U duality and α' corrections in three dimensions

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We consider the target space theory of bosonic and heterotic string theory to first order in α' compactified to three dimensions, using a formulation that is manifestly T duality invariant under $\mathrm{O}(d,d,\mathbb{R})$ with $d=23$ and $d=7$, respectively. While the two-derivative supergravity exhibits a symmetry enhancement to the U duality group $\mathrm{O}(d+1,d+1)$, the continuous group is known to be broken to $\mathrm{O}(d,d,\mathbb{R})$ by the first α' correction. We revisit this observation by computing the full effective actions in three dimensions to first order in α' by dualizing the vector gauge fields. We give a formally $\mathrm{O}(d+1,d+1)$ invariant formulation by invoking a vector compensator, and we observe a chiral pattern that allows one to reconstruct the bosonic action from the heterotic action. Furthermore, we obtain a particular massive deformation by integrating out the external B field. This induces a novel Chern-Simons term based on composite connections that, remarkably, is $\mathrm{O}(d+1,d+1)$ invariant to leading order in the deformation parameter.

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I. INTRODUCTION

Our goal in this paper is to explore some features arising in the interplay of higher-derivative α' corrections of the effective actions of string and M theory and the duality properties that these theories are expected to exhibit for certain backgrounds. Arguably the simplest duality property of string theory is T duality, which states that theories compactified on toroidal backgrounds T^d related by $\mathrm{O}(d,d,\mathbb{Z})$ transformations are physically equivalent, even though these backgrounds may be radically different as ordinary geometries. This means that conventional Einstein-Hilbert gravity looks quite different on these backgrounds, yet "stringy gravity" supposedly cannot tell the difference. Even larger duality groups arise for particular theories and backgrounds. In this paper, we consider compactifications to three spacetime dimensions, which are particularly interesting since in the supergravity limit the corresponding T duality group $\mathrm{O}(7,7)$ is enhanced to larger groups, and discrete subgroups of these so-called U duality groups are conjectured to be dualities of the full string/M theory [1,2]. Concretely, $\mathrm{O}(7,7)$ is generally enhanced to $\mathrm{O}(8,8)$, which for type II string theory or M theory is then further enhanced to $\mathrm{E}_{8,8}$ (a noncompact form of the largest finite-dimensional exceptional Lie group). The effect of α' corrections on the $\mathrm{E}_{8,8}$ enhancement is difficult to study, since in type II string theory, the corrections start at α'^3 with eight derivatives, but luckily in three dimensions, there is the smaller U duality group $\mathrm{O}(8,8)$ that can be discussed in bosonic and heterotic string theory whose α' corrections start with four derivatives. [For heterotic string theory, the U duality group is really $\mathrm{O}(8,24)$ but we truncate the vector fields, which reduces the group to $\mathrm{O}(8,8)$.]

At the level of the low-energy effective target space actions, the T duality property manifests itself in dimensional reduction (Kaluza-Klein compactification on T^d and subsequent truncation to massless modes), by exhibiting a global symmetry under the continuous group $\mathrm{O}(d,d,\mathbb{R})$.[1] The effective target space theories also receive an infinite

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[1]The enhancement to the continuous group can be understood as follows: The T duality group $\mathrm{O}(d,d,\mathbb{Z})$ is discrete for purely geometrical reasons because the symmetry transformations of fields need to be compatible with the periodicity conditions of the torus, but in dimensional reduction, all memory of the torus has disappeared, explaining the enhancement to the continuous group. In contrast, the discrete T duality group $\mathrm{O}(d,d,\mathbb{Z})$ is *not* visible in the low-energy effective actions compactified on tori without truncation. This requires a genuine double field theory [3,4].

number of higher-derivative corrections governed by the inverse string tension α', and it is known that the α' corrections preserve the continuous $\mathrm{O}(d,d,\mathbb{R})$ [5,6] (see Ref. [7] for a review). Generalizing previous work on cosmological reductions to one dimension (cosmic time) [8–10], we have recently determined the $\mathrm{O}(d,d,\mathbb{R})$ invariant effective actions to first order in α' for general reductions along d dimensions [11,12]. We apply this effective action to compactifications to three dimensions, with the goal to explore the fate of the duality enhancement to $\mathrm{O}(d+1,d+1)$. (Here, $d=7$ for heterotic string theory and $d=23$ for bosonic string theory, but for our discussion, d is really a free parameter, and so we sometimes only speak of $\mathrm{O}(8,8)$ for the sake of vividness.)

In contrast to T duality, which is a feature of classical string theory and preserved by all α' corrections, the U dualities capture features of the quantum theory. Therefore, one should perhaps not expect supergravity to exhibit U duality symmetries beyond zeroth order in α' without also including quantum corrections. A simple argument based on a scaling symmetry of the two-derivative theory in fact shows that the *continuous* U duality group is not preserved to higher order in α' [13,14]. Indeed, the Einstein-Hilbert term in string frame,

$$I_{\rm EH} = \int \mathrm{d}^D x\sqrt{g}e^{-\phi}R, \qquad (1.1)$$

has a global $\mathbb{R}\simeq \mathrm{O}(1,1)$ scaling symmetry with constant parameter λ, which acts as $g_{\mu\nu}\to e^{\lambda}g_{\mu\nu}$ and $\phi\to\phi+\frac{D-2}{2}\lambda$. This $\mathrm{O}(1,1)$ becomes part of the U duality group, but a typical higher-derivative coupling of the form

$$\alpha'\int \mathrm{d}^D x\sqrt{g}e^{-\phi}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \qquad (1.2)$$

is not invariant, but rather scales with $e^{-\lambda}$, hence breaking the symmetry. Intriguingly, however, the complete order α' action, including all matter couplings, scales homogeneously. There is hence a formal scaling invariance if one declares α' to scale with $e^{+\lambda}$. Below we will employ a similar scheme to establish a formal $\mathrm{O}(8,8)$ invariance.

The above scaling argument avoids the need to actually compute the effective action in, say, three dimensions, but by itself, it is not sufficient to show that the *discrete* subgroup is not realized in supergravity because for the discrete group, the scaling symmetry in fact trivializes, as we will discuss. We therefore revisit the problem and compute the complete order α' effective action in three dimensions, starting from the results of Refs. [11,12] and using, perturbatively in α', on shell transformations to dualize the vector gauge fields into scalar fields. In the two-derivative theory, this transformation shows the enhancement from $\mathrm{O}(7,7)$ to $\mathrm{O}(8,8)$ (or $\mathrm{E}_{8,8}$) because the new scalars organize into a larger coset matrix $\mathcal{M}_{\mathcal{M}\mathcal{N}}$, with $\mathrm{O}(8,8)$ indices $\mathcal{M},\mathcal{N}\in[\![1,16]\!]$. As expected, to first order in α', one finds that the continuous $\mathrm{O}(8,8)$ is no longer present. The manifest symmetry is $\mathrm{ISO}(7,7,\mathbb{R})$, i.e., $\mathrm{O}(7,7,\mathbb{R})$ times 14-dimensional translations that act as shifts on the scalars originating from dualization.

While $\mathrm{O}(8,8)$ is not a symmetry of the four-derivative action, we find it advantageous to employ a formulation that, as alluded to above, exhibits a formal invariance under this group upon introducing a nondynamical compensator. Specifically, for this, we can choose a constant vector $\mathsf{u}^{\mathcal{M}}$ in the fundamental representation, in terms of which the effective action takes an $\mathrm{O}(8,8)$ invariant form. The true theory then arises for a fixed vector $\mathsf{u}^{\mathcal{M}}$ pointing in a particular direction. Even though it is in principle always possible to restore a broken symmetry in a formal manner by introducing an unphysical tensor compensator whose fictitious transformations absorb the failure of the actual theory to be invariant, this approach turns out to be technically useful. In particular, it allows us to observe an intriguing "chiral pattern" of the four-derivative action, namely that only one chiral projection of $\mathsf{u}^{\mathcal{M}}$ is needed to write the action. To explain this, recall that given the $\mathrm{O}(8,8)$, coset matrix $\mathcal{M}$ and invariant metric η one can define the projection operators

$$P_{\mathcal{M}\mathcal{N}} = \frac{1}{2}(\eta_{\mathcal{M}\mathcal{N}}-\mathcal{M}_{\mathcal{M}\mathcal{N}}), \quad \bar{P}_{\mathcal{M}\mathcal{N}} = \frac{1}{2}(\eta_{\mathcal{M}\mathcal{N}}+\mathcal{M}_{\mathcal{M}\mathcal{N}}), \qquad (1.3)$$

onto two subspaces of opposite "chirality." The four-derivative action turns out to be fully determined by an $\mathrm{O}(8,8)$ invariant function $\mathcal{F}$ of a two-tensor and a vector argument, respectively, as follows

$$I_1 = \frac{1}{4}\int \mathrm{d}^3x\sqrt{-g_{\rm E}}\Big\{a\mathcal{F}[\mathcal{M},P\mathsf{u}] + b(\mathcal{F}[\mathcal{M},P\mathsf{u}])^{\star}\Big\}. \qquad (1.4)$$

Here * denotes a $\mathbb{Z}_2$ action, which is implemented on the coset matrix as $\mathcal{M}\to\mathcal{Z}^{\rm t}\mathcal{M}\mathcal{Z}$, where $\mathcal{Z}$ obeys $\mathcal{Z}^2=\mathbf{1}$ but is *not* an $\mathrm{O}(8,8)$ matrix. Under this $\mathbb{Z}_2$, the projectors (1.3) are interchanged:

$$P\to\mathcal{Z}^{\rm t}\bar{P}\mathcal{Z}, \qquad \bar{P}\to\mathcal{Z}^{\rm t}P\mathcal{Z}. \qquad (1.5)$$

The parameters a, b determine the theory: The heterotic action is obtained for $(a,b)=(-\alpha',0)$ and the bosonic action for $(a,b)=(-\alpha',-\alpha')$. The $\mathbb{Z}_2$ action, which exchanges a and b, has a higher-dimensional analog, sending the B field $B\to -B$, which is a symmetry of the bosonic action but not of the heterotic action. Since it is the same function $\mathcal{F}$ that determines the "$\mathbb{Z}_2$ dual" terms in the action, it follows that the bosonic action can be reconstructed from the heterotic action.

The above parametrization in terms of (a, b), together with the $\mathbb{Z}_2$ action, mimic the structure of double field theory at order α' [15,16]. The crucial "experimental" observation provided by our computation is that the compensator $\mathsf{u}^{\mathcal{M}}$ in Eq. (1.4) appears only in a "chiral" or projected form or, alternatively, that there is the formal "gauge invariance" under $\mathsf{u} \to \mathsf{u} + \eta \bar{P} \Lambda$. (Note that this comes very close to an actual symmetry enhancement, with an O(8) acting only on indices with a "barred" projection, but this viewpoint is not quite consistent as the projectors are field dependent and hence not compatible with global symmetries.) We do not have an explanation for this chiral pattern, but it would be interesting to explore whether it persists, at least for subsectors, to higher orders in α'. This would provide indirect constraints on the allowed higher-derivative couplings.

As the second main result of our paper, we consider a particular massive deformation, as a window into more general gauged supergravities in presence of α' corrections. The latter would be important in order to study, for instance, the fate of Kaluza-Klein truncations on spheres in presence of higher derivatives. The massive deformation we consider is obtained by integrating out the external B field. In three dimensions, its field strength is on shell a constant that is usually set to zero in dimensional reduction. Keeping instead this constant m while integrating out the B field leads to a massive deformation, which for the two-derivative theory, includes a potential term for the dilaton and a Chern-Simons term for the Kaluza-Klein vectors $A_\mu{}^M$ [17]. Including then the order α' corrections, one obtains additional couplings, which include a Chern-Simons term based on composite connections, with the latter originating from the scalar-dependent Green-Schwarz deformation uncovered in Refs. [11,12] (and given a worldsheet interpretation in Ref. [18]). Specifically, introducing an $O(8,8)$ frame field $\mathcal{V}$, the compact part of its Maurer-Cartan form $\mathcal{V}^{-1}\mathrm{d}\mathcal{V} = \mathcal{P} + \mathcal{Q}$ defines *composite* $O(8) \times O(8)$ connections $\mathcal{Q}$. The topological or Chern-Simons terms of the massive deformation at order α' and order m then read

$$I_{(1)\mathrm{top}} = (a+b)m \int \mathrm{tr}\left(\mathcal{Q} \wedge \mathrm{d}\mathcal{Q} + \frac{2}{3}\mathcal{Q} \wedge \mathcal{Q} \wedge \mathcal{Q}\right) - \frac{a-b}{4} m \int \mathrm{tr}\left(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) + \mathcal{O}(m^2), \tag{1.6}$$

where ω denotes the Levi-Civita spin connection (so that for $a - b \neq 0$, this action includes topologically massive gravity as a subsector [19]). Unexpectedly, the Chern-Simons terms are hence $O(8,8)$ invariant to leading order in m, although the full theory is not. This is remarkable, for there are now two parameters expected to break U duality, α' and m, yet for the leading Chern-Simons terms, $O(8,8)$ is restored. Again, we do not know what the physical significance of this observation is, and it remains to explore more general gaugings.

The remainder of this paper is organized as follows. In Sec. II, we give a short review of the duality enhancement in three dimensions for the two-derivative theory, with a particular focus on the scaling symmetries before and after dimensional reduction since these feature prominently in the subsequent discussion of α' corrections. In Sec. III, we compute the effective action in three dimensions to first order in α' by perturbatively dualizing the vector gauge fields into scalars, and we exhibit the chiral pattern explained above. We then turn in Sec. IV to a massive deformation, which is obtained by integrating out the external B field, with a focus on the resulting Chern-Simons terms for composite connections that exhibit an enhancement to the full U duality group to first order in the mass parameter. We conclude with a short outlook in Sec. V, while various identities and intermediate results are collected in appendices.

II. DUALITY ENHANCEMENT IN THREE DIMENSIONS

In this section, we discuss some general aspects of the bosonic and heterotic string effective actions dimensionally reduced to three spacetime dimensions. We review the scaling symmetries in higher dimensions (prior to any dimensional reduction) and in three dimensions, as a preparation for the discussion of duality enhancement from $O(d,d)$ to $O(d+1,d+1)$ that is expected to be a feature of string theory in three dimensions. We close the section with a discussion of field redefinitions, which are needed once higher-derivative corrections are included.

A. Scaling symmetries

The bosonic parts of the bosonic and heterotic string effective actions coincide at the two-derivative order upon truncating the Yang-Mills gauge fields of the heterotic theory. They describe the dynamics of a metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, a two-form $\hat{B}_{\hat{\mu}\hat{\nu}}$ and a dilaton $\hat{\phi}$ in $D = 26$ and $D = 10$ dimensions, respectively. In the string frame, the two-derivative action is given by

$$I_0^{(D)} = \int \mathrm{d}^D X \sqrt{-\hat{g}} e^{-\hat{\phi}} \left(\hat{R} + \partial_{\hat{\mu}}\hat{\phi}\partial^{\hat{\mu}}\hat{\phi} - \frac{1}{12}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}\right), \tag{2.1}$$

with the Ricci scalar $\hat{R}$ and the field-strength $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\partial_{[\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}]}$. It features several global scaling symmetries, with group $\mathbb{R}^+ \simeq O(1,1)$ and constant parameter λ, that we list in the following.

TABLE I. Scaling behavior of the fields (2.4) under the transformations (2.2), (2.3) and (2.5). We display the charges q of each field φ, representing the transformation $\varphi \to e^{q\lambda}\varphi$. For convenience, we have indicated the shifted dilaton Φ and Einstein frame metric $g_{\mathrm{E}\mu\nu} = e^{-2\Phi} g_{\mu\nu}$ rather than the dilaton $\hat{\phi}$ and the string frame metric $g_{\mu\nu}$.

	e^{Φ}	$g_{\mathrm{E}\mu\nu}$	$B_{\mu\nu}$	G_{mn}	B_{mn}	$A_\mu^{(1)m}$	$A_{\mu m}^{(2)}$
$(\mathrm{Dilaton}^D)$	$1/(D-2)$	0	$2/(D-2)$	$2/(D-2)$	$2/(D-2)$	0	$2/(D-2)$
$(\mathrm{Trombone}^D)$	$3-D$	$2D-4$	2	2	2	0	2
(Volume^{D-3})	$3-D$	$2D-6$	0	2	2	-1	1

(i) Constant dilaton shift:

$$(\mathrm{Dilaton}^D)\colon\ \hat{\phi} \to \hat{\phi} + \lambda, \qquad \hat{g}_{\hat{\mu}\hat{\nu}} \to e^{2\lambda/(D-2)} \hat{g}_{\hat{\mu}\hat{\nu}}, \qquad \hat{B}_{\hat{\mu}\hat{\nu}} \to e^{2\lambda/(D-2)} \hat{B}_{\hat{\mu}\hat{\nu}}. \tag{2.2}$$

(ii) On shell "trombone" symmetry:

$$(\mathrm{Trombone}^D)\colon\ \hat{\phi} \to \hat{\phi}, \qquad \hat{g}_{\hat{\mu}\hat{\nu}} \to e^{2\lambda} \hat{g}_{\hat{\mu}\hat{\nu}}, \qquad \hat{B}_{\hat{\mu}\hat{\nu}} \to e^{2\lambda} \hat{B}_{\hat{\mu}\hat{\nu}}, \tag{2.3}$$

which leaves invariant the equations of motion but rescales uniformly the action.

