$T\bar{T}$ deformations and the *pp*-wave correspondence

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In this paper we consider $T\bar{T}$ deformations in the context of pp waves obtained from gravity duals. We propose a deformation of $\mathrm{AdS}_5 \times S^5$ similar to the deformation of the single trace $T\bar{T}$ deformation of $\mathrm{AdS}_3 \times S^3 \times T^4$ with NS-NS flux, and study it through the Penrose limit, concluding that it must correspond to some dipole theory, probably noncommutative. We $T\bar{T}$ deform the world sheet string for the $\mathrm{AdS}_5 \times S^5$ pp wave, and find a corresponding spin chain Hamiltonian. Finally, directly $T\bar{T}$ deforming the spin chain Hamiltonian obtained from the pp wave, we find that it corresponds to an equivalent Berenstein-Maldacena-Nastase (BMN) sector of the $\mathcal{N} = 4$ super Yang-Mills.

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I. INTRODUCTION

The $T\bar{T}$ deformations of quantum field theories in two dimensions, defined by Zamolodchikov in [1,2], which are of special importance for integrable theories, have attracted a lot of interest recently. Although originally defined as deformations of renormalized theories, through the normal ordered product of the energy-momentum tensors T and $\bar{T}$ [more precisely, the $\det T_{\mu\nu}$ operator, so $\frac{1}{8}(T_{\alpha\beta}T^{\alpha\beta} - (T^{\alpha}{}_{\alpha})^2)$], in point splitting regularization, it was soon realized that one can equivalently define the theory by deforming the classical Lagrangian density by $\det T_{\mu\nu}$ at each step of the deformation, thus finding closed forms for the Lagrangian, at least in the scalar case [3,4].[1] This also leads to deformations of the classical solutions of the theory, see [6–9]. In higher dimensions, the equivalent of the $T\bar{T}$ deformations is less understood, and there are several proposals of deformations [4,10,11] (although one can also simply extend the $T\bar{T}$ deformed actions to higher dimensions [9]).

Given that the most interesting applications are to conformal and integrable field theories, a natural question is this: Can one define a gravity dual that corresponds to the $T\bar{T}$ deformations? A first such proposal was considered in [12], though it just stated that the deformation amount to giving Dirichlet boundary conditions at a finite cutoff position from the original boundary, in the Renormalization Group (RG) direction, $r = r_c$. This, however, is not quite the definition one would like, namely to have a deformed gravity dual of the deformed boundary field theory.

That has been obtained, in a certain sense, in the case of string theory in the $\mathrm{AdS}_3 \times S^3 \times T^4$ with NS-NS flux, considered in [13,14], with a boundary theory given by $\mathcal{M}^p/\mathfrak{S}_p$, where $\mathcal{M}$ is a conformal field theory (CFT) with central charge $c = 6k$, and k is the number of NS5 branes, p is the number of fundamental strings, with the duality to the symmetric product space known to be valid for $k = 1$. In that case, one can construct string world sheet vertex operators that correspond to the operators in the boundary CFT that are "single trace," $\sum_{i=1}^{p} \mathcal{O}_i$, where $\mathcal{O}_i$ is an $\mathcal{O}(x) \in \mathcal{M}$ that lives in the ith factor in the CFT product space. Then the usual $T\bar{T}$ deformation of the CFT would be of a "double trace type," namely the product of $T(x)$ and $\bar{T}(x)$, both of which are single traces $\sum_{i=1}^{p} T_i(x)$ in the CFT, and each have a corresponding string world sheet (integrated) vertex operator for the bulk theory. Yet what is easier to understand in the gravity dual is instead the "single trace" deformation,

$$D(z, \bar{z}) = A \sum_{i=1}^{p} T_i(z)\bar{T}_i(\bar{z}), \tag{1.1}$$

that has a corresponding (integrated) vertex operator for the bulk string world sheet, and it has been shown [15] to be

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[1]Note that there are arguments [5] that the $T\bar{T}$ deformations are generically thermodynamically unstable at high temperature.

given by a TsT transformation of the bulk in the CFT_2 directions x and t, and to have many of the properties of the usual $T\bar{T}$ deformation of CFT_2.

