subtraction constants). When $\Pi(Q^2)$ is computed through the OPE, the dispersion relation (43) connects QCD to hadronic physics, i.e., quark-hadron duality.

To reduce the contribution to the right-hand side of (43) from the high-energy behavior of $\rho(t)$ (such as from excited states) as well as to eliminate subtraction constants and local field-theory divergences, a transform is typically applied, leading to QCD sum rules. Two examples of QCD sum rules are Laplace sum rules [26–28,30] and Gaussian sum rules [29,30,61,62].² The (order-0) LSR, $\mathcal{R}(\sigma)$, is defined as

$$\mathcal{R}(\sigma) = \frac{1}{\sigma^N} \lim_{Q^2 \rightarrow \infty} \frac{(-Q^2)^N}{\Gamma(N)} \left(\frac{d}{dQ^2} \right)^N \Pi(Q^2), \quad (44)$$

where $\sigma = \frac{N}{Q^2}$, the Borel parameter, is in GeV^{-2} . The (order-0) GSR, $\mathcal{G}(\hat{s}, \tau)$, is defined as

$$\mathcal{G}(\hat{s}, \tau) = \sqrt{\tau} \lim_{\Delta^2 \rightarrow \infty} \frac{(-\Delta^2)^N}{\Gamma(N)} \left(\frac{d}{d\Delta^2} \right)^N \times \left(\frac{\Pi(-\hat{s} - i\Delta) - \Pi(\hat{s} + i\Delta)}{i\Delta} \right), \quad (45)$$

where $\tau = \frac{\Delta^2}{N}$ is in GeV^4 . Combining (43) with $\Pi \rightarrow \Pi^{(\text{OPE})}$ and (44) gives (see [26,29] for details)

$$\mathcal{R}(\sigma) = \int_0^\infty e^{-\sigma t} \rho^{(\text{OPE})}(t) dt = \int_0^\infty e^{-\sigma t} \rho(t) dt. \quad (46)$$

Similarly, combining (43) with $\Pi \rightarrow \Pi^{(\text{OPE})}$ and (45) gives (see [29,61,62] for details)

$$\begin{aligned} \mathcal{G}(\hat{s}, \tau) &= \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{(t-\hat{s})^2}{4\tau}} \rho^{(\text{OPE})}(t) dt \\ &= \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{(t-\hat{s})^2}{4\tau}} \rho(t) dt. \end{aligned} \quad (47)$$

For massless quarks, we set $\bar{\mu} = 1/\sqrt{\sigma}$ in the LSR [34] and $\bar{\mu} = \sqrt[4]{\tau}$ in the GSR [29,61]. These RG-improvement results are based on an RG equation free of anomalous-dimension contributions; hence, the anomalous-dimension factor $\alpha_s^{2\gamma_1/\beta_1}$ from (41) should be included as an additional positive multiplicative factor. However, the phenomenological analysis presented below is based on the sign of the GSR and an LSR ratio; in both cases, the anomalous-dimension factor has no effect and so can be ignored (see a similar argument in Ref [38]). In Fig. 8, we plot $\mathcal{R}(\sigma)$, and, in Fig. 9, we plot $\mathcal{G}(\hat{s}, \tau)$ at $\tau = 1 \text{ GeV}^4$. In both figures, the solid curve includes both LO and NLO perturbation theory

²Regarding arguments to sum rules, we follow the notation of [29].

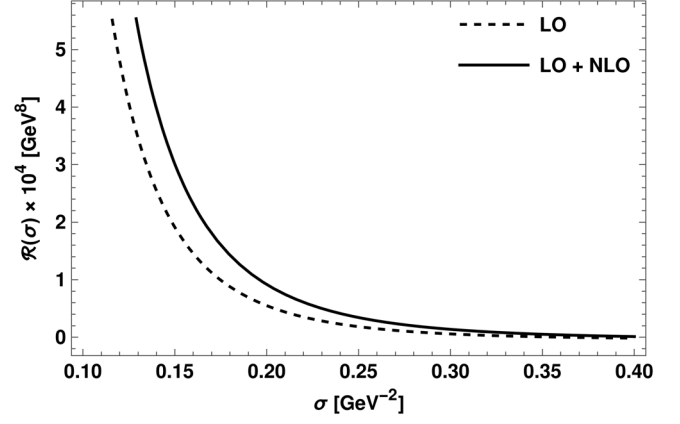


FIG. 8. The LSR $\mathcal{R}(\sigma)$ with NLO perturbation theory (the solid line) and without (the dashed line).

