Quantum dynamics of Lagrange multipliers

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When implementing a nonlinear constraint in quantum field theory by means of a Lagrange multiplier, $\lambda(x)$, it is often the case that quantum dynamics induce quadratic and even higher-order terms in $\lambda(x)$, which then does not enforce the constraint anymore. This is illustrated in the case of unimodular gravity, where the constraint is that the metric tensor has to be unimodular ($g(x) \equiv \det g_{\mu\nu}(x) = -1$).

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I. INTRODUCTION

Unimodular gravity (UG) is a variant of General Relativity (GR) originally conceived by Einstein in 1919 (cf. Ref. [1] for a recent review) in which the zero-mode component of the vacuum energy does not weigh. This is the only fully satisfactory solution to at least part of the well-known cosmological constant (CC) problem.

The theory rests on the assumption that only unimodular metrics $[g(x) \equiv \det g_{\mu\nu}(x) = -1]$ are admissible. This poses a formidable problem in practice because it is a nonlinear constraint. There have been at least three attempts to implement this constraint, namely:

(1) The simplest one consists in supplementing the GR Lagrangian with a Lagrange multiplier term implementing the constraint either as

$$\Delta S_{\rm UG} \equiv \int d^d x \frac{1}{\kappa}\lambda(x)(g(x)+1) \quad \text{or as}$$
$$\Delta S_{\rm UG} \equiv \int d^d x \frac{1}{\kappa}\lambda(x)(\sqrt{-g(x)}-1), \tag{1}$$

along with the standard linear splitting $g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \kappa h_{\mu\nu}(x)$. Examples of this are to be found in [2,3]. Note that the mass dimension of the multiplier is $[\lambda(x)] = 3$ in $d = 4$.

(2) The second one just defines an auxiliary unrestricted metric $g_{\mu\nu}$ in terms of which the unimodular metric is obtained

$$\gamma_{\mu\nu}(x) \equiv g^{-1/d}(x) g_{\mu\nu}(x). \tag{2}$$

This introduces a new Weyl gauge symmetry under

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x). \tag{3}$$

This formalism has been used extensively in [4].

(3) Finally, use can be made of the theorem on the effect that any unimodular metric is the exponential of a traceless one

$$g_{\mu\nu}(x) = (e^{G(x)})_{\mu\nu}, \tag{4}$$

with

$$\operatorname{tr} G_{\mu\nu}(x) = 0. \tag{5}$$

This has been used in Refs. [5–8].

The purpose of the present work is to examine the consistency of the first alternative under quantum corrections, although some comments will also be made on the exponential parametrization.

It is quite intuitive that the coupling of the multiplier with the graviton can induce divergent diagrams of higher order in the multipliers and even, in some cases kinetic energy terms for them; see Figs. 1 and 2 below. Examples of related phenomena appear in the principal chiral model [9] and in the physics of gravitons in a codimension-one brane [10], among others.

The conclusion is that any consistent renormalization[1] must include finite values for the coupling constants in front

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[1]Somebody could be tempted to put by hand all these coupling constants to zero. This would be most unnatural; it is more or less equivalent to putting to zero all coupling constants in front of the higher-dimensional operators in quantum gravity—no known symmetry principle supports this.