We want then first to ask this: Can we generalize this construction to the CFT_4 case, i.e., to the $\mathrm{AdS}_5\times S^5$ background? We would take the agnostic point of view, and construct a gravity dual, then try to understand the deformation of $\mathcal{N}=4$ SYM (super Yang-Mills) it corresponds to, by taking a Penrose limit, which usually simplifies a lot the analysis [16].

Reversely, the second question we ask is this: What is the $T\bar{T}$ deformation of the string worldsheet for the $\mathrm{AdS}_5\times S^5$ (maximally supersymmetric) pp wave, and what does it correspond for the spin chain in $\mathcal{N}=4$ SYM? We can either deform the string world sheet theory, then discretize it to obtain a corresponding spin chain Hamiltonian, or consider the deformation of the original pp wave spin chain Hamiltonian directly, employing a $T\bar{T}$ deformation procedure for one-dimensional (quantum mechanics) Hamiltonians, which we do using the procedure defined in [17].

The paper is organized as follows. In Sec. II we review the two-dimensional case, for the gravity dual to the single trace $T\bar{T}$ deformation, and its TsT construction. In Sec. III we apply the procedure to the four-dimensional case. In Sec. IV we take the Penrose limit of the gravity duals to the single trace deformation for $\mathrm{AdS}_3\times S^3\times T^4$, and then to the proposed $\mathrm{AdS}_5\times S^5$ deformation, and finally propose an interpretation for the deformation in $\mathcal{N}=4$ SYM. In Sec. V, we consider the $T\bar{T}$ deformation of the maximally supersymmetric pp wave obtained in the Penrose limit of $\mathrm{AdS}_5\times S^5$, first as a deformation of the string world sheet, discretized to give us a deformed spin chain Hamiltonian, and then finally as a deformation of the quantum mechanical spin chain Hamiltonian (obtained from the pp wave) itself, and propose a dual interpretation for this spin chain from $\mathcal{N}=4$ SYM. In Sec. VI we summarize and list open questions. In the Appendix we review the derivation of the $T\bar{T}$ deformation of a general quantum mechanical model.

II. TST TRANSFORMATIONS AND $T\bar{T}$ DEFORMATIONS: TWO-DIMENSIONAL CASE

In the context of the AdS/CFT correspondence, in particular in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ case, there is a proposed relation between the $T\bar{T}$ deformation of the CFT_2 at the boundary and a "single-trace" $T\bar{T}$ deformation on the string world sheet in the bulk [13,14], of the type

$$T(z)\bar{T}(\bar{z})\equiv A\sum_i T_i(z)\bar{T}_i(\bar{z}). \tag{2.1}$$

Specifically, considering the $\mathrm{AdS}_3\times S^3\times T^4$ solution with NS-NS flux (using the notation in [15]),

$$\begin{aligned}R^{-2}ds^2&=e^{2\rho}(-dt^2+dx^2)+d\rho^2+\frac{1}{4}(\sigma_1^2+\sigma_2^2+\sigma_3^2)\\&\quad+ds^2(T^4)\\H&=-2e^{2\rho}dt\wedge dx\wedge d\rho+\frac{1}{4}\sigma_1\wedge\sigma_2\wedge\sigma_3,\end{aligned} \tag{2.2}$$

and a constant dilaton $\Phi=\Phi_0$, the boundary CFT is given by $\mathcal{M}^p/\mathfrak{S}_p$, where $\mathcal{M}$ is a CFT with central charge $c=6k$, and k is the number of NS5-branes and p is the number of fundamental strings (we restrict to $k=1$ if we want to be sure that we have the symmetric product boundary CFT). In the $B_{\rho 0}=B_{\rho 1}=0$ gauge, we have $B_{01}=-e^{2\rho}$.

Note that in this paper we will restrict to solutions with NS-NS flux; the case of Ramond-Ramond (R-R) flux is not known to be related to a single-trace $T\bar{T}$ deformation of a tensor product CFT, hence we will not analyze it; its pp wave limit will also be considerably different due to the different fluxes.