(iii) Scaling of the internal volume: as we are interested in dimensional reductions down to three dimensions, we consider a splitting of the D-dimensional coordinates $X^{\hat{\mu}}$ into $\{x^\mu, y^m\}$, with $\mu \in [\![1,3]\!]$ and $m \in [\![1, D-3]\!]$ and decompose the fields as in Ref. [20]:

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + A_\mu^{(1)p} G_{pq} A_\nu^{(1)q} & A_\mu^{(1)p} G_{pn} \\ G_{mp} A_\nu^{(1)p} & G_{mn} \end{pmatrix}, \tag{2.4a}$$

$$\hat{B}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} B_{\mu\nu} - A_{[\mu}^{(1)m} A_{\nu]m}^{(2)} + A_\mu^{(1)m} B_{mn} A_\nu^{(1)n} & A_{\mu n}^{(2)} - B_{np} A_\mu^{(1)p} \\ -A_{\nu m}^{(2)} + B_{mp} A_\nu^{(1)p} & B_{mn} \end{pmatrix}, \tag{2.4b}$$

$$e^{\hat{\phi}} = \sqrt{\det(G_{mn})}\, e^{\Phi}. \tag{2.4c}$$

With this decomposition, and keeping the dependence on all coordinates, the action (2.1) features the additional scaling symmetry

$$(\mathrm{Volume}^{D-3})\colon\ \begin{cases} y^m \to e^{-\lambda} y^m, \\ \Phi \to \Phi + (3-D)\lambda, \\ g_{\mu\nu} \to g_{\mu\nu}, \\ B_{\mu\nu} \to B_{\mu\nu}, \end{cases} \qquad \begin{cases} G_{mn} \to e^{2\lambda} G_{mn}, \\ B_{mn} \to e^{2\lambda} B_{mn}, \\ A_\mu^{(1)m} \to e^{-\lambda} A_\mu^{(1)m}, \\ A_{\mu m}^{(2)} \to e^{\lambda} A_{\mu m}^{(2)}, \end{cases} \tag{2.5}$$

corresponding to the GL(1) subgroup of the $\mathrm{GL}(D-3)$ action on the internal coordinates.

These three scaling symmetries are summarized, in the three-dimensional variables of Eq. (2.4), in Table I. These scaling symmetries are at the origin of three-dimensional symmetries essential to the duality enhancement.

B. Scaling symmetries in three dimensions

We now consider the three-dimensional theory, which follows from toroidal compactification of the action (2.1) on T^{D-3} and a subsequent truncation to the zero modes. Using the parametrization (2.4), this amounts to neglecting the dependence on the internal coordinates y^m. We furthermore move to the Einstein frame by rescaling the metric, $g_{\mu\nu} \to g_{\mathrm{E}\mu\nu} = e^{-2\Phi} g_{\mu\nu}$, which yields the action [20]

$$I_0 = \int \mathrm{d}^3x \sqrt{-g_{\mathrm{E}}} \Big(R_{\mathrm{E}} - \partial_\mu \Phi \partial^\mu \Phi - \frac{e^{-4\Phi}}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \mathrm{Tr}(\partial_\mu G \partial^\mu G^{-1}) + \frac{1}{4} \mathrm{Tr}(G^{-1} \partial_\mu B G^{-1} \partial^\mu B) - \frac{e^{-2\Phi}}{4} F_{\mu\nu}^{(1)m} G_{mn} F^{(1)\mu\nu n} - \frac{e^{-2\Phi}}{4} H_{\mu\nu m} G^{mn} H^{\mu\nu}{}_n \Big), \tag{2.6}$$

TABLE II. Scaling behavior of the fields (2.4) under the transformations (2.8)–(2.10). We display the charges q of each field φ, representing the transformation $\varphi \to e^{q\lambda}\varphi$.

	e^{Φ}	$g_{\mathrm{E}\mu\nu}$	$B_{\mu\nu}$	G_{mn}	B_{mn}	$A_\mu^{(1)m}$	$A_{\mu m}^{(2)}$
(Dilaton^3)	1	0	2	0	0	1	1
(Trombone^3)	0	2	2	0	0	1	1
(T duality)	0	0	0	2	2	−1	1

with

$$
\begin{aligned}
H_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} - (3/2)(A_{[\mu}^{(1)m}F_{\nu\rho]m}^{(2)} + F_{[\mu\nu}^{(1)m}A_{\rho]m}^{(2)}), \\
H_{\mu\nu m} &= F_{\mu\nu m}^{(2)} - B_{mn}F_{\mu\nu}^{(1)n},
\end{aligned} \tag{2.7}
$$

where $F_{\mu\nu}^{(1)m} = \partial_\mu A_\nu^{(1)m} - \partial_\nu A_\mu^{(1)m}$ is the abelian field strength for $A_\mu^{(1)m}$, and similarly for $A_{\mu m}^{(2)}$. This action is invariant under the following scaling symmetries:

(i) Constant dilaton shift:

$$
(\text{Dilaton}^3):\ \begin{cases} \Phi \to \Phi + \lambda, \\ g_{\mathrm{E}\mu\nu} \to g_{\mathrm{E}\mu\nu}, \\ B_{\mu\nu} \to e^{2\lambda}B_{\mu\nu}, \end{cases} \quad \begin{cases} G_{mn} \to G_{mn}, \\ B_{mn} \to B_{mn}, \\ A_\mu^{(1)m} \to e^{\lambda}A_\mu^{(1)m}, \\ A_{\mu m}^{(2)} \to e^{\lambda}A_{\mu m}^{(2)}. \end{cases} \tag{2.8}
$$

(ii) On shell “trombone” symmetry:

$$
(\text{Trombone}^3):\ \begin{cases} \Phi \to \Phi, \\ g_{\mathrm{E}\mu\nu} \to e^{2\lambda}g_{\mathrm{E}\mu\nu}, \\ B_{\mu\nu} \to e^{2\lambda}B_{\mu\nu}, \end{cases} \quad \begin{cases} G_{mn} \to G_{mn}, \\ B_{mn} \to B_{mn}, \\ A_\mu^{(1)m} \to e^{\lambda}A_\mu^{(1)m}, \\ A_{\mu m}^{(2)} \to e^{\lambda}A_{\mu m}^{(2)}. \end{cases} \tag{2.9}
$$

(iii) Internal rescaling:

$$
(T\ \text{duality}):\ \begin{cases} \Phi \to \Phi, \\ g_{\mathrm{E}\mu\nu} \to g_{\mathrm{E}\mu\nu}, \\ B_{\mu\nu} \to B_{\mu\nu}, \end{cases} \quad \begin{cases} G_{mn} \to e^{2\lambda}G_{mn}, \\ B_{mn} \to e^{2\lambda}B_{mn}, \\ A_\mu^{(1)m} \to e^{-\lambda}A_\mu^{(1)m}, \\ A_{\mu m}^{(2)} \to e^{\lambda}A_{\mu m}^{(2)}, \end{cases} \tag{2.10}
$$

which corresponds to the $\mathrm{O}(1,1)$ subgroup of the T duality group $\mathrm{O}(D-3, D-3)$.

Table II summarizes these symmetries. They do not directly arise from the reduction of the higher-dimensional scaling symmetries of Sec. II A. Rather, the scaling symmetries in three dimensions originate from the mixing of the higher-dimensional ones:

$$
\begin{aligned}
(\text{Dilaton}^3) &= (\text{Dilaton}^D) + \frac{D-3}{D-2}(\text{Trombone}^D) - (\text{Volume}^{D-3}), \\
(\text{Trombone}^3) &= (\text{Trombone}^D) - (\text{Volume}^{D-3}), \\
(T\ \text{duality}) &= (D-3)(\text{Dilaton}^D) - \frac{D-3}{D-2}(\text{Trombone}^D) + (\text{Volume}^{D-3}).
\end{aligned} \tag{2.11}
$$

C. $\mathrm{O}(d+1, d+1)$ enhancement

T duality and $\mathrm{O}(d,d)$ As already mentioned, the scaling symmetry (2.10) is part of the bigger T duality symmetry group $\mathrm{O}(D-3, D-3) = \mathrm{O}(d,d)$ (with $d = 23$ and $d = 7$ in the bosonic and heterotic cases, respectively). The invariance under $\mathrm{O}(d,d)$ is best displayed upon packaging the d^2 scalar fields G_{mn} and B_{mn} into the $\mathrm{O}(d,d)$ matrix

$$
\mathcal{H}_{MN} = \begin{pmatrix} G_{mn} - B_{mp}G^{pq}B_{qn} & B_{mp}G^{pn} \\ -G^{mp}B_{pn} & G^{mn} \end{pmatrix}, \tag{2.12}
$$

parametrizing the coset space $\mathrm{O}(d,d)/(\mathrm{O}(d)\times\mathrm{O}(d))$, and regrouping the vector fields $A_\mu^{(1)m}$ and $A_{\mu m}^{(2)}$ into a single $\mathrm{O}(d,d)$ vector

$$\mathcal{A}_\mu{}^M = \begin{pmatrix} A_\mu^{(1)m} \\ A_{\mu m}^{(2)} \end{pmatrix}. \tag{2.13}$$

The action (2.6) then takes the form [20]

$$I_0 = \int \mathrm{d}^3x\sqrt{-g_\mathrm{E}}\bigg(R_\mathrm{E} - \partial_\mu\Phi\partial^\mu\Phi + \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}) - \frac{1}{12}e^{-4\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\bigg), \tag{2.14}$$

with the field strengths $\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^M$ and $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} - (3/2)\mathcal{A}_{[\mu}{}^M\mathcal{F}_{\nu\rho]M}$. We choose the metric signature to be $(-1, 1, 1)$. The $\mathrm{O}(d,d)$ indices are raised and lowered using the $\mathrm{O}(d,d)$ -invariant metric

$$\eta^{MN} = \begin{pmatrix} 0 & \delta^m{}_n \\ \delta_m{}^n & 0 \end{pmatrix}. \tag{2.15}$$

In particular, $\mathcal{H}^{MN} = \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$ is the inverse of the generalised metric $\mathcal{H}_{MN}$.

The action (2.14) is invariant under the $\mathrm{O}(d,d)$ transformation

$$g_{\mathrm{E}\mu\nu} \to g_{\mathrm{E}\mu\nu}, \qquad \Phi \to \Phi, \qquad \mathcal{H}_{MN} \to L_M{}^P L_N{}^Q \mathcal{H}_{PQ},$$
$$\mathcal{A}_\mu{}^M \to L^M{}_N\mathcal{A}_\mu{}^N, \qquad B_{\mu\nu} \to B_{\mu\nu}, \tag{2.16}$$

with $L_M{}^N \in \mathrm{O}(d,d)$, *i.e.* $L_M{}^P L_N{}^Q \eta_{PQ} = \eta_{MN}$. In three dimensions, the three-form field strength $H_{\mu\nu\rho}$ is on shell determined by a constant, which we set to zero for now. In Sec. IV, we will explore the massive deformations arising for a non-vanishing three-form.

From $\mathrm{O}(d,d)$ *to* $\mathrm{O}(d+1,d+1)$ The action (2.14) hides a symmetry enhancement from $\mathrm{O}(d,d)$ to $\mathrm{O}(d+1,d+1)$, thanks to the duality between vector and scalar fields in three dimensions [1]. Contrary to T duality, this enhanced symmetry does not leave the dilaton invariant: It combines in particular the $\mathrm{O}(1,1)$ scaling symmetry (2.8) to the $\mathrm{O}(d,d)$ symmetry (2.16) as $\mathrm{O}(d,d)\times\mathrm{O}(1,1) \subset \mathrm{O}(d+1,d+1)$. The enhancement is made manifest by dualizing the two-form field strengths $\mathcal{F}_{\mu\nu}{}^M$ into gradients $\partial_\mu\xi_M$ of scalar fields through the introduction of a Lagrange multiplier term in the action[2]:

$$\tilde{I}_0 = I_0 + \int \mathrm{d}^3x\frac{1}{2}\varepsilon^{\mu\nu\rho}\mathcal{F}_{\mu\nu}{}^M\partial_\rho\xi_M. \tag{2.17}$$

The equations of motion of ξ_M give the Bianchi identity for $\mathcal{F}_{\mu\nu}{}^M$. There is therefore no need for the vector fields $\mathcal{A}_\mu{}^M$, and we can consider $\mathcal{F}_{\mu\nu}{}^M$ as independent fields. Their equations of motion, given by

$$\mathcal{F}_{\mu\nu}{}^M = e^{2\Phi}\epsilon_{\mu\nu\rho}\partial^\rho\xi_N\mathcal{H}^{NM}, \tag{2.18}$$

are algebraic: We can eliminate the two-forms $\mathcal{F}_{\mu\nu}{}^M$ from the action in favor of the scalars ξ_M. The action then reads

$$\tilde{I}_0 = \int \mathrm{d}^3x\sqrt{-g_\mathrm{E}}\bigg(R_\mathrm{E} - \partial_\mu\Phi\partial^\mu\Phi + \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}) - \frac{1}{2}e^{2\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\bigg). \tag{2.19}$$

The $\mathrm{O}(1,1)$ and $\mathrm{O}(d,d)$ transformations (2.8) and (2.16) of $\mathcal{A}_\mu{}^M$ imply, by use of Eq. (2.18), the following transformations of ξ_M:

$$\mathrm{O}(1,1)\colon\ \xi_M \to e^{-\lambda}\xi_M, \qquad \mathrm{O}(d,d)\colon\ \xi_M \to L_M{}^N\xi_N. \tag{2.20}$$

The action $\tilde{I}_0$ depends on the metric and on $1 + d^2 + 2d = (d+1)^2$ scalar fields. The scalar fields can be organized into the $\mathrm{O}(d+1,d+1)$ matrix

$$\mathcal{M}_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \mathcal{H}_{MN} + e^{2\Phi}\xi_M\xi_N & e^{2\Phi}\xi_M & -\mathcal{H}_{MP}\xi^P - \frac{1}{2}e^{2\Phi}\xi_M\xi_P\xi^P \\ e^{2\Phi}\xi_N & e^{2\Phi} & -\frac{1}{2}e^{2\Phi}\xi_P\xi^P \\ -\mathcal{H}_{NP}\xi^P - \frac{1}{2}e^{2\Phi}\xi_N\xi_P\xi^P & -\frac{1}{2}e^{2\Phi}\xi_P\xi^P & e^{-2\Phi} + \xi_P\mathcal{H}^{PQ}\xi_Q + \frac{1}{4}e^{2\Phi}(\xi_P\xi^P)^2 \end{pmatrix}. \tag{2.21}$$

Here, the $\mathrm{O}(d+1,d+1)$ indices are split as $\mathcal{M} = \{M, +, -\}$ with respect to $\mathrm{O}(d,d)\times\mathrm{O}(1,1)$, while the $\mathrm{O}(d+1,d+1)$-invariant metric takes the form

$$\eta_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \eta_{MN} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.22}$$

Then, the action (2.19) becomes

$$\tilde{I}_0 = \int \mathrm{d}^3x\sqrt{-g_\mathrm{E}}\bigg(R_\mathrm{E} + \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1})\bigg). \tag{2.23}$$

[2] Here and in the following, $\varepsilon_{\mu\nu\rho}$ denotes the Levi-Civita symbol, and $\epsilon_{\mu\nu\rho}$ is the associated tensor.

It is manifestly invariant under the $\mathrm{O}(d+1,d+1)$ transformation

$$g_{\mathrm{E}\mu\nu} \to g_{\mathrm{E}\mu\nu}, \qquad \mathcal{M}_{\mathcal{M}\mathcal{N}} \to L_{\mathcal{M}}{}^{\mathcal{P}} L_{\mathcal{N}}{}^{\mathcal{Q}} \mathcal{M}_{\mathcal{P}\mathcal{Q}}, \tag{2.24}$$

with $L_{\mathcal{M}}{}^{\mathcal{N}} \in \mathrm{O}(d+1,d+1)$.

$\mathrm{O}(d+1,d+1)$ *transformations* Let us have a closer look at the $\mathrm{O}(d+1,d+1)$ symmetry. The generators $T^{\mathcal{M}\mathcal{N}}$ of $\mathrm{O}(d+1,d+1)$ can be decomposed into $\mathrm{O}(d,d)$ components as

$$T^{\mathcal{M}\mathcal{N}} = \{T^{MN}, T^{M+}, T^{M-}, T^{+-}\}. \tag{2.25}$$

The T^{MN} generate the $\mathrm{O}(d,d)$ transformation

$$g_{\mathrm{E}\mu\nu} \to g_{\mathrm{E}\mu\nu}, \qquad \Phi \to \Phi, \qquad \mathcal{H}_{MN} \to L_M{}^P L_N{}^Q \mathcal{H}_{PQ}, \\ \xi_M \to L_M{}^N \xi_N, \tag{2.26}$$

while T^{+-} generates the $\mathrm{O}(1,1)$ scaling symmetry

$$g_{\mathrm{E}\mu\nu} \to g_{\mathrm{E}\mu\nu}, \qquad \Phi \to \Phi + \lambda, \qquad \mathcal{H}_{MN} \to \mathcal{H}_{MN}, \\ \xi_M \to e^{-\lambda} \xi_M. \tag{2.27}$$

The charged generators T^{M+} generate the constant shifts

$$\xi_M \to \xi_M + c_M \tag{2.28}$$

and the components T^{M-} lead to complicated non-linear transformations.

