2D black holes, Bianchi I cosmologies, and α'

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We report two surprising results on α' corrections in string theory restricted to massless fields. First, for critical dimension Bianchi type I cosmologies with q scale factors only $q-1$ of them have nontrivial α' corrections. In particular, for Friedmann-Robertson-Walker backgrounds all α' corrections are trivial. Second, in noncritical dimensions, all terms in the spacetime action other than the cosmological term are field redefinition equivalent to terms with arbitrarily many derivatives, with the latter generally of the same order. Assuming an α' expansion with coefficients that fall off sufficiently fast, we consider field redefinitions consistent with this falloff and classify the higher-derivative terms for two-dimensional string theory with one timelike isometry. This most general duality-invariant theory permits black-hole solutions, and we provide perturbative and nonperturbative tools to explore them.

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I. INTRODUCTION

String theory features a fundamental length scale, which is the square root of the inverse string tension α' and expected to be roughly $\sqrt{\alpha'} \sim 10^{-32}$ cm. Some of the mysteries of string theory revolve around the precise ramifications of this possibly minimal length scale (see, for example, [1]). In this paper we report on two surprising results (at least surprising to the authors) regarding the higher-derivative corrections of Einstein gravity that come from string theory and that are governed by α'. One pertains to cosmologies in critical string theory [2–4] and one to black holes in two-dimensional string theory [5–7].

In order to determine the higher-derivative corrections, the first step is to find, up to a given order in α', a basis of higher-derivative invariants, i.e., to classify the independent higher-derivative terms up to field redefinitions. Apart from assumptions of locality and invertibility, the field redefinitions to be investigated are those that respect the symmetries one assumes the higher-derivative terms to have. The effective field theory of strings can be written consistent with manifest diffeomorphism invariance, and so one considers field redefinitions that preserve the tensor character of fields. For instance, the Riemann-squared term of bosonic or heterotic string theory in critical dimensions cannot be changed by field redefinitions, and hence its coefficient has an invariant meaning.[1]

In this paper we consider string theories in critical and noncritical dimensions. For simplicity, we focus on classical bosonic strings and we will examine cosmological backgrounds in the critical dimension as well as the familiar $D=2$ (two spacetime dimensional) black hole.[2] Our work will consider low-energy limits of these string backgrounds. We will discuss situations where the total number of spacetime dimensions D is written as $D=d+1$, with d Abelian isometries and with fields depending only on the one remaining coordinate, which may be timelike or spacelike. In this situation, the effective field theory of classical strings possesses a global $O(d,d,\mathbb{R})$ duality invariance [13–15]. Therefore, it should be possible to write the higher-derivative terms in a manifestly $O(d,d,\mathbb{R})$ invariant form, and the field redefinitions should respect this structure. Interestingly, as implied by the seminal work of Meissner [16], manifest $O(d,d,\mathbb{R})$ invariance is in

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[1]It should be emphasized, however, that field redefinitions that do *not* preserve the tensor character are perfectly legal and may, in fact, be necessary when additional symmetry principles are imposed, as for instance in double field theory with α' corrections [8–12].

[2]In string theory usage, the black hole background is generally considered a critical string theory, as one is working directly with a theory of matter central charge 26. The name noncritical strings is reserved for nonconformal field theories coupled to two-dimensional gravity, in which case the Liouville mode of the metric helps restore conformal invariance.

conflict with manifest diffeomorphism invariance in D dimensions, a fact that reveals itself forcefully in the general setting of double field theory [11,12]. Perhaps somewhat surprisingly, in the context of dimensional reduction to one dimension a complete classification of all duality invariant higher-derivative terms can be obtained and only first-order derivatives are needed [17,18], see also [19,20].

Any classification of higher-derivative terms depends on the space of backgrounds or fields to be included. The above mentioned "cosmological" classification with fields depending only on time includes the g_{00} component of the metric, the purely spatial components of the metric and B-field, and the dilaton. Restricting this space of backgrounds further, there will generally be a more refined classification. For instance, we may assume the consistent truncation where the B-field vanishes and the spatial metric is diagonal, with d "scale factors" on the diagonal that may or may not be equal. For this smaller space of backgrounds there are fewer higher-derivative terms that one can write, but also fewer field redefinitions, so that the classification problem has to be reconsidered. More restrictive backgrounds provide computational advantages given the smaller configuration space, but also provide less options to consider deformations of solutions and to explore their stability. As one of the two main technical results of this paper we show that for one of the scale factors (that can be picked arbitrarily) all higher-derivative terms can be removed by field redefinitions. In particular, specializing further to the case that all scale factors are equal, corresponding to Friedmann-Robertson-Walker (FRW) backgrounds, it follows that *all* higher-derivative terms are removable by field redefinitions. The nonperturbative cosmological FRW backgrounds with a single scale factor explored in [17,18] were obtained in the context of α' corrections that cannot be removed for general time-dependent backgrounds. In that context, general time-dependent perturbations of the solution can be consistently analyzed. Moreover, it should also be emphasized that the removal of higher-derivative terms is strictly perturbative, so that there may be nonperturbative solutions that are not accessible for classifications of very restrictive backgrounds. More generally, perturbations or fluctuations away from a background may not preserve any conditions, as in cosmological perturbation theory, where the fluctuations depend on all coordinates. In order to use dualities one then requires a genuine double field theory [21].

As the second main result of this paper we revisit the subject of higher-derivative modifications of string theory in noncritical dimension, with a particular focus on the black hole solution in two dimensions [5–7]. In fact, one of the continuing appeals of this black hole solution is that it is based on an exact CFT, hence giving rise to an exact string background. Since the access to this exact background is quite limited, α' corrections of the two-derivative solution were considered in [22], and analyzed in some detail by Tseytlin in [23,24] who asserted that a well-defined α' expansion in noncritical dimensions does not exist. Moreover, while string loop corrections can be made arbitrarily small, the α' corrections cannot [25], hence making it particularly urgent to get a handle on these corrections.

We point out some surprising problems with the interpretation of higher-derivative terms as perturbative corrections to the leading order action or solution. To explain this let us recall the spacetime action for the metric and dilaton of string theory in D dimensions:

$$I[g,\phi] = \frac{1}{2\kappa_0^2}\int d^D x\sqrt{-g}e^{-2\phi}\bigg(-\frac{2(D-26)}{3\alpha'} + R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \sum_{n=1}^{\infty}(\alpha')^n F^{(n)}(g,\phi)\bigg), \tag{1.1}$$

where the $F^{(n)}$ denote possible terms of order $2n+2$ in derivatives, and we will sometimes refer to the term proportional to $\frac{1}{\alpha'}$ as the cosmological term. We will show that for $D \neq 26$ *any* term in the action other than the cosmological term can be traded, by means of field redefinitions involving derivatives, for a term with higher derivatives. This includes the two-derivative Einstein-Hilbert term that could be traded for a term with four derivatives. Iterating the redefinitions, the Einstein-Hilbert term could be traded for terms with, say, 42 derivatives. Alternatively, field redefinitions allow us to rewrite the theory with the original cosmological term and interactions, as a theory with the cosmological term and interactions having $2n$ or more derivatives, for any $n \geq 2$. In contrast to string theory in the critical dimension, there are *no* invariant terms apart from the cosmological term.

This result is puzzling, because adopting the usual perturbative mindset one would view terms with large numbers of derivatives as subleading compared to a term with two derivatives, and hence one would feel free to drop them. This is indeed the standard procedure of bringing higher-derivative terms to a minimal form, but using this procedure literally for string theory away from the critical dimension, one would conclude that only the cosmological term is nontrivial. What is really happening is that in such string backgrounds generic higher-derivative term are not actually subleading relative to terms with lesser number of derivatives. Thus, while the field redefinitions to be discussed are perfectly legal, it is the *second step* of dropping induced terms with more derivatives that is generally illegal.

To elaborate this point let us recall that while informally one refers to the higher-derivative corrections of string theory as "α' corrections," α' itself, being dimensionful, is not a small expansion parameter. In terms of the fundamental length scale $\sqrt{\alpha'}$ of string theory, α' is just one.

In fact, α' can be eliminated from the action (1.1) by defining the dimensionless derivative operator

$$\bar{\partial}_\mu \coloneqq \sqrt{\alpha'}\frac{\partial}{\partial x^\mu}, \tag{1.2}$$

and rescaling the action by α':

$$\bar{I} \coloneqq \alpha' I = \frac{1}{2\kappa_0^2}\int d^D x\sqrt{-g}e^{-2\phi}\bigg(-\frac{2(D-26)}{3} + \bar{R} + 4g^{\mu\nu}\bar{\partial}_\mu\phi\bar{\partial}_\nu\phi + \sum_{n=1}^{\infty}\bar{F}^{(n)}(g,\phi)\bigg), \tag{1.3}$$

where the bar over R or F indicates that all derivatives ∂_μ have been replaced by $\bar{\partial}_\mu$. In this formulation there is no α' left, and there is no expansion in α'. Rather, one should think of the higher-derivative corrections as an expansion in terms of small derivatives of the fields. While this can make sense in critical-dimension string theory, in noncritical dimension string theory generic solutions feature fields whose dimensionless derivatives are of order $\bar{\partial} \sim \mathcal{O}(1)$, so that all higher-derivative terms can have significant effects.

Indeed, the 2D black-hole solution is obtained by balancing the effects of the order $1/\alpha'$ cosmological term and the two derivative terms, the result being a configuration where dimensionless derivatives are of order one. This sheds doubt on attempts to find a more accurate black hole solution by means of higher-derivative corrections. Nevertheless, we show that if the higher-derivative terms are suppressed in a particular way, some simplifications of the effective action are valid. To see this, suppose an oracle gives us an action of the form (1.1) with infinitely many higher-derivative terms. *A priori*, general higher-derivative terms are all of the same order. Let us suppose, however, that the higher-derivative terms come with numerical coefficients that fall off in such a way that terms with four or more derivatives are sub-leading compared to terms with less derivatives. Since two-derivative terms come with order one coefficients, this could happen if the terms of order $(\alpha')^n$, with $n \geq 1$, come with coefficients of order ϵ^n with $\epsilon < 1$. In this situation we can ask and answer the following question: What are the most general field redefinitions that preserve this pattern, and what are the most general higher-derivative corrections modulo these restricted field redefinitions? We will show that these additional requirements eliminate those field redefinitions that allow one to remove arbitrary terms, and we will arrive at a minimal nontrivial set of higher-derivative terms that resembles the cosmological classification for critical string theory. Given the current knowledge of derivative corrections, we cannot know if such a classification applies to the 2D black hole. The general action in this setup, where fields are time independent, is obtained in Sec. II D and takes the form

$$I = \int dx n e^{-\Phi}\left[Q^2 + (D\Phi)^2 - M^2 + \sum_{i\geq 1}\frac{\epsilon_i}{Q^{2i}}M^{2i+2}\right]. \tag{1.4}$$

Here the metric is $ds^2 = -m^2(x)dt^2 + n^2(x)dx^2$, we defined $M = \frac{1}{n}\partial_x \ln m$, $Q^2 = 16/\alpha'$, and Φ is the duality-invariant dilaton. The first three terms in brackets define the action up to two derivatives; the last term, with arbitrary coefficients, represents the possible inequivalent higher-derivative terms. General field redefinitions of the "lapse" function $n(x)$ are not allowed for the classification—those are redefinitions that can remove any higher-derivative term. A linear combination of the lapse and dilaton can be redefined, and so can the metric component m. The work in [26] assumed an expansion analogous to that above and discussed possible solutions of the resulting equations, aiming to resolve the black hole singularity (the case when the metric depends on a single spatial coordinate is found in [27]).

This paper is organized as follows. In Sec. II we point out and discuss the subtleties that arise for field redefinitions of the string effective action in noncritical dimensions, i.e., in presence of the cosmological term. We do this both in general dimensions and in dimensional reduction to one spatial dimension where, helped by duality symmetry and the simplicity of the theory, we classify higher-derivative terms up to the above mentioned class of field redefinitions. In Sec. III we revisit the two-dimensional black hole solution and discuss possible perturbative and nonperturbative α' modifications. We also discuss the proposal of Dijkgraaf, Verlinde, and Verlinde [22] for a possibly exact background. In Sec. IV we revisit string cosmologies of type Bianchi I in critical dimensions and classify all higher-derivative corrections. We offer some concluding remarks in Sec. V.

II. STRING THEORY IN $D \neq 26$ AND IN $D = 2$

In this section we begin by showing how in the $D \neq 26$ string effective action for massless fields, having a cosmological term of order $1/\alpha'$, any interaction term can be removed by a field redefinition of the metric. We then turn to the important case when the spacetime dimension is two ($D = 2$) and examine the theory with the assumption that fields do not depend on time—they only depend on the spatial coordinate x. The field variables are the "lapse" function $n(x)$ whose square multiplies dx^2 in the metric, a metric component $m(x)$, whose square multiplies dt^2, and a duality invariant dilaton Φ. We use a simplified "dilaton-lapse" model to discuss possible field redefinitions, noting that they fall into two classes, one in which the cosmological term varies, and one in which it does not. We find that solutions, like the black hole or those of the dilaton-lapse model, do not lend themselves to an α' expansion. Instead we discuss a possible suppression of derivative corrections that could allow for a consistent set of field redefinitions—those in the second class above. We use this to finally give a classification of higher-derivative interactions for the $D=2$ backgrounds that have no time dependence.

A. Field redefinitions for $D \neq 26$

Let us reconsider the effective spacetime action I_D for strings in D dimensions (1.1) including its higher-derivative corrections:

$$I_D[g,\phi] = \int d^D x\sqrt{-g}e^{-2\phi}\Big(-\Lambda + R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \sum_{n=1}^{\infty}(\alpha')^n F^{(n)}(g,\phi)\Big), \tag{2.1}$$

where we have set the B-field to zero, we have introduced the constant Λ defined to be

$$\Lambda = \frac{2(D-26)}{3\alpha'}, \tag{2.2}$$

and the $F^{(n)}$ are arbitrary functions of g and ϕ of order $2n+2$ in derivatives. Note that the cosmological term is of order $1/\alpha'$, the two-derivative terms are of zeroth order in α', and the higher-derivative terms begin at order α'.

Consider now the *exact* field redefinition of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \Delta g_{\mu\nu}, \tag{2.3}$$

where we will take $\Delta g_{\mu\nu}$ to be a local function given by a derivative expansion.[3] This implies

$$\begin{aligned} g^{\mu\nu} &\to g^{\mu\nu} - \Delta g^{\mu\nu} + \mathcal{O}((\Delta g)^2),\\ \sqrt{-g} &\to \sqrt{-g}\Big(1 + \frac{1}{2}g^{\mu\nu}\Delta g_{\mu\nu} + \mathcal{O}((\Delta g)^2)\Big)\\ R &\to R + \Delta g^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big)\\ &\quad + g^{\mu\nu}(\nabla_\rho\Delta\Gamma^\rho_{\mu\nu} - \nabla_\mu\Delta\Gamma^\rho_{\rho\nu}) + \mathcal{O}((\Delta g)^2). \end{aligned} \tag{2.4}$$

Indices on $\Delta g_{\mu\nu}$ are raised with the unperturbed $g^{\mu\nu}$. We take $\Delta g_{\mu\nu}$ to be given by a derivative expansion:

$$\Delta g_{\mu\nu} = \Delta^{(1)}g_{\mu\nu} + \Delta^{(2)}g_{\mu\nu} + \cdots, \tag{2.5}$$

where $\Delta^{(n)}g_{\mu\nu}$ is of order $2n+2$ in derivatives. The first term in the redefinition above has four derivatives.