Then, as observed in [15], the $T\bar{T}$ deformed solution (corresponding to the single trace $T\bar{T}$ on the world sheet) in the notation in [18] (the volume of T^4 is $(2\pi l_s)^4 v$),

$$\begin{aligned}\frac{ds^2}{l_s^2}&=\frac{k(-dt^2+dx^2)}{\frac{l_s^2}{R^2}+e^{-2\phi}}+kd\phi^2+kds^2_{S^3}+ds^2_{T^4},\\e^{2\Phi}&=\frac{vk}{p}\frac{e^{-2\phi}}{\frac{l_s^2}{R^2}+e^{-2\phi}}\equiv e^{2\Phi_0}\frac{e^{-2\phi}}{\frac{l_s^2}{R^2}+e^{-2\phi}},\\H&=-\frac{2e^{2\phi}}{(1+\frac{l_s^2}{R^2}e^{2\phi})^2}dt\wedge dx\wedge d\phi+\frac{kl_s^2}{4}\sigma_1\wedge\sigma_2\wedge\sigma_3,\end{aligned} \tag{2.3}$$

can be obtained as a TsT transformation in the CFT_2 directions x and t (though there is no proof of the relation to TsT outside this specific case).

To see that, write the TsT transformations from [19], with $-2\gamma=l_s^2/R^2$ (note that the overall scale of the metric is $R_{\mathrm{AdS}}=kl_s^2$). We have a T duality on ϕ_1 (isometry direction), then $\phi_2\to\phi_2+\gamma\phi_1$, then T duality back on ϕ_1. In the shift step, this changes the metric as

$$\begin{aligned}ds^2&=\hat{g}_{11}d\phi_1^2+\hat{g}_{22}d\phi_2^2+2\hat{g}_{12}d\phi_1d\phi_2+2\hat{g}_{23}d\phi_2d\phi_3+2\hat{g}_{24}d\phi_2d\phi_4+\cdots\to\\ds^2&=\hat{g}_{11}d\phi_1^2+\hat{g}_{22}(d\phi_2+\gamma d\phi_1)^2+2\hat{g}_{12}d\phi_1(d\phi_2+\gamma d\phi_1)+2\hat{g}_{23}(d\phi_2+\gamma d\phi_1)d\phi_3+\cdots\Rightarrow\\\hat{G}_{11}&=\hat{g}_{11}+\gamma^2\hat{g}_{22}+2\gamma\hat{g}_{12},\\\hat{G}_{1i}&=\hat{g}_{1i}+\gamma\hat{g}_{2i},\quad\forall\ \ i\neq 1,\end{aligned} \tag{2.4}$$

and similar relations for other fields (if there are any). On the other hand, for T duality, the Buscher rules are

$$\begin{aligned} \tilde{G}_{11} &= \frac{1}{G_{11}}, \qquad \tilde{G}_{1i} = \frac{B_{1i}}{G_{11}}, \\ \tilde{G}_{ij} &= G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}, \\ \tilde{B}_{1i} &= \frac{G_{1i}}{G_{11}}, \\ \tilde{B}_{ij} &= B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}, \\ \tilde{\Phi} &= \Phi - \frac{1}{2}\log G_{11}. \end{aligned} \tag{2.5}$$

Then, indeed, after a T duality in time t, we get (dropping the R factors)

$$\begin{aligned} ds^2 &= -e^{-2\rho}dt^2 + 2dtdx + d\rho^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &\quad + ds^2(T^4), \\ \Phi &= \Phi_0 - \frac{1}{2}\log e^{2\rho}, \\ B_{01} &= 0. \end{aligned} \tag{2.6}$$

After the shift $x \to x + \gamma t$, we find

$$\begin{aligned} ds^2 &= dt^2(-e^{-2\rho} + 2\gamma) + 2dtdx + d\rho^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &\quad + ds^2(T^4), \\ \Phi &= \Phi_0 - \frac{1}{2}\log e^{2\rho}, \\ B_{01} &= 0. \end{aligned} \tag{2.7}$$