Frame formalism For later convenience, let us define the frame fields

$$\mathcal{V}_{\mathcal{M}}{}^{\mathcal{A}} = \begin{pmatrix} E_M{}^A & e^{\Phi}\xi_M & 0 \\ 0 & e^{\Phi} & 0 \\ -\xi^P E_P{}^A & -\frac{1}{2} e^{\Phi} \xi_P \xi^P & e^{-\Phi} \end{pmatrix}, \tag{2.29}$$

so that $\mathcal{M}_{\mathcal{M}\mathcal{N}} = \mathcal{V}_{\mathcal{M}}{}^{\mathcal{A}} \delta_{\mathcal{A}\mathcal{B}} \mathcal{V}_{\mathcal{N}}{}^{\mathcal{B}}$, with the $(2d+2)\times(2d+2)$ identity matrix $\delta_{\mathcal{A}\mathcal{B}}$. $E_M{}^A$ is the frame field associated to the $\mathrm{O}(d,d)$ generalized metric, i.e., $\mathcal{H}_{MN} = E_M{}^A \delta_{AB} E_N{}^B$, and we denote the inverse by $E_A{}^M$. As for the "curved" $\mathrm{O}(d+1,d+1)$ indices $\mathcal{M}$, we split the flat indices as $\mathcal{A} = \{A, \hat{+}, \hat{-}\}$. These flat indices are raised and lowered using the flat version of the invariant tensor (2.22):

$$\eta_{\mathcal{A}\mathcal{B}} = \mathcal{V}_{\mathcal{A}}{}^{\mathcal{M}} \eta_{\mathcal{M}\mathcal{N}} \mathcal{V}_{\mathcal{B}}{}^{\mathcal{N}} = \begin{pmatrix} \eta_{AB} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.30}$$

with $\mathcal{V}_{\mathcal{A}}{}^{\mathcal{M}}$ the inverse frame field and $\eta_{AB} = E_A{}^M \eta_{MN} E_B{}^N$. The frame fields can be used to define the Maurer-Cartan form $(\mathcal{V}^{-1}\partial_\mu \mathcal{V})_{\mathcal{A}}{}^{\mathcal{B}} = \mathcal{P}_{\mu\mathcal{A}}{}^{\mathcal{B}} + \mathcal{Q}_{\mu\mathcal{A}}{}^{\mathcal{B}}$, where

$$\mathcal{P}_{\mu\mathcal{A}}{}^{\mathcal{B}} = \begin{pmatrix} P_{\mu A}{}^B & \frac{1}{2} e^{\Phi} E_A{}^M \partial_\mu \xi_M & -\frac{1}{2} e^{\Phi} \partial_\mu \xi^M E_M{}^C \delta_{CA} \\ \frac{1}{2} e^{\Phi} \partial_\mu \xi_M E_C{}^M \delta^{CB} & \partial_\mu \Phi & 0 \\ -\frac{1}{2} e^{\Phi} \partial_\mu \xi^M E_M{}^B & 0 & -\partial_\mu \Phi \end{pmatrix}, \tag{2.31}$$

$$\mathcal{Q}_{\mu\mathcal{A}}{}^{\mathcal{B}} = \begin{pmatrix} Q_{\mu A}{}^B & \frac{1}{2} e^{\Phi} E_A{}^M \partial_\mu \xi_M & \frac{1}{2} e^{\Phi} \partial_\mu \xi^M E_M{}^C \delta_{CA} \\ -\frac{1}{2} e^{\Phi} \partial_\mu \xi_M E_C{}^M \delta^{CB} & 0 & 0 \\ -\frac{1}{2} e^{\Phi} \partial_\mu \xi^M E_M{}^B & 0 & 0 \end{pmatrix}, \tag{2.32}$$

such that $(\mathcal{P}_\mu \delta)_{\mathcal{A}\mathcal{B}}$ and $(\mathcal{Q}_\mu \delta)_{\mathcal{A}\mathcal{B}}$ are symmetric and antisymmetric, respectively. Sometimes it is also convenient to use the basis in which $\eta_{\mathcal{A}\mathcal{B}}$ is diagonal and has the form

$$\eta_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} -\delta_{ab} & 0 \\ 0 & \delta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{2.33}$$

with the index split $\mathcal{A} \to \{a, \bar{a}\}, a, \bar{a} \in [\![0, d]\!]$. In this basis, the Maurer-Cartan components $\mathcal{P}_\mu{}^{\mathcal{A}\mathcal{B}}$ can be expressed in terms of an $\mathrm{O}(d+1)$ bivector $\mathsf{P}_\mu{}^{a\bar{a}}$:

$$\mathcal{P}_\mu{}^{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & \mathsf{P}_\mu{}^{a\bar{b}} \\ \mathsf{P}^{\mathrm{t}}_\mu{}^{\bar{a}b} & 0 \end{pmatrix}. \tag{2.34}$$

D. Field redefinitions

The presence of higher-derivative corrections makes it possible to perform field redefinitions that are perturbative in α', and previous works showed that they are necessary to exhibit duality symmetries [8]. We denote by $\tilde{I}_1$ the part of the action of order α' (after dualization of the vector fields), so that $\tilde{I} = \tilde{I}_0 + \tilde{I}_1 + \mathcal{O}(\alpha'^2)$ is the total action. In the same manner as in Refs. [9,10], we consider field redefinitions of the form

$$\varphi \to \varphi + \alpha' \delta\varphi, \tag{2.35}$$

where φ denotes a generic field. Under such redefinitions, the variation of $\tilde{I}_0$ is

TABLE III. Replacement rules for the terms carrying the leading two-derivative contribution from the field equations descending from the two-derivative action (2.19) and associated field redefinitions. The explicit replacement rules are given in Eq. (2.38).

Term in the action	Field redefinitions	Replacement
$\alpha' X\Box\Phi$	$\delta\Phi = -\frac{1}{2}X$	$\alpha' X Q_\Phi$
$\alpha' X^{\mu\nu} R_{\mathrm{E}\mu\nu}$	$\delta g_{\mathrm{E}}^{\mu\nu} = -X^{(\mu\nu)} + g_{\mathrm{E}}^{\mu\nu} X_\rho{}^\rho$	$\alpha' X^{\mu\nu} Q_{g\mu\nu}$
$\alpha' \mathrm{Tr}(X\Box\mathcal{H})$	$\delta\mathcal{H}^{MN} = 4X^{MN}$	$\alpha' \mathrm{Tr}(XQ_{\mathcal{H}})$
$\alpha' X^M \Box\xi_M$	$\delta\xi_M = -e^{-2\Phi}\mathcal{H}_{MN}X^N$	$\alpha' X^M Q_{\xi M}$

$$\delta\tilde{I}_0 = \alpha' \int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}\Big[\delta\Phi E_\Phi + \delta g_{\mathrm{E}}^{\mu\nu} E_{g\mu\nu} + \mathrm{Tr}(\delta\mathcal{H}^{-1}E_{\mathcal{H}}) + E_\xi{}^M \delta\xi_M\Big], \tag{2.36}$$

where[3]

$$E_\Phi = 2\Box\Phi - e^{2\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N, \tag{2.37a}$$

$$E_{g\mu\nu} = R_{\mathrm{E}\mu\nu} - \partial_\mu\Phi\partial_\nu\Phi + \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}) - \frac{1}{2}e^{2\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N \\ - \frac{1}{2}g_{\mathrm{E}\mu\nu}\Big(R_{\mathrm{E}} - \partial_\rho\Phi\partial^\rho\Phi + \frac{1}{8}\mathrm{Tr}(\partial_\rho\mathcal{H}\partial^\rho\mathcal{H}^{-1}) - \frac{1}{2}e^{2\Phi}\partial_\rho\xi_M\mathcal{H}^{MN}\partial^\rho\xi_N\Big), \tag{2.37b}$$

$$E_{\mathcal{H}MN} = -\frac{1}{4}\Big[\Box\mathcal{H}_{MN} + (\mathcal{H}\partial_\mu\mathcal{H}^{-1}\partial^\mu\mathcal{H})_{MN} + e^{2\Phi}\partial_\mu\xi_M\partial^\mu\xi_N - e^{2\Phi}\mathcal{H}_{MP}\partial_\mu\xi^P\partial^\mu\xi^Q\mathcal{H}_{QN}\Big], \tag{2.37c}$$

$$E_\xi{}^M = e^{2\Phi}\Big(\Box\xi_N\mathcal{H}^{NM} + 2\partial_\mu\Phi\partial^\mu\xi_N\mathcal{H}^{NM} + \partial_\mu\xi_N\partial^\mu\mathcal{H}^{NM}\Big), \tag{2.37d}$$

and $\Box = \nabla_\mu\nabla^\mu$. As we will not consider orders in α' higher than one, there is no need to compute how the redefinition affects corrections $\tilde{I}_1$ to $\tilde{I}_0$ of order $\mathcal{O}(\alpha')$: This variation will generate $\mathcal{O}(\alpha'^2)$ terms. The expressions of the shifts $\delta\varphi$ can then be chosen to cancel given terms in $\tilde{I}_1$. In the following, we will use these redefinitions to cancel terms that contain as factors the leading two-derivative contributions from the field equations, as was done in Ref. [12]. These factors can be replaced as follows:

$$\begin{aligned} \Box\Phi &\longrightarrow Q_\Phi = \frac{1}{2}e^{2\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N, \\ R_{\mathrm{E}\mu\nu} &\longrightarrow Q_{g\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}) + \frac{1}{2}e^{2\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N, \\ \Box\mathcal{H}_{MN} &\longrightarrow Q_{\mathcal{H}MN} = -(\mathcal{H}\partial_\mu\mathcal{H}^{-1}\partial^\mu\mathcal{H})_{MN} - e^{2\Phi}\partial_\mu\xi_M\partial^\mu\xi_N + e^{2\Phi}\mathcal{H}_{MP}\partial_\mu\xi^P\partial^\mu\xi^Q\mathcal{H}_{QN}, \\ \Box\xi_M &\longrightarrow Q_{\xi M} = -2\partial_\mu\Phi\partial^\mu\xi_M - \partial_\mu\xi_N(\partial^\mu\mathcal{H}^{-1}\mathcal{H})^N{}_M. \end{aligned} \tag{2.38}$$

These replacements and the associated field redefinitions are summed up in Table III.

[3]Note that, as $\mathcal{H}\eta\mathcal{H} = \eta$, $\mathcal{H}$ is a constrained field and that the derivation of $E_{\mathcal{H}}$ by variation of the action must be done carefully [10].

III. FOUR-DERIVATIVE ACTION IN THREE DIMENSIONS

We now revisit the symmetry enhancement of Sec. II in the presence of first order α' corrections. To this end, we start from the manifestly $\mathrm{O}(d,d)$ invariant four-derivative action of Ref. [12] and perform the dualization of the vector fields as in Eq. (2.18) above. We then express the resulting action in terms of $\mathrm{O}(d+1,d+1)$ quantities upon introduction of a nondynamical compensator, which reveals an interesting structure of the action. Higher-order corrections obstruct the $\mathrm{O}(d+1,d+1)$ symmetry enhancement of the two-derivative action as is most straightforwardly seen by tracking the fate of the scaling symmetries discussed in Sec. II. Typical four-derivative corrections to the action (2.1) are of the form [21]

$$I_1 \propto \alpha' \int \mathrm{d}^D X \sqrt{-\hat{g}} e^{-\hat{\phi}} \times \left(\hat{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{R}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} - \frac{1}{2} \hat{R}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\lambda}} \hat{H}^{\hat{\rho}\hat{\sigma}}{}_{\hat{\lambda}} + \dots \right). \tag{3.1}$$

Under the transformations (2.2), (2.3), and (2.5), the action I_1 transform homogeneously with the charges[4]

	(Dilaton^D)	(Trombone^D)	(Volume^{D-3})
I_1	$-2/(D-2)$	-2	0

(3.2)

Both the dilaton shift and the trombone symmetries are broken by α' corrections. With the translation (2.11), the charges of I_1 under the three-dimensional symmetries (2.8)–(2.10) are

	(Dilaton^3)	(Trombone^3)	$(T\text{ duality})$
I_1	-2	-2	0

(3.3)

In particular, the T duality scaling $\mathrm{O}(1,1) \subset \mathrm{O}(d,d)$ is preserved in presence of first order α' corrections, in agreement with the arguments of Ref. [5]. The presence of $\mathrm{O}(d,d)$ was explicitly verified in Refs. [11,12] at first order in α'. The symmetry under the dilaton shift (2.8), however, is broken, and so is the symmetry enhancement from $\mathrm{O}(d,d)$ to $\mathrm{O}(d+1,d+1)$. In the following, we formulate the α' corrections to the three-dimensional action (2.23) in terms of the $\mathrm{O}(d+1,d+1)$ objects defined in the previous section, by introduction of a nondynamical compensator.

A. Dualization of the vector fields

We will treat the cases of the bosonic and heterotic string effective actions at the same time, using the notations of Ref. [16]. Our starting point is the manifestly $\mathrm{O}(d,d)$ invariant action of Ref. [12]

$$\begin{aligned} I = \int \mathrm{d}^3x \sqrt{-g} e^{-\Phi} \Bigg[& R + \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{8} \mathrm{Tr}(\partial_\mu \mathcal{H} \partial^\mu \mathcal{H}^{-1}) - \frac{1}{4} \mathcal{F}_{\mu\nu}{}^M \mathcal{H}_{MN} \mathcal{F}^{\mu\nu N} \\ & - \frac{a+b}{8} \Bigg(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{16} \mathrm{Tr}(\partial_\mu \mathcal{H} \partial_\nu \mathcal{H}^{-1} \partial^\mu \mathcal{H} \partial^\nu \mathcal{H}^{-1}) - \frac{1}{32} \mathrm{Tr}(\partial_\mu \mathcal{H} \partial_\nu \mathcal{H}^{-1}) \mathrm{Tr}(\partial^\mu \mathcal{H} \partial^\nu \mathcal{H}^{-1}) \\ & + \frac{1}{8} \mathcal{F}_{\mu\nu}{}^M \mathcal{H}_{MN} \mathcal{F}^N_{\rho\sigma} \mathcal{F}^{\mu\rho P} \mathcal{H}_{PQ} \mathcal{F}^{\nu\sigma Q} - \frac{1}{2} \mathcal{F}_{\mu\nu}{}^M \mathcal{H}_{MN} \mathcal{F}^{\mu\rho N} \mathcal{F}^{\nu\sigma P} \mathcal{H}_{PQ} \mathcal{F}_{\rho\sigma}{}^Q \\ & + \frac{1}{8} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}_{\rho\sigma M} \mathcal{F}^{\mu\rho N} \mathcal{F}^{\nu\sigma}{}_N - \frac{1}{2} R_{\mu\nu\rho\sigma} \mathcal{F}^{\mu\nu M} \mathcal{H}_{MN} \mathcal{F}^{\rho\sigma N} \\ & - \frac{1}{2} \mathcal{F}_{\mu\nu}{}^M (\mathcal{H} \partial_\rho \mathcal{H}^{-1} \partial^\nu \mathcal{H})_{MN} \mathcal{F}^{\mu\rho N} + \frac{1}{4} \mathcal{F}^{\mu\rho M} \mathcal{H}_{MN} \mathcal{F}^\nu{}_\rho{}^N \mathrm{Tr}(\partial_\mu \mathcal{H} \partial_\nu \mathcal{H}^{-1}) \Bigg) \\ & + \frac{a-b}{4} \Bigg(-\frac{1}{16} \mathrm{Tr}(\partial_\mu \mathcal{H} \partial^\mu \mathcal{H}^{-1} \partial_\nu \mathcal{H} \partial^\nu \mathcal{H}^{-1} \mathcal{H} \eta) - \frac{1}{16} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}_{\rho\sigma M} \mathcal{F}^{\mu\nu P} \mathcal{H}_{PQ} \mathcal{F}^{\rho\sigma Q} \\ & + \frac{1}{4} R^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}_{\rho\sigma M} + \frac{1}{8} \mathcal{F}_{\mu\nu}{}^M (\partial_\rho \mathcal{H} \partial^\rho \mathcal{H}^{-1})_M{}^N \mathcal{F}^{\mu\nu}{}_N \\ & + \frac{1}{4} \mathcal{F}_{\mu\nu}{}^M (\partial^\mu \mathcal{H} \partial_\rho \mathcal{H}^{-1})_M{}^N \mathcal{F}^{\nu\rho}{}_N \Bigg) + \mathcal{O}(\alpha'^2) \Bigg], \end{aligned} \tag{3.4}$$

having eliminated the two-form $B_{\mu\nu}$ by virtue of its three-dimensional field equations as in (2.19) above. The bosonic and heterotic actions correspond to $(a,b) = (-\alpha', -\alpha')$ and $(a,b) = (-\alpha', 0)$, respectively [16].

In order to express this action in terms of $\mathrm{O}(d+1,d+1)$ covariant objects, we first switch to the Einstein frame

$$g_{\mu\nu} \to g_{\mathrm{E}\mu\nu} = e^{-2\Phi} g_{\mu\nu}, \tag{3.5}$$

and use the fact that in three dimensions, the Riemann tensor can be expressed as

[4] In Eq. (3.2) and (3.3), we give the charges of I_1 under the trombone symmetries using as convention that the charges of $I_0^{(D)}$ and I_0 under these transformations are 0; i.e., we drop the global factor under which the lowest order equations of motion are rescaled.

$$R_{\mu\nu\rho\sigma} = S_{\mu\rho}g_{\nu\sigma} + S_{\nu\sigma}g_{\mu\rho} - S_{\mu\sigma}g_{\nu\rho} - S_{\nu\rho}g_{\mu\sigma}, \quad (3.6)$$

in terms of the Schouten tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}$.