The redefined action I'_D is given by (2.1) with g replaced by $g+\Delta g$:

$$\begin{aligned} I'_D &:= I_D[g+\Delta g,\phi]\\ &= \int d^D x\sqrt{-g}e^{-2\phi}(-\Lambda + R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)\\ &\quad + \frac{1}{2}\int d^D x\sqrt{-g}e^{-2\phi}g^{\mu\nu}\Delta^{(1)}g_{\mu\nu}(-\Lambda)\\ &\quad + \alpha'\int d^D x\sqrt{-g}e^{-2\phi}F^{(1)}(g,\phi) + \cdots, \end{aligned} \tag{2.6}$$

where we used (2.4) and where the ellipsis denote terms with *more than four derivatives*. This follows from $\Delta^{(1)}g_{\mu\nu}$ being already of fourth order in derivatives, so that in particular new terms induced from the two-derivative action are already of order six in derivatives. We can cancel the four-derivative $F^{(1)}$ term by choosing

$$\Delta^{(1)}g_{\mu\nu} = \frac{2}{D}\frac{\alpha'}{\Lambda}g_{\mu\nu}F^{(1)}(g,\phi). \tag{2.7}$$

Since Λ is of order $1/\alpha'$, the above right-hand side is of order α'^2. The six-derivative terms encoded in $F^{(2)}$ receive further contributions from the field redefinition, and we denote the totality of all such terms by $\tilde{F}^{(2)}$. The redefined action then reads

$$I'_D = \int d^D x\sqrt{-g}e^{-2\phi}\Big(-\Lambda + R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (\alpha')^2\tilde{F}^{(2)}(g,\phi) + \cdots\Big), \tag{2.8}$$

where the ellipsis denotes all terms with more than six derivatives. Of course, we could have instead cancelled the Einstein-Hilbert term R by including a two-derivative term $\Delta^{(0)}g$ in the Δg expansion, and setting $\Delta^{(0)}g_{\mu\nu} = \frac{2}{D}\frac{1}{\Lambda}g_{\mu\nu}R$. In that case the action would have the cosmological term followed by terms with four derivatives.

The procedure above can be iterated. Looking at the action (2.8) we can just repeat the procedure by setting

$$\Delta^{(2)}g_{\mu\nu} = \frac{2}{D}\frac{\alpha'^2}{\Lambda}g_{\mu\nu}\tilde{F}^{(2)}(g,\phi), \tag{2.9}$$

so as to cancel the terms $\tilde{F}^{(2)}$ with six derivatives. Thus, all higher-derivative corrections can be moved to arbitrary high order in α'.

B. Dimensional reduction

We want to analyze the duality properties and α' corrections of the black hole solution in 2D string theory [5,6]. The two-derivative action for string theory in D dimensions, taken with vanishing B-field, reads

[3]As a field redefinition one must view this replacement as setting $g_{\mu\nu} = g'_{\mu\nu} + \Delta g_{\mu\nu}(g')$ so that the action becomes $I(g) = I(g' + \Delta g(g')) \equiv I'(g')$. One can solve for g' in terms of g by inverting the expression for g in terms of g'.

$$I_D = \int d^D x\sqrt{-g}e^{-2\phi}\left(-\frac{2(D-26)}{3\alpha'} + R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right). \tag{2.10}$$

To focus on the string theory black hole we set $D = 2$, with coordinates $x^\mu = (x^0, x^1) = (t, x)$ and fields that do not depend on time t. We thus make the ansatz

$$g_{\mu\nu} = \begin{pmatrix} -m^2(x) & 0 \\ 0 & n^2(x) \end{pmatrix}, \qquad \phi = \phi(x). \tag{2.11}$$

It is worthwhile to compare with the cosmological case, where all fields depend on time and are independent of d internal spatial coordinates. In this case one has a global $O(d, d)$ duality symmetry. Here, spacetime is two dimensional, and the fields do not depend on time. Time is then the one "internal" coordinate and the duality group is just $O(1, 1)$. In the cosmological setting, the component of the metric in the time-time direction is the lapse function. Here, the component $n(x)$ of the metric in the space-space direction is the analog of the cosmological lapse function. The resulting action will be x-reparametrization invariant. With reparametrizations $x \to x - \lambda(x)$ we have that scalars A transform as $\delta_\lambda A = \lambda\partial_x A$. The field $m^2(x) = -g_{00}(x)$ is a scalar under x-reparametrizations, and so is $m(x)$. The field $n(x)$ transforms as a density: $\delta_\lambda n = \partial_x(\lambda n)$. Since only x derivatives exist, we do not need partial derivatives, and ordinary x derivatives will be denoted with primes $' \equiv \frac{d}{dx}$. When multiplied with n^{-1}, x derivatives then give the covariant derivative

$$D \equiv \frac{1}{n}\frac{d}{dx}. \tag{2.12}$$

If A is a scalar, then DA is also a scalar. The group of dualities here is $O(1, 1)$. In the component connected to the identity, group elements h are of the form

$$h = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix}, \qquad h^t\eta h = \eta,$$
$$\text{with } \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \eta^t = \eta^{-1}. \tag{2.13}$$

Here α is any real number, and η is the $O(1, 1)$ metric. In the component disconnected to the identity, we have

$$h = \begin{pmatrix} 0 & e^\alpha \\ e^{-\alpha} & 0 \end{pmatrix} \in O(1, 1). \tag{2.14}$$

Note that for $\alpha = 0$ we have $h = \eta$, as a group element.

In this setup, the generalized metric $\mathcal{S}$ is a two-by-two matrix that involves the internal metric components, that is, the component m^2 introduced above, and the internal B-field, which vanishes in one dimension. We thus get

$$\mathcal{S} = \begin{pmatrix} 0 & m^2 \\ m^{-2} & 0 \end{pmatrix} \in O(1, 1), \tag{2.15}$$

where we noted that $\mathcal{S}$ is an element of the group $O(1, 1)$, in fact, an element belonging in the component disconnected to the identity, as it is clear from its determinant being equal to minus one. Under duality one has

$$\mathcal{S} \to h\mathcal{S}h^{-1}. \tag{2.16}$$

For the duality transformations connected to the identity, the field m is scaled by a constant: $m \to me^\alpha$. The field $n(x)$ is duality invariant, and we have $\phi \to \phi + \frac{\alpha}{2}$. As a result, we have the duality-invariant dilaton Φ given as

$$e^{-\Phi(x)} \equiv m(x)e^{-2\phi(x)}. \tag{2.17}$$

Since both $\phi(x)$ and $m(x)$ are scalars under x-reparametrizations, Φ is also a scalar under such reparametrizations. The above equation makes the duality invariance of the measure clear:

$$\sqrt{-g}e^{-2\phi} = nme^{-2\phi} = ne^{-\Phi}. \tag{2.18}$$

For the disconnected dualities we take $h = \eta$ and then have

$$\mathcal{S} \to \eta\mathcal{S}\eta = \begin{pmatrix} 0 & m^{-2} \\ m^2 & 0 \end{pmatrix}, \qquad \text{or} \qquad m \to \frac{1}{m}, \tag{2.19}$$

while invariance of (2.17) yields $\phi \to \phi - \ln|m|$. This is the familiar discrete $\mathbb{Z}_2$ duality.

The effective action can be written ignoring time, as no quantity is time dependent. The resulting one-dimensional action, setting $D = 2$ in (2.10) and on account of the above comments, takes the form

$$I = \int dxne^{-\Phi}(Q^2 + R + 4n^{-2}(\phi')^2), \tag{2.20}$$

where we defined

$$Q^2 \equiv \frac{16}{\alpha'}. \tag{2.21}$$

To get a useful expression in terms of the m and Φ fields, we need to work out the Ricci scalar for the above metric ansatz. The nonvanishing Christoffel symbols are given by

$$\Gamma^1_{00} = \frac{1}{n^2}mm', \quad \Gamma^0_{10} = m^{-1}m', \quad \Gamma^1_{11} = n^{-1}n'. \tag{2.22}$$

The Ricci tensor $R_{\nu\sigma} = \partial_\mu\Gamma^\mu_{\nu\sigma} - \partial_\nu\Gamma^\mu_{\mu\sigma} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$ then yields the components

$$R_{00} = \frac{1}{n^2}(mm'' - n^{-1}n'mm'),$$
$$R_{11} = -m^{-1}m'' + m^{-1}m'n^{-1}n'. \tag{2.23}$$

The Ricci scalar is therefore

$$R = -\frac{2m''}{mn^2} + \frac{2}{mn^3}n'm' = -\frac{2}{mn}\left(\frac{m'}{n}\right)' = -\frac{1}{mn}\left(\frac{(m^2)'}{mn}\right)'. \tag{2.24}$$

Inserting this into the action, using (2.17) to pass from ϕ to Φ and m, and integrating by parts in order to have only first-order derivatives gives

$$I = \int dx \frac{1}{n} e^{-\Phi}(n^2Q^2 + (\Phi')^2 - (m^{-1}m')^2). \tag{2.25}$$

The action can also be written in terms of the generalized metric $\mathcal{S}$

$$I = \int dx \frac{1}{n} e^{-\Phi}\left(n^2Q^2 + (\Phi')^2 + \frac{1}{8}\mathrm{tr}(\mathcal{S}')^2\right). \tag{2.26}$$

$O(1,1)$ global duality invariance is manifest because n and Φ are invariant and $\mathcal{S}$ transforms as indicated in (2.16). Finally, defining

$$M \equiv m^{-1}Dm = \frac{m'}{mn}, \tag{2.27}$$

we can also write the action as

$$I = \int dxne^{-\Phi}(Q^2 + (D\Phi)^2 - M^2). \tag{2.28}$$

The equations of motion follow from the general variation

$$\delta I = \int dxne^{-\Phi}\left[\frac{\delta n}{n}E_n + 2\frac{\delta m}{m}E_m + \delta\Phi E_\Phi\right], \tag{2.29}$$

for which we find

$$E_n = -(D\Phi)^2 + M^2 + Q^2 = 0,$$
$$E_m = DM - (D\Phi)M = 0,$$
$$E_\Phi = -2D^2\Phi + (D\Phi)^2 + M^2 - Q^2 = 0. \tag{2.30}$$

As a consequence of coordinate-reparametrization invariance, these equations are not all independent but satisfy the Bianchi identity

$$D\Phi(E_n + E_\Phi) + 2ME_m - DE_n = 0. \tag{2.31}$$

We note that the second equation in (2.30) implies

$$(e^{-\Phi}M)' = e^{-\Phi}(M' - \Phi'M) = 0, \tag{2.32}$$

i.e., that $e^{-\Phi}M$ is constant and does not depend on x. This is a manifestation of Noether's theorem for $O(1,1)$ invariance.

C. Two classes of field redefinitions

In this section we will explore field redefinitions of a dilaton-lapse model with a cosmological term. In this model, obtained by setting $M = 0$ in the action (2.28), one can easily find solutions of the equations of motion, even after including a large class of higher-derivative terms. This allows us to see explicitly the effect of field redefinitions. Moreover, the results obtained here apply almost without change to the case of the $D = 2$ black hole.

We will see clearly in this dilaton-lapse model that solutions of the lowest order nontrivial equations imply that a conventional derivative expansion is problematic: there is no obvious suppression of the higher-derivative terms. All terms seem equally important and field redefinitions used to classify interactions do not operate as usual.

Our analysis below will assume theories in which there is an *in-built* suppression of the higher-derivative terms, a suppression due to constants that multiply the interactions and become smaller as the number of derivatives increase.

While we cannot justify such an assumption in the case of the 2D black hole, the assumption does seem to hold for certain field theories arising from string theory, as is the case of tachyon dynamics, where, as discussed in [28], the Lagrangian L takes the form

$$L = \frac{1}{2}\phi(\partial^2 + 1)\phi + \frac{1}{3}(e^{\xi^2\partial^2}\phi)^3. \tag{2.33}$$

This Lagrangian, written in terms of unit-free fields and derivatives, depends on a constant ξ, a nonlocality parameter. Roughly, a term with $2k$ derivatives comes with a factor ξ^{2k}. If $\xi < 1$ there is some suppression, and for $\xi \ll 1$ a strong suppression.

Returning to our dilaton-lapse model, even assuming an in-built suppression, complications arise due to the cosmological term. We will argue that there are two classes of field redefinitions:

(1) Separate field redefinitions of the lapse or dilaton function, which generate new interactions via the variation of the cosmological term as well as the variation of other terms.
(2) Simultaneous field redefinitions of the lapse and dilaton for which no terms arise from the variation of the cosmological term.

We will argue that redefinitions of type 1 do not preserve the structure of in-built suppression: higher-derivative terms induced by the redefinitions are not suppressed appropriately and thus cannot be neglected. Redefinitions of type 2, however, respect the structure of in-built suppression and thus can be used in the conventional setting of effective field theory to classify interactions. We will consider the issue first on a simplified model and then we will generalize.

1. Dilaton-lapse model and redefinitions

Consider, therefore, the two-derivative theory obtained by setting $M = 0$ in the action (2.28):

$$I^{(0)} = \int dxne^{-\Phi}[Q^2 + (D\Phi)^2]. \tag{2.34}$$

Its equations of motion are given by

$$\begin{aligned} E_n^{(0)} &\equiv Q^2 - (D\Phi)^2 = 0, \\ E_\Phi^{(0)} &\equiv -Q^2 + (D\Phi)^2 - 2D^2\Phi = 0. \end{aligned} \tag{2.35}$$

These are solved by the linear dilaton background

$$n^{(0)} = 1, \qquad \Phi^{(0)} = Qx. \tag{2.36}$$

The simplest higher-derivative extension to (2.34) is given by

$$I^{(1)} = \int dxne^{-\Phi}[Q^2 + (D\Phi)^2 + c\alpha'(D\Phi)^4]. \tag{2.37}$$

Note that for the solution $\Phi^{(0)} = Qx$, all three terms in the action are of the same order Q^2 (since $\alpha' \sim 1/Q^2$). The lapse equation following from the above action is given by

$$E_n \equiv Q^2 - (D\Phi)^2 - 3c\alpha'(D\Phi)^4 = 0, \tag{2.38}$$

and the dilaton equation is automatically satisfied when the lapse equation holds due to the Bianchi identity [see (2.31) specialized to $E_m = 0$]. The above equation admits a unique real solution of the form

$$n = 1, \quad \Phi = \omega x, \quad \omega^2 \equiv \frac{\sqrt{1 + 12c\alpha' Q^2} - 1}{6c\alpha'}. \tag{2.39}$$

This solution can be considered a small correction to (2.36) when ω^2 has a convergent perturbative expansion in powers of $c\alpha' Q^2$. This happens when $|12c\alpha' Q^2| < 1$. With $Q^2 = \frac{16}{\alpha'}$, the only option is a small coefficient in the action, namely $|c| < \frac{1}{192}$. More generally, we need

$$\epsilon \equiv c\alpha' Q^2 \ll 1. \tag{2.40}$$

If this condition is satisfied, ω^2 can be expanded in powers of ϵ and so

$$\Phi = \omega x = \left(1 - \frac{3}{2}\epsilon + \mathcal{O}(\epsilon^2)\right) Qx = \Phi^{(0)} - \frac{3}{2}\epsilon Qx + \mathcal{O}(\epsilon^2), \tag{2.41}$$

where the leading term is the two-derivative solution (2.36) and the rest are truly small corrections to it. If (2.40) does not hold, the four-derivative term in the action contributes terms comparable to the two-derivative solution, making a derivative expansion meaningless. We assume that (2.40) holds.