And after the T duality back on t, we find

$$\begin{aligned} ds^2 &= \frac{e^{2\rho}(-dt^2 + dx^2)}{1 - 2\gamma e^{2\rho}} + d\rho^2 + \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + ds^2(T^4), \\ \Phi &= \Phi_0 - \frac{1}{2}\log e^{2\rho} - \frac{1}{2}\log(e^{-2\rho} - 2\gamma) \\ &\Rightarrow e^{2\Phi} = e^{2\Phi_0}\frac{e^{-2\rho}}{e^{-2\rho} - 2\gamma}, \\ B_{01} &= -\frac{e^{2\rho}}{1 - 2\gamma e^{2\rho}} \Rightarrow H_{01\rho} = -\frac{2e^{2\rho}}{(1 - 2\gamma e^{2\rho})^2}. \end{aligned} \tag{2.8}$$

The TsT background above is an example of a Yang-Baxter transformation. It corresponds to the classical r matrix

$$r = \frac{1}{2}P_0 \wedge P_1, \tag{2.9}$$

where P_0 and P_1 are generators of the Poincaré group. In general

$$r = A \wedge B, \tag{2.10}$$

where we say that the transformation is said to be *Abelian* if $[A, B] = 0$. See [20] to see how we can build the deformed backgrounds.

For a background of the form $\mathrm{AdS}_d \times M_{10-d}$, TsT deformations of the inner space M_{10-d} give marginal deformations of the CFT_{d-1}, while TsT deformations of the AdS (anti–de Sitter) space give massive theories.

In the $\mathrm{AdS}_3 \times \mathbb{S}^3 \times T^4$ case, the Yang-Baxter deformation $r = \frac{1}{2}P_0 \wedge P_1$ gives the dual of the $T\bar{T}$ deformation.

III. TST TRANSFORMATIONS AND $T\bar{T}$ DEFORMATIONS: FOUR-DIMENSIONAL CASE

In dimensions higher than two, the definition of $T\bar{T}$ deformations is not unique: there are several proposals.

The first paper to deal with this, [4], notes that the two-dimensional deformation

$$\partial_t S = \frac{1}{2}\int d^2x\sqrt{g}[(\epsilon^{\mu\nu}\epsilon^{\rho\sigma})T_{\mu\rho}T_{\nu\sigma}] = \int d^2x\sqrt{g}\det T_{\mu\nu} \tag{3.1}$$

can be generalized by noting that in two dimensions $\epsilon^{\mu\nu}\epsilon^{\rho\sigma} = g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}$, and then the above deformation can be trivially generalized to any dimension to the quadratic form

$$\partial_t S = \frac{1}{2}\int d^2x\sqrt{g}[(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma})T_{\mu\rho}T_{\nu\sigma}], \tag{3.2}$$

which now however is not a determinant anymore. They, however, do not explore this further, other than to say that, for a single scalar field (with $X = (\partial_\mu\phi)^2$), one gets

$$\partial_t\mathcal{L} = (D-1)\left[\frac{D}{2}\mathcal{L}^2 - 2X\partial_X\mathcal{L}\mathcal{L}\right]. \tag{3.3}$$

Instead, they consider the generalization with a determinant to an arbitrary power α, $(-\det T)^{\frac{1}{\alpha}}$,

$$\partial_t S = \frac{1}{\alpha - D}\int\sqrt{g}\left[-\frac{1}{D!}\epsilon^{\mu_1\ldots\mu_D}\epsilon^{\nu_1\ldots\nu_D}T_{\mu_1\nu_1}\ldots T_{\mu_D\nu_D}\right]^{\frac{1}{\alpha}}, \tag{3.4}$$

and find that they can write a closed form expression of the deformed Lagrangian of a free scalar field for any D for $\alpha = 1$, $\{\frac{1}{2t}[\sqrt{1 + 4t(X/2)^{D-1}} - 1]\}^{\frac{1}{D-1}}$, but otherwise the deformation can be considered for any α and any potential V.

The subsequent paper of Marika Taylor [10] considers, instead of the two possibilities above, another one, a quadratic form similar to the above, but where the trace part has the coefficient $1/(D-1)$,

$$\partial_t S = \int d^D x \sqrt{g}\left[T^{\mu\nu}T_{\mu\nu} - \frac{1}{D-1}T^{\mu}{}_{\mu}T^{\nu}{}_{\nu}\right], \qquad (3.5)$$

and argues that it is the correct generalization in view of the holographic proposal of McGough *et al.* [12].[2]

But we can consider that the same arguments used by [15] to find that the single-trace deformation on the world sheet, giving before a CFT_2 $T\bar{T}$ deformation, is equal to a TsT transformation in the boundary directions, can be used to argue that the same is true for $\mathrm{AdS}_5\times S^5$: the $T\bar{T}$ deformation is defined by the TsT transformation.