Next, we dualize the vectorial degrees of freedom to scalar ones. To this end, we proceed as we did for the two-derivative action, and introduce a Lagrange multiplier term to the action:

$$\tilde{I} = I + \frac{1}{2}\int \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\mathcal{F}_{\mu\nu}{}^M\partial_\rho\xi_M. \quad (3.7)$$

The equations of motion of ξ_M still gives the Bianchi identities for $\mathcal{F}_{\mu\nu}{}^M$. But, considering $\mathcal{F}_{\mu\nu}{}^M$ as an independent field, its equations of motion $\delta\tilde{I}/\delta\mathcal{F} = 0$ now contain corrections of order α'. These equations are algebraic in $\mathcal{F}$ and can be solved perturbatively in α'. The solution takes the form

$$\mathcal{F}_{\mu\nu}{}^M = \mathcal{F}^{(0)M}_{\mu\nu} + \alpha'\mathcal{F}^{(1)M}_{\mu\nu}, \quad (3.8)$$

where $\mathcal{F}^{(0)M}_{\mu\nu}$ is a solution of $\delta\tilde{I}_0/\delta\mathcal{F} = 0$, as given in Eq. (2.18). The exact expression of $\mathcal{F}^{(1)M}_{\mu\nu}$ is not necessary for our purpose, as we now show. Equation (3.8) is algebraic and can be introduced in the action (3.7), which, schematically, takes the following form:

$$\tilde{I}(\mathcal{F}^{(0)} + \alpha'\mathcal{F}^{(1)}) = \tilde{I}_0(\mathcal{F}^{(0)}) + I_1(\mathcal{F}^{(0)}) + \alpha'\mathcal{F}^{(1)}\frac{\delta\tilde{I}_0}{\delta\mathcal{F}}(\mathcal{F}^{(0)}) + \mathcal{O}(\alpha'^2), \quad (3.9)$$

where I_1 is the four-derivative action. The dependence on $\mathcal{F}^{(1)}$ is thus proportional to the equation of motion of $\mathcal{F}$ at order α'^0, evaluated at its solution $\mathcal{F}^{(0)}$. Then, at first order in α', the corrections to the duality relation (2.18) cancel out in the action, and the lowest order relation can be used to dualize the vectors.

Applying this procedure, after some computation, the four-derivative part of the action (3.4) turns into

$$\begin{aligned}
\tilde{I}_1 = \int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}e^{-2\Phi}\Big[&-\frac{a+b}{8}\Big(\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1}) - \frac{1}{32}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&+ 4\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi + \frac{1}{4}e^{4\Phi}\partial_\mu\xi^M\partial_\nu\xi_M\partial^\mu\xi^N\partial^\nu\xi_N\\
&- \frac{1}{4}e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q - \frac{1}{2}e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\partial_\nu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q\\
&+ \frac{1}{2}e^{2\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})^{MN}\partial^\mu\xi_N - \frac{1}{2}e^{2\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})^{MN}\partial^\nu\xi_N\\
&- \frac{1}{4}e^{2\Phi}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1})\partial_\nu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N + \frac{1}{4}e^{2\Phi}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N\\
&- 2e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N + 4R_{\mathrm{E}\mu\nu}R_{\mathrm{E}}^{\mu\nu} - R_{\mathrm{E}}^2 + 4\Box\Phi\Box\Phi + 8\Box\Phi\partial_\mu\Phi\partial^\mu\Phi\\
&+ 4\nabla_\mu\nabla_\nu\Phi\nabla^\mu\nabla^\nu\Phi - 8\nabla_\mu\nabla_\nu\Phi\partial^\mu\Phi\partial^\nu\Phi - 8R_{\mathrm{E}\mu\nu}\nabla^\mu\nabla^\nu\Phi + 8R_{\mathrm{E}\mu\nu}\partial^\mu\Phi\partial^\nu\Phi\\
&- 4R_{\mathrm{E}}\partial_\mu\Phi\partial^\mu\Phi - 2e^{2\Phi}R_{\mathrm{E}\mu\nu}\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N + e^{2\Phi}R_{\mathrm{E}}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\\
&- 2e^{2\Phi}\Box\Phi\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N + 2e^{2\Phi}\nabla_\mu\nabla_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N\Big)\\
&+ \frac{a-b}{4}\Big(-\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}\mathcal{H}\eta) - \frac{1}{4}e^{4\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q\\
&- \frac{1}{2}e^{2\Phi}R_{\mathrm{E}}\partial_\mu\xi_M\partial^\mu\xi^M + e^{2\Phi}R_{\mathrm{E}\mu\nu}\partial^\mu\xi_M\partial^\nu\xi^M - e^{2\Phi}\nabla_\mu\nabla_\nu\Phi\partial^\mu\xi_M\partial^\nu\xi^M\\
&+ e^{2\Phi}\Box\Phi\partial_\mu\xi_M\partial^\mu\xi^M + e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\partial^\nu\xi^M - \frac{1}{4}e^{2\Phi}\partial_\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H})^M{}_N\partial^\nu\xi^N\Big)\Big]. \quad (3.10)
\end{aligned}$$

We now convert all second order derivatives in Eq. (3.10) into products of first order derivatives to allow comparison with the basis of $\mathrm{O}(d+1, d+1)$-invariant four-derivative terms of Ref. [12]. To do so, we first use integrations by parts so that all the second order derivatives appear in the leading two-derivative contribution of the equations of motion (2.37). Using furthermore the field redefinitions of Sec. II D, we obtain the action

$$
\begin{aligned}
\tilde{I}_1 = \int \mathrm{d}^3x\sqrt{-g_\mathrm{E}}e^{-2\Phi}\bigg[&-\frac{a+b}{8}\bigg(-\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi+\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&+\frac{1}{32}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-\frac{1}{64}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1})\mathrm{Tr}(\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&+\frac{1}{4}e^{4\Phi}\partial_\mu\xi^M\partial_\nu\xi_M\partial^\mu\xi^N\partial^\nu\xi_N-\frac{1}{4}e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q\\
&+\frac{1}{4}e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\partial_\nu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q+\frac{1}{2}e^{2\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})^{MN}\partial^\mu\xi_N\\
&-\frac{1}{2}e^{2\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})^{MN}\partial^\nu\xi_N-2e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N\\
&+\frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-\frac{1}{4}\partial_\mu\Phi\partial^\mu\Phi\mathrm{Tr}(\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})+e^{2\Phi}\partial_\mu\Phi\partial_\nu\xi_M\partial^\mu\mathcal{H}^{MN}\partial^\nu\xi_N\bigg)\\
&+\frac{a-b}{4}\bigg(-\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}\mathcal{H}\eta)+\frac{1}{4}e^{4\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q\\
&-\frac{1}{4}e^{2\Phi}\partial_\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H})^M{}_N\partial^\nu\xi^N-\frac{1}{2}e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\xi_M\partial^\nu\xi^M-\frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\partial^\mu\xi_M\partial^\nu\xi^M\\
&+\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1})\partial_\nu\xi_M\partial^\nu\xi^M+e^{2\Phi}\partial_\mu\Phi\partial^\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\mathcal{H})^M{}_N\partial^\nu\xi^N\bigg)\bigg]. \qquad (3.11)
\end{aligned}
$$

Explicitly, we have used the following order α' field redefinitions:

$$
\begin{aligned}
\delta g_\mathrm{E}^{\mu\nu} &= -\frac{a+b}{8\alpha'}\bigg(-4e^{-2\Phi}R_\mathrm{E}^{\mu\nu}+2e^{-2\Phi}g_\mathrm{E}^{\mu\nu}R_\mathrm{E}+8e^{-2\Phi}\partial^\mu\Phi\partial^\nu\Phi-2e^{-2\Phi}g_\mathrm{E}^{\mu\nu}\partial_\rho\Phi\partial^\rho\Phi\\
&\quad+\frac{1}{2}e^{-2\Phi}\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-\frac{1}{4}e^{-2\Phi}g_\mathrm{E}^{\mu\nu}\mathrm{Tr}(\partial_\rho\mathcal{H}\partial^\rho\mathcal{H}^{-1})-3g_\mathrm{E}^{\mu\nu}\partial_\rho\xi_M\mathcal{H}^{MN}\partial^\rho\xi_N\bigg)-\frac{a-b}{4\alpha'}\partial^\mu\xi_M\partial^\nu\xi^M,\\
\delta\Phi &= -\frac{a+b}{8\alpha'}\bigg(2e^{-2\Phi}R_\mathrm{E}-4e^{-2\Phi}\Box\Phi-\frac{3}{2}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\bigg)-\frac{a-b}{4\alpha'}\frac{1}{4}\partial_\mu\xi_M\partial^\mu\xi^M,\\
\delta\xi_M &= -\frac{a+b}{4\alpha'}e^{-2\Phi}\partial_\mu\Phi\partial^\mu\xi_M-\frac{a-b}{4\alpha'}e^{-2\Phi}\mathcal{H}_{MN}\partial_\mu\Phi\partial^\mu\xi^N, \qquad (3.12)
\end{aligned}
$$

in the notations of Eq. (2.35).

B. $\mathrm{O}(d+1,d+1)$-covariant formulation

We now aim to express the action (3.11) in terms of $\mathrm{O}(d+1,d+1)$-covariant objects. We define the $\mathrm{O}(d+1,d+1)$ currents $\mathcal{J}_\mu=\partial_\mu\mathcal{M}\mathcal{M}^{-1}$ for the matrix $\mathcal{M}$ from Eq. (2.21). Explicitly, this current takes the form

$$
\mathcal{J}_{\mu\mathcal{M}}{}^{\mathcal{N}} = \begin{pmatrix} \mathcal{J}_{\mu M}{}^N & \mathcal{J}_{\mu M}{}^+ & -e^{2\Phi}\partial_\mu\xi_P\mathcal{H}^P{}_M \\ e^{2\Phi}\partial_\mu\xi_P\mathcal{H}^{PN} & 2\partial_\mu\Phi-e^{2\Phi}\xi_P\mathcal{H}^{PQ}\partial_\mu\xi_Q & 0 \\ \mathcal{J}_{\mu-}{}^N & 0 & -2\partial_\mu\Phi+e^{2\Phi}\xi_P\mathcal{H}^{PQ}\partial_\mu\xi_Q \end{pmatrix}, \qquad (3.13)
$$

with

$$
\begin{aligned}
\mathcal{J}_{\mu M}{}^N &= \partial_\mu\mathcal{H}_{MP}\mathcal{H}^{PN}+e^{2\Phi}(\xi_M\partial_\mu\xi_P\mathcal{H}^{PN}-\mathcal{H}_{MP}\partial_\mu\xi^P\xi^N),\\
\mathcal{J}_{\mu M}{}^+ &= \partial_\mu\xi_M-\partial_\mu\mathcal{H}_{MP}\mathcal{H}^{PQ}\xi_Q+2\partial_\mu\Phi\xi_M-e^{2\Phi}\xi_P\mathcal{H}^{PQ}\partial_\mu\xi_Q\xi_M+\frac{1}{2}e^{2\Phi}\xi_P\xi^P\partial_\mu\xi_Q\mathcal{H}^Q{}_M,\\
\mathcal{J}_{\mu-}{}^N &= -\partial_\mu\xi^N-\xi^P\partial_\mu\mathcal{H}_{PQ}\mathcal{H}^{QN}-2\partial_\mu\Phi\xi^N+e^{2\Phi}\xi_P\mathcal{H}^{PQ}\partial_\mu\xi_Q\xi^N-\frac{1}{2}e^{2\Phi}\xi_P\xi^P\partial_\mu\xi_Q\mathcal{H}^{QN}. \qquad (3.14)
\end{aligned}
$$

In terms of this object and using the $\mathrm{O}(d,d)$ decomposition of Appendix A, the four-derivative action (3.11) can be cast into the rather compact form

$$\begin{aligned}\tilde{I}_1 = \int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}e^{-2\Phi}\Big\{&+\frac{a}{4}\Big[-\frac{1}{32}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\mu\mathcal{J}^\nu)-\frac{1}{16}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{M}\eta)\\
&-\frac{1}{64}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu)\mathrm{Tr}(\mathcal{J}^\mu\mathcal{J}^\nu)+\frac{1}{128}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu)\mathrm{Tr}(\mathcal{J}_\nu\mathcal{J}^\nu)\\
&+e^{-2\Phi}\Big(-\frac{1}{2}(\mathsf{u}P\eta\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\mu\mathcal{J}^\nu P\mathsf{u})+\frac{1}{2}(\mathsf{u}P\eta\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu P\mathsf{u})-\frac{1}{2}(\mathsf{u}P\eta\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{J}^\mu P\mathsf{u})\\
&-\frac{1}{4}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu)(\mathsf{u}P\eta\mathcal{J}^\mu\mathcal{J}^\nu P\mathsf{u})+\frac{1}{8}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu)(\mathsf{u}P\eta\mathcal{J}_\nu\mathcal{J}^\nu P\mathsf{u})\Big)\\
&+e^{-4\Phi}(-2(\mathsf{u}P\eta\mathcal{J}_\mu\mathcal{J}_\nu P\mathsf{u})(\mathsf{u}P\eta\mathcal{J}^\mu\mathcal{J}^\nu P\mathsf{u})+(\mathsf{u}P\eta\mathcal{J}_\mu\mathcal{J}^\mu P\mathsf{u})(\mathsf{u}P\eta\mathcal{J}_\nu\mathcal{J}^\nu P\mathsf{u}))\Big]\\
&+\frac{b}{4}\Big[-\frac{1}{32}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\mu\mathcal{J}^\nu)+\frac{1}{16}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{M}\eta)\\
&-\frac{1}{64}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu)\mathrm{Tr}(\mathcal{J}^\mu\mathcal{J}^\nu)+\frac{1}{128}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu)\mathrm{Tr}(\mathcal{J}_\nu\mathcal{J}^\nu)\\
&+e^{-2\Phi}\Big(\frac{1}{2}(\mathsf{u}\bar{P}\eta\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\mu\mathcal{J}^\nu\bar{P}\mathsf{u})-\frac{1}{2}(\mathsf{u}\bar{P}\eta\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu\bar{P}\mathsf{u})+\frac{1}{2}(\mathsf{u}\bar{P}\eta\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{J}^\mu\bar{P}\mathsf{u})\\
&+\frac{1}{4}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu)(\mathsf{u}\bar{P}\eta\mathcal{J}^\mu\mathcal{J}^\nu\bar{P}\mathsf{u})-\frac{1}{8}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu)(\mathsf{u}\bar{P}\eta\mathcal{J}_\nu\mathcal{J}^\nu\bar{P}\mathsf{u})\Big)\\
&+e^{-4\Phi}(-2(\mathsf{u}\bar{P}\eta\mathcal{J}_\mu\mathcal{J}_\nu\bar{P}\mathsf{u})(\mathsf{u}\bar{P}\eta\mathcal{J}^\mu\mathcal{J}^\nu\bar{P}\mathsf{u})+(\mathsf{u}\bar{P}\eta\mathcal{J}_\mu\mathcal{J}^\mu\bar{P}\mathsf{u})(\mathsf{u}\bar{P}\eta\mathcal{J}_\nu\mathcal{J}^\nu\bar{P}\mathsf{u}))\Big]\Big\}. \qquad (3.15)\end{aligned}$$

Here, we have defined the projectors[5]

$$P_{\mathcal{M}\mathcal{N}} = \frac{1}{2}(\eta_{\mathcal{M}\mathcal{N}}-\mathcal{M}_{\mathcal{M}\mathcal{N}}) \quad \text{and} \quad \bar{P}_{\mathcal{M}\mathcal{N}} = \frac{1}{2}(\eta_{\mathcal{M}\mathcal{N}}+\mathcal{M}_{\mathcal{M}\mathcal{N}}), \qquad (3.16)$$

and the $\mathrm{O}(d+1,d+1)$ compensator vector

$$\mathsf{u}^{\mathcal{M}} = \{0,1,0\}, \qquad (3.17)$$

which parametrizes the breaking of the symmetry group $\mathrm{O}(d+1,d+1)$ at the four-derivative order. The action (3.15) enjoys a formal $\mathrm{O}(d+1,d+1)$ invariance, which is broken by the explicit choice (3.17) for the vector $\mathsf{u}^{\mathcal{M}}$. This is manifest in the above form for all terms except the explicit dilaton prefactor, but it even holds for these factor thanks to the relations (3.22) below.

It is interesting to note that the above action (3.15) has the chiral structure

$$\tilde{I}_1 = \frac{1}{4}\int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}\{a\mathcal{F}[\mathcal{M},P\mathsf{u}]+b(\mathcal{F}[\mathcal{M},P\mathsf{u}])^*\}, \qquad (3.18)$$

with a fixed function $\mathcal{F}$ that depends only on $\mathcal{M}$ and a projection of u. The * in the second term indicates the $\mathbb{Z}_2$ action, under which $\mathcal{M}$ transforms as

$$\mathcal{M}\to\mathcal{Z}^{\mathrm{t}}\mathcal{M}\mathcal{Z}, \qquad \mathcal{Z}=\begin{pmatrix} Z & 0\\ 0 & \sigma_3\end{pmatrix}, \qquad Z=\begin{pmatrix} \mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}, \qquad (3.19)$$

where

$$\mathcal{Z}^{\mathrm{t}}\eta\mathcal{Z} = -\eta. \qquad (3.20)$$

Consequently, the projectors transform as

$$P\to\mathcal{Z}^{\mathrm{t}}\bar{P}\mathcal{Z}, \qquad \bar{P}\to\mathcal{Z}^{\mathrm{t}}P\mathcal{Z}, \qquad (3.21)$$

while furthermore,

$$e^{2\Phi} = \mathrm{u}^M\mathcal{M}_{MN}\mathrm{u}^N = -2\mathrm{u}^M P_{MN}\mathrm{u}^N = 2\mathrm{u}^M\bar{P}_{MN}\mathrm{u}^N. \qquad (3.22)$$

In particular, the last equation shows that the dilaton Φ is invariant under the $\mathbb{Z}_2$ action. The chiral form (3.18) of the

[5] If we had not truncated the vector fields from the 10-dimensional heterotic action, $P_{\mathcal{M}\mathcal{N}}$ and $\bar{P}_{\mathcal{M}\mathcal{N}}$ would correspond to projection on O(24) and O(8), respectively.

action implies that the four-derivative action of the heterotic string ($b = 0$) encodes the function $\mathcal{F}$, and thereby the entire action. In particular, the result for the bosonic string ($a = b$) follows or can be reconstructed from the heterotic result. Moreover, in the heterotic case, we note the formal "gauge invariance"

$$\mathsf{u} \to \mathsf{u} + \eta \bar{P} \Lambda, \tag{3.23}$$

under which the action (3.15) (for $b = 0$) is invariant.