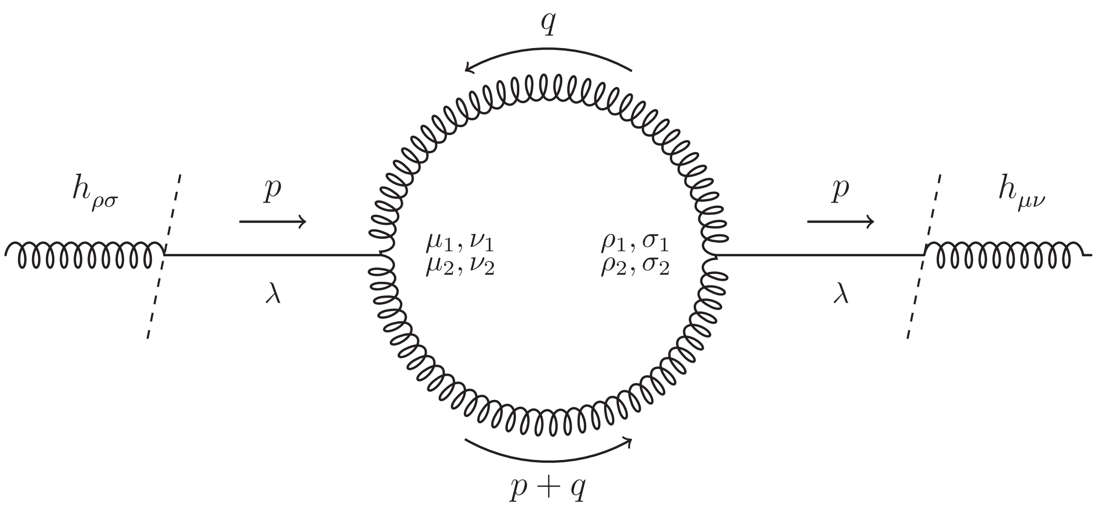

FIG. 1. Feynman diagram yielding a divergent $\lambda^2(x)$ part.

of these operators, which in turn conveys the fact that the Lagrange multiplier does not work as a multiplier anymore: it has become a full-fledge field with its own dynamics.

We shall, in fact, demonstrate in detail how the operator

$$\mathcal{O}_1 \equiv \lambda(x)^2, \tag{6}$$

is generated by one-loop diagrams. We further argue, without explicit calculation, that a two-loop diagram will induce the kinetic energy correction

$$\mathcal{O}_2 \equiv (\partial_\mu \lambda(x))^2. \tag{7}$$

To be specific, let $g(x)$ denote, as usual, the determinant of the metric. Assume that the unimodularity condition; $g(x) = -1$, is enforced with the first alternative just exposed, i.e., by including in the path integral the additional contribution,

$$\int \mathcal{D}\lambda e^{\frac{i}{\kappa}\int d^4x\lambda(x)(g(x)+1)}. \tag{8}$$

We want to explore the implications of such a gauge-fixing procedure when radiative corrections are taken into account in the covariant perturbation theory of quantum gravity. Consider gravitons, $h_{\mu\nu}(x)$ propagating on a flat background, $\eta_{\mu\nu}$, such that,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x). \tag{9}$$

Since,

$$-g(x) = 1 + \kappa h(x) + \frac{\kappa^2}{2}(h^2(x) - h_\mu{}^\nu(x) h^\mu{}_\nu(x)) + O(\kappa)^3, \tag{10}$$

where $h(x) = h_{\mu\nu}(x)\eta^{\mu\nu}$, the term in Eq. (8) gives rise to an interaction