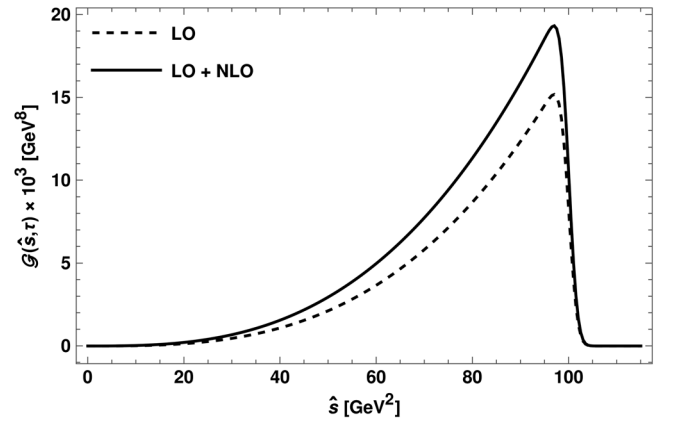


FIG. 9. The GSR $\mathcal{G}(\hat{s}, \tau)$ with NLO perturbation theory (the solid line) and without (the dashed line) at $\tau = 1 \text{ GeV}^4$.

whereas the dashed curve includes LO perturbation theory only. It can be seen that NLO perturbation theory makes significant contributions to both the LSR and GSR. Note that, for $\sigma \gtrsim 0.4 \text{ GeV}^{-2}$, $\mathcal{R}(\sigma)$ becomes negative and, therefore, is unphysical for this region of Borel parameter.

We split the hadronic spectral function into hadronic and QCD continuum contributions

$$\rho(t) = \rho^{(\text{had})}(t) + \rho^{(\text{OPE})}(t)\theta(t - s_0), \quad (48)$$

where $\rho^{(\text{had})}(t)$ contains the resonance(s) content of $\rho(t)$, s_0 is the continuum threshold parameter, and $\theta(t - s_0)$ is a Heaviside step function. Plugging (48) into (47) gives

$$\begin{aligned} \mathcal{G}(\hat{s}, \tau, s_0) &= \frac{1}{\sqrt{4\pi\tau}} \int_0^{s_0} e^{-\frac{(t-\hat{s})^2}{4\tau}} \rho^{(\text{OPE})}(t) dt \\ &= \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty e^{-\frac{(t-\hat{s})^2}{4\tau}} \rho^{(\text{had})}(t) dt, \end{aligned} \quad (49)$$

where $\mathcal{G}(\hat{s}, \tau, s_0)$ is a (continuum-)subtracted GSR.

Using (49), we extract a lower bound on s_0 . Integrating (49) over $-\infty < \hat{s} < \infty$ gives

$$\mathcal{F}(s_0) = \int_0^{s_0} \rho^{(\text{OPE})}(t) dt = \int_0^\infty \rho^{(\text{had})}(t) dt, \quad (50)$$

a finite-energy sum rule (FESR). In the FESR, we set $\bar{\mu} = \sqrt{s_0}$ [29]. As $\rho^{(\text{had})}(t) \geq 0$, it follows that $\mathcal{F}(s_0) \geq 0$, and, as discussed above, omitting the anomalous-dimension factor has no effect on the sign of the FESR. In Fig. 10, we plot $\mathcal{F}(s_0)$ with and without NLO perturbation theory. Only values of s_0 that lead to positive $\mathcal{F}(s_0)$ are physically allowed. Therefore, with NLO perturbation theory, we find that

$$s_0 > 6.47 \text{ GeV}^2, \quad (51)$$

and, without NLO perturbation theory, we find that

$$s_0 > 7.64 \text{ GeV}^2. \quad (52)$$

To reiterate, the bounds (51) and (52) are constraints on s_0 that follow directly from the physical requirement that the FESR be positive. The optimized value $s_0 = 4.50 \text{ GeV}^2$ used in [22] does not satisfy (52).