We may also express the result in terms of the $\mathrm{O}(d+1, d+1)$ coset currents (2.31). In the basis (2.34), the four-derivative action (3.15) takes the form

$$\begin{aligned}\tilde{I}_1 = \frac{\alpha'}{4}\int d^3x\sqrt{-g}e^{-2\Phi}\Bigg\{&+\frac{a}{4}\Big[\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^{t\mu}\mathsf{P}_\nu \mathsf{P}^{t\nu}) - \mathrm{Tr}(\mathsf{P}^t_\mu \mathsf{P}^\mu \mathsf{P}^t_\nu \mathsf{P}^\nu) - \mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^t_\nu \mathsf{P}^\mu \mathsf{P}^{t\nu}) - \mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^t_\nu)\mathrm{Tr}(\mathsf{P}^\mu \mathsf{P}^{t\nu})\\
&+\frac{1}{2}\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^{t\mu})\mathrm{Tr}(\mathsf{P}_\nu \mathsf{P}^{t\nu}) + 4(\mathsf{P}_\mu \mathsf{P}^t_\nu \mathsf{P}^\mu \mathsf{P}^{t\nu})^{00} - 4(\mathsf{P}_\mu \mathsf{P}^{t\mu}\mathsf{P}_\nu \mathsf{P}^{t\nu})^{00} + 4(\mathsf{P}_\mu \mathsf{P}^t_\nu \mathsf{P}^\nu \mathsf{P}^{t\mu})^{00}\\
&+4\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^t_\nu)(\mathsf{P}^\mu \mathsf{P}^{t\nu})^{00} - 2\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^{t\mu})(\mathsf{P}_\nu \mathsf{P}^{t\nu})^{00} - 8(\mathsf{P}_\mu \mathsf{P}^t_\nu)^{00}(\mathsf{P}^\mu \mathsf{P}^{t\nu})^{00}\\
&+4(\mathsf{P}_\mu \mathsf{P}^{t\mu})^{00}(\mathsf{P}_\nu \mathsf{P}^{t\nu})^{00}\Big]\\
&+\frac{b}{4}\Big[-\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^{t\mu}\mathsf{P}_\nu \mathsf{P}^{t\nu}) + \mathrm{Tr}(\mathsf{P}^t_\mu \mathsf{P}^\mu \mathsf{P}^t_\nu \mathsf{P}^\nu) - \mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^t_\nu \mathsf{P}^\mu \mathsf{P}^{t\nu}) - \mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^t_\nu)\mathrm{Tr}(\mathsf{P}^\mu \mathsf{P}^{t\nu})\\
&+\frac{1}{2}\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^{t\mu})\mathrm{Tr}(\mathsf{P}_\nu \mathsf{P}^{t\nu}) + 4(\mathsf{P}^t_\mu \mathsf{P}_\nu \mathsf{P}^{t\mu} \mathsf{P}^\nu)^{\bar{0}\bar{0}} - 4(\mathsf{P}^t_\mu \mathsf{P}^\mu \mathsf{P}^t_\nu \mathsf{P}^\nu)^{\bar{0}\bar{0}} + 4(\mathsf{P}^t_\mu \mathsf{P}_\nu \mathsf{P}^{t\nu}\mathsf{P}^\mu)^{\bar{0}\bar{0}}\\
&+4\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^t_\nu)(\mathsf{P}^{t\mu}\mathsf{P}^\nu)^{\bar{0}\bar{0}} - 2\mathrm{Tr}(\mathsf{P}_\mu \mathsf{P}^{t\mu})(\mathsf{P}^t_\nu \mathsf{P}^\nu)^{\bar{0}\bar{0}} - 8(\mathsf{P}^t_\mu \mathsf{P}_\nu)^{\bar{0}\bar{0}}(\mathsf{P}^{t\mu}\mathsf{P}^\nu)^{\bar{0}\bar{0}}\\
&+4(\mathsf{P}^t_\mu \mathsf{P}^\mu)^{\bar{0}\bar{0}}(\mathsf{P}^t_\nu \mathsf{P}^\nu)^{\bar{0}\bar{0}}\Big]\Bigg\}.\end{aligned} \tag{3.24}$$

Here, the ^{0} and $^{\bar{0}}$ components are defined by contracting out $\mathrm{O}(d+1)\times\mathrm{O}(d+1)$ vectors as

$$\mathsf{P}^{0\bar{a}}_\mu = e^{-\Phi}\tilde{\mathsf{u}}_a \mathsf{P}^{a\bar{a}}_\mu, \qquad \mathsf{P}^{a\bar{0}}_\mu = \mathsf{P}^{a\bar{a}}_\mu \tilde{\mathsf{u}}_{\bar{a}} e^{-\Phi}, \tag{3.25}$$

with $\tilde{\mathsf{u}}_a = \mathsf{u}^M \mathcal{V}_{Ma}$, $\tilde{\mathsf{u}}_{\bar{a}} = \mathsf{u}^M \mathcal{V}_{M\bar{a}}$. Spelling out Eq. (3.25) brings the action into manifestly H gauge invariant form. In these notations, P and P^t are interchanged under the $\mathbb{Z}_2$ action.

More compactly, and using the notations of Refs. [22,23], the result (3.24) can be rewritten in the form

$$\begin{aligned}\tilde{I}_1 = \frac{\alpha'}{4}\int d^3x\sqrt{-g}e^{-2\Phi}\Big[&\frac{a}{4}F_{abcd}((\mathsf{P}_\mu \mathsf{P}^{t\mu})^{ab}(\mathsf{P}_\nu \mathsf{P}^{t\nu})^{cd}\\
&-2(\mathsf{P}_\mu \mathsf{P}^t_\nu)^{ab}(\mathsf{P}^\mu \mathsf{P}^{t\nu})^{cd}) + \frac{b}{4}\bar{F}_{\bar{a}\bar{b}\bar{c}\bar{d}}((\mathsf{P}^t_\mu \mathsf{P}^\mu)^{\bar{a}\bar{b}}(\mathsf{P}^t_\nu \mathsf{P}^\nu)^{\bar{c}\bar{d}}\\
&-2(\mathsf{P}^t_\mu \mathsf{P}_\nu)^{\bar{a}\bar{b}}(\mathsf{P}^{t\mu}P^\nu)^{\bar{c}\bar{d}})\Big],\end{aligned} \tag{3.26}$$

with

$$F_{abcd} = \frac{3}{2}\delta_{(ab}\delta_{cd)} - 6\delta_{0(a}\delta_{bc}\delta_{d)0} + 4\delta_{0a}\delta_{0b}\delta_{0c}\delta_{0d}, \tag{3.27}$$

and $\bar{F}_{\bar{a}\bar{b}\bar{c}\bar{d}}$ defined in the same way, exchanging unbared indices for bared ones. In the case of the heterotic supergravity, $(a, b) = (-\alpha', 0)$, Eq. (3.26) consistently reproduces the weak coupling limit of the U duality invariant modular integrals conjectured to describe the exact four-derivative couplings; cf. Refs. [22–24]. (See in particular Eqs. (4.16) and (4.34) of Ref. [23].[6])

We close this subsection with some general remarks on the symmetry group after inclusion of the first α' correction. First, note that the scaling symmetry denoted (Dilaton[3]) above, which acts as $\Phi \to \Phi + \lambda$, $\lambda \in \mathbb{R}$, actually trivializes for the discrete subgroup. To see this, we use that these transformations are embedded into $\mathrm{O}(8, 8)$ as

$$\lambda \mapsto L_{(\lambda)} = \begin{pmatrix} \mathbf{1}_7 & 0 & 0 \\ 0 & e^\lambda & 0 \\ 0 & 0 & e^{-\lambda} \end{pmatrix} \in \mathrm{O}(8, 8, \mathbb{R}), \tag{3.28}$$

but the requirement that this transformation actually belongs to $\mathrm{O}(8, 8, \mathbb{Z})$; i.e., that all its matrix entries are integers, then implies that $e^\lambda = e^{-\lambda} = 1$ or $\lambda = 0$, reducing (Dilaton[3]) to the trivial group. Thus, by itself, the fact that the continuous scaling symmetry is broken by the explicit $e^{-2\Phi}$ prefactors at order α' is *not* in conflict with an $\mathrm{O}(8, 8, \mathbb{Z})$ symmetry.

We now turn to to the computation of the symmetry group, confirming our above conclusion that the

[6]We thank Guillaume Bossard for helpful explanations on this relation.

continuous $O(8,8)$ duality group is broken to its geometric subgroup ISO(7,7). In order to determine the symmetry group, we note that since the above formulation is manifestly $O(8,8)$ invariant provided the compensator $\mathsf{u}^{\mathcal{M}}$ transforms as a vector, the actual symmetry group is given by the invariance group of $\mathsf{u}^{\mathcal{M}} = (0,1,0)$. Recalling the index split $\mathcal{M} = (M,+,-)$, an $O(8,8)$ matrix reads

$$L_{\mathcal{M}}{}^{\mathcal{N}} = \begin{pmatrix} L_M{}^N & L_M{}^+ & L_M{}^- \\ L_+{}^N & L_+{}^+ & L_+{}^- \\ L_-{}^N & L_-{}^+ & L_-{}^- \end{pmatrix} \in O(8,8), \tag{3.29}$$

and is subject to

$$L_{\mathcal{M}}{}^{\mathcal{K}} \eta_{\mathcal{K}\mathcal{L}} L_{\mathcal{N}}{}^{\mathcal{L}} = \eta_{\mathcal{M}\mathcal{N}}. \tag{3.30}$$

The condition that $\mathsf{u}^{\mathcal{M}}$ is invariant; i.e., that $\mathsf{u}'^{M} = 0$, $\mathsf{u}'_+ = 0$ and $\mathsf{u}'_- = 1$, is quickly seen to imply

$$L_M{}^- = 0, \qquad L_+{}^- = 0, \qquad L_-{}^- = 1. \tag{3.31}$$

Using this in Eq. (3.30) yields furthermore

$$L_M{}^N \in O(7,7), \quad L_+{}^M = 0, \quad (L^{-1})_N{}^M L_-{}^N = -\eta^{MN} L_N{}^+,$$
$$L_+{}^+ = 1, \quad L_-{}^+ = -\frac{1}{2} L_-{}^M \eta_{MN} L_-{}^N. \tag{3.32}$$

Thus, a general $O(8,8)$ matrix that leaves u invariant is parametrized in terms of a general $O(7,7)$ matrix $L_M{}^N$ and a vector c_M:

$$L_{\mathcal{M}}{}^{\mathcal{N}} = \begin{pmatrix} L_M{}^N & L_M{}^K c_K & 0 \\ 0 & 1 & 0 \\ -c^N & -\frac{1}{2} c_K c^K & 1 \end{pmatrix}. \tag{3.33}$$

One may verify that the c_M, for $L_M{}^N = \delta_M{}^N$, acts as

$$\xi'_M = \xi_M + c_M, \tag{3.34}$$

and thus precisely parametrize the constant shifts of the scalars dual to vectors. As expected, we recovered precisely the geometric subgroup ISO(7,7) of $O(8,8)$.

IV. A MASSIVE DEFORMATION

So far, the three form field strength $H_{\mu\nu\rho}$ has been set to zero. We now integrate out the B field explicitly and explore the deformations induced by a nonvanishing three-form flux. At the two-derivative level, this results in a topological mass for the vectors and a potential for the dilaton [17]. We review this massive deformation and show how it fits in the more general framework of gauged supergravity. We then extend the analysis to the four-derivative corrections. In particular, we show that the resulting massive deformation induces Chern-Simons terms for composite connections featuring an enhancement to the full $O(d+1,d+1)$ to first order in the mass parameter.

A. Two-derivative action

We first consider the two-derivative action (2.14), which we rewrite in the string frame as

$$I_0 = \int d^3x \sqrt{-g} e^{-\Phi} \times \left(\mathcal{L}^{(0)}(g_{\mu\nu}, \Phi, \mathcal{H}, \mathcal{A}_\mu{}^M) - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \tag{4.1}$$

with $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} + \Omega^{(\mathrm{MS})}_{\mu\nu\rho}$ and $\Omega^{(\mathrm{MS})}_{\mu\nu\rho} = -(3/2) \times \mathcal{A}_{[\mu}{}^M \mathcal{F}_{\nu\rho]M}$ the abelian Chern-Simons deformation of the three-form field-strength [20]. In three dimensions, we can rewrite the field-strength as[7]

$$H_{\mu\nu\rho} = -\frac{1}{6} h \epsilon_{\mu\nu\rho}, \tag{4.2}$$

and I_0 becomes

$$I_0 = \int d^3x \sqrt{-g} e^{-\Phi} \left(\mathcal{L}^{(0)}(g_{\mu\nu}, \Phi, \mathcal{H}, \mathcal{A}_\mu{}^M) + \frac{1}{72} h^2 \right). \tag{4.3}$$

In the following, we do not specify explicitly the fields on which $\mathcal{L}^{(0)}$ depends. To dualize the degrees of freedom related to $B_{\mu\nu}$, we introduce an auxiliary field f and consider the action

$$I'_0 = \int d^3x \sqrt{-g} e^{-\Phi} \left(\mathcal{L}^{(0)} - \frac{1}{2} f^2 - \frac{1}{6} f h \right). \tag{4.4}$$

Taking its variation with respect to f, we get the algebraic equation of motion

$$f = -\frac{1}{6} h, \tag{4.5}$$

which can be used in Eq. (4.4) to get back Eq. (4.3), hence the equivalence between I_0 and I'_0. We now consider I'_0, which gives the following equation of motion after varying with respect to $B_{\mu\nu}$:

$$\nabla_\mu (e^{-\Phi} f) = 0 \Rightarrow f = m e^{\Phi}, \qquad m \in \mathbb{R}. \tag{4.6}$$

Unlike Eq. (4.5), this equation is not algebraic and, in general, could not simply be inserted back into the action.

[7]The global factor is chosen so that $h = \epsilon^{\mu\nu\rho} H_{\mu\nu\rho}$, with $g_{\mu\nu}$ of signature $(-1,1,1)$.

One needs to work at the level of the equations of motion. It can however be checked that, in the particular case we are considering here, it is consistent to use Eq. (4.6) directly in the action (4.4) to get

$$I''_0 = \int \mathrm{d}^3x \left[\sqrt{-g}\left(e^{-\Phi}\mathcal{L}^{(0)} - \frac{1}{2}m^2 e^{\Phi}\right) - \frac{1}{6}m\varepsilon^{\mu\nu\rho}\Omega^{(\mathrm{MS})}_{\mu\nu\rho}\right], \tag{4.7}$$

where we ignored a total derivative. I''_0 is equivalent to I_0, with the degrees of freedom of the two-form $B_{\mu\nu}$ dualized into m. For $m = 0$, we recover the action we started with in Sec. II C.

Let us move to the Einstein frame to discuss further the properties of the action:

$$\begin{aligned} I''_0 = \int \mathrm{d}^3x \Bigg[\sqrt{-g_\mathrm{E}}\Bigg(& R_\mathrm{E} - \partial_\mu\Phi\partial^\mu\Phi + \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}^{-1}\partial^\mu\mathcal{H}) \\ & - \frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N} - \frac{1}{2}m^2e^{4\Phi}\Bigg) \\ & + \frac{1}{4}m\varepsilon^{\mu\nu\rho}\mathcal{A}_\mu{}^M\mathcal{F}_{\nu\rho M}\Bigg]. \end{aligned} \tag{4.8}$$

For $m \neq 0$, the $\mathrm{O}(d+1, d+1)$ symmetry of the action (2.23) is broken down to $\mathrm{O}(d,d)$ and constant shifts in ξ_M [the scaling symmetry (2.8) is broken]. The term quadratic in m acts as a potential, and the vectors $\mathcal{A}_\mu{}^M$ acquire a topological mass proportional to m, leading to a topologically massive Yang-Mills theory [19,25] for $\mathrm{U}(1)^{2d}$, with equations of motion

$$\nabla_\mu(e^{-2\Phi}\mathcal{F}^{\mu\nu N}\mathcal{H}_{NM} - m\epsilon^{\mu\nu\rho}\mathcal{A}_{\rho M}) = 0. \tag{4.9}$$

The Chern-Simons coupling prevents us from dualizing the vectors as done in Sec. II C. We can however rewrite the Yang-Mills gauging of Eq. (4.8) as a pure Chern-Simons type gauging with gauge group $\mathrm{U}(1)^{2d} \ltimes T_{2d}$ using the on shell equivalence of Ref. [26], where T_{2d} is a $2d$-dimensional translation group. Consider the action

$$\begin{aligned} \tilde{I}''_0 = \int \mathrm{d}^3x \Bigg[\sqrt{-g_\mathrm{E}}\Bigg(& R_\mathrm{E} + \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{M}^{-1}\partial^\mu\mathcal{M}) \\ & - \frac{1}{2}m^2e^{2\Phi}\mathcal{A}_\mu{}^M\mathcal{H}_{MN}\mathcal{A}^{\mu N} - me^{2\Phi}\mathcal{A}_\mu{}^M\mathcal{H}_{MN}\partial^\mu\xi^N \\ & - \frac{1}{2}m^2e^{4\Phi}\Bigg) - \frac{1}{4}m\varepsilon^{\mu\nu\rho}\mathcal{A}_\mu{}^M\mathcal{F}_{\nu\rho M}\Bigg]. \end{aligned} \tag{4.10}$$

The induced equations of motion for the vectors are

$$\mathcal{F}_{\mu\nu}{}^M = e^{2\Phi}\epsilon_{\mu\nu\rho}(\partial^\rho\xi^N + m\mathcal{A}^{\rho N})\mathcal{H}_N{}^M, \tag{4.11}$$

which imply Eq. (4.9). It can be checked similarly that all the equations of motion of $\tilde{I}''_0$ are identical on shell to those of I''_0, upon systematically eliminating $\partial_\mu\xi_M$ using Eq. (4.11). Thus, I''_0 and $\tilde{I}''_0$ are equivalent.