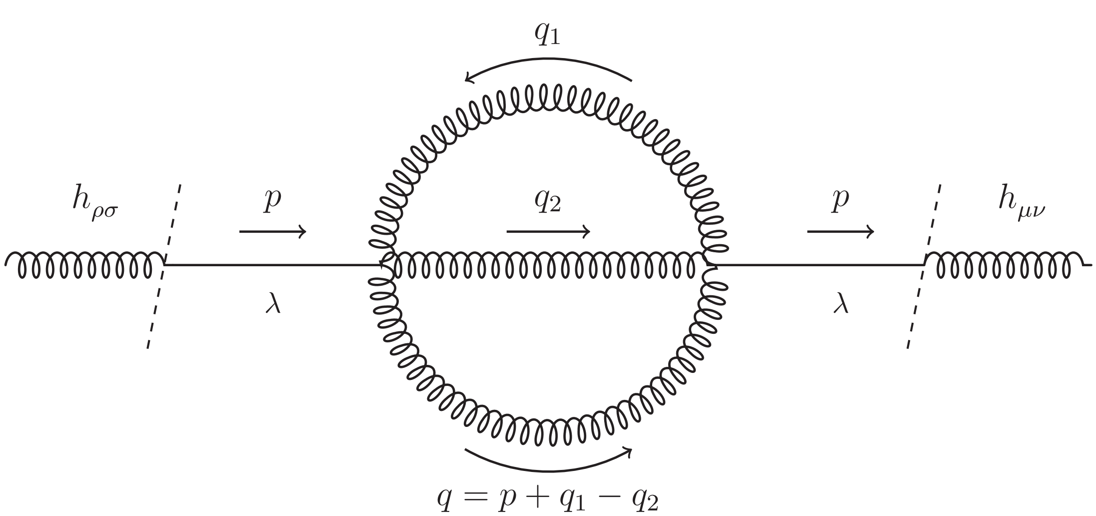

FIG. 2. Kinetic contributions to $\lambda(x)$.

$$i\frac{\kappa}{2}\int d^4x\lambda(x)h_{\mu_1\nu_1}(x)h_{\mu_2\nu_2}(x)V_L{}^{\mu_1\nu_1\mu_2\nu_2}, \qquad (11)$$

$$V_L{}^{\mu_1\nu_1\mu_2\nu_2} = \frac{1}{2}(2\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} - \eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}). \qquad (12)$$

At the tree level, the contribution to the 1PI function is linear in $\lambda(x)$ for it is given by Eq. (8). However, radiative corrections yield, in fact, a one-loop contribution to the 1PI functional quadratic in $\lambda(x)$. Such contribution is given by the 1PI diagram in Fig. 1, whose value in dimensional regularization is given by the following Feynman integral:

$$\Gamma_L(p) = \frac{\kappa^2}{2}\int\frac{d^dq}{(2\pi)^d}V_L{}^{\mu_1\nu_1\mu_2\nu_2}G_{\mu_1\nu_1\rho_1\sigma_1}(p) \times G_{\mu_2\nu_2\rho_2\sigma_2}(p+q)V_L{}^{\rho1\sigma1\rho2\sigma2}, \qquad (13)$$

where the vertices V_L are given by Eq. (12) and $G_{\mu_1\nu_1\rho_1\sigma_1}(p)$ denotes the graviton propagator discussed in Sec. II.

II. COMPUTING THE DIAGRAM

In this section we shall work out $\Gamma_L(p)$ in Eq. (13) for quantum unimodular gravity as defined in [4] on the one side and [3] on the other.

The general structure of the propagator is of the form,

$$G_{\mu_1\nu_1\mu_2\nu_2}(q) = -\frac{1}{q^2}\bigg(A_1(\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}+\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2}) + A_2\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + A_3\frac{1}{q^2}(\eta_{\mu_1\nu_1}q_{\mu_2}q_{\nu_2}+\eta_{\mu_2\nu_2}q_{\mu_1}q_{\nu_1}) + A_4\frac{1}{q^2}(\eta_{\mu_1\mu_2}q_{\nu_1}q_{\nu_2}+\eta_{\mu_1\nu_2}q_{\nu_1}q_{\mu_2}+\eta_{\nu_1\mu_2}q_{\mu_1}q_{\nu_2}+\eta_{\nu_1\nu_2}q_{\mu_1}q_{\mu_2}) + A_5\frac{1}{q^4}q_{\mu_1}q_{\nu_1}q_{\mu_2}q_{\nu_2}\bigg). \qquad (14)$$

The value of the coefficients A_1 to A_5 depends, in general, on the set of fields introduced by the Becchi-Rouet-Stora-Tyutin (BRST) quantization procedure. The set of fields introduced in [4] is quite different from the set of fields in [3]. Let us then begin by adapting the BRST formalism of [4], which was developed for the unrestricted metric in Eq. (2), to our case.