Using the GSR, we motivate a lower bound on the $T_{dud\bar{u}}^{0^{++}}$ tetraquark ground-state mass M . We employ a single narrow resonance model in (48),

$$\rho^{(\text{had})}(t) = f^2 \delta(t - M^2), \quad (53)$$

where M is the mass of the lightest $T_{dud\bar{u}}^{0^{++}}$ tetraquark that couples to (1) and f is its corresponding coupling strength. In using (53), it is assumed that excited states are sufficiently suppressed relative to the ground state by the Gaussian kernel of (45) that they can be ignored. Plugging (53) into (49) gives

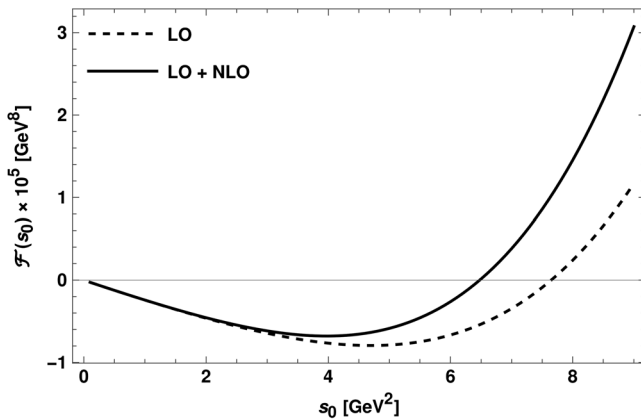


FIG. 10. The FESR $\mathcal{F}(s_0)$ with NLO perturbation theory (the solid line) and without (the dashed line).

$$\begin{aligned} \mathcal{G}(\hat{s}, \tau, s_0) &= \frac{1}{\sqrt{4\pi\tau}} \int_0^{s_0} e^{-\frac{(t-\hat{s})^2}{4\tau}} \rho^{(\text{OPE})}(t) dt \\ &= \frac{f^2}{\sqrt{4\pi\tau}} e^{-\frac{(\hat{s}-M^2)^2}{4\tau}}. \end{aligned} \quad (54)$$

In general, in (54), τ should be restricted to some interval $\tau_{\min} \leq \tau \leq \tau_{\max}$. For instance, τ_{\min} should be chosen such that the running coupling $\alpha_s(\sqrt[4]{\tau})$ [see (33)] is not too large. As in [61], we choose $\tau_{\min} = 1 \text{ GeV}^4$. Regarding τ_{\max} , the width of the Gaussian kernel on the rhs of (49) is $\sqrt{2\tau}$, and, as τ increases, so too does the sensitivity of the GSR to excited states, eventually violating (53). Fortunately, for our purposes, we do not actually need a specific value for τ_{\max} . (For a more rigorous discussion of τ_{\min} and τ_{\max} based on Hölder inequalities, see [52].) Since the rhs of (54) is positive, it follows that $\mathcal{G}(\hat{s}, \tau, s_0)$ must also be positive; however, since $\rho^{(\text{OPE})}(t) < 0$ for $t \lesssim 4 \text{ GeV}^2$, we find that there are regions of (\hat{s}, τ, s_0) parameter space where $\mathcal{G}(\hat{s}, \tau, s_0) < 0$. This is shown in Fig. 11 for $\tau = 1 \text{ GeV}^4$ and $s_0 = 10 \text{ GeV}^2$. Such regions of parameter space are unphysical. We denote the zero of $\mathcal{G}(\hat{s}, \tau, s_0)$ with respect to \hat{s} as $\hat{s}_{\text{crit}}(\tau, s_0)$. We note that the single narrow resonance contribution to the rhs of (54) has width $\sqrt{2\tau}$. If, for self-consistency, we require that the full width of the resonance contribution be contained in the region where $\mathcal{G}(\hat{s}, \tau, s_0) > 0$, then

$$M^2 \gtrsim \sqrt{2\tau} + \hat{s}_{\text{crit}}(\tau, s_0) \quad (55)$$

for all allowed τ at physical s_0 . Numerically, we find that the rhs of (55) is a decreasing function of s_0 . Thus,

$$M^2 \gtrsim \sqrt{2\tau} + \hat{s}_{\text{crit}}(\tau, \infty). \quad (56)$$

In Fig. 12, we plot the square root of the rhs of (56) versus τ and find that $M \gtrsim 2.2 \text{ GeV}$. An analogous analysis that omits NLO perturbation theory finds that $M \gtrsim 2.4 \text{ GeV}$.