Furthermore, in the $\mathrm{AdS}_5\times \mathbb{S}^5$ case, the TsT transformation is (conjectured to be) dual to noncommutative deformations of the $\mathcal{N}=4$ SYM theory [21–25]. In order to obtain a metric that is Lorentz invariant (though, of course, the B field is not), one has to consider noncommutativity in the (01) and (23) directions, corresponding to two successive TsT transformations, one in the (01) directions, followed by another in the (23) directions. As in [21,22], the construction must be done in Euclidean signature, so that the results in the (01) and (23) directions are the same, and that there are no apparent singularities (in Minkowski signature, the (01) TsT transformation results in a factor of $1-\gamma^2 e^{4\rho}$ instead of $1+\gamma^2 e^{4\rho}$, even though TsT is a symmetry of the string theory).

For a single TsT transformation, in the (01) Euclidean directions, we start with $B_{ij}=0$, $\Phi=\Phi_0$ and

$$ds^2 = e^{2\rho}(dt^2 + d\vec{x}^2) + d\rho^2 + ds^2_{S^5}. \qquad (3.6)$$

Then after T duality on t, we still find $B=0$, and

$$ds^2 = e^{-2\rho}dt^2 + e^{2\rho}d\vec{x}^2 + d\rho^2 + ds^2_{S^5},$$
$$\Phi = \Phi_0 - \frac{1}{2}\log e^{2\rho}. \qquad (3.7)$$

After the shift $x\to x+\gamma t$, where $x=x_1$, we still have $B=0$, and

$$ds^2 = dt^2(e^{-2\rho}+\gamma^2 e^{2\rho}) + 2\gamma e^{2\rho}dt dx + e^{2\rho}d\vec{x}^2 + d\rho^2 + ds^2_{S^5},$$
$$\Phi = \Phi_0 - \frac{1}{2}\log e^{2\rho}. \qquad (3.8)$$

Finally, after the T-duality back on t, we have

$$ds^2 = \frac{e^{2\rho}(dt^2+dx^2)}{1+\gamma^2 e^{4\rho}} + e^{2\rho}(dx_2^2+dx_3^2) + d\rho^2 + ds^2_{S^5},$$
$$B_{01} = \frac{\gamma e^{2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}} \Rightarrow H_{01\rho} = \partial_\rho B_{01} = -\frac{4\gamma}{(e^{-2\rho}+\gamma^2 e^{2\rho})^2},$$
$$\Phi = \Phi_0 - \frac{1}{2}\log e^{2\rho} - \frac{1}{2}\log(e^{-2\rho}+\gamma^2 e^{2\rho}) \Rightarrow e^{2\Phi} = e^{2\Phi_0}\frac{e^{-2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}}. \qquad (3.9)$$

But rather, for a TsT transformation in the directions (01) and one in the directions (23), but with the same parameter, we obtain, like in [21,22], a Lorentz invariant metric, though with a non-Lorentz invariant B field,

$$ds^2 = \frac{e^{2\rho}(-dt^2+d\vec{x}^2)}{1+\gamma^2 e^{4\rho}} + d\rho^2 + ds^2_{S^5},$$
$$B_{01} = B_{23} = \frac{\gamma e^{2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}} \Rightarrow H_{23\rho} = H_{01\rho} = \partial_\rho B_{01} = -\frac{4\gamma}{(e^{-2\rho}+\gamma^2 e^{2\rho})^2},$$
$$\Phi = \Phi_0 - \log e^{2\rho} - \log(e^{-2\rho}+\gamma^2 e^{2\rho}) \Rightarrow e^{2\Phi} = e^{2\Phi_0}\left(\frac{e^{-2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}}\right)^2. \qquad (3.10)$$

This is consistent with the results in [21,22]. We can now Wick rotate back to Minkowski signature.