For simplicity, we rewrite the theory in terms of a unit-free derivative $\bar{\partial}_x \equiv (1/Q)\partial_x$, so that $\bar{D} = (1/Q)D$. In terms of these derivatives, the action (2.37) becomes

$$\bar{I}^{(1)} \equiv \frac{1}{Q^2} I^{(1)} = \int dxne^{-\Phi}[1 + (\bar{D}\Phi)^2 + \epsilon(\bar{D}\Phi)^4]. \tag{2.42}$$

In this notation a variation of the lapse $n \to n + \delta n$ gives to leading order

$$\begin{aligned} \bar{I}^{(1)} &\to \bar{I}^{(1)} + \int dxne^{-\Phi}\frac{\delta n}{n}\bar{E}_n + \mathcal{O}((\delta n)^2), \\ \text{with} \quad \bar{E}_n &= 1 - (\bar{D}\Phi)^2 - 3\epsilon(\bar{D}\Phi)^4. \end{aligned} \tag{2.43}$$

We now perform a lapse redefinition to remove the four-derivative term. We take

$$\frac{\delta n}{n} = -\epsilon(\bar{D}\Phi)^4. \tag{2.44}$$

The associated field redefinition is $n = n' + \delta n(n')$ and $\Phi = \Phi'$. The redefined action, called $\bar{I}'$, and written in terms of the new (primed) fields is given by

$$\begin{aligned} \bar{I}' = \int dxn'e^{-\Phi'}[&1 + (\bar{D}\Phi')^2 + \epsilon(\bar{D}\Phi')^6 + \mathcal{O}(\epsilon^2) \\ &+ \mathcal{O}((\delta n)^2)]. \end{aligned} \tag{2.45}$$

The field redefinition eliminated the four-derivative term at the cost of introducing a six-derivative term, *also* at order ϵ. Since derivatives $\bar{D}$ in the unperturbed theory are of order one, this new term is not parametrically smaller than the original four-derivative term. This shows that pure lapse transformations do not allow us to classify interactions in the sense of effective field theory. They are redefinitions of type 1, and as claimed do not respect the structure of in-built suppression. We would have wanted the new six-derivative interaction to appear at a higher order in ϵ.

We now discuss type 2 transformations; those that preserve the structure of in-built suppression and thus can be used to remove higher-derivative terms without inducing same-order effects. In this case, it will allow us to eliminate the four-derivative term consistently. The redefinition is of the form

$$\begin{aligned} \Phi &= \Phi' + \delta\Phi(n', \Phi'), \\ n &= n' + \delta n(n', \Phi'), \\ \text{with } \delta\Phi &= \frac{\delta n}{n}. \end{aligned} \tag{2.46}$$

For such a correlated redefinition of the dilaton and the lapse, the variation of the action to linearized order is, from (2.29),

$$\delta I = \int dx n e^{-\Phi}[(E_n + E_\Phi)\delta\Phi],$$
$$E_n + E_\Phi = -2\bar{D}^2\Phi + \mathcal{O}(\epsilon). \tag{2.47}$$

An important fact is that this linear combination of equations of motion has no constant term. We will choose, to order ϵ, the following variation in order to remove the four-derivative term from (2.42):

$$\frac{\delta n}{n} = \delta\Phi = \frac{3}{2}\epsilon(\bar{D}\Phi')^2. \tag{2.48}$$

Under this redefinition, (2.42) changes to a new action, with primed fields

$$\begin{aligned}\bar{I}' &= \int dx n' e^{-\Phi'}\bigg[1 + (\bar{D}\Phi')^2 + \epsilon(\bar{D}\Phi')^4 \\ &\quad + (-2\bar{D}^2\Phi')\left[\frac{3}{2}\epsilon(\bar{D}\Phi')^2\right] + \mathcal{O}(\epsilon^2)\bigg] \\ &= \int dx n' e^{-\Phi'}[1 + (\bar{D}\Phi')^2 + \epsilon[(\bar{D}\Phi')^4 \\ &\quad - 3(\bar{D}\Phi')^2\bar{D}^2\Phi'] + \mathcal{O}(\epsilon^2)] \\ &= \int dx n' e^{-\Phi'}[1 + (\bar{D}\Phi')^2 + \mathcal{O}(\epsilon^2)]. \end{aligned}\tag{2.49}$$

In the last equality we used integration by parts to see that the $\mathcal{O}(\epsilon)$ term is identically zero. This time we succeed in eliminating the four-derivative term since the only additional terms generated are truly higher-order $\mathcal{O}(\epsilon^2)$ effects.

2. The general dilaton-lapse model and field redefinitions

The lessons of the above discussion can be refined by considering the general version of the dilaton model:

$$I = \int dx n e^{-\Phi}\left[Q^2 + (D\Phi)^2 + \sum_{i\geq 1} c_i \alpha'^i \mathcal{L}^{(2i+2)}(D;\Phi)\right], \tag{2.50}$$

which includes infinitely many α' corrections depending only on covariant derivatives of Φ, with $\mathcal{L}^{(2i+2)}(D;\Phi)$ containing $2i+2$ derivatives. The same action can be rewritten in terms of the dimensionless derivative $\bar{D}$, absorbing a factor of Q^2:

$$\bar{I} = \int dx n e^{-\Phi}\left[1 + (\bar{D}\Phi)^2 + \sum_{i\geq 1}\epsilon_i \bar{L}^{(2i+2)}\right],$$
$$\epsilon_i \equiv c_i(\alpha' Q^2)^i. \tag{2.51}$$

Again, this action has no meaningful derivative expansion unless the coefficients ϵ_i decay fast enough. This can be formalized by extending the condition $\epsilon \ll 1$ considered before to the condition:

$$\epsilon \equiv \epsilon_1 \ll 1, \qquad \epsilon_i \sim (\epsilon)^i, \qquad i \geq 1, \tag{2.52}$$

where the symbol $\sim$ denotes proportionality up to factors of order one. The above condition guarantees that each term in the derivative expansion is parametrically smaller than the previous one. In order to explore the effect of perturbative field redefinitions, we will assume that this condition is satisfied. Note that the condition above also implies that

$$\epsilon_p \epsilon_k \sim \epsilon_{p+k}. \tag{2.53}$$

A pure lapse transformation or a pure dilaton transformation will break condition (2.52), the correlation between the number of derivatives and the power of ϵ. This happens because such redefinitions generate variations from the zero derivative term in the action and the two-derivative term in the action, and the powers of ϵ are no longer correlated with the number of derivatives.

The solution is clear, to leave the correlation between derivatives and powers of ϵ we must perform (as we did before) a redefinition of both the lapse and the dilaton:

$$\Phi = \Phi' + \delta\Phi(\Phi', n'), \qquad n = n' + \delta n(\Phi', n'), \tag{2.54}$$

where the variations are related as follows:

$$\frac{\delta n}{n'} = e^{\delta\Phi} - 1. \tag{2.55}$$

This is constructed such that $n e^{-\Phi}$ is kept invariant:

$$\begin{aligned} n e^{-\Phi} &= (n' + n'(e^{\delta\Phi} - 1))e^{-\Phi'}e^{-\delta\Phi} \\ &= n' e^{\delta\phi} e^{-\Phi'} e^{-\delta\Phi} = n' e^{-\Phi'}. \end{aligned}\tag{2.56}$$

As a result, the cosmological term does not generate variations, and this will allow us to preserve condition (2.52). In the above we take

$$\delta\Phi = \sum_{i\geq 1}\epsilon_i F^{(2i)}, \tag{2.57}$$

with $F^{(2i)}$ generic gauge and duality invariant terms depending on $\bar{D}\Phi$ and containing $2i$ derivatives, and the ϵ_i are the constants introduced earlier. A completely analogous expression holds for the variation of n, with a related set of functions $G^{(2i)}$.

Applying the variations to the action (2.51), we note that we must only vary the terms inside the brackets. Beginning with the two-derivative term and using the above $\delta\Phi$, each term in the variation will have an ϵ_i accompanied with $2i+2$ derivatives, $2i$ of them from $F^{(2i)}$, and the other two from the two-derivative term being varied. This is indeed consistent with the structure of the suppression. Continuing with the higher-derivative terms, varying $\bar{\mathcal{L}}^{(2j+2)}$ in the action with the $F^{(2k)}$ term of the dilaton variation, one gets the product $\epsilon_j \epsilon_k \sim \epsilon_{j+k}$ multiplying terms with $(2j+2k)+2$ derivatives, which is also consistent with the constraint. Of course, identical remarks hold for the variation of n. This shows how the claimed redefinitions are consistent with the falloff conditions.

Therefore, we can use (2.55) order by order in ϵ so to remove terms consistently. This is the procedure developed in [17] for critical strings, with the role of α' played here by the ϵ_i's satisfying (2.52). Using that logic we can implement field redefinitions as simple substitution rules in the action. We can in fact conclude that the all-order theory (2.51) is totally equivalent to the lowest order one (2.34). Indeed, the substitution rule follows from the equation of motion from part of the theory up to two-derivatives

$$\bar{E}_n + \bar{E}_\Phi = -2\bar{D}^2\Phi \quad \Rightarrow \quad \bar{D}^2\Phi \simeq 0 + \mathcal{O}(\epsilon), \qquad (2.58)$$

and so we can recursively eliminate any term containing higher-derivatives of the dilaton (using integration by parts and dropping total derivatives).

D. Classification of higher-derivatives in $D = 2$

The final conclusion of the previous subsection can be easily extended to the case when m propagates. To see this, we consider the analogue to (2.51) in the presence of M

$$\bar{I} = \int dxne^{-\Phi}\left[1 + (\bar{D}\Phi)^2 - \bar{M}^2 + \sum_{i\geq 1}\epsilon_i\bar{\mathcal{L}}^{(2i+2)}(n, \bar{D}\Phi, \bar{M})\right], \qquad (2.59)$$

where $\bar{\mathcal{L}}^{(2i+2)}$ contains $2i+2$ derivatives, and the in-built suppression conditions $\epsilon_i \sim (\epsilon)^i$ holds. We also extended the bar notation to $\bar{M} = \frac{1}{Q}M$. Then, we notice that by extending (2.55) to

$$\Phi = \Phi' + \delta\Phi(n', \Phi', m'), \qquad n = n' + \delta n(n', \Phi', m'),$$
$$m = m' + \delta m(n', \Phi', m'), \qquad \frac{\delta n}{n} = e^{\delta\Phi} - 1,$$
$$\delta\Phi = \sum_{i\geq 1}\epsilon_i F_\Phi^{(2i)}, \qquad \frac{\delta m}{m} = \sum_{i\geq 1}\epsilon_i F_m^{(2i)}, \qquad (2.60)$$

the in-built suppression feature is satisfied for the induced terms since transformations of m do not affect the measure $ne^{-\Phi}$ and so it remains invariant under the redefinitions (2.60). Finally, in the same way we could use (2.58) to perform a classification for the dilaton model (2.51), here we can apply the rules

$$E_n + E_\Phi = 0 \quad \Rightarrow \quad \bar{D}^2\Phi \simeq \bar{M}^2 + \mathcal{O}(\epsilon), \qquad (2.61a)$$

$$E_m = 0 \quad \Rightarrow \quad \bar{D}\,\bar{M} \simeq \bar{D}\Phi\bar{M} + \mathcal{O}(\epsilon), \qquad (2.61b)$$

where we used the lowest order equations of motion given in (2.30). The classification of [17] goes through up to the point where redefinitions of the lapse function are needed, which in this case are not allowed because of the extra condition $n'e^{-\Phi'} = ne^{-\Phi}$. Specifically, the same step-by-step proof of Sec. 2.2 in [17], with the same itemization, proceeds as follows. We assume that to any order in ϵ any term in the action is writable as a product of factors $\bar{D}^k\Phi$ and $\bar{D}^l\bar{M}$. We can now perform field redefinitions of the form (2.60), which in practice consist of applying the rules (2.61) in the action, in order to establish:

(1) *A factor in an action including $\bar{D}^2\Phi$ can be replaced by a factor with only first derivatives.* This follows directly from (2.61a).
(2) *A factor in an action including $\bar{D}\,\bar{M}$ can be replaced by a factor with only first derivatives.* This follows directly from (2.61b).
(3) *Any action can be reduced so that it only has first derivatives of* Φ. The proof proceeds as in [17]: We write any higher-derivative as $\bar{D}^{p+2}\Phi = \bar{D}^p(\bar{D}^2\Phi)$, and then integrate by parts the $\bar{D}^p$. Then we substitute $\bar{D}^2\Phi \to \bar{M}^2$, after which we integrate the derivatives back one-by-one, eliminating any second derivative created, using (1) or (2). At the end we are left with only first-order derivatives of Φ.
(4) *Any action can be reduced so that it only contains $\bar{M}$, not its derivatives.* The proof is identical to the previous one.
(5) *Any higher-derivative term is equivalent to one without any appearance of $\bar{D}\Phi$.* So far we have shown that a generic higher-derivative term in the action is

$$I = \int dxne^{-\Phi}(\bar{D}\Phi)^{2p}\bar{M}^{2l}, \qquad (2.62)$$

where duality invariance demands even powers of $\bar{M}$ and thus even powers of $\bar{D}\Phi$, since the total number of derivatives must be even. This means that $p, l = 0, 1, 2, \ldots$ and $p + l > 1$, to have at least four derivatives. Using $e^{-\Phi}\bar{D}\Phi = -\bar{D}e^{-\Phi}$ for one of the $\bar{D}\Phi$ factors, and then integrating by parts we find

$$\begin{aligned} I &= -\int dxn\bar{D}(e^{-\Phi})(\bar{D}\Phi)^{2p-1}\bar{M}^{2l}, \\ &= \int dxne^{-\Phi}((2p-1)(\bar{D}\Phi)^{2p-2}\bar{D}^2\Phi\bar{M}^{2l} \\ &\quad + (\bar{D}\Phi)^{2p-1}2l\bar{M}^{2l-1}\bar{D}\,\bar{M}), \\ &\simeq \int dxne^{-\Phi}((2p-1)(\bar{D}\Phi)^{2p-2}\bar{M}^{2l+2} \\ &\quad + 2l(\bar{D}\Phi)^{2p}\bar{M}^{2l}), \end{aligned} \qquad (2.63)$$

where we used (2.61) in the last line. The second term in the last line is a multiple of the original term. Thus, bringing it to the left-hand side,

$$(1-2l)I \simeq (2p-1)\int dxne^{-\Phi}(\bar{D}\Phi)^{2p-2}\bar{M}^{2l+2}, \tag{2.64}$$

and so, since $l \neq \frac{1}{2}$,

$$I \simeq \frac{2p-1}{1-2l}\int dxne^{-\Phi}(\bar{D}\Phi)^{2p-2}\bar{M}^{2l+2}. \tag{2.65}$$

Thus, we can systematically reduce the powers of $\bar{D}\Phi$ in steps of two until removing all $\bar{D}\Phi$ factors.[4]

The above chain of arguments proved that there is a field basis in which all higher-derivative terms involve only powers of $\bar{M}^2$ and so the most general action is given by

$$I = \int dxne^{-\Phi}\left[1+(\bar{D}\Phi)^2-\bar{M}^2+\sum_{i\geq 1}\epsilon_i\bar{M}^{2i+2}\right], \tag{2.69}$$

where, if truncated at order ϵ^{N-1} we know from (2.52) that the remaining terms are of order $\mathcal{O}(\epsilon^N\bar{D}^{2N+2})$ and therefore contribute small corrections to the solutions of the truncated theory.