The action of the quantum theory is not invariant under the full diffeomorphism group but only under the TDiff subgroup,

$$\delta g_{\mu\nu} = \nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu, \qquad \nabla_\lambda c^{T\lambda} = 0. \qquad (15)$$

The nilpotent BRST operator $s_D^2 = 0$ is defined by

$$s_D g_{\mu\nu} = 0,$$
$$s_D h_{\mu\nu} = \partial_\mu c^T_\nu + \partial_\nu c^T_\mu + c^{T\rho}\partial_\rho h_{\mu\nu} + h_{\rho\nu}\partial_\mu c^{T\rho} + h_{\rho\mu}\partial_\nu c^{T\rho}, \qquad (16)$$

and

$$s_D g(x) = 2g(x)\nabla_\mu c^{T\mu} = c^{T\mu}\partial_\mu g(x),$$
$$s_D\lambda(x) = c^{T\rho}\partial_\rho\lambda(x), \qquad (17)$$

where $c^{T\mu}$ is the ghost fields for transverse diffeomorphisms, the transverse condition implies $\partial_\mu c^{T\mu} = 0$. The BRST transformations are then defined in such a way that the BRST algebra closes.

First of all, let us note that

$$\int d^dx s_D^2\lambda(x) = -\int d^dx\partial_\rho(c^{T\rho}c^{T\sigma}\partial_\sigma\lambda(x)) = 0. \qquad (18)$$

The action is invariant under s_D because

$$s_D\int d^dx\lambda(x)(g(x)+1) = \int d^dx\partial_\rho[c^{T\rho}(g(x)+1)\lambda(x)]. \qquad (19)$$

Introduce now the following set of fields

$$h^{(0,0)}_{\mu\nu}, \quad c^{(1,1)}_\mu, \quad b^{(1,-1)}_\mu, \quad f^{(0,0)}_\mu, \quad \phi^{(0,2)},$$
$$\pi^{(1,-1)}, \quad \pi'^{(1,1)}, \quad \bar{c}^{(0,-2)}, \quad c'^{(0,0)},$$
$$c^{(1,1)}, \quad b^{(1,-1)}, \quad f^{(0,0)}, \qquad (20)$$

where $c^{(1,1)}_\mu$ denotes c_μ and $h^{(0,0)}_{\mu\nu}$ is $h_{\mu\nu}$ and the superscript (n,m) carries the Grassmann number, n (defined modulo two) and ghost number, m. Also we need to introduce a projector $\Theta_{\mu\nu}$

$$c^T_\mu = \Theta_{\mu\nu}c^\nu = (\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)c^{\nu(1,1)}, \qquad (21)$$

and

$$\partial_\mu(c^{\rho T}\partial_\rho c^{T\mu}) = 0, \qquad \partial_\mu[(Q^{-1})^\mu_\nu(c^{\rho T}\partial_\rho c^{T\mu})] = 0, \qquad (22)$$

where

$$(Q^{-1})^\mu_\nu = \frac{1}{\Box}\delta^\mu_\nu. \qquad (23)$$

To summarize, the action of s_D over the fields (20)

$$\begin{aligned} s_D g_{\mu\nu} &= 0, \\ s_D h_{\mu\nu} &= \partial_\mu c^T_\nu + \partial_\nu c^T_\mu + c^{\rho T}\partial_\rho h_{\mu\nu} + h_{\rho\nu}\partial_\mu c^{T\rho} + h_{\rho\mu}\partial_\nu c^{T\rho}, \\ s_D c^{(1,1)\mu} &= (Q^{-1})^\mu_\nu (c^{\rho T}\partial_\rho c^{T\nu}) + \partial^\mu \phi^{(0,2)}, \\ s_D \phi^{(0,2)} &= 0, \\ s_D b^{(1,-1)}_\mu &= f^{(0,0)}_\mu, \\ s_D f^{(0,0)}_\mu &= 0, \\ s_D \bar{c}^{(0,-2)} &= \pi^{(1,-1)}, \\ s_D \pi^{(1,-1)} &= 0, \\ s_D c'^{(0,0)} &= \pi'^{(1,1)}, \\ s_D \pi'^{(1,1)} &= 0, \\ s_D c^{(1,1)} &= c^{T\rho}\partial_\rho c^{(1,1)}, \\ s_D b^{(1,-1)} &= c^{T\rho}\partial_\rho b^{(1,-1)}, \\ s_D f^{(0,0)} &= c^{T\rho}\partial_\rho f^{(0,0)}. \end{aligned} \tag{24}$$