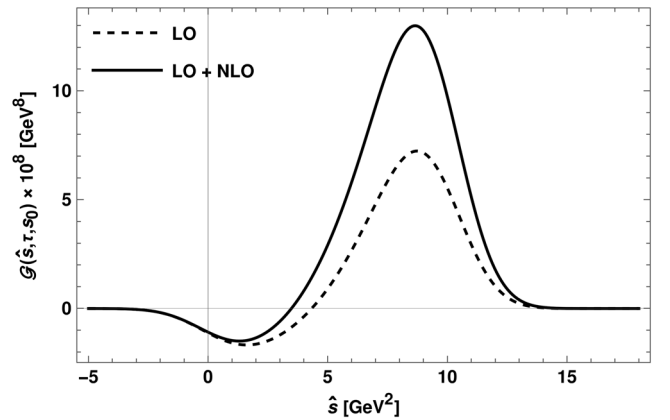


FIG. 11. The subtracted GSR $\mathcal{G}(\hat{s}, \tau, s_0)$ at $\tau = 1 \text{ GeV}^4$ and $s_0 = 10 \text{ GeV}^2$ with NLO perturbation theory (the solid line) and without (the dashed line).

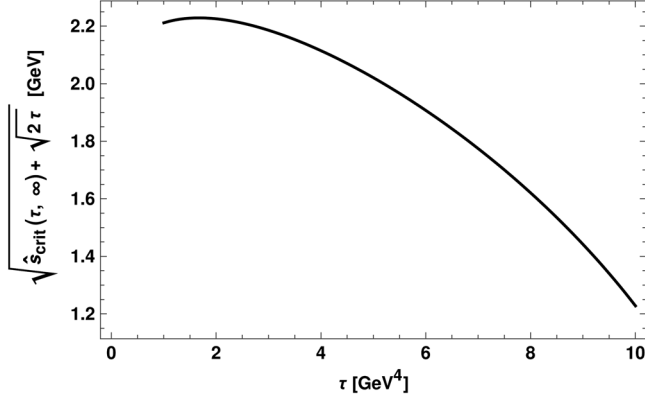


FIG. 12. The square-root of the rhs of (56) versus τ .

Using the LSR, we determine an upper bound on the $T_{dud\bar{u}}^{0^{++}}$ tetraquark ground-state mass M . As shown in [63],

$$M \leq \sqrt{\frac{-\frac{d}{d\sigma}\mathcal{R}(\sigma)}{\mathcal{R}(\sigma)}}. \quad (57)$$

Inequality (57) follows from positivity of the hadronic spectral function and applies to an extensive class of resonance models [63]. In Fig. 13, we plot the right-hand side of (57) with and without NLO perturbation theory. Without NLO perturbation theory, we find that $M \leq 4.6$ GeV. With NLO perturbation theory, we find that $M \leq 4.2$ GeV. We note that, at the minimum value of the solid curve in Fig. 13, the LSR $\mathcal{R}(\sigma)$ is positive and its perturbative contributions are greater than three times the (magnitude of the) condensate contributions, i.e., the extracted upper bound comes from a region of OPE convergence.

Uncertainty in our results is dominated by the value of κ used in (7). For the central value $\kappa = 2$ used in the above analysis, our NLO mass bounds are 2.2 GeV $\lesssim M \leq 4.2$ GeV. Both upper and lower mass bounds increase with increasing κ . For $\kappa = 3$ (the upper range of Refs. [30,56,57]),