III. BLACK HOLE SOLUTIONS

This section begins by reviewing the construction of the black hole solution in the field variables and action with simple duality properties (Sec. II B). We point out that the dilaton-lapse theory can be viewed as giving rise to a zeroth order solution—a linear dilaton profile, such that the black hole arises as a perturbation of this solution. We then begin to examine the solutions of the general two-dimensional theory with spatial dependence only, using the classification in (2.69). The general theory is specified by a power series in $\bar{M}$, with just even powers, which we call $F(\bar{M})$. We demonstrate that the general solution for the fields $m(x)$ and $\Phi(x)$ can be obtained in terms of an integral involving functions of $\bar{M}$ easily constructed from $F(\bar{M})$. The two-derivative black hole solution, in the gauge $n=1$, is then given by $\bar{M}=\mathrm{csch}\bar{x}$ and $e^{\Phi}=\mathrm{csch}\bar{x}$, and $m(x)$ can be obtained by suitable integration of $\bar{M}$.

The integral formulation is the basis for a perturbative solution of the equations of motion. The perturbation is relative to the two-derivative black hole solution which is taken to be the zeroth order solution, with the perturbative parameter $\epsilon<1$ that controls the falloff of the higher-derivative terms in the action. We find that at each order of the perturbation the contributions to $\bar{M}$ and e^{Φ} are given by finite polynomials in $\mathrm{csch}\bar{x}$.

We conclude with some analysis of the systematics of nonperturbative solutions, motivated by the perturbative results. We use an ansatz where $\bar{M}$ and e^{Φ} are written as series expansions in terms of $\mathrm{csch}\bar{x}$ and find that the series for $\bar{M}$ appears to fix both the series for e^{Φ} and for $F(\bar{M})$. We also discuss the ansatz for an "exact" solution by Dijkgraaf, Verlinde, and Verlinde [22]. There is no simple method, however, to analyze this solution in our framework since in our field basis the T-duality transformation takes the form $m\to 1/m$, which is not the case in their formulation, except in the $k\to\infty$ limit ($k=9/4$ for the black hole). More concretely, we see that their ansatz cannot be fit into our formulation.

A. Black holes in the two-derivative theory

For completeness, we begin by rederiving the black hole solution in the standard coordinates resembling the Schwarzschild solution in four dimensions. The equations of motion are given in (2.30) and $E_m=0$ implies that $e^{-\Phi}M$ is constant, i.e.,

$$M = e^{\Phi}q, \qquad q=\text{const}. \tag{3.1}$$

The two remaining equations we put in the equivalent form $E_n=0$, $E_\Phi+E_n=0$, and write out the covariant derivatives:

$$\frac{1}{n^2}(\Phi')^2-\left(\frac{m'}{mn}\right)^2 = Q^2,$$
$$\frac{1}{n}(n^{-1}\Phi')'-\left(\frac{m'}{mn}\right)^2 = 0. \tag{3.2}$$

We now set

$$mn=1. \tag{3.3}$$

This constraint can be viewed as a gauge condition. Indeed, mn is a density since the lapse n is a density and m is a scalar. Instead of gauge fixing to $n=1$, which we will do at some point, here we gauge fix to have $mn=1$. Note that in this gauge $M=m'$. We now make the ansatz

[4]Let us justify (2.64), in which we solve an equivalence relation between actions as if it was an actual equation. Suppose we establish an on-shell equivalence of an action term I of the form

$$I \simeq \alpha I + K, \tag{2.66}$$

where α is a numerical coefficient and K another action term. In order to solve $I\simeq\frac{1}{1-\alpha}K$ for $\alpha\neq 1$ we write

$$I = (1-\beta)I+\beta I, \tag{2.67}$$

where β is for now an undetermined parameter. Using (2.66) in the second term, we have

$$I \simeq (1-\beta)I+\beta(\alpha I+K) = (1-\beta(1-\alpha))I+\beta K. \tag{2.68}$$

Since β is arbitrary we can choose $\beta=\frac{1}{1-\alpha}$, so that the terms proportional to I cancel, and $I\simeq\frac{1}{1-\alpha}K$, as anticipated by solving (2.66) naively.

$$e^{-\Phi} = \frac{m}{f}. \tag{3.4}$$

This relation is just a convenient way to parametrize the dilaton in terms of a function f to be determined. It implies by differentiation

$$\Phi' = -\frac{m'}{m} + \frac{f'}{f}. \tag{3.5}$$

The relation (3.1) then implies

$$e^{-\Phi}M = e^{-\Phi}m' = \frac{m}{f}m' = q = \text{const.}, \tag{3.6}$$

and hence

$$mm' = \frac{1}{2}(m^2)' = qf \quad \Rightarrow \quad mm'' + (m')^2 = qf'. \tag{3.7}$$

Therefore, given f, m can be determined by integration:

$$m^2 = 2qF, \quad \text{where } F' = f. \tag{3.8}$$

We next insert the ansatz (3.4) into the second equation in (3.2) and obtain

$$\begin{aligned} 0 &= m(m\Phi')' - (m')^2 = m\left(-m' + m\frac{f'}{f}\right)' - (m')^2 \\ &= m\left(-m'' + m'\frac{f'}{f} + m\left(\frac{f'}{f}\right)'\right) - (m')^2 = m^2\left(\frac{f'}{f}\right)', \end{aligned} \tag{3.9}$$

where we used (3.5) in the first line and both relations in (3.7) in the second line. Since $m^2 \neq 0$ this equation implies that $\frac{f'}{f}$ is constant, hence

$$f(x) = e^{\gamma x + \delta}, \qquad \gamma, \delta = \text{const.} \tag{3.10}$$

Its integral then determines F and hence m^2 via (3.8):

$$m^2 = \frac{2q}{\gamma}e^{\gamma x + \delta} + c, \tag{3.11}$$

with c a new integration constant. Finally, we turn to the equation of motion of the lapse function n, the first equation in (3.2). Inserting the above ansatz one finds

$$\begin{aligned} Q^2 &= m^2(\Phi')^2 - (m')^2 = m^2\left(-\frac{m'}{m} + \frac{f'}{f}\right)^2 - (m')^2 \\ &= -2mm'\frac{f'}{f} + m^2\gamma^2 = -2qf' + \gamma^2\left(\frac{2q}{\gamma}e^{\gamma x+\delta} + c\right) \\ &= \gamma^2 c. \end{aligned} \tag{3.12}$$

Thus, the equation just places a relation between the constants in the problem.

Imposing the boundary condition that the metric approaches the Minkowski metric far away from the black hole, which we will choose to corresponds to $x \to -\infty$, implies that $c = 1$. Then picking $\gamma = Q$ for Q positive we have from (3.11) and (3.4):

$$m^2 = 1 + \frac{2q}{Q}e^{Qx+\delta}, \qquad e^{-\Phi} = me^{-Qx-\delta}. \tag{3.13}$$

The second relation here makes it clear that m is, at all points, a real positive number. The above is the well-known black hole solution of 2D string theory and agrees, for instance, with the form given in [6] for

$$a \equiv -\frac{2q}{Q}, \qquad \delta = 0. \tag{3.14}$$

Summarizing, the black hole metric and dilaton read

$$\begin{aligned} ds^2 &= -m^2(x)dt^2 + \frac{1}{m^2(x)}dx^2, \\ e^{-\Phi} &= m(x)e^{-Qx}, \end{aligned} \tag{3.15}$$

where

$$m^2(x) = 1 - ae^{Qx}, \qquad a > 0, \qquad Q > 0. \tag{3.16}$$

1. Comments on black hole solution

Let us close with some brief comments on this BH solution. We first note that only if $a > 0$ does $m^2(x)$ vanish for some value of x, as we would expect for a genuine black hole at the event horizon. Since $m^2 \to 1$ as $x \to -\infty$ we see that the latter is indeed the asymptotically flat region. For the dilaton we have

$$\Phi = Qx - \frac{1}{2}\ln m^2 = Qx - \frac{1}{2}\ln|1 - ae^{Qx}|. \tag{3.17}$$

Absolute values are needed here for the region beyond the horizon, where $m^2 < 0$. The need for absolute values also follows because m must be defined as real and positive, as we mentioned above. In the region $x \to -\infty$ we have

$$\Phi = Qx + \frac{1}{2}ae^{Qx} + \frac{1}{4}a^2e^{2Qx} + \mathcal{O}(e^{3Qx}). \tag{3.18}$$

Here $\Phi \to -\infty$ in the asymptotically flat region, which corresponds to weak coupling.

Horizon, singularity, and duality.—The Ricci scalar (2.24) encodes in 2D the full Riemann curvature. With $mn = 1$ it yields for the above metric

$$R = -(m^2(x))'' = aQ^2 e^{Qx}. \tag{3.19}$$

The metric is singular at the zero of $m^2(x)$, the location of the horizon:

$$\text{horizon:}\; m^2(x_0) = 0 \;\;\rightarrow\;\; a e^{Qx_0} = 1 \;\;\rightarrow\;\; x_0 = -\frac{1}{Q}\ln a, \tag{3.20}$$

and in analogy to the Schwarzschild solution in 4D we expect this to be a coordinate singularity. Indeed, the curvature (3.19) at this point is regular

$$\text{curvature at horizon:}\; R(x_0) = Q^2. \tag{3.21}$$

The black hole singularity is at the point where the curvature R diverges, namely as $x \to \infty$:

$$\text{BH singularity:}\; x \to \infty. \tag{3.22}$$

According to [7] a duality transformation exchanges the horizon of the above black hole with the singularity, which we confirm here. The T-duality transformation keeps n invariant while sending $m \to m^{-1}$. Therefore, the dual metric obeys

$$ds^2_{\text{dual}} = -\frac{1}{m^2(x)}dt^2 + \frac{1}{m^2(x)}dx^2 = \frac{1}{m^2(x)}(-dt^2 + dx^2). \tag{3.23}$$

The new curvature is obtained from (2.24) letting $m \to 1/m$ and $n \to 1/m$:

$$R_{\text{dual}} = (m^2)'' - \frac{(m^2)'(m^2)'}{m^2}. \tag{3.24}$$

This shows that the zero of m^2, the former horizon location, now corresponds to a curvature singularity:

$$R_{\text{dual}}(x_0) = \infty, \tag{3.25}$$

all other points having finite curvature. The horizon is now the curvature singularity. According to [7] the original BH singularity $x \to \infty$ turns into the horizon of the second one. Evaluating the curvature explicitly,

$$R_{\text{dual}}(x) = -aQ^2 e^{Qx} + \frac{aQ^2 e^{Qx}}{\left(1 - \frac{1}{ae^{Qx}}\right)} = \frac{Q^2}{1 - \frac{1}{a}e^{-Qx}}. \tag{3.26}$$

Indeed, for $x \to \infty$, the former position of the curvature singularity, consistent with (3.21) we find that

$$\lim_{x\to\infty} R_{\text{dual}}(x) = Q^2. \tag{3.27}$$

Black hole as deformation of the lapse-dilaton model solution.—Consider the lapse-dilaton model action:

$$I = \int dx n e^{-\Phi}(Q^2 + (D\Phi)^2), \tag{3.28}$$

Following (2.30), the equations of motion are

$$\begin{aligned} E_n &= -(D\Phi)^2 + Q^2 = 0, \\ E_\Phi &= -2D^2\Phi + (D\Phi)^2 - Q^2 = 0. \end{aligned} \tag{3.29}$$

The equations imply $D\Phi = Q$, up to an irrelevant sign, and $D^2\Phi = 0$. We take $n = 1$ and obtain $\Phi = Qx$:

$$n = 1, \qquad \Phi = Qx. \tag{3.30}$$

This can be viewed as a zeroth-order solution. Once we restore the M^2 term to the action, we have the full black hole equations of motion. In the asymptotic region $x \to -\infty$ there are *small* corrections to the above zeroth order solution that can be written in the form

$$\begin{aligned} \Phi &= Qx + \alpha_1 e^{Qx} + \alpha_2 e^{2Qx} + \cdots, \\ n &= 1 + \beta_1 e^{Qx} + \beta_2 e^{2Qx} + \cdots, \\ m &= 1 + \gamma_1 e^{Qx} + \gamma_2 e^{2Qx} + \cdots. \end{aligned} \tag{3.31}$$

Working with $mn = 1$ fixes the expansion of m in terms of that of n, and to zeroth order one has $m = 1$. Moreover $M = m'$. We have checked that the equations of motion of the full theory now reproduce the terms of the solution for the black hole—expanded in the asymptotic region [see (3.18)].

B. Black holes in the α'-corrected theory

We have already found a canonical presentation for the action following our classification of possible duality invariant terms up to field redefinitions. We can therefore examine how the black hole solution changes when higher-derivative terms are included. In order to do so we rewrite the general form (2.69) of the action as

$$I = \int d\bar{x} n e^{-\Phi}(1 + (\bar{D}\Phi)^2 + F(\bar{M})), \tag{3.32}$$

where we are using

$$\begin{aligned} \bar{x} &\equiv Qx, & \bar{\partial} &\equiv \partial_{\bar{x}} = \frac{1}{Q}\partial_x, \\ \bar{D} &\equiv \frac{1}{n}\bar{\partial}, & \bar{M} &\equiv \frac{1}{n}\bar{\partial}\ln m = \frac{1}{Q}M. \end{aligned} \tag{3.33}$$

We set the general expansion

$$F(\bar{M}) \equiv \sum_{i=0}^{\infty} c_i \epsilon^i \bar{M}^{2i+2} = -\bar{M}^2 + \ldots, \quad c_0 = -1, \tag{3.34}$$

where we rewrote the ϵ dependence in a way that the structure of in-built suppression (which is always assumed) is made manifest. In this case $\epsilon_i = c_i \epsilon^i$ where all coefficients c_i are of order one and so condition (2.52) is satisfied.