IV. PENROSE LIMIT OF SINGLE-TRACE $T\bar{T}$ DEFORMATIONS OF $\mathrm{AdS}_3\times S^3\times T^4$ AND $\mathrm{AdS}_5\times S^5/\mathcal{N}=4$ SYM CORRESPONDENCE

In this section, we want to understand the proposed single-trace $T\bar{T}$ deformation of $\mathrm{AdS}_5\times S^5$ from the point of view of $\mathcal{N}=4$ SYM. When a gauge/gravity duality is difficult to understand, it helps to consider the Penrose

[2]Note that in [11] a deformation for which the trace part has coefficient $2/D$ was considered as the correct one for deforming a Maxwell theory of Abelian p forms to a Born-Infeld-type one, including cases with S duality, similar to what happens in $D=2$ and $D=4$.

limit, which will simplify both sides of the duality. Moreover, it is interesting to understand what, if any, is the deformation of the spin chain in the BMN limit?

A. Review of Penrose limit for $\mathrm{AdS}_5 \times S^5$ in Poincaré coordinates

Since the metric (3.10) is written in Poincaré coordinates, it is worth first reviewing the method for dealing with Penrose limits in Poincaré coordinates, since it is somewhat unusual. Note that we could consider transforming the metric in the equivalent of AdS global coordinates, since then the map to the CFT side is better understood. It is not clear if that is equivalent to what we do here.

Unlike the Penrose limit in global coordinates, where the limit is easy to do, and the null geodesic sits in the center of AdS, at $\rho = 0$, the Penrose limit in AdS in Poincaré coordinates involves motion in an extra coordinate. The method was developed in [26] and later, in more specific detail, in [27], where it was done for the (near horizon limit of) Dp-branes in Poincaré coordinates, that includes the $\mathrm{AdS}_5 \times S^5$ case. It was further used in similar cases in [28]. In both cases, the essential point is that the null geodesic involves, besides t, motion in *two* spatial coordinates, one being the isometry direction and the other the radial direction.

That is different from the case of motion along a single isometry direction, discussed for instance in [29]. In this latter case, for motion in x^λ, $\frac{dx^i}{d\lambda} = \delta^i_\lambda$, so we need no acceleration in that direction, leading to the conditions (imposed on the geodesic)

$$\Gamma^i_{\lambda\lambda} = 0. \tag{4.1}$$

For static and diagonal metrics, $\partial_t g_{ij} = 0$ and $g_{0i} = g^{0i} = 0$, the geodesic equation for $i = t$ is always satisfied, and for motion in an *isometry* direction, $\partial_\lambda g_{\mu\nu} = 0$, we obtain the simple condition

$$g^{il}\partial^l g_{\lambda\lambda} = \partial^i g_{\lambda\lambda} = 0, \quad \forall \ i. \tag{4.2}$$

Anyway, back to our more general case, for motion in two directions, following [27], for the case $p = 3$, i.e., for D3-branes giving $\mathrm{AdS}_5 \times S^5$ in Poincaré coordinates, with metric

$$ds^2 = R^2 \frac{(-dt^2 + d\vec{x}_3^2 + dz^2)}{z^2} + R^2 d\Omega_5^2, \tag{4.3}$$

where we write the metric in S^5 by separating in an angle, plus an S^4,

$$d\Omega_5^2 = d\psi^2 + \sin^2\psi \, d\Omega_4^2. \tag{4.4}$$

We consider the null geodesic for motion in (t, z, ψ). It is not completely obvious that this will give the same as the motion at fixed $\rho = 0$ in global coordinates (in the center of AdS), but we find this is true after the fact, since we get the same pp wave.

The method involves the following: (1) writing an effective Lagrangian for motion in (t, z, ψ). From it, we find the geodesic in terms of a null affine parameter λ that will be x^+ in the end. (2) Then, in terms of that motion on the geodesic, we change coordinates from (t, z, ψ) to (λ, β, ϕ) defined such that we end up with the metric in the form stated by Penrose as always possible, and leading to the pp wave in Rosen coordinates. (3) We take the limit to get the pp wave in Rosen coordinates, and then (4) apply the usual transformation to get the Brinkmann coordinates.