The fermion X_{TD} performing the gauge fixing of the TDiff symmetry reads

$$\begin{aligned} X_{TD} = b^{(1,-1)}_\mu [F^\mu + \rho_1 f^{\mu(0,0)}] + \bar{c}^{(0,-2)}[F^\mu_2 c^{(1,1)}_\mu + \rho_2 \pi'^{(1,1)}] \\ + c'^{(0,0)}[F^\mu_1 b^{(1,-1)}_\mu + \rho_3 \pi^{(1,-1)}], \end{aligned} \tag{25}$$

where F^μ is a function containing the graviton field and F^μ_1, F^μ_2, and the three operators ρ_i can be freely chosen.

For the computation at hand, one can choose

$$\begin{aligned} F_\mu &= \gamma_1 \partial^\nu h_{\mu\nu} + \gamma_2 \partial_\mu h, \\ F^\mu_1 &= \alpha_1 \partial^\mu, \\ F^\mu_2 &= \alpha_2 \partial^\mu, \\ (\rho_2 - \rho_3)^{-1} &= \gamma \Box. \end{aligned} \tag{26}$$

Applying the s_D operator over the gauge fixing term using Eq. (24)

$$\begin{aligned} \int d^d x s_D X_{TD} = \int d^d x \{ f^{(0,0)}_\mu [F^\mu + \rho_1 f^{\mu(0,0)}] - b^{(1,-1)}_\mu s_D F^\mu + \pi^{(1,-1)}[F^\mu_2 c^{(1,1)}_\mu + \rho_2 \pi'^{(1,1)}] \\ + \bar{c}^{(0,-2)} F^\mu_2 \partial_\mu \phi^{(0,2)} + \pi'^{(1,1)}[F^\mu_1 b^{(1,-1)}_\mu + \rho_3 \pi^{(1,-1)}] + c'^{(0,0)} F^\mu_1 f^{(0,0)}_\mu \}, \end{aligned} \tag{27}$$

in which,

$$s_D F^\mu = \gamma_1 \Box c^{T\mu} + \gamma_1 \partial_\nu [c^{\rho T}\partial_\rho h^{\mu\nu} + h^{\rho\nu}\partial^\mu c^{T\rho} + h^{\rho\mu}\partial^\nu c^{T\rho}] + \gamma_2 \partial^\mu [c^{T\rho}\partial_\rho h + 2h_{\rho\sigma}\partial^\rho c^{T\sigma}]. \tag{28}$$

The bosonic quadratic action in our problem reads,

$$\begin{aligned} S_2 - S^{TD}_{\rm hf} = -\frac{1}{2}\int d^d x \bigg\{ h^{\alpha\beta}\left[\frac{1}{4}\eta_{\beta\nu}\eta_{\alpha\mu}\bar{\Box} - \frac{1}{4}\eta_{\alpha\beta}\eta_{\mu\nu}\bar{\Box} + \frac{1}{2}\eta_{\alpha\beta}\partial_\mu\partial_\nu - \frac{1}{2}\eta_{\beta\nu}\partial_\alpha\partial_\mu\right] h^{\mu\nu} - 2h\lambda \\ - 2h^{\mu\nu}\left[\frac{\gamma_1}{2}(\partial_\nu f^{(0,0)}_\mu + \partial_\mu f^{(0,0)}_\nu) + \gamma_2 \eta_{\mu\nu}\partial^\lambda f^{(0,0)}_\lambda\right] + 2\rho_1 f^{(0,0)}_\mu f^{\mu(0,0)} + 2\alpha_1 c'^{(0,0)}\partial^\mu f^{(0,0)}_\mu \bigg\}, \end{aligned} \tag{29}$$