The equations arising from variation of m, n, and Φ give

$$\begin{aligned} \bar{D}(e^{-\Phi} f(\bar{M})) &= 0, \\ 1 - (\bar{D}\Phi)^2 - g(\bar{M}) &= 0, \\ -2\bar{D}^2\Phi + (\bar{D}\Phi)^2 - 1 - F(\bar{M}) &= 0. \end{aligned} \tag{3.35}$$

Here,

$$\begin{aligned} f(\bar{M}) &\equiv F'(\bar{M}) = \sum_{i=0}^{\infty} (2i+2) c_i \epsilon^i \bar{M}^{2i+1} \\ &= -2\bar{M} + 4c_1 \epsilon \bar{M}^3 + \mathcal{O}(\epsilon^2), \\ g(\bar{M}) &\equiv \sum_{i=0}^{\infty} (2i+1) c_i \epsilon^i \bar{M}^{2i+2} = -\bar{M}^2 + 3c_1 \epsilon \bar{M}^4 + \mathcal{O}(\epsilon^2), \end{aligned} \tag{3.36}$$

with primes denoting derivative with respect to the argument. The following relations are easily checked:

$$g'(\bar{M}) = \bar{M} f'(\bar{M}), \qquad g(\bar{M}) + F(\bar{M}) = \bar{M} f(\bar{M}). \tag{3.37}$$

Adding the second and third equations in (3.35) we get

$$\bar{D}^2\Phi + \frac{1}{2}\bar{M} f(\bar{M}) = 0, \tag{3.38}$$

which can replace the third equation to find:

$$\begin{aligned} \bar{D}(e^{-\Phi} f(\bar{M})) &= 0, \\ 1 - (\bar{D}\Phi)^2 - g(\bar{M}) &= 0, \\ \bar{D}^2\Phi + \frac{1}{2}\bar{M} f(\bar{M}) &= 0. \end{aligned} \tag{3.39}$$

We now define

$$\Omega \equiv e^{-\Phi}. \tag{3.40}$$

The first two equations above are readily rewritten. The third gives a more complicated equation that combined with the second simplifies

$$\begin{aligned} \bar{D}(\Omega f(\bar{M})) &= 0, \\ (\bar{D}\Omega)^2 + (g(\bar{M}) - 1)\Omega^2 &= 0, \\ \bar{D}^2\Omega - (1 + h(\bar{M}))\Omega &= 0, \end{aligned} \tag{3.41}$$

where

$$h(\bar{M}) \equiv \frac{1}{2}\bar{M} f(\bar{M}) - g(\bar{M}). \tag{3.42}$$

A nonperturbative approach to solutions.—We can focus on the first two equations in (3.41), then solve for $\bar{D}\Omega/\Omega$ in both and equate the result. This is the procedure used in [17]. We get

$$\frac{f'(\bar{M})}{f(\bar{M})} \bar{D}\,\bar{M} = \pm\sqrt{1 - g(\bar{M})}. \tag{3.43}$$

Equivalently, we have, with $\bar{D}\,\bar{M} = \frac{1}{n}\frac{d\bar{M}}{d\bar{x}}$,

$$\frac{f'(\bar{M}) d\bar{M}}{f(\bar{M})\sqrt{1 - g(\bar{M})}} = \pm n d\bar{x}. \tag{3.44}$$

By integration we have

$$\int^{\bar{M}} \frac{f'(M) dM}{f(M)\sqrt{1 - g(M)}} = \pm \int^{\bar{x}} n(x') dx' + C. \tag{3.45}$$

Here C is a constant of integration. This equation fixes a relation between a function of $\bar{M}$ (the left-hand side) and $\int^{\bar{x}} n(x') dx'$. We now adopt the gauge

$$n(\bar{x}) = 1, \tag{3.46}$$

in which the integral condition becomes the simpler

$$\int^{\bar{M}} \frac{f'(M) dM}{f(M)\sqrt{1 - g(M)}} = \pm(\bar{x} - \bar{x}_0). \tag{3.47}$$

Here $\bar{x}_0$ is an integration constant. If we have a function $W(M)$ such that

$$dW \equiv \frac{f'(M)}{f(M)\sqrt{1 - g(M)}} dM, \tag{3.48}$$

the general solution to (3.47) is given by

$$W(\bar{M}) = \pm(\bar{x} - \bar{x}_0), \tag{3.49}$$

a relation that can be inverted to determine $\bar{M}(\bar{x})$. With $\bar{M} = \bar{\partial} \log m$, in the $n = 1$ gauge, this determines $m(\bar{x})$. The dilaton is then found from the first equation in (3.41),

$$\Omega f(\bar{M}) = q, \tag{3.50}$$

with q some constant. By the Bianchi identity, the last equation in (3.41) holds when the first two hold.

Application to the standard black hole solution.—Let us now rederive the lowest order black hole solution from the general formula (3.47). For the two-derivative action we have [see (3.36)]

$$f(\bar{M}) = -2\bar{M}, \quad f'(\bar{M}) = -2, \quad g(\bar{M}) = -\bar{M}^2, \quad (3.51)$$

and so (3.48) takes the form

$$dW = \frac{d\bar{M}}{\bar{M}\sqrt{1+\bar{M}^2}} \quad \to \quad W(\bar{M}) = -\mathrm{arcsch}\bar{M}. \quad (3.52)$$

By inserting this result into (3.49) and inverting $W(\bar{M})$ we end up with

$$\bar{M} = \mathrm{csch}\bar{x} = \bar{\partial}\ln m, \quad (3.53)$$

where we chose a suitable sign and fixed $\bar{x}_0 = 0$, without loss of generality. This is easily integrated and we obtain

$$m(x) = \tanh\frac{\bar{x}}{2}. \quad (3.54)$$

The positivity of m requires $\bar{x} = Qx \geq 0$, which determines the allowed space region (we always assume $Q > 0$). This is the *exterior* region to the black hole, with $x \to \infty$ the asymptotically flat region. The horizon is at $x = 0$.

The dilaton solution is obtained from (3.50). Using $f(\bar{M}) = -2\bar{M}$ and (3.53) we get

$$e^{\Phi(\bar{x})} = -\frac{2}{q}\mathrm{csch}\bar{x}. \quad (3.55)$$

Since $\bar{x} \geq 0$, the constant q must be negative.

The above is a solution in the $n = 1$ gauge. In order to connect with the zeroth order solution (3.16) we need to perform a coordinate transformation. The metric we considered here takes the form

$$ds^2 = -m^2(x)dt^2 + dx^2, \quad \text{with } m(x) = \tanh\frac{\bar{x}}{2}. \quad (3.56)$$

We now introduce a coordinate $x'(x)$ via

$$dx' = -m(x)dx, \quad (3.57)$$

so that the metric would be of the expected form in the gauge $mn = 1$:

$$ds^2 = -m^2(x)dt^2 + \frac{dx'^2}{m^2(x)}, \quad (3.58)$$

where in this form $m^2(x)$ must be written in terms of x'. The sign in (3.57) was chosen with hindsight, as the asymptotic region of the $n = 1$ solution is at plus infinity, and the one in the $mn = 1$ gauge is at minus infinity. Equation (3.57) can be easily integrated:

$$x' = -\frac{2}{Q}\ln\cosh\frac{Qx}{2} + C \quad \to \quad \cosh^2\frac{Qx}{2} = \frac{1}{a}e^{-Qx'}, \quad a > 0. \quad (3.59)$$

Here a is an integration constant. As a result,

$$m(x) = \tanh\frac{Qx}{2} = \sqrt{1 - \cosh^{-2}\frac{Qx}{2}} = \sqrt{1 - ae^{Qx'}}, \quad (3.60)$$

in exact agreement with (3.16), obtained in the $mn = 1$ gauge. The dilatons in the two theories also agree, since they are scalars. Indeed, from (3.15)

$$e^{-\Phi(x')} = m(x')e^{-Qx'} = \tanh\frac{Qx}{2}\cdot a\cosh^2\frac{Qx}{2} = \frac{a}{2}\sinh Qx. \quad (3.61)$$

This coincides with $e^{-\Phi}$ in (3.55) if we take $q = -a$. This is possible since $a > 0$ and $q < 0$.

1. A curious case

Let us consider a particular nonperturbative case, where the action contains the two-derivative term and one single correction, a four-derivative term with a fixed coefficient $c_1\epsilon = \epsilon_1 = -\frac{1}{12}$:

$$F(\bar{M}) = -\bar{M}^2 - \frac{\bar{M}^4}{12} \quad \to \quad f(\bar{M}) = -2\bar{M} - \frac{1}{3}\bar{M}^3,$$
$$g(\bar{M}) = -\bar{M}^2 - \frac{1}{4}\bar{M}^4. \quad (3.62)$$

In this case

$$\sqrt{1 - g(\bar{M})} = 1 + \frac{\bar{M}^2}{2} = -\frac{1}{2}f'(\bar{M}), \quad (3.63)$$

and Eq. (3.47) simplifies notably:

$$\int^{\bar{M}} \frac{dM}{M\left(1 + \frac{M^2}{6}\right)} = \pm\bar{x}. \quad (3.64)$$

Absorbing the constant of integration into a finite shift of $\bar{x}$ we find the solution

$$\bar{M}^2 = \frac{6e^{\pm 2\bar{x}}}{1 - e^{\pm 2\bar{x}}}. \quad (3.65)$$

In order for the solution to describe the exterior region of a black hole with the asymptotically flat region is at $x \to \infty$, we must have $\bar{M} \to 0$ as $x \to \infty$ and that requires choosing the bottom sign. We thus have

$$\bar{M}^2 = \frac{6e^{-2\bar{x}}}{1-e^{-2\bar{x}}} = 3(\coth\bar{x}-1) = 3(\sqrt{1+\mathrm{csch}^2\bar{x}}-1). \tag{3.66}$$

With $\bar{M}$ known, using (3.50) we can read the dilaton profile

$$\begin{aligned} e^{\Phi(\bar{x})} &= -\frac{\sqrt{3}}{q}\sqrt{\coth\bar{x}-1}(\coth\bar{x}+1) \\ &= -\frac{\sqrt{3}}{q}\mathrm{csch}\bar{x}\sqrt{1+\coth\bar{x}}. \end{aligned} \tag{3.67}$$

Finally, we can determine m from $\bar{M} = \bar{\partial}\ln m$ to find

$$m(\bar{x}) = e^{\int^{\bar{x}}\bar{M}(x)dx} = e^{-\sqrt{6}\arcsin e^{-\bar{x}}}, \tag{3.68}$$

where the constant of integration has been chosen to have $m\to 1$ as $x\to\infty$. This solution is valid down to $x=0$. While in the two-derivative theory $m(x)$ vanishes at $x=0$, a point identified as the horizon, here $m(x=0) = \exp(-\sqrt{6}\frac{\pi}{2}) \simeq 0.02$. It may be of interest to investigate more completely this and related solutions.

C. Systematics of perturbative solutions

An analytic expression for the nonperturbative $W(\bar{M})$ may not exist in general. However, when considering perturbative solutions in ϵ, Eq. (3.48) becomes a power series in ϵ, where each term is easier to integrate than the nonperturbative dW. In this perturbative regime, a systematic approach exists such that solutions to any order in ϵ can be obtained from lowest order ones. This algorithm takes (3.48) as the starting point and expands it around small ϵ to get

$$dW = \frac{f'(\bar{M})d\bar{M}}{f(\bar{M})\sqrt{1-g(\bar{M})}} = dW_0 + \epsilon dW_1 + \epsilon^2 dW_2 + \mathcal{O}(\epsilon^3). \tag{3.69}$$

Integrating each of these terms we arrive at the perturbative version of (3.49),

$$W(\bar{M}) = W_0(\bar{M}) + \epsilon W_1(\bar{M}) + \epsilon^2 W_2(\bar{M}) + \mathcal{O}(\epsilon^3) = -\bar{x}, \tag{3.70}$$

where we pick the minus sign option, and we are working in the $n=1$ gauge. From (3.52) we already know that $W_0(\bar{M}) = -\mathrm{arcsch}\bar{M}$, which clearly can be inverted. By doing so, (3.70) becomes

$$\begin{aligned} \bar{M} &= \mathrm{csch}(\bar{x} + \epsilon W_1(\bar{M}) + \epsilon^2 W_2(\bar{M})) + \mathcal{O}(\epsilon^3) \\ &= \mathrm{csch}\bar{x} + \epsilon\,\mathrm{csch}'\bar{x}W_1(\bar{M}) \\ &\quad + \epsilon^2\left[\frac{1}{2}\mathrm{csch}''\bar{x}W_1^2(\bar{M}) + \mathrm{csch}'\bar{x}W_2(\bar{M})\right] + \mathcal{O}(\epsilon^3), \end{aligned} \tag{3.71}$$

where $'$ means derivative with respect to the argument. Then, we expand

$$\bar{M} = \bar{M}_0 + \epsilon\bar{M}_1 + \epsilon^2\bar{M}_2 + \mathcal{O}(\epsilon^3), \tag{3.72}$$

on both sides of the last equality to read the solution order by order in ϵ

$$\bar{M}_0(\bar{x}) = \mathrm{csch}\bar{x}, \tag{3.73a}$$

$$\bar{M}_1(\bar{x}) = \mathrm{csch}'\bar{x}W_1(\bar{M}_0(\bar{x})), \tag{3.73b}$$

$$\begin{aligned} \bar{M}_2(\bar{x}) &= \frac{1}{2}\mathrm{csch}''\bar{x}W_1^2(\bar{M}_0(\bar{x})) + \mathrm{csch}'\bar{x}W_2(\bar{M}_0(\bar{x})) \\ &\quad + \mathrm{csch}'\bar{x}W_1'(\bar{M}_0(\bar{x}))\bar{M}_1(\bar{x}). \end{aligned} \tag{3.73c}$$

We can see that each order $\bar{M}_i(\bar{x})$ is determined from the lowest order ones. From $\bar{M} = \bar{\partial}\ln m$ and (3.50) we can get the perturbative solutions for $m(\bar{x})$ and $\Phi(\bar{x})$. The resulting solution will be in the $n=1$ gauge but it can be mapped to the standard form by the use of (3.57) with the now-corrected $m(\bar{x})$.

Just as a demonstration of how the above algorithm works in practice, we work out the first ϵ order explicitly. Up to first order we have [see (3.36)]

$$\begin{aligned} f(\bar{M}) &= -2\bar{M} + 4c_1\epsilon\bar{M}^3, \\ f'(\bar{M}) &= -2 + 12c_1\epsilon\bar{M}^2, \\ g(\bar{M}) &= -\bar{M}^2 + 3c_1\epsilon\bar{M}^4. \end{aligned} \tag{3.74}$$

Inserting these quantities into (3.69), and expanding up to first order in ϵ we can read

$$\begin{aligned} dW_0 &= \frac{1}{\bar{M}\sqrt{1+\bar{M}^2}}d\bar{M}, \\ dW_1 &= -\frac{c_1}{2}\frac{\bar{M}(8+5\bar{M}^2)}{(1+\bar{M}^2)^{\frac{3}{2}}}d\bar{M}. \end{aligned} \tag{3.75}$$

Each order can be integrated independently to obtain

$$\begin{aligned} W_0(\bar{M}) &= -\mathrm{arcsch}\bar{M}, \\ W_1(\bar{M}) &= -\frac{c_1}{2}\frac{2+5\bar{M}^2}{\sqrt{1+\bar{M}^2}}. \end{aligned} \tag{3.76}$$

Finally, by using (3.73b) we can read $\bar{M}_1(\bar{x})$:

$$\begin{aligned}\bar{M}_1(\bar{x}) &= \mathrm{csch}'\,\bar{x}\,W_1(\bar{M}_0(\bar{x}))\\ &= (-\coth\,\bar{x}\,\mathrm{csch}\,\bar{x})\left(-\frac{c_1}{2}\frac{2+5\mathrm{csch}^2\,\bar{x}}{\sqrt{1+\mathrm{csch}^2\,\bar{x}}}\right)\\ &= c_1\left(\mathrm{csch}\,\bar{x}+\frac{5}{2}\mathrm{csch}^3\,\bar{x}\right). \end{aligned} \tag{3.77}$$

All in all, up to order ϵ, $\bar{M}(\bar{x})$ is given by

$$\bar{M}(\bar{x}) = \mathrm{csch}\,\bar{x} + \epsilon c_1\left(\mathrm{csch}\,\bar{x}+\frac{5}{2}\mathrm{csch}^3\,\bar{x}\right)+\mathcal{O}(\epsilon^2). \tag{3.78}$$

We easily find $m(\bar{x})$ from

$$\begin{aligned}m(\bar{x}) &= e^{\int^{\bar{x}}\bar{M}(x')dx'}\\ &= e^{\int^{\bar{x}}\bar{M}_0(x')dx'}\left(1+\epsilon\int^{\bar{x}}\bar{M}_1(x')dx'\right)+\mathcal{O}(\epsilon^2)\\ &= \tanh\,\frac{\bar{x}}{2}\left(1+\epsilon c_1\int^{\bar{x}}\left(\mathrm{csch}\,x'+\frac{5}{2}\mathrm{csch}^3\,x'\right)dx'\right)\\ &\quad+\mathcal{O}(\epsilon^2), \end{aligned} \tag{3.79}$$

and performing the integral,

$$\begin{aligned}m(\bar{x}) &= \tanh\frac{\bar{x}}{2}\left(1-\frac{1}{4}c_1\epsilon\left[\ln\tanh\frac{\bar{x}}{2}+5\coth\,\bar{x}\,\mathrm{csch}\,\bar{x}\right]\right)\\ &\quad+\mathcal{O}(\epsilon^2). \end{aligned} \tag{3.80}$$

Finally, the dilaton profile comes from combining (3.50), (3.74), and (3.78) and is given by

$$\begin{aligned}\Omega^{-1} = e^{\Phi(\bar{x})} &= \frac{1}{q}f(\bar{M})\\ &= -\frac{2}{q}\left[\mathrm{csch}\,\bar{x}+c_1\epsilon\left(\mathrm{csch}\,\bar{x}+\frac{1}{2}\mathrm{csch}^3\,\bar{x}\right)\right]\\ &\quad+\mathcal{O}(\epsilon^2). \end{aligned} \tag{3.81}$$

In view of the pattern emerging at order ϵ in (3.78) and (3.81), it feels natural to ask whether such structure persists perturbatively to all orders. Indeed, by following an inductive procedure, explained briefly in the Appendix, we confirmed that the following ansatz can be used to solve (3.41) to all orders in ϵ:

$$\bar{M} = \sum_{p\geq 0}\bar{M}^{(p)}\epsilon^p,$$

$$\bar{M}^{(p)} = \sum_{k=0}^{p}a_k^{(p)}\mathrm{csch}^{2k+1}\bar{x}, \tag{3.82a}$$

$$\Omega^{-1} = \sum_{p\geq 0}[\Omega^{-1}]^{(p)}\epsilon^p,$$

$$[\Omega^{-1}]^{(p)} = \sum_{k=0}^{p}b_k^{(p)}\mathrm{csch}^{2k+1}\bar{x}. \tag{3.82b}$$

Here $a_k^{(p)}$ and $b_k^{(p)}$ are some order-one coefficients determined completely from the c_i coefficients in the action. For instance, from (3.78) we can read $a_0^{(0)}=1$, $a_0^{(1)}=c_1, a_1^{(1)}=\frac{5}{2}c_1$.