Applying it for the metric (4.3), we get (1) the effective Lagrangian for (t, z, ψ) motion (taking out the irrelevant R^2 factor)

$$L = -z^{-2}\dot{t}^2 + z^{-2}\dot{z}^2 + \dot{\psi}^2, \tag{4.5}$$

which must be taken together with the constraint $L = 0$, for a null geodesic ($ds^2 = 0$). Since L is independent of t, ψ (only on the derivatives), the corresponding equations of motion are integrated to integrals of motion,

$$\frac{\partial L}{\partial \dot{t}} = -2E = \text{constant}, \qquad \frac{\partial L}{\partial \dot{\psi}} = 2\mu = \text{constant}, \tag{4.6}$$

which together with the constraint $L = 0$ give three differential equations for $t(\lambda)$, $z(\lambda)$, $\psi(\lambda)$,

$$\begin{aligned} &\dot{t} = z^2 E, \qquad \dot{\psi} = \mu, \\ &\dot{z}^2 = \dot{t}^2 - z^2\dot{\psi}^2 \Rightarrow \dot{z} = \pm z\sqrt{z^2E^2 - \mu^2}. \end{aligned} \tag{4.7}$$

Then we get

$$\begin{aligned} \mu\lambda &= \int^{Ez/\mu} \frac{dy}{y\sqrt{y^2 - 1}}, \\ \frac{d\psi}{dz} &= \pm \frac{\mu}{z\sqrt{z^2E^2 - \mu^2}}, \\ \frac{dt}{dz} &= E\frac{z}{\sqrt{z^2E^2 - \mu^2}}, \end{aligned} \tag{4.8}$$

integrated to

$$\begin{aligned} z &= z_\lambda, \\ t &= t_\lambda - \frac{\beta}{E} + \mu\phi, \\ \psi &= \psi_\lambda + E\phi, \end{aligned} \tag{4.9}$$

where t_λ and z_λ refer to $\int \frac{dt}{d\lambda} d\lambda$ and $\int \frac{dz}{d\lambda} d\lambda$, respectively, and the coefficients of the β, ϕ extra terms (independent of λ, so "constants" from the point of view of the previous integration) are chosen such that this change of coordinates gives $g_{\lambda\beta}=1$, $g_{\lambda\phi}=0$, as we will shortly see.

Indeed, by differentiation we obtain

$$\begin{aligned} dt &= \frac{dt}{d\lambda}d\lambda - \frac{d\beta}{E} + \mu d\phi = z^2 E d\lambda - \frac{d\beta}{E} + \mu d\phi, \\ d\psi &= \frac{d\psi}{d\lambda}d\lambda + E d\phi = \mu d\lambda + E d\phi, \\ dz &= \frac{dz}{d\lambda}d\lambda = z\sqrt{z^2E^2-\mu^2}\,d\lambda, \end{aligned} \tag{4.10}$$

and (2) by substituting this in the metric (4.3), we obtain

$$\begin{aligned} R^{-2}ds^2 &= 2d\lambda d\beta - \frac{d\beta^2}{E^2z^2} + \frac{2\mu}{E}\frac{d\beta d\phi}{z^2} + d\phi^2\left(E^2-\frac{\mu^2}{z^2}\right) \\ &\quad + \frac{d\vec{x}^2}{z^2} + \sin^2(\psi_\lambda + E\phi)d\Omega_4^2, \end{aligned} \tag{4.11}$$

which is exactly in the form stated by the Penrose theorem $(2dVdU + \alpha dV^2 + \sum_i \beta_i dV dY^i + \sum_{i,j} C_{i,j} dY^i dY^j)$.

We can therefore (3) take the Penrose limit by rescaling with R and taking $R\to\infty$,

$$U = \lambda = u, \qquad V = \beta = \frac{v}{R^2}, \qquad Y^i = \frac{y^i}{R}, \tag{4.12}$$

where Y^i stands for $\Omega_4, \phi, \vec{x}_3$. Specifically then

$$\vec{\Omega}_4 = \frac{\vec{y}_4}{R}, \qquad \phi = \frac{\varphi}{R}, \qquad \vec{x}_3 = \frac{\vec{x}'_3}{R}. \tag{4.13}$$

The resulting pp wave in Rosen coordinates is

$$ds^2 = 2dudv + d\varphi^2\left(E^2 - \frac{\mu^2}{z^2}\right) + \frac{d\vec{x}'^2_3}{z^2} + \sin^2\psi_\lambda d\vec{y}_4^2, \tag{4.14}$$

and where $z = z(\lambda = u), \psi_\lambda = \psi_\lambda(\lambda = u)$.