The action (4.10) can be nicely rewritten using the embedding tensor formalism [27,28] of three-dimensional half-maximal gauged supergravity [29,30] (see also Ref. [31] for a review and the notations used here). The bosonic part of the action describes the dynamics of the metric $g_{\mathrm{E}\mu\nu}$, scalars $\mathcal{M}_{\mathcal{MN}} \in \mathrm{O}(d+1, d+1)$ and vectors $A_\mu{}^{[\mathcal{MN}]}$ via the action

$$\int \mathrm{d}^3x \left[\sqrt{-g_\mathrm{E}}\left(R_\mathrm{E} + \frac{1}{8}\mathrm{Tr}(D_\mu\mathcal{M}^{-1}D^\mu\mathcal{M}) - V\right) + \mathcal{L}_\mathrm{CS}\right]. \tag{4.12}$$

The covariant derivative on $\mathcal{M}_{\mathcal{MN}}$ is

$$D_\mu\mathcal{M}_{\mathcal{MN}} = \partial_\mu\mathcal{M}_{\mathcal{MN}} + 4A_\mu{}^{\mathcal{PQ}}\Theta_{\mathcal{PQ}|(\mathcal{M}}{}^{\mathcal{K}}\mathcal{M}_{\mathcal{N})\mathcal{K}}, \tag{4.13}$$

with the gauging given by the embedding tensor

$$\Theta_{\mathcal{MN}|\mathcal{PQ}} = \frac{1}{2}(\eta_{\mathcal{M}[\mathcal{P}}\theta_{\mathcal{Q}]\mathcal{N}} - \eta_{\mathcal{N}[\mathcal{P}}\theta_{\mathcal{Q}]\mathcal{M}}), \tag{4.14}$$

where $\theta_{\mathcal{MN}}$ is symmetric. In full generality, the embedding tensor contains more representations that we will not need here. The vectors are described by a Chern-Simons term of the form

$$\begin{aligned} \mathcal{L}_\mathrm{CS} = -\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{MN}|\mathcal{PQ}}A_\mu{}^{\mathcal{MN}}\Bigg(&\partial_\nu A_\rho{}^{\mathcal{PQ}} \\ & + \frac{1}{3}\Theta_{\mathcal{RS}|\mathcal{UV}}f^{\mathcal{PQ},\mathcal{RS}}{}_{\mathcal{XY}}A_\nu{}^{\mathcal{UV}}A_\rho{}^{\mathcal{XY}}\Bigg), \end{aligned} \tag{4.15}$$

with $f^{\mathcal{MN},\mathcal{PQ}}{}_{\mathcal{KL}} = 4\delta_{[\mathcal{K}}{}^{[\mathcal{M}}\eta^{\mathcal{N}][\mathcal{P}}\delta_{\mathcal{L}]}{}^{\mathcal{Q}]}$ the structure constants of $\mathfrak{so}(d+1, d+1)$. Finally, the potential is given by

$$\begin{aligned} V = \frac{1}{8}\theta_{\mathcal{MN}}\theta_{\mathcal{PQ}}(&2\mathcal{M}^{\mathcal{MP}}\mathcal{M}^{\mathcal{NQ}} - 2\eta^{\mathcal{MP}}\eta^{\mathcal{NQ}} \\ & - \mathcal{M}^{\mathcal{MN}}\mathcal{M}^{\mathcal{PQ}}). \end{aligned} \tag{4.16}$$

The action (4.10) then results from the restriction to an embedding tensor with only nonvanishing component θ_{--}. More precisely, a formally $\mathrm{O}(d+1, d+1)$-covariant form of I''_0 is given by

$$\begin{aligned}\tilde{I}_0'' = \int \mathrm{d}^3x \Big[& -\varepsilon^{\mu\nu\rho}\theta_{\mathcal{MN}}A_{\mu\mathcal{P}}{}^{\mathcal{N}}\partial_\nu A_\rho{}^{\mathcal{PM}} \\ & + \sqrt{-g_{\mathrm{E}}}\Big(R_{\mathrm{E}} + \frac{1}{8}\mathrm{Tr}(D_\mu\mathcal{M}^{-1}D^\mu\mathcal{M}) \\ & - \frac{1}{8}\theta_{\mathcal{MN}}\theta_{\mathcal{PQ}}(2\mathcal{M}^{\mathcal{MP}}\mathcal{M}^{\mathcal{NQ}} - 2\eta^{\mathcal{MP}}\eta^{\mathcal{NQ}} \\ & - \mathcal{M}^{\mathcal{MN}}\mathcal{M}^{\mathcal{PQ}})\Big)\Big], \end{aligned} \tag{4.17}$$

with the specific parametrization

$$A_\mu{}^{M-} = \frac{1}{2}\mathcal{A}_\mu{}^M, \qquad \theta_{--} = 2m. \tag{4.18}$$

The covariant derivative is then given by $D_\mu\xi_M = \partial_\mu\xi_M + m\mathcal{A}_{\mu M}$, $D_\mu\Phi = \partial_\mu\Phi$, and $D_\mu\mathcal{H}_{MN} = \partial_\mu\mathcal{H}_{MN}$. The symmetry breaking induced by $m \neq 0$ is now translated into the choice of embedding tensor with θ_{--} as the only nonvanishing component, which breaks $\mathrm{O}(d+1, d+1)$ to $\mathrm{O}(d,d)$ and shifts in ξ_M.

B. Four-derivative action

The two-form degrees of freedom can be integrated out in the four-derivative action using the same procedure as the one used in Sec. IV A. With Eq. (4.2), the action (7.16) of Ref. [12] is given by

$$\begin{aligned}\tilde{I} = I + \int \mathrm{d}^3x\sqrt{-g}e^{-\Phi} \\ \times \left[\frac{1}{72}h^2 + \alpha'\left(-\frac{1}{6}\mathfrak{a}h + \frac{1}{36}\mathfrak{b}h^2 + \frac{1}{6^4}\mathfrak{c}h^4\right)\right],\end{aligned} \tag{4.19}$$

where I is the action (3.4) and[8]

$$\begin{aligned}\mathfrak{a} &= \epsilon^{\mu\nu\rho}\Big[-\frac{a+b}{8\alpha'}\left(\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho} - \frac{1}{2}\mathcal{F}_{\mu\sigma}{}^M(\mathcal{H}\partial_\nu\mathcal{H}^{-1})_M{}^N\mathcal{F}_\rho{}^\sigma{}_N\right) \\ &\quad + \frac{a-b}{4\alpha'}\left(\Omega^{(\omega)}_{\mu\nu\rho} + \frac{1}{4}\mathcal{F}_{\mu\sigma}{}^M\partial^\sigma\mathcal{H}_{MN}\mathcal{F}_{\nu\rho}{}^N\right)\Big], \\ \mathfrak{b} &= -\frac{a+b}{8\alpha'}\left(R - \frac{1}{4}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}) + \frac{3}{4}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\right) \\ &\quad + \frac{a-b}{4\alpha'}\frac{1}{8}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M, \\ \mathfrak{c} &= \frac{a+b}{8\alpha'}\frac{5}{4}.\end{aligned} \tag{4.20}$$

$\Omega^{(\omega)}_{\mu\nu\rho}$ is the gravitational Chern-Simons form of heterotic supergravity, and $\Omega^{(\mathrm{GS})}_{\mu\nu\rho}$ is the three-form needed in the Green-Schwarz type mechanism of Ref. [11], which satisfies[9]

$$4\partial_{[\mu}\Omega^{(\mathrm{GS})}_{\nu\rho\sigma]} = \frac{3}{8}\mathrm{Tr}(\partial_{[\mu}\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial_\rho\mathcal{H}\partial_{\sigma]}\mathcal{H}^{-1}\mathcal{H}\eta). \tag{4.21}$$

As previously, we can equivalently write $\tilde{I}$ as

$$\begin{aligned}\tilde{I}' = I + \int \mathrm{d}^3x\sqrt{-g}e^{-\Phi} \\ \times\left[-\frac{1}{2}f^2 - \frac{1}{6}fh + \alpha'(\mathfrak{a}f + \mathfrak{b}f^2 + \mathfrak{c}f^4)\right],\end{aligned} \tag{4.22}$$

with an auxiliary field f. The equations of motion for f are

$$f = -\frac{1}{6}h + \alpha'(\mathfrak{a} + 2\mathfrak{b}f + 4\mathfrak{c}f^3), \tag{4.23}$$

and can be solved perturbatively in α':

$$f = f^{(0)} + \alpha' f^{(1)}, \quad \text{with} \quad \begin{cases} f^{(0)} = -\frac{1}{6}h, \\ f^{(1)} = \mathfrak{a} - \frac{1}{3}\mathfrak{b}h - \frac{1}{54}\mathfrak{c}h^3 \end{cases}. \tag{4.24}$$

As in the two-derivative case, Eq. (4.22) with Eq. (4.24) gives $\tilde{I}$. The equations of motion for $B_{\mu\nu}$ are unchanged and given by Eq. (4.6) and, again, it is consistent to use them directly in the action, leading to

$$\begin{aligned}\tilde{I}'' = I + \int \mathrm{d}^3x\sqrt{-g}\Big[-\frac{1}{6}m\epsilon^{\mu\nu\rho}\Omega^{(\mathrm{MS})}_{\mu\nu\rho} - \frac{1}{2}m^2e^{\Phi} \\ + \alpha'(\mathfrak{a}m + \mathfrak{b}m^2e^{\Phi} + \mathfrak{c}m^4e^{3\Phi})\Big].\end{aligned} \tag{4.25}$$

As a byproduct, observe that we can safely consider the case $m = 0$ and recover the actions considered in Secs. II and III.

Writing this action in terms of $\mathrm{O}(d+1, d+1)$ fields and an embedding tensor breaking the symmetry to $\mathrm{O}(d,d)\times\mathrm{O}(1,1)$, as we did for the two-derivative action in Sec. IV A, requires one to reproduce the analysis of Sec. III with modified rules for the field redefinitions (given the modified two-derivative equations of motion) and with the additional terms in Eq. (4.25), as detailed in Appendix B. All computations done, we get

$$\begin{aligned}\tilde{I}_1'' = \hat{I}_1 + \int \mathrm{d}^3x\Big\{ & \sqrt{-g_{\mathrm{E}}}\frac{a+b}{8}(4m^2e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi + m^4e^{6\Phi}) \\ & + m\epsilon^{\mu\nu\rho}\Big[-\frac{a+b}{8}\Big(\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho} \\ & - \frac{1}{2}e^{2\Phi}D_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H})^M{}_N D_\rho\xi^N\Big) + \frac{a-b}{4}\Omega^{(\omega_{\mathrm{E}})}_{\mu\nu\rho}\Big]\Big\},\end{aligned} \tag{4.26}$$

where $\hat{I}_1$ is given by the action (3.15) with all currents covariantized: $\hat{\mathcal{J}}_\mu = D_\mu\mathcal{M}\mathcal{M}^{-1}$. Observe that the

---

[8] Remember that a and b are of order α'.

[9] Normalized as in Eq. (14) of Ref. [11]; i.e., $\tilde{H}_{\mu\nu\rho} = H_{\mu\nu\rho} + \frac{a+b}{2}\Omega^{(\mathrm{GS})}_{\mu\nu\rho}$.

Green-Schwarz type mechanism of Ref. [11] generates, once restricted to three dimensions, a Chern-Simons term based on composite gauge fields. The properties of this term are best displayed using the frame formalism of Sec. II C. In Ref. [11], the Green-Schwarz three-from $\Omega^{(\mathrm{GS})}_{\mu\nu\rho}$ has been written as a Chern-Simons form for $\mathrm{O}(d)\times O(d)$ composite gauge fields:

$$\Omega^{(\mathrm{GS})}_{\mu\nu\rho}=\frac{3}{2}\mathrm{CS}_{\mu\nu\rho}(Q)=\frac{3}{2}\mathrm{Tr}\left(Q_{[\mu}\partial_\nu Q_{\rho]}\delta\eta+\frac{2}{3}Q_{[\mu}Q_\nu Q_{\rho]}\delta\eta\right), \tag{4.27}$$

with $Q_{\mu A}{}^B$ defined in Eq. (2.32) and δ_{AB} the identity matrix.[10] Generalizing this definition to the covariantization $\hat{\mathcal{Q}}_{\mu\mathcal{A}}{}^{\mathcal{B}}$ of the $\mathrm{O}(d+1)\times\mathrm{O}(d+1)$ connection $\mathcal{Q}_{\mu\mathcal{A}}{}^{\mathcal{B}}$ of Eq. (2.32) (i.e., defining $\hat{\mathcal{Q}}_{\mu\mathcal{A}}{}^{\mathcal{B}}$ from the covariantized Maurer-Cartan form $\mathcal{V}^{-1}D_\mu\mathcal{V}$), we get

$$\begin{aligned}\mathrm{CS}_{\mu\nu\rho}(\hat{\mathcal{Q}})=\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho}-\frac{1}{2}e^{2\Phi}D_{[\mu}\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H})^M{}_N D_{\rho]}\xi^N\\ -\frac{m}{2}e^{2\Phi}D_{[\mu}\xi_M\mathcal{F}_{\nu\rho]}{}^M,\end{aligned} \tag{4.28}$$

of exterior derivative[11]

$$\begin{aligned}&\partial_{[\mu}\mathrm{CS}_{\nu\rho\sigma]}(\hat{\mathcal{Q}})\\ &=\frac{1}{16}\mathrm{Tr}(D_{[\mu}\mathcal{M}D_\nu\mathcal{M}^{-1}D_\rho\mathcal{M}D_{\sigma]}\mathcal{M}^{-1}\mathcal{M}\eta)\\ &\quad-2\Theta_{\mathcal{MN}|\mathcal{PQ}}F_{[\mu\nu}{}^{\mathcal{MN}}(D_\rho\mathcal{M}^{-1}D_{\sigma]}\mathcal{M}\mathcal{M}^{-1})^{\mathcal{PQ}}\\ &\quad+16\Theta_{\mathcal{MN}|\mathcal{PQ}}\Theta_{\mathcal{KL}|\mathcal{R}}{}^{\mathcal{P}}F_{[\mu\nu}{}^{\mathcal{MN}}F_{\rho\sigma]}{}^{\mathcal{KL}}\mathcal{M}^{\mathcal{RQ}}.\end{aligned} \tag{4.29}$$

Putting this term in the action and eliminating the field-strength using the dualization relation (4.11) gives

$$\begin{aligned}&\int d^3x m\varepsilon^{\mu\nu\rho}\mathrm{CS}_{\mu\nu\rho}(\hat{\mathcal{Q}})\\ &=\int d^3x m\Big[\varepsilon^{\mu\nu\rho}\left(\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho}-\frac{1}{2}e^{2\Phi}D_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H})^M{}_N D_\rho\xi^N\right)\\ &\quad+\sqrt{-g_{\mathrm{E}}}m^2e^{4\Phi}D_\mu\xi_M\mathcal{H}^{MN}D^\mu\xi_N\Big].\end{aligned} \tag{4.30}$$

Thus, we can write the action (4.26) in terms of the three-form (4.28):

$$\begin{aligned}\tilde{I}_1''=\hat{I}_1+\int d^3x\Big[\sqrt{-g_{\mathrm{E}}}\frac{a+b}{8}(m^2(4e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi\\ +e^{4\Phi}D_\mu\xi_M\mathcal{H}^{MN}D^\mu\xi_N)+m^4e^{6\Phi})\\ +m\varepsilon^{\mu\nu\rho}\left(-\frac{a+b}{8}\mathrm{CS}_{\mu\nu\rho}(\hat{\mathcal{Q}})+\frac{a-b}{4}\Omega^{(\omega_{\mathrm{E}})}_{\mu\nu\rho}\right)\Big].\end{aligned} \tag{4.31}$$

We can furthermore express the first line in terms of the $\mathrm{O}(d+1,d+1)$ currents, as done in Sec. III B, and the embedding tensor (4.18):

$$\begin{aligned}\tilde{I}_1''=\hat{I}_1+\int d^3xm\Big[\sqrt{-g_{\mathrm{E}}}\frac{a+b}{8}\Big(\frac{1}{2}\theta_{\mathcal{MN}}(\hat{\mathcal{J}}_\mu\hat{\mathcal{J}}^\mu\mathcal{M})^{\mathcal{MN}}\\ +\frac{1}{8}(\theta_{\mathcal{MN}}\mathcal{M}^{\mathcal{MN}})^3\Big)\\ +\varepsilon^{\mu\nu\rho}\left(-\frac{a+b}{8}\mathrm{CS}_{\mu\nu\rho}(\hat{\mathcal{Q}})+\frac{a-b}{4}\Omega^{(\omega_{\mathrm{E}})}_{\mu\nu\rho}\right)\Big].\end{aligned} \tag{4.32}$$

Thus, the massive deformation following from integrating out the B field induces a gauging of the action (3.15) and additional couplings. Remarkably, these couplings break the $\mathrm{O}(d+1,d+1)$ symmetry only due to the gauging (4.18) of the shift symmetry; no vector compensator is needed, in contrast to the term $\hat{I}_1$. The first line in Eq. (4.32) features a deformation of the two-derivative action and a potential. The second line of Eq. (4.32) is given by Chern-Simons terms based on composite and spin connections. Most interestingly, to leading order in m, these Chern-Simons terms are given by

$$\begin{aligned}&\int d^3x\, m\varepsilon^{\mu\nu\rho}\left(-\frac{a+b}{8}\mathrm{CS}_{\mu\nu\rho}(\hat{\mathcal{Q}})+\frac{a-b}{4}\Omega^{(\omega_{\mathrm{E}})}_{\mu\nu\rho}\right)\\ &=\int d^3x\, m\varepsilon^{\mu\nu\rho}\left(-\frac{a+b}{8}\mathrm{CS}_{\mu\nu\rho}(\mathcal{Q})+\frac{a-b}{4}\Omega^{(\omega_{\mathrm{E}})}_{\mu\nu\rho}\right)\\ &\quad+\mathcal{O}(m^2),\end{aligned} \tag{4.33}$$

which is invariant under $\mathrm{O}(d+1,d+1)$. Although the full theory exhibits a breaking of the $\mathrm{O}(d+1,d+1)$ U duality both by the higher-derivative parameter α' and the mass deformation m, for the leading Chern-Simons terms, the full $\mathrm{O}(d+1,d+1)$ is restored. The physical meaning of this observation has to be investigated, as well as its extension to more general gaugings.