Let us point out that there is an apparent incompatibility between this perturbative all-order formula for $\bar{M}$ and the nonperturbative result for the curious case in (3.66). Expanding the latter in powers of csch$\bar{x}$ yields $\bar{M}=\sqrt{\frac{3}{2}}\,\mathrm{csch}\,\bar{x}+\mathcal{O}(\mathrm{csch}^3\,\bar{x})$, which disagrees with the first term in (3.78). This comparison, however, is not meaningful since in this section we are expanding in small ϵ and *not* in small csch$\bar{x}$, as we do for the curious case. In fact, the perturbative expansion in ϵ contains an infinite number of contributions to each power of csch$\bar{x}$. If we had the full expansion, the coefficient of csch$\bar{x}$ would be $\sum_{p=0}^{\infty}a_0^{(p)}\epsilon^p$ (with $c_1\epsilon=-\frac{1}{12}$ for the curious case), but we only know $a_0^{(0)}$ and $a_0^{(1)}$ from (3.78). Moreover, complicating the comparison, Eqs. (3.66) and (3.82a) may be expressed in different coordinate systems. In passing from (3.64) to (3.65) we could choose a different constant of integration, and thus replace x by $x-x_0$ in (3.66), for some value of x_0 that could also be adjusted and would affect the expansion of $\bar{M}$ in powers of csch$\bar{x}$. We believe both expansions should match once all coefficients are resummed and the coordinates are changed appropriately.

D. Systematics of nonperturbative solutions

Recall that for the two-derivative action the black hole solution took the form

$$\Omega^{-1} = e^{\Phi} = -\frac{2}{q}\mathrm{csch}\,\bar{x}, \qquad \bar{M} = \mathrm{csch}\,\bar{x}. \tag{3.83}$$

Of course, associated to $\bar{M}=\bar{\partial}\ln m$ we have $m=\tanh\frac{\bar{x}}{2}$. Moreover, in perturbative solutions, we found that both e^{Φ} and $\bar{M}$ are written as infinite series in terms of csch$\bar{x}$, beginning with a linear term. We therefore consider a generalization that we write as follows:

$$\begin{aligned}e^{\Phi} &= p_{\Phi}\circ\mathrm{csc}\,h,\\ \bar{M} &= p_M^{-1}\circ\mathrm{csc}\,h, \end{aligned} \tag{3.84}$$

where we use $\circ$ to denote composition of functions, and both sides of the equalities are functions of $\bar{x}$. We have introduced two polynomials, p_{Φ} and p_M, with p_M^{-1} the inverse function to p_M, so that $p_M\circ p_M^{-1}=I$, with I the identity function. We write

$$\begin{aligned}p_{\Phi}(u) &= a_1u+a_2u^2+\mathcal{O}(u^3),\\ p_M(u) &= b_1u+b_2u^2+\mathcal{O}(u^3). \end{aligned} \tag{3.85}$$

From the equation $e^{-\Phi}f(\bar{M}) = q$ we have $f(\bar{M}) = qe^{\Phi}$ and therefore, as functions of $\bar{x}$

$$f \circ \bar{M} = qp_{\Phi} \circ \text{csch}. \tag{3.86}$$

Composing $\bar{M}^{-1}$ (the inverse function of $\bar{M}$) from the right on both sides of the above equation we have

$$f = qp_{\Phi} \circ \text{csch} \circ \bar{M}^{-1} = qp_{\Phi} \circ \text{csch} \circ \text{arcsch} \circ p_M, \tag{3.87}$$

and therefore we conclude that

$$f = qp_{\Phi} \circ p_M. \tag{3.88}$$

We conventionally take the normalization of the m field to be determined in the action by $F(\bar{M}) = -\bar{M}^2 + \mathcal{O}(\bar{M}^4)$. Therefore we have $f(\bar{M}) = F'(\bar{M}) = -2\bar{M} + \mathcal{O}(\bar{M}^3)$, and $g(\bar{M}) = -\bar{M}^2 + \mathcal{O}(\bar{M}^4)$. We thus require that

$$f = -2I + \mathcal{O}(I^3). \tag{3.89}$$

This requirement, given the expression for f in (3.88) and the expansions in (3.85), means the constraints are

$$\begin{aligned} qa_1b_1 &= -2, \\ a_1b_2 + a_2b_1^2 &= 0. \end{aligned} \tag{3.90}$$

This means, in particular that $a_1 \neq 0$ and $b_1 \neq 0$.

We must now calculate $g(\bar{M})$. For this we use the middle equation in (3.39) which fixes $g = 1 - (\bar{D}\Phi)^2$. The derivative of Φ is calculated from (3.84) and we find

$$e^{\Phi}\bar{D}\Phi = (p'_{\Phi} \circ \text{csch}) \cdot (-\coth \cdot \text{csch}). \tag{3.91}$$

Squaring and dividing by $e^{2\Phi}$ gives

$$(\bar{D}\Phi)^2 = \frac{(I^2 \circ p'_{\Phi} \circ \text{csch}) \cdot (\text{csch}^2 + \text{csch}^4)}{I^2 \circ p_{\Phi} \circ \text{csch}}. \tag{3.92}$$

To view this equation as a function of $\bar{M}$ we can use the second equation in (3.84) to write $\text{csch} = p_M(\bar{M})$ and thus

$$(\bar{D}\Phi)^2 = \frac{(I^2 \circ p'_{\Phi} \circ p_M) \cdot (p_M^2 + p_M^4)}{I^2 \circ p_{\Phi} \circ p_M}. \tag{3.93}$$

We therefore have

$$g(\bar{M}) = 1 - \frac{(p'_{\Phi}(p_M))^2 p_M^2 (1 + p_M^2)}{(p_{\Phi}(p_M))^2}. \tag{3.94}$$

Here the right-hand side is evaluated with $p_M(\bar{M})$. It is now possible to check what constraint this result gives given that $g(\bar{M}) = -\bar{M}^2 + \mathcal{O}(\bar{M}^4)$. Indeed keeping just linear terms on the polynomials we have that the above equation gives

$$\begin{aligned} g &= 1 - \frac{(a_1 + 2a_2p_M)^2}{(a_1 + a_2p_M)^2} + \mathcal{O}(\bar{M}^2) \\ &= 1 - \frac{(1 + 2\frac{a_2}{a_1}b_1\bar{M})^2}{(1 + \frac{a_2}{a_1}b_1\bar{M})^2} + \mathcal{O}(\bar{M}^2), \\ &= -\frac{2a_2b_1}{a_1}\bar{M} + \mathcal{O}(\bar{M}^2). \end{aligned} \tag{3.95}$$

Since this linear term must vanish and both b_1 and a_1 are nonzero, we conclude that $a_2 = 0$. The second equation in (3.90) then implies that $b_2 = 0$. The quadratic terms in the polynomials vanish. Our perturbative results also hinted in this direction. Therefore, we refine the ansatz in (3.85) to read

$$\begin{aligned} p_{\Phi}(u) &= a_1u + a_3u^3 + a_5u^5 + \mathcal{O}(u^7), \\ p_M(u) &= b_1u + b_3u^3 + b_5u^5 + \mathcal{O}(u^7). \end{aligned} \tag{3.96}$$

The solution now proceeds by first finding $f(\bar{M})$ using (3.88). Then, using $g' = Mf'$ one finds the associated $g(\bar{M})$. Finally, this result is compared with the result for $g(\bar{M})$ from (3.94). We have found by solving this system on a computer that the series p_M, defining the metric $\bar{M}(x)$, fixes the other polynomial p_{Φ} as well as $F(\bar{M})$, the action. We get, for example,

$$a_1 = -\frac{2}{b_1q}, \quad a_3 = \frac{b_1^2 - 1}{2b_1^3q}, \quad a_5 = \frac{4b_3 - 8b_1^5 + 11b_1^3 - 3b_1}{32b_1^6q}. \tag{3.97}$$

We also find

$$f(\bar{M}) = -2\bar{M} + \left(\frac{b_1^2}{2} - \frac{1}{2} - \frac{2b_3}{b_1}\right)\bar{M}^3 + \cdots. \tag{3.98}$$

1. The simplest dilaton profile

Suppose the dilaton profile is now fixed to be the one for the black hole in the two-derivative approximation:

$$e^{\Phi} = -\frac{2}{q}\text{csch}\bar{x}. \tag{3.99}$$

In the language of (3.84) this corresponds to the choice $p_{\Phi} = -\frac{2}{q}I$. In this situation, we find that $f = -2p_M(\bar{M})$ as it follows from (3.88). Finally, from (3.94) one finds $g(\bar{M}) = -p_M^2(\bar{M})$. This means that

$$g(\bar{M}) = -\frac{1}{4}(f(\bar{M}))^2. \tag{3.100}$$

Taking the derivative with respect to $\bar{M}$, we have

$$g'(\bar{M}) = -\frac{1}{2} f(\bar{M}) f'(\bar{M}). \tag{3.101}$$

Recall now that by the common origin of f and g from F, we have $g' = \bar{M} f'$, and so comparing with the above we conclude that $f(\bar{M}) = -2\bar{M}$, as expected. This is the two-derivative theory, with a quadratic $F(\bar{M}) = -\bar{M}^2$.

2. On the ansatz of Dijkgraaf, Verlinde, and Verlinde

Based on the form of the Virasoro operator L_0 in the CFT description of the black hole background, Dijkgraaf, Verlinde, and Verlinde (DVV) conjectured that certain metric and dilaton profiles could represent the *exact* black hole solution in the α' expansion [22] (Sec. IV. 1). Their solution requires some careful translation, for their dilaton ϕ multiplies the action as e^{ϕ} and ours is the duality invariant dilaton. With this taken into account, their ansatz for the dilaton is

$$e^{-\Phi} = \sinh \bar{x}. \tag{3.102}$$

This is actually the dilaton profile of the two-derivative theory (with $q = -2$), and as proven above, this means that within our framework the only possible solution is that of the two-derivative theory. The DVV profile cannot be seen as a solution of the α' corrected theory in the presentation we have chosen. As we mentioned in the introduction to this section, this result was expected, as the duality transformations in the DVV profile do not correspond to those of our manifestly dual formulation.

In fact, this is not the only complication. The proposal also says that the metric profile takes the form

$$m^2(\bar{x}) = \frac{1}{\coth^2(\frac{\bar{x}}{2}) - \frac{2}{k}} = \frac{1}{\frac{\cosh \bar{x}+1}{\cosh \bar{x}-1} - \frac{2}{k}}. \tag{3.103}$$

With this we now compute:

$$\bar{M} = \frac{1}{2}\bar{\partial} m^2(\bar{x}) = \frac{\sinh \bar{x}}{(\cosh \bar{x} - 1)(1 + \frac{2}{k} + (1 - \frac{2}{k})\cosh \bar{x})}. \tag{3.104}$$

Given that $\cosh \bar{x} = \sqrt{1 + \mathrm{csch}^2 \bar{x}}/\mathrm{csch}\, \bar{x}$ we quickly find that

$$\bar{M} = \frac{\mathrm{csch}\, \bar{x}}{(\sqrt{1 + \mathrm{csch}^2 \bar{x}} - \mathrm{csch}\, \bar{x})((1 + \frac{2}{k})\mathrm{csch}\, \bar{x} + (1 - \frac{2}{k})\sqrt{1 + \mathrm{csch}^2 \bar{x}})}. \tag{3.105}$$

This has a Taylor series in the variable $\mathrm{csch}\, \bar{x}$, as required for our setup. Moreover the leading term is linear in $\mathrm{csch}\, \bar{x}$ as required for $f(\bar{M})$ leading term to be linear in $\bar{M}$. In fact, with $\bar{M} = p_M^{-1} \circ \mathrm{csch}$, the Taylor series is

$$\begin{aligned} p_M^{-1}(u) &= \frac{u}{1 - \frac{2}{k}} - \frac{4u^2}{k(1 - \frac{2}{k})^2} + \mathcal{O}(u^3) \\ &= 9u - 144u^2 + \mathcal{O}(u^3), \quad \text{for } k = 9/4. \end{aligned} \tag{3.106}$$

We then have $p_M(u) = \frac{1}{9}u + \frac{16}{81}u^2 + \cdots$. This also violates the present framework, as we showed above that the polynomial p_M cannot have a quadratic term. All in all, our framework does not give any evidence that the DVV ansatz is a solution. A solution is meaningful if we also have the associated equations of motion. Those are missing in the DVV conjecture.

IV. BIANCHI I COSMOLOGY

Following the approach of [17], in this section we go back to strings in the critical dimension (so there is no cosmological term) and we classify the α' corrections for backgrounds known as Bianchi type-I cosmologies. These backgrounds were also studied in [29], with a focus on nonperturbative α'-complete solutions with matter sources. Bianchi type-I cosmologies feature a diagonal metric with *a priori* independent "scale factors" on the diagonal. In the first subsection we define these backgrounds and determine the corresponding two-derivative action and equations of motion. In the second subsection we classify the most general higher-derivative terms, up to field redefinitions, thereby arriving at the classification of α' corrections.

A. Bianchi I ansatz

Bianchi type-I (BI) cosmologies are given by a homogeneous but generically anisotropic metric, where the B-field vanishes:

$$g_{mn}(t) = a_m(t)^2 \delta_{mn}, \qquad b_{mn}(t) = 0. \tag{4.1}$$

Here $m, n = 1, \ldots, d$ are internal indices, and the indices are not summed over. In general, the a_m are d independent scale factors, but we will consider the case where there are only $q \leq d$ different scale factors. We then have groups of N_i scale factors a_i with $i = 1, \ldots, q$ such that $\sum_{i=1}^{q} N_i = d$. By definition all N_i are nonzero positive integers. The case where all scale factors are different is included for $q = d$ and $N_i = 1$ for all i, while the fully isotropic case (FRW) is included for $q = 1$ and $N_1 = d$. For each of these q scale factors a_i we define the corresponding Hubble parameter H_i as follows:

$$H_i \equiv \frac{Da_i}{a_i}, \quad i = 1, \dots, q, \tag{4.2}$$

where $D \equiv \frac{1}{n}\frac{\partial}{\partial t}$ is a covariant derivative under one-dimensional diffeomorphisms.