which has the general structure

$$S_2 - S^{TD}_{\rm hf} = -\frac{1}{2}\int d^d x \psi^A K_{AB} \psi^B, \tag{30}$$

where we have written the quadratic operator corresponding to the generalized field, ψ^A, defined as a vector

$$\psi^A \equiv \begin{pmatrix} h_{\mu\nu} \\ f^{(0,0)}_\mu \\ c'^{(0,0)} \\ \lambda \end{pmatrix}. \tag{31}$$

After going into momentum space, the graviton propagator can be obtained by finding the inverse of the $K_{AB}(p)$ operator,

$$K_{AB} G^{BC} = I^C_A. \tag{32}$$

By doing so, one finds that

$$G_{\rho\sigma\mu\nu}(p) = -\frac{1}{p^2}\left(2(\eta_{\rho\mu}\eta_{\sigma\nu}+\eta_{\rho\nu}\eta_{\sigma\mu}) - \frac{4}{(d-2)}\eta_{\rho\sigma}\eta_{\mu\nu} + \frac{8}{(d-2)}\frac{1}{p^2}(\eta_{\mu\nu}p_\rho p_\sigma + \eta_{\rho\sigma}p_\mu p_\nu) + -\frac{2(\gamma_1^2-\rho_1)}{\gamma_1^2}\frac{1}{p^2}(\eta_{\rho\mu}p_\sigma p_\nu + \eta_{\rho\nu}p_\sigma p_\mu + \eta_{\sigma\mu}p_\rho p_\nu + \eta_{\sigma\nu}p_\rho p_\mu) + -\frac{8(2\gamma_1^2+(d-2)\rho_1)}{(d-2)\gamma_1^2}\frac{1}{p^4}p_\rho p_\sigma p_\mu p_\nu\right). \tag{33}$$

We are now ready to calculate the 1PI diagram in Fig. 1 in dimensional regularization. After substituting in Eq. (13) for the obtained vertex and propagator, the resulting integral can be expressed as a linear combination of integrals of the type,

$$\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q+p)^2}\frac{P^{(\alpha,\beta)}(q,q+p)}{(q^2)^\alpha((p+q)^2)^\beta}, \tag{34}$$

where, α and β are non-negative integers such that $0 \le \alpha+\beta \le 4$ and $P^{(\alpha,\beta)}(q,q+p)$ is a polynomial in q^μ and $(p+q)^\mu$ of dimension $2(\alpha+\beta)$.

The value of such integrals can be consulted in [11] or elsewhere, and setting $d = 4+2\epsilon$ yields,

$$\Gamma_L(p) = -\frac{3i(3\gamma_1^4+\rho_1^2)}{2\pi^2\gamma_1^4\epsilon} + \frac{i(9(1+4\gamma_E)\gamma_1^4 - 6\gamma_1^2\rho_1 + 12(3\gamma_1^4+\rho_1^2)\ln(\frac{-p^2}{\mu^2}) + (5+12\gamma_E)\rho_1^2)}{8\pi^2\gamma_1^4} + O(\epsilon), \tag{35}$$

where, in Eq. (35), we have set $\kappa = 1$, and γ_E is the Euler-Mascheroni constant.