V. CONCLUSIONS

In this paper, we have computed the effective action of bosonic and heterotic string theory in three dimensions to first order in α' starting from the known four-derivative result in manifestly $\mathrm{O}(d,d)$-invariant form and perturbatively dualizing the vector gauge fields into scalars. We have cast the result into a formally $\mathrm{O}(d+1,d+1)$-invariant form upon introduction of a non-dynamical

[10]The $\delta\eta$ in Eq. (4.27) is needed to reproduce the right relative sign in Eq. (34) of Ref. [11].

[11]Here, we used that $[D_\mu,D_\nu]\mathcal{M}_{\mathcal{MN}}=4F_{\mu\nu}{}^{\mathcal{PQ}}\Theta_{\mathcal{PQ}|(\mathcal{M}}{}^{\mathcal{K}}\mathcal{M}_{\mathcal{N})\mathcal{K}}$ and $F_{\mu\nu}{}^{\mathcal{MN}}=2\partial_{[\mu}A_{\nu]}{}^{\mathcal{MN}}+4A_{[\mu}{}^{\mathcal{PQ}}\Theta_{\mathcal{PQ}|\mathcal{K}}{}^{[\mathcal{M}}A_{\nu]}{}^{\mathcal{N}]\mathcal{K}}$.

compensator $\mathfrak{u}^{\mathcal{M}}$. The resulting action reveals the intriguing chiral pattern (3.18), showing that in particular, the action of the bosonic string can be reconstructed from the heterotic action. One may expect that an extension of this structure to higher orders in α' will constrain the potential higher order corrections in a similar way.

Another interesting application of the formally $\mathrm{O}(d+1,d+1)$-invariant formulation will be the study of $\mathrm{O}(4,4)$ triality rotations on the resulting action. This would, for instance, allow one to relate the inequivalent higher order corrections obtained from T^3 compactification of chiral $\mathcal{N}=(2,0)$ and nonchiral $\mathcal{N}=(1,1)$ supergravity in six dimensions, respectively. With the latter corresponding to the standard heterotic corrections (see [32] for a recent discussion), this may be turned into a prediction of the α' corrections of the chiral theory in six dimensions. As this theory arises from 10-dimensional type IIB supergravity compactified on the complex surface K3, this computation would give profitable insights into the higher-derivative corrections in ten dimensions.

The other main result of this paper is the massive deformation identified in Sec. IV upon integrating out the B field in three dimensions, keeping a constant three-form flux. As exhibited in Eq. (4.32) above, this gives rise to a deformation of the target space metric and the scalar potential, as well as to new Chern-Simons terms for composite connections which remarkably exhibit an enhancement to the full $\mathrm{O}(d+1,d+1)$ to first order in the mass parameter. Within the general framework of gauged supergravities, such massive deformations take the form of a particular and somewhat degenerate example of a general gauging. In this respect, our results may be viewed as a glimpse into the structure of the α' corrections of more general gauged supergravities in three dimensions. Of particular interest for holographic applications would be the study of these deformations around the $\mathrm{AdS}_3\times S^3$ background. We hope to come back to these issues in the future.

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APPENDIX A: O(d,d) DECOMPOSITIONS OF O($d+1$, $d+1$)

We list in the following the $\mathrm{O}(d,d)$ decomposition of the $\mathrm{O}(d+1,d+1)$ terms used in Sec. III B and of the basis of $\mathrm{O}(d+1,d+1)$-invariant terms carrying four derivatives [11].

$$\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu)=-\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})+4e^{2\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N+8\partial_\mu\Phi\partial_\nu\Phi. \tag{A1}$$

$$(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{M})_{++}=e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N+4e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi. \tag{A2}$$

$$(\mathcal{J}_\mu\mathcal{J}_\nu\eta)_{++}=-e^{4\Phi}\partial_\mu\xi_M\partial_\nu\xi^M. \tag{A3}$$

$$\begin{aligned}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\mu\mathcal{J}^\nu)=&\,\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-8e^{2\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\mu\mathcal{H}^{-1})^{MN}\partial^\nu\xi_N\\&-16e^{2\Phi}\partial_\mu\Phi\partial_\nu\xi_M\partial^\mu\mathcal{H}^{MN}\partial^\nu\xi_N+32\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\mu\Phi+4e^{4\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_N\partial^\nu\xi^N\\&+4e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi^Q+32e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N.\end{aligned} \tag{A4}$$

$$\begin{aligned}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu)=&\,\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-4e^{2\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})^{MN}\partial^\mu\xi_N\\&-4e^{2\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})^{MN}\partial^\nu\xi_N-16e^{2\Phi}\partial_\mu\Phi\partial^\mu\xi_M\partial_\nu\mathcal{H}^{MN}\partial^\nu\xi_N\\&+32\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\mu\Phi+2e^{4\Phi}\partial_\mu\xi_M\partial^\mu\xi^M\partial_\nu\xi_N\partial^\nu\xi^N+2e^{4\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_N\partial^\nu\xi^N\\&+2e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\partial_\nu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi^Q+2e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi^Q\\&+16e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N+16e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N.\end{aligned} \tag{A5}$$

$$\begin{aligned}\mathrm{Tr}(\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{M}\eta)=&\,\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}\mathcal{H}\eta)-4e^{2\Phi}\partial_\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\partial^\nu\mathcal{H})^M{}_N\partial^\mu\xi^N\\&+4e^{2\Phi}\partial_\mu\xi_M(\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H})^M{}_N\partial^\nu\xi^N-16e^{2\Phi}\partial_\mu\Phi\partial^\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H})^M{}_N\partial^\nu\xi^N\\&+4e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N\partial^\mu\xi_P\partial^\nu\xi^P-4e^{4\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\partial_\nu\xi_P\partial^\nu\xi^P\\&-16e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\xi_M\partial^\nu\xi^M+16e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\partial^\nu\xi^M.\end{aligned} \tag{A6}$$

$$
\begin{aligned}
(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\mu\mathcal{J}^\nu\mathcal{M})_{++} = & -e^{4\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\mu\mathcal{H}^{-1})^{MN}\partial^\nu\xi_N - 4e^{4\Phi}\partial_\mu\Phi\partial_\nu\xi_M\partial^\mu\mathcal{H}^{MN}\partial_\nu\xi_N \\
& + 16e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi + e^{6\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q \\
& + e^{6\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_N\partial^\nu\xi^N + 12e^{4\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N.
\end{aligned}
\tag{A7}
$$

$$
(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\mu\mathcal{J}^\nu\eta)_{++} = e^{4\Phi}\partial_\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H})^M{}_N\partial^\nu\xi^N - 2e^{6\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q - 4e^{4\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\partial^\nu\xi^M.
\tag{A8}
$$

$$
\begin{aligned}
(\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{M})_{++} = & -e^{4\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})^{MN}\partial_\nu\xi_N - 4e^{4\Phi}\partial_\mu\Phi\partial^\mu\xi_M\partial_\nu\mathcal{H}^{MN}\partial^\nu\xi_N \\
& + 16e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi + e^{6\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial^\mu\xi_N\partial_\nu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q \\
& + e^{6\Phi}\partial_\mu\xi_M\partial^\mu\xi^M\partial_\nu\xi_N\partial^\nu\xi^N + 4e^{4\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N \\
& + 8e^{4\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N.
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
(\mathcal{J}_\mu\mathcal{J}^\mu\mathcal{J}_\nu\mathcal{J}^\nu\eta)_{++} = & \, e^{4\Phi}\partial_\mu\xi_M(\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H})^M{}_N\partial^\nu\xi^N + 4e^{4\Phi}\partial_\mu\Phi\partial^\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\mathcal{H})^M{}_N\partial^\nu\xi^N \\
& - 2e^{6\Phi}\partial_\mu\xi_M\partial^\mu\xi^M\partial_\nu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q + 4e^{4\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\partial^\nu\xi^M - 8e^{4\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\xi_M\partial^\nu\xi^M.
\end{aligned}
\tag{A10}
$$

$$
\begin{aligned}
(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{J}^\mu\mathcal{M})_{++} = & -e^{4\Phi}\partial_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})^{MN}\partial^\mu\xi_N - 4e^{4\Phi}\partial_\mu\Phi\partial^\mu\xi_M\partial_\nu\mathcal{H}^{MN}\partial^\nu\xi_N \\
& + 16e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi + e^{6\Phi}\partial_\mu\xi_M\mathcal{H}^{MN}\partial_\nu\xi_N\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q \\
& + e^{6\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_N\partial^\nu\xi^N + 8e^{4\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N + 4e^{4\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\xi_M\mathcal{H}^{MN}\partial^\nu\xi_N.
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
(\mathcal{J}_\mu\mathcal{J}_\nu\mathcal{J}^\nu\mathcal{J}^\mu\eta)_{++} = & \, e^{4\Phi}\partial_\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\partial^\nu\mathcal{H})^M{}_N\partial^\mu\xi^N - 4e^{4\Phi}\partial_\mu\Phi\partial^\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\mathcal{H})^M{}_N\partial^\nu\xi^N \\
& - 2e^{6\Phi}\partial_\mu\xi_M\partial_\nu\xi^M\partial^\mu\xi_P\mathcal{H}^{PQ}\partial^\nu\xi_Q - 8e^{4\Phi}\partial_\mu\Phi\partial_\nu\Phi\partial^\mu\xi_M\partial^\nu\xi^M + 4e^{4\Phi}\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\xi_M\partial^\nu\xi^M.
\end{aligned}
\tag{A12}
$$

APPENDIX B: FOUR-DERIVATIVE ACTION WITH MASS DEFORMATION

We detail here the computation of the action (4.26). We first give the rules for field redefinitions as modified by the mass deformation of the two-derivative action. We then move to Einstein frame, use field redefinitions to convert all second order derivatives into product of first order derivatives, and finally dualize the vector fields.

1. Field redefinitions

The two-derivative action in Einstein frame with B field integrated out is given in Eq. (4.8). Its equations of motion are

$$
\delta I_0'' = \alpha' \int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}[\delta\Phi E_\Phi + \delta g_{\mathrm{E}}^{\mu\nu}E_{g\mu\nu} + \mathrm{Tr}(\delta\mathcal{H}^{-1}E_{\mathcal{H}}) + \delta\mathcal{A}_\mu{}^M E_{\mathcal{A}}{}^\mu{}_M],
\tag{B1}
$$

where

$$
E_\Phi = 2\Box\Phi + \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N} - 2m^2e^{4\Phi},
\tag{B2a}
$$

$$
\begin{aligned}
E_{g\mu\nu} = & \, R_{\mathrm{E}\mu\nu} - \partial_\mu\Phi\partial_\nu\Phi + \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}) - \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\rho}{}^M\mathcal{H}_{MN}\mathcal{F}_\nu{}^{\rho N} \\
& - \frac{1}{2}g_{\mathrm{E}\mu\nu}\left(R_{\mathrm{E}} - \partial_\rho\Phi\partial^\rho\Phi + \frac{1}{8}\mathrm{Tr}(\partial_\rho\mathcal{H}\partial^\rho\mathcal{H}^{-1}) - \frac{1}{4}e^{-2\Phi}\mathcal{F}_{\rho\sigma}{}^M\mathcal{H}_{MN}\mathcal{F}^{\rho\sigma N} - \frac{1}{2}m^2e^{4\Phi}\right),
\end{aligned}
\tag{B2b}
$$

$$
E_{\mathcal{H}MN} = -\frac{1}{4}\left[\Box\mathcal{H}_{MN} + (\mathcal{H}\partial_\mu\mathcal{H}^{-1}\partial^\mu\mathcal{H})_{MN} + \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu M}\mathcal{F}^{\mu\nu}{}_N - \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^P\mathcal{H}_{PM}\mathcal{F}^{\mu\nu Q}\mathcal{H}_{QN}\right],
\tag{B2c}
$$

$$
E_{\mathcal{A}}{}^\mu{}_M = e^{-2\Phi}\nabla_\nu\mathcal{F}^{\nu\mu N}\mathcal{H}_{NM} - 2e^{-2\Phi}\nabla_\nu\Phi\mathcal{F}^{\nu\mu N}\mathcal{H}_{NM} + e^{-2\Phi}\mathcal{F}^{\nu\mu N}\nabla_\nu\mathcal{H}_{NM} + \frac{m}{2}\epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho M}.
\tag{B2d}
$$

The resulting field redefinitions are as follows:

$$
\begin{aligned}
\Box\Phi &\longrightarrow Q_{\Phi} = -\frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N} + m^2 e^{4\Phi},\\
R_{\mathrm{E}\mu\nu} &\longrightarrow Q_{g\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}) + \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\rho}{}^{M}\mathcal{H}_{MN}\mathcal{F}_{\nu}{}^{\rho N}\\
&\qquad - \frac{1}{4}g_{\mathrm{E}\mu\nu}e^{-2\Phi}\mathcal{F}_{\rho\sigma}{}^{M}\mathcal{H}_{MN}\mathcal{F}^{\rho\sigma N} + \frac{1}{2}g_{\mathrm{E}\mu\nu}m^2 e^{4\Phi},\\
\Box\mathcal{H}_{MN} &\longrightarrow Q_{\mathcal{H}MN} = -(\mathcal{H}\partial_\mu\mathcal{H}^{-1}\partial^\mu\mathcal{H})_{MN} - \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu M}\mathcal{F}^{\mu\nu}{}_{N} + \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^{P}\mathcal{H}_{PM}\mathcal{F}^{\mu\nu Q}\mathcal{H}_{QN},\\
\nabla_\nu\mathcal{F}^{\nu\mu M} &\longrightarrow Q_{\mathcal{A}}{}^{\mu M} = 2\partial_\nu\Phi\mathcal{F}^{\nu\mu M} - \mathcal{F}^{\nu\mu N}(\partial_\nu\mathcal{H}\mathcal{H}^{-1})_{N}{}^{M} - \frac{m}{2}e^{2\Phi}\epsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho N}\mathcal{H}^{NM}.
\end{aligned}
\tag{B3}
$$

2. Einstein frame

We write the action (4.25) in Einstein frame ($g_{\mu\nu} \to g_{\mathrm{E}\mu\nu} = e^{-2\Phi}g_{\mu\nu}$):

$$
\begin{aligned}
I_1'' = \int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}\Bigg[&-\frac{a+b}{8}e^{-2\Phi}\Bigg(4R_{\mathrm{E}\mu\nu}R_{\mathrm{E}}^{\mu\nu} - R_{\mathrm{E}}^2 - 8R_{\mathrm{E}\mu\nu}\nabla^\mu\nabla^\nu\Phi + 8R_{\mathrm{E}\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - 4R_{\mathrm{E}}\partial_\mu\Phi\partial^\mu\Phi\\
&+ 4\nabla_\mu\nabla_\nu\Phi\nabla^\mu\nabla^\nu\Phi + 4\Box\Phi\Box\Phi - 8\nabla_\mu\nabla_\nu\Phi\partial^\mu\Phi\partial^\nu\Phi + 8\Box\Phi\partial_\mu\Phi\partial^\mu\Phi + 4\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi\\
&+ \frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1}) - \frac{1}{32}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&+ \frac{1}{8}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}_{\rho\sigma M}\mathcal{F}^{\mu\rho N}\mathcal{F}^{\nu\sigma}{}_{N} + \frac{1}{8}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{H}_{MN}\mathcal{F}_{\rho\sigma}{}^{N}\mathcal{F}^{\mu\rho P}\mathcal{H}_{PQ}\mathcal{F}^{\nu\sigma Q}\\
&- \frac{1}{2}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{H}_{MN}\mathcal{F}^{\mu\rho N}\mathcal{F}^{\nu\sigma P}\mathcal{H}_{PQ}\mathcal{F}_{\rho\sigma}{}^{Q} + \frac{1}{2}e^{-2\Phi}(R_{\mathrm{E}} + 2\partial_\mu\Phi\partial^\mu\Phi)\mathcal{F}_{\rho\sigma}{}^{M}\mathcal{H}_{MN}\mathcal{F}^{\rho\sigma N}\\
&- 2e^{-2\Phi}(R_{\mathrm{E}\mu\nu} - \nabla_\mu\nabla_\nu\Phi + \partial_\mu\Phi\partial_\nu\Phi)\mathcal{F}^{\mu\rho M}\mathcal{H}_{MN}\mathcal{F}^{\nu}{}_{\rho}{}^{N}\\
&- \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^{M}(\mathcal{H}\partial_\rho\mathcal{H}^{-1}\partial^\nu\mathcal{H})_{MN}\mathcal{F}^{\mu\rho N} + \frac{1}{4}e^{-2\Phi}\mathcal{F}^{\mu\rho M}\mathcal{H}_{MN}\mathcal{F}^{\nu}{}_{\rho}{}^{N}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\\
&+ me^{2\Phi}\epsilon^{\mu\nu\rho}\left(\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho} - \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\sigma}{}^{M}(\mathcal{H}\partial_\nu\mathcal{H}^{-1})_{M}{}^{N}\mathcal{F}_{\rho}{}^{\sigma}{}_{N}\right) - \frac{5}{4}m^4e^{8\Phi}\\
&+ m^2e^{4\Phi}\left(R_{\mathrm{E}} - 4\Box\Phi - 2\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{4}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}) + \frac{3}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\right)\Bigg)\\
&+ \frac{a-b}{4}e^{-2\Phi}\Bigg(-\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}\mathcal{H}\eta) - \frac{1}{16}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}_{\rho\sigma M}\mathcal{F}^{\mu\nu P}\mathcal{H}_{PQ}\mathcal{F}^{\rho\sigma Q}\\
&- \frac{1}{4}e^{-2\Phi}R_{\mathrm{E}}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}^{\mu\nu}{}_{M} + e^{-2\Phi}R_{\mathrm{E}\mu\nu}\mathcal{F}^{\mu\rho M}\mathcal{F}^{\nu}{}_{\rho M} - e^{-2\Phi}\nabla_\mu\nabla_\nu\Phi\mathcal{F}^{\mu\rho M}\mathcal{F}^{\nu}{}_{\rho M}\\
&- \frac{1}{2}e^{-2\Phi}\partial_\rho\Phi\partial^\rho\Phi\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}^{\mu\nu}{}_{M} + e^{-2\Phi}\partial_\mu\Phi\partial_\nu\Phi\mathcal{F}^{\mu\rho M}\mathcal{F}^{\nu}{}_{\rho M}\\
&+ \frac{1}{8}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^{M}(\partial_\rho\mathcal{H}\partial^\rho\mathcal{H}^{-1})_{M}{}^{N}\mathcal{F}^{\mu\nu}{}_{N} + \frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^{M}(\partial^\mu\mathcal{H}\partial_\rho\mathcal{H}^{-1})_{M}{}^{N}\mathcal{F}^{\nu\rho}{}_{N}\\
&+ me^{2\Phi}\epsilon^{\mu\nu\rho}\left(\Omega^{(\omega_{\mathrm{E}})}_{\mu\nu\rho} + \frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\sigma}{}^{M}\partial^\sigma\mathcal{H}_{MN}\mathcal{F}_{\nu\rho}{}^{N}\right) + \frac{m^2}{8}e^{2\Phi}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}^{\mu\nu}{}_{M}\Bigg)\Bigg].
\end{aligned}
\tag{B4}
$$