Following the notation and conventions of [17], the generalized metric and its derivative take the form

$$\mathcal{S}_M{}^N = \begin{pmatrix} 0 & a_m^2 \delta_{mn} \\ a_m^{-2}\delta^{mn} & 0 \end{pmatrix},$$
$$(D\mathcal{S})_M{}^N = 2\begin{pmatrix} 0 & H_m a_m^2 \delta_{mn} \\ -H_m a_m^{-2}\delta^{mn} & 0 \end{pmatrix}, \tag{4.3}$$

where there is no sum of repeated indices. Even powers of $D\mathcal{S}$ are given by

$$((D\mathcal{S})^{2k})_M{}^N = (-)^k 2^{2k} \begin{pmatrix} H_m^{2k}\delta_m{}^n & 0 \\ 0 & H_m^{2k}\delta^m{}_n \end{pmatrix}, \tag{4.4}$$

and its trace is given by

$$\mathrm{Tr}((D\mathcal{S})^{2k}) = (-)^k 2^{2k+1}\mathrm{tr}(H_m^{2k}\delta_m{}^n) = (-)^k 2^{2k+1}\sum_{i=1}^{q} N_i H_i^{2k}, \tag{4.5}$$

where we noted that there are only q different directions and each of them is repeated N_i times. In particular, for $k = 1$ we have

$$\mathrm{Tr}((D\mathcal{S})^2) = -8\sum_{i=1}^{q} N_i H_i^2. \tag{4.6}$$

The two derivative action in the cosmological setting is [17]

$$I = \int dtne^{-\Phi}\left[-(D\Phi)^2 - \frac{1}{8}\mathrm{Tr}((D\mathcal{S})^2)\right], \tag{4.7}$$

with n the lapse function, Φ the dilaton, and $\mathcal{S}$ the generalized metric that we evaluated above for the BI setup with scale factors a_i. Thus the BI two-derivative action reads

$$I_{\mathrm{BI}}^{(0)} = \int dtne^{-\Phi}\left[-(D\Phi)^2 + \sum_{i=1}^{q} N_i H_i^2\right]. \tag{4.8}$$

The equations of motion are given by

$$a_i \frac{\delta I_{\mathrm{BI}}^{(0)}}{\delta a_i} = 0 \quad \Rightarrow \quad DH_i = D\Phi H_i, \quad i = 1, \dots, q, \tag{4.9a}$$

$$\frac{\delta I_{\mathrm{BI}}^{(0)}}{\delta \Phi} = 0 \quad \Rightarrow \quad 2D^2\Phi = (D\Phi)^2 + \sum_{i=1}^{q} N_i H_i^2, \tag{4.9b}$$

$$n\frac{\delta I_{\mathrm{BI}}^{(0)}}{\delta n} = 0 \quad \Rightarrow \quad (D\Phi)^2 = \sum_{i=1}^{q} N_i H_i^2. \tag{4.9c}$$

Using (4.9c) in (4.9b) we get a more useful version of the equations:

$$DH_i = D\Phi H_i, \quad i = 1, \dots, q, \tag{4.10a}$$

$$D^2\Phi = \sum_{i=1}^{q} N_i H_i^2, \tag{4.10b}$$

$$(D\Phi)^2 = \sum_{i=1}^{q} N_i H_i^2. \tag{4.10c}$$

In the context of BI backgrounds, $O(d,d)$ symmetry reduces to a $(\mathbb{Z}_2)^q$ invariance under the transformations

$$a_i \to a_i^{-1} \quad \Rightarrow \quad H_i \to -H_i, \qquad i = 1, \dots, q,$$
$$\Phi \to \Phi, \qquad n \to n. \tag{4.11}$$

As expected, (4.8) and (4.9) are duality invariant. Note that the duality transformations act on each different scale factor individually. Because of this, while H_1^2 or H_2^2 are duality invariant, terms like $H_1 H_2$ are not. The latter, however, is invariant under a subset of duality transformations, full-factorized T-dualities, which transform all scale factors simultaneously.

B. Classification

In the context of string low energy effective actions, Eq. (4.8) is just the leading contribution to the full action, containing an infinite expansion in α', namely

$$I = \sum_{p\geq 0} \alpha'^p I^{(p)}, \tag{4.12}$$

where each order $I^{(p)}$ can contain a sum of different terms

$$I^{(p)} = \sum_k c_{p,k} I_k^{(p)}, \tag{4.13}$$

where the coefficients $c_{p,k}$ are constant real numbers. Our goal now is to use field redefinitions to reduce these higher-order terms to a minimal set of couplings in the spirit of [17]. To this end we will use the lowest-order equations of motion (4.10) simply as substitution rules in the action which we rewrite here for convenience:

$$DH_i \simeq D\Phi H_i, \quad i = 1, \dots, q, \tag{4.14a}$$

$$D^2\Phi \simeq \sum_{i=1}^{q} N_i H_i^2, \tag{4.14b}$$

$$H_q^2 \simeq \frac{1}{N_q}\left((D\Phi)^2 - \sum_{i=1}^{q-1} N_i H_i^2\right). \tag{4.14c}$$

The way in which we reordered the last rule distinguishes one particular Hubble parameter over the others. The reason of this split is that, in what follows, we will prove that one of the scale factors can be completely removed from higher-order terms, appearing only in the two-derivative theory (4.8). Without loss of generality, we chose the scale factor a_q.

For BI universes, the first steps of the classification are similar to those applied in [17]. Specifically, the same step-by-step proof of Sec. II D with the same itemization, proceeds as follows. We assume that to any order in α' any term in the action is writable as a product of factors $D^k\Phi$ and $D^l H_i$ with $i = 1, \dots, q$. We can now perform field redefinitions of a_i and the combination of Φ and n that yields rules (4.14a) and (4.14b) in order to establish the following:

(1) *A factor in an action including $D^2\Phi$ can be replaced by a factor with only first derivatives.*

This follows directly from the substitution rule in (4.14b).

(2) *A factor in an action including DH_i can be replaced by a factor with only first derivatives.*

This follows directly from the first substitution rule (4.14a).

(3) *Any action can be reduced so that it only has first derivatives of Φ.*

The proof proceeds as in [17]: We write any higher derivative as $D^{p+2}\Phi = D^p(D^2\Phi)$, and then integrate by parts the D^p. Then we substitute $D^2\Phi \to \sum_{i=1}^q N_i H_i^2$, after which we integrate back one-by-one, eliminating any second derivative created, using (1) or (2). At the end we are left with only first-order derivatives of Φ.

(4) *Any action can be reduced so that it only contains products of H_i, not their derivatives.*

The proof is identical to the previous one.

(5) *Any higher-derivative term is equivalent to one without any appearance of $D\Phi$.*

Up to this point, any higher-order term in the action is of the form

$$I = \int dtne^{-\Phi}(D\Phi)^p \prod_{i=1}^q H_i^{l_i}. \tag{4.15}$$

For this term to be a higher-derivative one, the total number of derivatives must be larger or equal to four:

$$p + l \geq 4, \qquad \text{with } l \equiv \sum_{i=1}^q l_i. \tag{4.16}$$

Because of duality invariance we must have $l_i \in 2\mathbb{Z}$, and therefore l is also even. Some l_i could be zero, and the analysis should hold even if all l_i are zero. Since the total number of derivatives must be even, it follows that $p \in 2\mathbb{Z}$. So $p \geq 2$.

We proceed to show that any appearance of $D\Phi$ can be removed. To this end, consider the generic term (4.15) with the conditions above and manipulate it as follows:

$$\begin{aligned} I &= -\int dtnD(e^{-\Phi})(D\Phi)^{p-1}\prod_{i=1}^q H_i^{l_i} \\ &= \int dtne^{-\Phi}\left((p-1)(D\Phi)^{p-2}D^2\Phi\prod_{i=1}^q H_i^{l_i} + (D\Phi)^{p-1}\sum_{j=1}^q(l_j H_j^{-1}DH_j)\prod_{i=1}^q H_i^{l_i}\right) \\ &\simeq \int dtne^{-\Phi}\left((p-1)(D\Phi)^{p-2}\sum_{j=1}^q(N_j H_j^2)\prod_{i=1}^q H_i^{l_i} + l(D\Phi)^p\prod_{i=1}^q H_i^{l_i}\right). \end{aligned} \tag{4.17}$$

To pass to the second line we integrated by parts, and to pass to the third line we used (4.14a) and (4.14b). The last term of the third line is in fact proportional to I. Bringing it to the left-hand side we end up with

$$I \simeq \frac{p-1}{1-l}\int dtne^{-\Phi}(D\Phi)^{p-2}\sum_{j=1}^q(N_j H_j^2)\prod_{i=1}^q H_i^{l_i}. \tag{4.18}$$

Since l is even, $1 - l \neq 0$, and the right-hand side is well defined. Using (4.18) recursively we can reduce the number of $D\Phi$'s in steps of two, ending up with zero since p is even. The above formula works fine if all l_i are zero; one simply sets $l = 0$ and the rightmost product is replaced by one.

The above chain of arguments proved that there is a field basis in which all higher-derivative terms are of the form

$$I = \int dtne^{-\Phi}\prod_{i=1}^q H_i^{l_i}. \tag{4.19}$$

Here $l = \sum_i l_i \geq 4$ is still even. Using the three rules (4.14) we now show that any appearance of H_q can be removed from these higher-order terms. We begin by rewriting I in a convenient way,

$$I = \int dtne^{-\Phi} \prod_{i=1}^{q} H_i^{l_i} = \int dtne^{-\Phi} H_q^2 H_q^{l_q-2} \prod_{i=1}^{q-1} H_i^{l_i}. \tag{4.20}$$

Using (4.14c) we now have

$$\begin{aligned} I &\simeq \frac{1}{N_q} \int dtne^{-\Phi} \left((D\Phi)^2 - \sum_{j=1}^{q-1} N_j H_j^2 \right) H_q^{l_q-2} \prod_{i=1}^{q-1} H_i^{l_i}, \\ &\simeq \frac{1}{N_q} \int dtne^{-\Phi} \left(\frac{1}{3-l} \sum_{j=1}^{q} N_j H_j^2 - \sum_{j=1}^{q-1} N_j H_j^2 \right) H_q^{l_q-2} \prod_{i=1}^{q-1} H_i^{l_i}, \\ &= \frac{1}{N_q} \int dtne^{-\Phi} \left(\frac{l-2}{3-l} \sum_{j=1}^{q-1} N_j H_j^2 + \frac{1}{3-l} N_q H_q^2 \right) H_q^{l_q-2} \prod_{i=1}^{q-1} H_i^{l_i}, \\ &= \frac{1}{N_q} \frac{l-2}{3-l} \int dtne^{-\Phi} H_q^{l_q-2} \prod_{i=1}^{q-1} H_i^{l_i} \sum_{j=1}^{q-1} N_j H_j^2 + \frac{1}{3-l} I. \end{aligned} \tag{4.21}$$

In passing to the second line we used (4.18) on the dilaton term with $p = 2$ and with $l \to l - 2$. In passing to the third line we separated the qth direction from the first sum. Finally, in the last line, we recognized the original term (4.19). Since l is even, the denominators are different from zero. Taking the latter to the left-hand side and since $l \neq 2$, we end up with

$$\begin{aligned} I &= \int dtne^{-\Phi} \prod_{i=1}^{q} H_i^{l_i} \\ &\simeq -\frac{1}{N_q} \int dtne^{-\Phi} H_q^{l_q-2} \left(\prod_{i=1}^{q-1} H_i^{l_i} \right) \sum_{j=1}^{q-1} N_j H_j^2. \end{aligned} \tag{4.22}$$

This shows that we can reduce by two units the power of H_q, at the expense of increasing the powers of the other H's. Using the result recursively, we are able to redefine away any appearance of H_q at higher orders in α', and its only appearance is in the two-derivative action (4.8)! The most general higher-derivative term is therefore given by

$$I = \int dtne^{-\Phi} \prod_{i=1}^{q-1} H_i^{l_i}. \tag{4.23}$$

As a corollary of this result, we get the absence of α' corrections for two particular cases of Bianchi Type-I universes:

(i) FRW: In this case we have only one independent scale factor:

$$q = 1, \quad N_1 = d, \quad a_1(t) \equiv a(t), \quad H_1(t) \equiv H(t). \tag{4.24}$$

Since from (4.23) we can always remove one Hubble parameter completely, there is a scheme where there are no α' corrections in the action at all. The action to all orders in α' is just given by the two-derivative theory:

$$I_{\rm FRW} = \int dtne^{-\Phi} [-(D\Phi)^2 + d \cdot H^2]. \tag{4.25}$$

(ii) Isotropic and static directions: A slightly more general case than FRW corresponds to the case

$$\begin{aligned} &q = 2, \qquad N_1 \equiv N, \qquad N_2 = d - N, \\ &a_1(t) \equiv a(t), \qquad H_1(t) \equiv H(t), \\ &a_2(t) = \text{const.} \quad \Rightarrow \quad H_2 = 0. \end{aligned} \tag{4.26}$$

This is one of the simplest anisotropic backgrounds where we have N isotropic directions and $d - N$ static ones. As in FRW, there is only one Hubble parameter, that we can redefine away. From (4.23) we see that there are no higher-order corrections at all. In this case the full action is given by the lowest order one:

$$I_{\rm static} = \int dtne^{-\Phi} [-(D\Phi)^2 + N \cdot H^2]. \tag{4.27}$$

A few comments are in order concerning the FRW case, which appears to be in conflict with [17]. There it was shown that in terms of a generic generalized metric $\mathcal{S}$, encoding a general time-dependent spatial metric and B-field, there is a minimal field basis in terms of which all α' corrections are of the form

$$\begin{aligned} I'[\mathcal{S}] = \int dtne^{-\Phi} \{ &\alpha' c_{2,0} \text{Tr}((D\mathcal{S})^4) \\ &+ \alpha'^2 c_{3,0} \text{Tr}((D\mathcal{S})^6) + \cdots \}, \end{aligned} \tag{4.28}$$

with the ellipsis denoting higher-order single-trace and multitrace terms of even powers of $(D\mathcal{S})$, but without factors $\mathrm{Tr}((D\mathcal{S})^2)$. The coefficients $c_{2,0}$, $c_{3,0}$, etc., *cannot* be changed by field redefinitions and hence have an invariant meaning (and are certainly nonzero). Specializing (4.28) then to FRW backgrounds with a single scale factor $a(t)$ one obtains corrections of (4.25) with higher powers of H^2. Depending on the coefficients the resulting theory may exhibit, for instance, nonperturbative de Sitter vacua, which are not visible in (4.25). So how is this result consistent with our above statement that for FRW background all higher-derivative corrections are removable by field redefinitions?