Let us now move on and obtain $\Gamma_L(p)$ in Eq. (13) for the BRST formulation of [3]. Now the vertex $V_L{}^{\mu_1\nu_1\mu_2\nu_2}$ reads

$$V_L^{\mu_1\nu_1\mu_2\nu_2} = \frac{1}{4}(\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} - \eta^{\nu_1\mu_2}\eta^{\mu_1\nu_2}), \tag{36}$$

for it is $\sqrt{-g(x)}$ rather than $g(x)$ which couples to $\lambda(x)$. The graviton propagator of [3] [see (3.15), therein] is retrieved by setting

$$A_1 = 1, \quad A_2 = -1, \quad A_3 = 2, \quad A_4 = -1, \quad A_5 = -4,$$

in (14). For this choice of A_is and the vertex in (36), $\Gamma_L(p)$ turns out to be equal to

$$-\frac{9i}{32\pi^2\epsilon} - \frac{3(11+12\gamma)i}{128\pi^2} - \frac{9i}{32\pi^2}\ln\left(\frac{p^2}{\mu^2}\right) + O(\epsilon), \tag{37}$$

where $d = 4+2\epsilon$ and metric signature is $(-,+,+,+)$. Again, we have set $\kappa = 1$.

Had we chosen to define the theory *à la* Wilson by assuming, for example, that all fields vanish when their momentum is bigger than the UV cutoff scale, Λ_{UV}, then the integration of the fast graviton modes with momentum

$$p \in [\Lambda, \Lambda_{\mathrm{UV}}], \tag{38}$$

would give rise for the low-energy modes $\lambda_{\mathrm{low}}(x)$ (that is, those whose Fourier transform vanishes when $p > \Lambda$) to

$$c\ \ln\left(\frac{\Lambda_{\mathrm{UV}}}{\Lambda}\right)\kappa^2 \int d^4x \frac{1}{2}\lambda_{\mathrm{low}}(x)^2, \tag{39}$$

where $c = \frac{3(3\gamma_1^4+\rho_1^2)}{\pi^2\gamma_1^4}$ and $c = \frac{9}{16\pi^2}$, respectively, for the two BRST formulations of unimodular gravity we have discussed. The result that we have obtained above means that the low-energy modes $\lambda_{\mathrm{low}}(x)$ do not work as a Lagrange multiplier so that the unimodularity condition is not imposed on the corresponding graviton low-energy modes.

III. A COMMENT ON GR IN THE UNIMODULAR GAUGE

The phenomena discussed here also affect GR in the unimodular gauge as discussed in [12]. The propagator of GR has the same general form discussed above, with

$$A_1 = -1, \qquad A_2 = \frac{2}{d-2}, \qquad A_3 = -\frac{4}{d-2}, \qquad A_4 = 1-\alpha, \qquad A_5 = \frac{4(2+\alpha(d-2))}{d-2}.$$

This leads to the UV divergent result for the corresponding diagram

$$-i\frac{3(3+\alpha^2)}{32\pi^2\epsilon} + i\frac{9+36\gamma+\alpha(-6+\alpha(5+12\gamma))}{128\pi^2} + i\frac{3(3+\alpha^2)}{32\pi^2}\ln\left(\frac{p^2}{\mu^2}\right) + O(\epsilon). \tag{40}$$

The MS renormalization implies a counterterm

$$\frac{3(3+\alpha^2)}{32\pi^2\epsilon}\int d^4x \frac{1}{2}\lambda(x)^2, \tag{41}$$

spoiling the working of the Lagrange multiplier as such. It is worth pointing out that this counterterm does not jeopardize the BRST quantization of GR, owing to the fact that it is BRST exact. What it means is that the gauge-fixing fermion was not general enough, which is always a delicate issue for theories like GR that are not renormalizable by power counting.

A similar computation can be carried out for GR in the unimodular gauge as defined in [13]. The result that one obtains is Eq. (37), for it turns out that the graviton propagator of [13] is the same as the graviton propagator from the unimodular theory of [3].

IV. A REMARK ON THE EXPONENTIAL PARAMETRIZATION

Let us present a formal proof of the fact that this third approach of the Introduction does not suffer from this sickness, at least if the unimodularity condition is implemented by adding to the action the term

$$\int d^d x \frac{1}{\kappa} \lambda(x) h(x),$$

$\lambda(x)$ being a Lagrange multiplier and $h(x) = h^{\mu}_{\mu}(x)$.