3. Integrations by part and field redefinitions

We now convert all second order derivatives in Eq. (B4) into products of first order derivatives: We first use integrations by part so that all the second order derivatives appear in the leading two-derivative contribution of the equations of motion (B2), and obtain[12]

$$\begin{aligned}
I_1'' = \int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}\Big[&-\frac{a+b}{8}e^{-2\Phi}\Big(4\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi + \frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&-\frac{1}{32}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1}) + \frac{1}{8}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}_{\rho\sigma}{}^N\mathcal{F}^{\mu\rho P}\mathcal{H}_{PQ}\mathcal{F}^{\nu\sigma Q}\\
&+\frac{1}{8}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma M}\mathcal{F}^{\mu\rho N}\mathcal{F}^{\nu\sigma}{}_N - \frac{1}{2}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\rho N}\mathcal{F}^{\nu\sigma P}\mathcal{H}_{PQ}\mathcal{F}_{\rho\sigma}{}^Q\\
&-e^{-2\Phi}\partial_\mu\Phi\partial^\mu\Phi\mathcal{F}_{\rho\sigma}{}^M\mathcal{H}_{MN}\mathcal{F}^{\rho\sigma N} + 6e^{-2\Phi}\partial_\mu\Phi\partial_\nu\Phi\mathcal{F}^{\mu\rho M}\mathcal{H}_{MN}\mathcal{F}^\nu{}_\rho{}^N\\
&-\frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M(\mathcal{H}\partial_\rho\mathcal{H}^{-1}\partial^\nu\mathcal{H})_{MN}\mathcal{F}^{\mu\rho N} + \frac{1}{4}e^{-2\Phi}\mathcal{F}^{\mu\rho M}\mathcal{H}_{MN}\mathcal{F}^\nu{}_\rho{}^N\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\\
&-2e^{-2\Phi}\partial_\nu\Phi\mathcal{F}^{\mu\rho M}\partial_\mu\mathcal{H}_{MN}\mathcal{F}^\nu{}_\rho{}^N + \frac{1}{2}e^{-2\Phi}\partial_\rho\Phi\mathcal{F}_{\mu\nu}{}^M\partial^\rho\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\\
&+me^{2\Phi}\epsilon^{\mu\nu\rho}\Big(\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho} - \frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\sigma}{}^M(\mathcal{H}\partial_\nu\mathcal{H}^{-1})_M{}^N\mathcal{F}_\rho{}^\sigma{}_N\Big) - \frac{5}{4}m^4e^{8\Phi}\\
&+m^2e^{4\Phi}\Big(-2\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{4}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}) + \frac{3}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\Big)\\
&+4R_{\mathrm{E}\mu\nu}R_{\mathrm{E}}{}^{\mu\nu} - R_{\mathrm{E}}^2 - 12R_{\mathrm{E}\mu\nu}\partial^\mu\Phi\partial^\nu\Phi + 4R_{\mathrm{E}}\partial_\mu\Phi\partial^\mu\Phi - 4R_{\mathrm{E}}\Box\Phi + 8\Box\Phi\Box\Phi\\
&+\frac{1}{2}e^{-2\Phi}R_{\mathrm{E}}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N} - 2e^{-2\Phi}R_{\mathrm{E}\mu\nu}\mathcal{F}^{\mu\rho M}\mathcal{H}_{MN}\mathcal{F}^\nu{}_\rho{}^N + m^2e^{4\Phi}(R_{\mathrm{E}} - 4\Box\Phi)\\
&-2e^{-2\Phi}\partial_\nu\Phi\nabla_\mu\mathcal{F}^{\mu\rho M}\mathcal{H}_{MN}\mathcal{F}^\nu{}_\rho{}^N + \frac{1}{2}e^{-2\Phi}\Box\Phi\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\Big)\\
&+\frac{a-b}{4}e^{-2\Phi}\Big(-\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}\mathcal{H}\eta) - \frac{1}{16}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma M}\mathcal{F}^{\mu\nu P}\mathcal{H}_{PQ}\mathcal{F}^{\rho\sigma Q}\\
&-3e^{-2\Phi}\partial_\mu\Phi\partial_\nu\Phi\mathcal{F}^{\mu\rho M}\mathcal{F}^\nu{}_{\rho M} + \frac{1}{2}e^{-2\Phi}\partial_\rho\Phi\partial^\rho\Phi\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M\\
&+\frac{1}{8}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M(\partial_\rho\mathcal{H}\partial^\rho\mathcal{H}^{-1})_M{}^N\mathcal{F}^{\mu\nu}{}_N + \frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M(\partial^\mu\mathcal{H}\partial_\rho\mathcal{H}^{-1})_M{}^N\mathcal{F}^{\nu\rho}{}_N\\
&+me^{2\Phi}\epsilon^{\mu\nu\rho}\Big(\Omega^{(\omega_{\mathrm{E}})}_{\mu\nu\rho} + \frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\sigma}{}^M\partial^\sigma\mathcal{H}_{MN}\mathcal{F}_{\nu\rho}{}^N\Big) + \frac{m^2}{8}e^{2\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M\\
&-\frac{1}{4}e^{-2\Phi}R_{\mathrm{E}}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M + e^{-2\Phi}R_{\mathrm{E}\mu\nu}\mathcal{F}^{\mu\rho M}\mathcal{F}^\nu{}_{\rho M} - \frac{1}{4}e^{-2\Phi}\Box\Phi\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M + e^{-2\Phi}\nabla_\mu\mathcal{F}^{\mu\rho}\partial_\nu\Phi\mathcal{F}^\nu{}_{\rho M}\Big)\Big]. \qquad \text{(B5)}
\end{aligned}$$

Using the field redefinitions (B3), we get

$$\begin{aligned}
I_1'' = \int \mathrm{d}^3x\sqrt{-g_{\mathrm{E}}}\Big[&-\frac{a+b}{8}e^{-2\Phi}\Big(\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1}) + \frac{1}{32}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&-\frac{1}{64}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1})\mathrm{Tr}(\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}) + \frac{1}{8}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}_{\rho\sigma}{}^N\mathcal{F}^{\mu\rho P}\mathcal{H}_{PQ}\mathcal{F}^{\nu\sigma Q}\\
&+\frac{1}{8}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma M}\mathcal{F}^{\mu\rho N}\mathcal{F}^{\nu\sigma}{}_N - \frac{1}{2}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\rho N}\mathcal{F}^{\nu\sigma P}\mathcal{H}_{PQ}\mathcal{F}_{\rho\sigma}{}^Q\\
&+\frac{3}{16}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\mathcal{F}_{\rho\sigma}{}^P\mathcal{H}_{PQ}\mathcal{F}^{\rho\sigma Q} - \partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi
\end{aligned}$$

[12]Note that here, contrary to what we did in Sec. III, we use partial integrations and field redefinitions before dualizing the vector fields.

$$
\begin{aligned}
&+\frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-\frac{1}{4}\partial_\mu\Phi\partial^\mu\Phi\mathrm{Tr}(\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&+e^{-2\Phi}\partial_\mu\Phi\partial^\mu\Phi\mathcal{F}_{\rho\sigma}{}^M\mathcal{H}_{MN}\mathcal{F}^{\rho\sigma N}-2e^{-2\Phi}\partial_\mu\Phi\partial_\nu\Phi\mathcal{F}^{\mu\rho M}\mathcal{H}_{MN}\mathcal{F}^\nu{}_\rho{}^N\\
&-\frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M(\mathcal{H}\partial_\rho\mathcal{H}^{-1}\partial^\nu\mathcal{H})_{MN}\mathcal{F}^{\mu\rho N}+\frac{1}{2}e^{-2\Phi}\partial_\rho\Phi\mathcal{F}_{\mu\nu}{}^M\partial^\rho\mathcal{H}_{MN}\mathcal{F}^{\mu\nu N}\\
&+me^{2\Phi}\epsilon^{\mu\nu\rho}\left(\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho}-\frac{1}{2}e^{-2\Phi}\mathcal{F}_{\mu\sigma}{}^M(\mathcal{H}\partial_\nu\mathcal{H}^{-1})_M{}^N\mathcal{F}_\rho{}^\sigma{}_N+e^{-2\Phi}\partial_\sigma\Phi\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^\sigma{}_{\rho M}\right)-4m^2e^{4\Phi}\partial_\mu\Phi\partial^\mu\Phi-m^4e^{8\Phi}\Bigg)\\
&+\frac{a-b}{4}e^{-2\Phi}\Bigg(-\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}\mathcal{H}\eta)-\frac{1}{16}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma M}\mathcal{F}^{\mu\nu P}\mathcal{H}_{PQ}\mathcal{F}^{\rho\sigma Q}\\
&-\frac{1}{8}e^{-4\Phi}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M\mathcal{F}_{\rho\sigma}{}^P\mathcal{H}_{PQ}\mathcal{F}^{\rho\sigma Q}+\frac{1}{2}e^{-4\Phi}\mathcal{F}_{\mu\rho}{}^M\mathcal{F}_\nu{}^\rho{}_M\mathcal{F}^{\mu\sigma P}\mathcal{H}_{PQ}\mathcal{F}^\nu{}_\sigma{}^Q\\
&+\frac{1}{4}e^{-2\Phi}\partial_\rho\Phi\partial^\rho\Phi\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M+\frac{1}{32}e^{-2\Phi}\mathrm{Tr}(\partial_\rho\mathcal{H}\partial^\rho\mathcal{H}^{-1})\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu}{}_M\\
&-\frac{1}{8}e^{-2\Phi}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\mathcal{F}^{\mu\rho M}\mathcal{F}^\nu{}_{\rho M}+\frac{1}{8}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M(\partial_\rho\mathcal{H}\partial^\rho\mathcal{H}^{-1})_M{}^N\mathcal{F}^{\mu\nu}{}_N\\
&+\frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\nu}{}^M(\partial^\mu\mathcal{H}\partial_\rho\mathcal{H}^{-1})_M{}^N\mathcal{F}^{\nu\rho}{}_N-e^{-2\Phi}\partial_\mu\Phi\mathcal{F}_{\nu\rho}{}^M(\partial^\nu\mathcal{H}\mathcal{H}^{-1})_M{}^N\mathcal{F}^{\mu\rho}{}_N\\
&+me^{2\Phi}\epsilon^{\mu\nu\rho}\left(\Omega^{(\omega_\mathrm{E})}_{\mu\nu\rho}+\frac{1}{4}e^{-2\Phi}\mathcal{F}_{\mu\sigma}{}^M(\partial^\sigma\mathcal{H}_{MN}+2\partial^\sigma\Phi\mathcal{H}_{MN})\mathcal{F}_{\nu\rho}{}^N\right)\Bigg)\Bigg].
\end{aligned}
\tag{B6}
$$

4. Dualization of the vector fields

We now dualize the vector fields into scalars by using the two-derivative dualization Eq. (4.11) in the form

$$
\mathcal{F}_{\mu\nu}{}^M=e^{2\Phi}\epsilon_{\mu\nu\rho}D^\rho\xi_N\mathcal{H}^{NM},
\tag{B7}
$$

with $D_\mu\xi_M=\partial_\mu\xi_M+m\mathcal{A}_{\mu M}$. We get

$$
\begin{aligned}
\tilde{I}_1''=\int &\mathrm{d}^3x\sqrt{-g_\mathrm{E}}e^{-2\Phi}\Bigg[-\frac{a+b}{8}\Bigg(-\partial_\mu\Phi\partial^\mu\Phi\partial_\nu\Phi\partial^\nu\Phi+\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&+\frac{1}{32}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-\frac{1}{64}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1})\mathrm{Tr}(\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})\\
&+\frac{1}{4}e^{4\Phi}D_\mu\xi^MD_\nu\xi_MD^\mu\xi^ND^\nu\xi_N-\frac{1}{4}e^{4\Phi}D_\mu\xi_M\mathcal{H}^{MN}D_\nu\xi_ND^\mu\xi_P\mathcal{H}^{PQ}D^\nu\xi_Q\\
&+\frac{1}{4}e^{4\Phi}D_\mu\xi_M\mathcal{H}^{MN}D^\mu\xi_ND_\nu\xi_P\mathcal{H}^{PQ}D^\nu\xi_Q+\frac{1}{2}e^{2\Phi}D_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})^{MN}D^\mu\xi_N\\
&-\frac{1}{2}e^{2\Phi}D_\mu\xi_M(\mathcal{H}^{-1}\partial^\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})^{MN}D^\nu\xi_N-2e^{2\Phi}\partial_\mu\Phi\partial_\nu\Phi D^\mu\xi_M\mathcal{H}^{MN}D^\nu\xi_N\\
&+\frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi\mathrm{Tr}(\partial^\mu\mathcal{H}\partial^\nu\mathcal{H}^{-1})-\frac{1}{4}\partial_\mu\Phi\partial^\mu\Phi\mathrm{Tr}(\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1})+e^{2\Phi}\partial_\mu\Phi D_\nu\xi_M\partial^\mu\mathcal{H}^{MN}D^\nu\xi_N\\
&+me^{2\Phi}\epsilon^{\mu\nu\rho}\left(\frac{2}{3}\Omega^{(\mathrm{GS})}_{\mu\nu\rho}-\frac{1}{2}e^{2\Phi}D_\mu\xi_M(\mathcal{H}^{-1}\partial_\nu\mathcal{H})^M{}_ND_\rho\xi^N\right)-4m^2e^{4\Phi}\partial_\mu\Phi\partial^\mu\Phi-m^4e^{8\Phi}\Bigg)\\
&+\frac{a-b}{4}\Bigg(-\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1}\partial_\nu\mathcal{H}\partial^\nu\mathcal{H}^{-1}\mathcal{H}\eta)+\frac{1}{4}e^{4\Phi}D_\mu\xi_MD_\nu\xi^MD^\mu\xi_P\mathcal{H}^{PQ}D^\nu\xi_Q\\
&-\frac{1}{4}e^{2\Phi}D_\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\partial^\mu\mathcal{H})^M{}_ND^\nu\xi^N-\frac{1}{2}e^{2\Phi}\partial_\mu\Phi\partial^\mu\Phi D_\nu\xi_MD^\nu\xi^M-\frac{1}{8}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial_\nu\mathcal{H}^{-1})D^\mu\xi_MD^\nu\xi^M\\
&+\frac{1}{16}\mathrm{Tr}(\partial_\mu\mathcal{H}\partial^\mu\mathcal{H}^{-1})D_\nu\xi_MD^\nu\xi^M+e^{2\Phi}\partial_\mu\Phi D^\mu\xi_M(\partial_\nu\mathcal{H}^{-1}\mathcal{H})^M{}_ND^\nu\xi^N+me^{2\Phi}\epsilon^{\mu\nu\rho}\Omega^{(\omega_\mathrm{E})}_{\mu\nu\rho}\Bigg)\Bigg],
\end{aligned}
\tag{B8}
$$

which is equivalent to Eq. (4.26). Note that the terms $\frac{m}{4}\epsilon^{\mu\nu\rho}\mathcal{F}_{\mu\sigma}{}^{M}(\partial^{\sigma}\mathcal{H}_{MN}+2\partial^{\sigma}\Phi\mathcal{H}_{MN})\mathcal{F}_{\nu\rho}{}^{N}$ give, upon dualization,

$$-\frac{m}{2}e^{4\Phi}\epsilon^{\mu\nu\rho}D_{\mu}\xi_{M}(\partial_{\nu}\mathcal{H}^{MN}+2\partial_{\nu}\Phi\mathcal{H}^{MN})D_{\rho}\xi_{N}=0. \tag{B9}$$

Idem for $m\epsilon^{\mu\nu\rho}e^{-2\Phi}\partial_{\sigma}\Phi\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}^{\sigma}{}_{\rho M}$.

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