To understand the subtlety let us consider the first correction in α' and let us add a term proportional to $\mathrm{Tr}((D\mathcal{S})^2)$:

$$I^{(1)}[\mathcal{S}] = \int dtne^{-\Phi}\Big\{c_{2,0}\mathrm{Tr}((D\mathcal{S})^4) + \xi[\mathrm{Tr}((D\mathcal{S})^2)]^2\Big\}. \tag{4.29}$$

As recalled above and shown in [17], the new term in here can be removed by field redefinitions: the coefficient ξ has no invariant meaning and we may choose $\xi = 0$, as done in (4.28). We can, however, also choose it to be nonzero and adjust it so that for FRW backgrounds it cancels the contribution from the single trace term. Specifically,

$$\xi \equiv -\frac{c_{2,0}}{2d} \quad \Rightarrow \quad I^{(1)}[\mathcal{S}_{\mathrm{FRW}}] = 0, \tag{4.30}$$

where $\mathcal{S}_{\mathrm{FRW}}$ denotes the generalized metric (4.3) for a single scale factor. Thus, there is a field basis also for $\mathcal{S}$ so that, *when evaluated on FRW backgrounds*, the first-order α' corrections disappear. Similar remarks apply to all higher-derivative corrections.

So what, in view of the above discussion, is the fate of potential nonperturbative de Sitter vacua? It must be emphasized that the above manipulations using field redefinitions order by order in α' are strictly perturbative. There is no reason why a nonperturbative solution that is visible in one perturbative scheme must also be visible in another perturbative scheme, and one must await a better understanding of nonperturbative string theory even in this simple setting. Relatedly, whether a given solution physically exhibits the properties of de Sitter space depends on how one probes the spacetime with matter (or, more concretely, with "clocks") and in which field basis one couples the clock to the background fields.[5]

[5]We thank Robert Brandenberger for discussions on these points.

V. CONCLUSIONS AND OUTLOOK

The target space description of string theory contains Einstein gravity coupled to matter fields, universally including an antisymmetric tensor (B-field) and a scalar (dilaton), but importantly it also receives an infinite number of higher-derivative corrections. These corrections are only meaningful up to field redefinitions and can hence only be determined up to those redefinitions. Classifying the possible higher-derivative terms up to field redefinitions is hence the important first step of any attempts to determine these corrections. In this paper we have explored some surprising, and to the best of our knowledge largely unremarked, phenomena that arise when considering higher-derivative corrections for particular backgrounds and/or in noncritical dimensions. We have shown that for flat FRW backgrounds (i.e., spatially flat and homogeneous backgrounds with a single scale factor) *all* α' corrections are on-shell trivial. More generally, for so-called Bianchi type-I backgrounds governed by q scale factors only $q-1$ receive nontrivial higher-derivative corrections. Moreover, and perhaps more thought-provokingly, we have emphasized that for noncritical dimensions there are *no* invariant terms other than the cosmological term, as any term with two or more derivatives can be traded for terms with an arbitrary number of derivatives, hence invalidating the familiar setting of perturbative α' corrections. However, assuming that the numerical coefficients governing the terms in the action fall off in such a manner that terms with more derivatives are subleading relative to terms with less derivatives one can give a meaningful classification of higher-derivative corrections, as displayed in the main text and applied to the black hole solution of 2D string theory.

Our work is relevant to the understanding of effective field theory in gravitational theories with a cosmological term. We have argued that, although legal, perturbative redefinitions that generate variations of the order $1/\alpha'$ cosmological term do not respect the canonical structure of α' corrections. Higher-derivative terms in gravity are subtle in that their physical effect, small in the context of a derivative expansion, can sometimes be large. For the 2D string black hole, we have seen that they are always intrinsically large. As discussed recently by Horowitz *et al.* in [30] for near-extremal black holes, higher-derivative terms have an unusually strong effect allowing for a very highly curved geometry near the horizon for generic solutions.

It would be important to extend the research reported here in various directions, for instance:

(i) Based on the results of [17,18] various promising string cosmology proposals have already been explored, see for instance [27,29,31–37]. In view of the results presented here, these should be revisited for models with two or more scale factors, in particular with a focus on semirealistic embeddings into string theory.

(ii) While we have not been able to find the exact black hole solution proposed by Dijkgraaf, Verlinde, and Verlinde within our classification of higher-derivative corrections, there should be a different scheme in which it is a solution, as implied by [23,26]. The proposed solution will likely only be useful if the actual theory of which it is a solution can be written down.

(iii) The observation that in gravity theories with cosmological constant apparently any term in the action but the cosmological term can be removed by field redefinitions arguably deserves further investigation. The most extreme form of this effect can be displayed for an action with cosmological constant Λ of the form

$$S = -2\Lambda \int d^D x \sqrt{-g} \mathcal{L}(g), \qquad \text{with}$$
$$\mathcal{L} = 1 - \frac{1}{2\Lambda} R + \cdots, \tag{5.1}$$

which describes Einstein-Hilbert gravity with a cosmological constant, plus arbitrary higher-derivative terms implicit in the ellipsis. The field redefinition

$$g'_{\mu\nu} = [\mathcal{L}(g)]^{2/D} g_{\mu\nu} \tag{5.2}$$

maps the purely cosmological constant theory $S' = -2\Lambda \int d^D x \sqrt{-g'}$ into the action S above.[6] Of course, the fractional powers in (5.2) are generally problematic, but for functions $\mathcal{L}$ of the above form they can be defined as a power series.

(iv) It may be possible to investigate the possible field redefinitions of gravitational theories with cosmological term by exploring how observables of such theories are preserved. The entropy of black holes may provide a useful observable for this analysis.

(v) The arguably most exciting prospect of having an "α'-complete" theory governing black holes would be as a model for how string theory deals with the black hole singularity (see, e.g., [38]) and the information loss paradox. Tentative speculations along these lines are as old as string theory itself, but what has been lacking are methods that allow one to write down concrete theories in which such questions can be explored in a precise manner. The framework presented should get us closer to that goal.

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APPENDIX: ALL-ORDER PERTURBATIVE SOLUTION

In this appendix we prove there exist coefficients $a_k^{(p)}$ and $b_k^{(p)}$ such that (3.82) solve the all-order equations of motion (3.41). To see this, it is easier to work with the inverse of (3.82b), which, together with (3.82a), is given by

$$\bar{M} = \sum_{p\geq 0} \bar{M}^{(p)} \epsilon^p, \qquad \bar{M}^{(p)} = \sum_{k=0}^{p} a_k^{(p)} \mathrm{csch}^{2k+1} \bar{x}, \tag{A1a}$$

$$\Omega = \sum_{p\geq 0} \Omega^{(p)} \epsilon^p, \qquad \Omega^{(p)} = \sum_{k=0}^{p} d_k^{(p)} \mathrm{csch}^{2k-1} \bar{x}, \tag{A1b}$$

where the new coefficients $d_k^{(p)}$ depend on $b_k^{(p)}$. For instance, $d_0^{(0)} = \frac{1}{b_0^{(0)}} = -\frac{q}{2}$.

The proof follows by induction. Assuming (A1) holds up to an including order ϵ^{p-1}, we demonstrate that this structure is also a solution of the order p equations of motion, for some particular choice of coefficients.

To begin with, we take the second equation of (3.41) in the $n = 1$ gauge, expand it in powers of ϵ, and read the order p equation

$$\Omega^{(p)\prime\prime} - \Omega^{(p)} - [h(\bar{M})\Omega]^{(p)} = 0. \tag{A2}$$

In order to get the coefficients $[h(\bar{M})\Omega]^{(p)}$ we recall the definition of $h(\bar{M})$ (3.42), which, after using the definition of $f(\bar{M})$ and $g(\bar{M})$, reads

$$h(\bar{M}) = \frac{1}{2} \bar{M} f(\bar{M}) - g(\bar{M}) = -\sum_{i\geq 0} i c_i \epsilon^i \bar{M}^{2i+2}. \tag{A3}$$

Multiplying this expansion with Ω, and expanding

$$\bar{M}^{2i+2}\Omega = \sum_{p\geq 0} [\bar{M}^{2i+2}\Omega]^{(p)} \epsilon^p, \tag{A4}$$

we can use the Cauchy product identity for product of power series to read

[6] We thank Ashoke Sen for pointing out this formulation.

$$-[h(\bar{M})\Omega]^{(p)} = \sum_{i=0}^{p} i c_i [\bar{M}^{2i+2}\Omega]^{(p-i)} = \sum_{i=0}^{p-1} (i+1) c_{i+1} [\bar{M}^{2i+4}\Omega]^{(p-i-1)}, \quad (A5)$$

where for the second equality we used that the $i = 0$ term in the series was zero and then we just renamed the index $i \to i+1$. Equation (A5) shows explicitly that $[h(\bar{M})\Omega]^{(p)}$ depends on solutions to previous orders only, namely $\Omega^{(k)}$ and $\bar{M}^{(k)}$ with $k < p$, which is a direct consequence of (A3) starting at order ϵ. This last step is crucial in our proof since now, to get $[\bar{M}^{2i+4}\Omega]^{(k)}$ with $k < p$, we can use the expansion (A1). In order to do so we need a preliminary result: consider two fields A and B admitting an ϵ expansion with coefficients of the form

$$A^{(p)} = \sum_{k=0}^{p} a_k^{(p)} \mathrm{csch}^{2k}\bar{x}, \qquad B^{(p)} = \sum_{k=0}^{p} b_k^{(p)} \mathrm{csch}^{2k}\bar{x}. \quad (A6)$$

It can be shown that the product of them also admits an expansion of the same form, namely

$$[AB]^{(p)} = \sum_{k=0}^{p} f(a,b)_k^{(p)} \mathrm{csch}^{2k}\bar{x}, \quad (A7)$$

where each $f(a,b)_k^{(p)}$ depends on $a_i^{(j)}$ and $b_i^{(j)}$ with $i \le k$ and $j \le p$. Since now AB has the same expansion as A and B, we can repeat the previous step by multiplying AB with another A or B and use (A7) for the new product. By repeating this procedure iteratively, we end up with the extended result

$$[A^q B^l]^{(p)} = \sum_{k=0}^{p} f(a,b)_k^{(p)} \mathrm{csch}^{2k}\bar{x}, \quad (A8)$$

with q and l integers. Obviously the coefficients $f(a,b)_k^{(p)}$ here are not the same as in (A7), but they follow the same convention.

Coming back to our problem, we can use this intermediate result to get $[\bar{M}^{2i+4}\Omega]^{(j)}$ by noticing that, up to order $p-1$, $\frac{\bar{M}}{\mathrm{csch}\bar{x}}$ and $\mathrm{csch}\bar{x}\Omega$ has the same structure as A and B above. [As it can be seen from (A1).] Then, we can use (A8) directly with

$$A \to \frac{\bar{M}}{\mathrm{csch}\bar{x}}, \quad B = \mathrm{csch}\bar{x}\Omega, \quad q = 2i+4, \quad l = 1, \quad (A9)$$

so to get

$$[\bar{M}^{2i+4}\Omega]^{(j)} = \mathrm{csch}^{2i+3}\bar{x} \sum_{k=0}^{j} f(a,d)_k^{(j)} \mathrm{csch}^{2k}\bar{x}, \quad (A10)$$

which is only valid for $j < p$ and. With these coefficients, we can come back to (A5) to get

$$-[h(\bar{M})\Omega]^{(p)} = \sum_{i=0}^{p-1} \sum_{k=0}^{p-i-1} (i+1) c_{i+1} f(a,d)_k^{(p-i-1)} \times \mathrm{csch}^{2(k+i)+3}\bar{x} = \sum_{k=0}^{p-1} g(a,d)_k^{(p-1)} \mathrm{csch}^{2k+3}\bar{x}, \quad (A11)$$

where in the second equality we noticed that both sums can be merged into one, upon defining new combinations $g(a,d)_k^{(p-1)}$ which depend on coefficients that were determined from previous steps of the inductive procedure. Finally, we can insert this result back into the order-p equation (A2) to get

$$\Omega^{(p)\prime\prime} - \Omega^{(p)} + \sum_{k=0}^{p-1} g(a,d)_k^{(p-1)} \mathrm{csch}^{2k+3}\bar{x} = 0. \quad (A12)$$

Then, we can propose

$$\Omega^{(p)} = \sum_{k=0}^{p} d_k^{(p)} \mathrm{csch}^{2k-1}\bar{x} \Rightarrow \Omega^{(p)\prime\prime} = \sum_{k=0}^{p} d_k^{(p)} (2k-1)^2 \mathrm{csch}^{2k-1}\bar{x} + \sum_{k=0}^{p} d_k^{(p)} 2k \mathrm{csch}^{2k+1}\bar{x}, \quad (A13)$$

and check whether there exist coefficients $d_k^{(p)}$ such that (A12) is satisfied. By inserting (A13) into (A12), manipulating the limits of the series and renaming indices we arrive at

$$0 = \sum_{k=0}^{p-2} \left[d_{k+2}^{(p)} 4(k+2)(k+1) + d_{k+1}^{(p)} 2(k+1) + g(a,d)_k^{(p-1)} \right] \mathrm{csch}^{2k+3}\bar{x} + \left(2p d_p^{(p)} + g(a,d)_{p-1}^{(p-1)} \right) \mathrm{csch}^{2p+1}\bar{x}. \quad (A14)$$

By demanding that each term in the sum vanishes, we arrive at a linear system, involving p equations for the p coefficients $d_k^{(p)}$ with $1 \le k \le p$, while $d_0^{(p)}$ is unconstrained.[7] One can check that this is an independent system

[7]These undetermined coefficients just renormalize the zeroth-order integration constant q.

with a unique solution and so $d_k^{(p)}$ are completely determined from previous-order coefficients. This concludes the proof for the dilaton solution, and now we move into $\bar{M}$.

We start from the first equation of (3.41),

$$f(\bar{M}) = q\Omega^{-1} \quad \Rightarrow \quad [f(\bar{M})]^{(p)} = q\sum_{k=0}^{p} b_k^{(p)} \mathrm{csch}^{2k+1}\bar{x}, \tag{A15}$$

where we inserted (3.82b) to read the order p coefficient. The coefficients $b_k^{(p)}$ are all known from inverting Ω^p. They depend on the $d_k^{(p)}$ that we just determined in our previous step. Using the definition of $f(\bar{M})$, we read $[f(\bar{M})]^{(p)}$

$$\begin{aligned} [f(\bar{M})]^{(p)} &= \sum_{i=0}^{p} 2(i+1)c_i[\bar{M}^{2i+1}]^{(p-i)} \\ &= -2\bar{M}^{(p)} + \sum_{i=0}^{p-1} 2(i+2)c_{i+1}[\bar{M}^{2i+3}]^{(p-i-1)} \\ &= -2\bar{M}^{(p)} + \sum_{k=1}^{p} f(a)_{k-1}^{(p-1)} \mathrm{csch}^{2k+1}\bar{x}, \end{aligned} \tag{A16}$$

where in the second equality we separated the term $i = 0$ in the sum and renamed indices $i \to i+1$. For the third equality we used (A8) [which is possible because $[\bar{M}^{2i+3}]^{(p-i-1)}$ involves lower-order solutions of the form (A1a)], wrote everything as a power series in csch$\bar{x}$ and renamed indices one more time. The coefficients $f(a)_k^{(p-1)}$ are fully determined from previous orders and the c_i coefficients. Finally, by inserting this result into (A15) we can isolate $\bar{M}^{(p)}$ to get

$$\bar{M}^{(p)} = -\frac{q}{2}\mathrm{csch}\bar{x} - \frac{1}{2}\sum_{k=1}^{p}\left[q b_k^{(p)} - f(a)_{k-1}^{(p-1)}\right]\mathrm{csch}^{2k+1}\bar{x}, \tag{A17}$$

where we can see that, indeed, $\bar{M}^{(p)}$ has the desired form (A1) with the new coefficients $a_k^{(p)}$ completely determined from $d_k^{(p)}$ and $f(a)_k^{(p-1)}$.

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