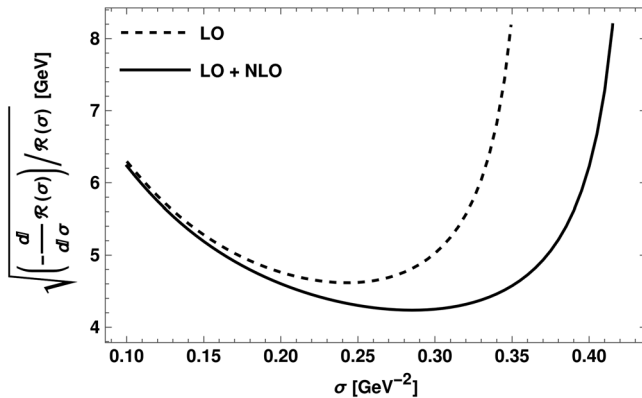


FIG. 13. $\sqrt{-\frac{d\mathcal{R}(\sigma)}{d\sigma}/\mathcal{R}(\sigma)}$ with NLO perturbation theory (the solid line) and without (the dashed line).

we find 2.4 GeV $\lesssim M \leq 4.6$ GeV. Because the prevailing evidence for violation of vacuum saturation indicates $\kappa > 1$ [30,56,57], our $\kappa = 1$ result of 1.9 GeV $\lesssim M \leq 4.0$ GeV provides a conservative lower-mass bound of $M > 1.9$ GeV.

IV. DISCUSSION

Motivated by the unphysical violation of positivity in the LO LSRs of [22] and the large NLO perturbative effects for 0^{++} light tetraquarks [37], we calculated NLO contributions to perturbation theory for a $T_{dud\bar{u}}^{0^{++}}$ tetraquark correlation function (2)–(3) in the limit of massless u and d quarks. Our results represent the first complete NLO perturbative calculation of light-quark, exotic- J^{PC} tetraquark sum rules. Instead of renormalizing the interpolating current (1), we eliminated nonlocal divergences using diagrammatic renormalization methods as outlined in Ref. [43]. Instead of evaluating dimensionally regularized integrals analytically, we evaluated them numerically using pySecDec [44]. We then fit the pySecDec-generated data to the known functional form of the imaginary part of the correlation function. The successful combination of pySecDec with diagrammatic renormalization establishes a valuable and efficient new methodology for computing radiative corrections to correlation functions of operators composed of light quarks. Furthermore, diagrammatic renormalization and pySecDec can, in principle, be applied to systems containing heavy quarks although, in our experience, computational runtimes and RAM requirements increase significantly when heavy quarks are introduced.

Relative to LO perturbation theory, the NLO corrections make significant contributions to $\rho^{(\text{OPE})}(t)$ (see Figs. 6–7) and to QCD sum rules (see Figs. 8–11). Although the NLO corrections mitigate the violation of positivity in the sum rules, there are still unphysical regions of s_0 and Borel-scale parameter space. Using positivity to constrain physical regions within a GSR, we motivated a lower bound on the $T_{dud\bar{u}}^{0^{++}}$ tetraquark ground-state mass M , and, using an LSR, we determined an upper bound. Taking into account both LO and NLO perturbation theory, we found that, for our central vacuum-saturation parameter $\kappa = 2$,

$$2.2 \text{ GeV} \lesssim M \leq 4.2 \text{ GeV} \quad (58)$$

which should be compared with a range determined by omitting NLO perturbation theory,

$$2.4 \text{ GeV} \lesssim M \leq 4.6 \text{ GeV}. \quad (59)$$

Increasing the vacuum saturation parameter to $\kappa = 3$ increases both the upper and lower bounds in (58). For the current (1), we found no evidence for the existence of a $T_{dud\bar{u}}^{0^{++}}$ tetraquark under 1.9 GeV, and note that 1.9 GeV was

obtained by decreasing the vacuum saturation parameter to $\kappa = 1$, thereby underestimating known violations of vacuum saturation [30,56,57]. This lower bound on the $T_{d\bar{u}\bar{u}}^{0^{+-}}$ mass is contrary to the results of [22]. The source of the discrepancy between Ref. [22] and our conservative mass bound $M > 1.9$ GeV can be traced to the value of s_0 used in [22] which violates the physical positivity constraint (52). Finally, we note that the $T_{d\bar{u}\bar{u}}^{0^{+-}}$ mass bounds (58) encompass the QCD Gaussian sum-rules mass predictions

for 0^{+-} , light-quark hybrids [52], suggesting the interesting possibility of hybrid-tetraquark mixing in light-quark systems.

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