Now, let us set $\kappa = 1$ and define the partition function as follows:

$$Z[J_{\mu\nu}(x), j(x), \cdots] = e^{iW[J_{\mu\nu}(x), j(x), \cdots]} = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\lambda \cdots e^{iS + i\int d^4x \lambda(x) h(x) + i(J^{\mu\nu}(x) h_{\mu\nu}(x) + j(x)\lambda(x)) + \cdots},$$

where $h(x) \equiv \bar{g}^{\mu\nu} h_{\mu\nu}(x)$ and the dots stand for contributions involving other fields and external sources but no $\lambda(x)$. An identity can be easily written as

$$0 = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\lambda \cdots \frac{\delta}{\delta\lambda(x)} \left(e^{iS + i\int d^4x \lambda(x) h(x) + i(J^{\mu\nu}(x) h_{\mu\nu}(x) + j(x)\lambda(x)) + \cdots}\right) = i\langle h(x) + j(x)\rangle, \tag{42}$$

where as usual

$$\langle F(x)\rangle \equiv \int \mathcal{D}h_{\mu\nu} \mathcal{D}\lambda \cdots F(x) e^{iS + i\int d^4x \lambda(x) h(x) + i(J^{\mu\nu}(x) h_{\mu\nu}(x) + j(x)\lambda(x) + \cdots)}. \tag{43}$$

This Ward identity can be written as

$$-i\bar{g}^{\mu\nu} \frac{\delta Z}{\delta J^{\mu\nu}} + j(x), \qquad Z = 0 = \bar{g}^{\mu\nu} \frac{\delta W}{\delta J^{\mu\nu}} + j(x). \tag{44}$$

Define now the 1PI effective action as

$$\Gamma[h_{\mu\nu}, \lambda, \cdots] \equiv W[J_{\mu\nu}, j, \cdots] - \int d^4x (J^{\mu\nu} h_{\mu\nu}(x) + j\lambda(x)) + \cdots). \tag{45}$$

Then,

$$\bar{g}^{\mu\nu} h_{\mu\nu}(x) - \frac{\delta\Gamma}{\delta\lambda(x)} = 0. \tag{46}$$

It follows that the full dependence on the Lagrange multiplier is captured by

$$\Gamma = \tilde{\Gamma} + \int d^4x \lambda(x) \bar{g}^{\mu\nu} h_{\mu\nu}(x), \tag{47}$$

where

$$\frac{\delta\tilde{\Gamma}}{\delta\lambda(x)} = 0. \tag{48}$$

V. COMMENTS AND CONCLUSIONS

Let us begin by pointing out that aside from the diagram here studied, one can expect higher-loop diagrams making other terms involving $\lambda(x)$ present in the action, following Gell-Mann's totalitarian principle; whatever is not forbidden is compulsory. In particular, we expect diagrams such as those in Fig. 2 give rise to kinetic terms for the $\lambda(x)$ field; the 1PI diagram in Fig. 2 is quadratically divergent by power counting.

At any rate, we have shown in this work that the imposition of the unimodular constraint in UG through a Lagrange multiplier does not survive, in general, quantum effects; unless the exponential parametrization is used appropriately. These generate, in general, (UV divergent) quadratic (and higher)-order terms in the multiplier, as well as kinetic-energy contributions, all of which spoil the work of the Lagrange multiplier as such.

A reasonable hypothetical cancellation of those diagrams would be in an appropriately quantized unimodular supergravity [14]. Of course, were we ready to pay the price of giving up unitarity to get a renormalizable theory of gravity—e.g., by considering quadratic gravity[2] [15]—with propagators falling off at infinity as $1/p^4$, then the relevant diagrams computed in this paper would be UV finite by power counting. Hence, the problem that we have unearthed in this paper for a unimodular gravity theory based solely on the Einstein-Hilbert would disappear. Whether a similar problem shows up again at a higher-loop level demands an analysis which lies outside the scope of this paper.

[2]We thank an anonymous referee for pointing this out to us.

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