For the (4) transformation to Brinkmann coordinates, we have in general

$$\begin{aligned} &g_{ij} \equiv e_i^a e_j^b \delta_{ab} \Rightarrow \\ &u = x^+, \\ &v = x^- + \frac{1}{2}\dot{e}_{ai} e^i_b x^a x^b, \\ &y^i = e^i_a x^a, \end{aligned} \tag{4.15}$$

leading to the Brinkmann form pp wave,

$$\begin{aligned} &ds^2 = 2dx^+dx^- + H(x^+, x^i)(dx^+)^2 + d\vec{x}^2 \\ &H = A_{ab}x^a x^b \\ &A_{ab} = \ddot{e}_{ai} e^i_b. \end{aligned} \tag{4.16}$$

In our case, we obtain the vielbeins

$$e^\varphi_\varphi = \sqrt{E^2 - \frac{\mu^2}{z^2}}, \qquad e^{x'}_{x'} = \frac{1}{z}, \qquad e^y_y = \sin\psi_\lambda, \tag{4.17}$$

so the new coordinates are

$$\begin{aligned} &\lambda = u = x^+, \qquad \tilde{\varphi} = \varphi\sqrt{E^2 - \frac{\mu^2}{z^2}}, \qquad \tilde{x} = \frac{x}{z}, \\ &\tilde{y} = y\sin\psi_\lambda, \qquad z = z(x^+), \end{aligned} \tag{4.18}$$

and x^- is not needed.

Then

$$\begin{aligned} A_{\tilde{y}\tilde{y}} &= \frac{1}{\sin\psi_\lambda}\frac{d^2}{d\lambda^2}\sin\psi_\lambda = \frac{1}{\sin\psi_\lambda}\frac{d}{d\lambda}\left(\cos\psi_\lambda \frac{d\psi}{d\lambda}\right) \\ &= \tan^{-1}\psi_\lambda \frac{d^2\psi_\lambda}{d\lambda} - \left(\frac{d\psi_\lambda}{d\lambda}\right)^2 = 0 - \mu^2, \end{aligned} \tag{4.19}$$

and given that

$$\frac{d}{d\lambda}\sqrt{E^2 - \frac{\mu^2}{z^2}} = \frac{\mu^2}{z}, \tag{4.20}$$

we obtain also

$$\begin{aligned} A_{\tilde{x}\tilde{x}} &= z\frac{d^2}{dz^2}\frac{1}{z} = -z\frac{d}{d\lambda}\left(\frac{1}{z^2}\frac{dz}{d\lambda}\right) = -z\frac{d}{d\lambda}\sqrt{E^2 - \frac{\mu^2}{z^2}} = -\mu^2, \\ A_{\tilde{\varphi}\tilde{\varphi}} &= \frac{1}{\sqrt{E^2 - \frac{\mu^2}{z^2}}}\frac{d^2}{d\lambda^2}\sqrt{E^2 - \frac{\mu^2}{z^2}} = \frac{1}{\sqrt{E^2 - \frac{\mu^2}{z^2}}}\frac{d}{d\lambda}\frac{\mu^2}{z} \\ &= -\frac{\mu^2}{z\sqrt{E^2 - \frac{\mu^2}{z^2}}}\frac{dz}{d\lambda} = -\mu^2, \end{aligned} \tag{4.21}$$

so in the end we obtain the usual pp wave,

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \mu^2(\tilde{\vec{x}}_3\tilde{\vec{x}}_3 + \tilde{\varphi}\tilde{\varphi} + \tilde{\vec{y}}_4\tilde{\vec{y}}_4)(dx^+)^2 \\ &\quad + d\tilde{\vec{x}}_3^2 + d\tilde{\varphi}^2 + d\tilde{\vec{y}}_4^2. \end{aligned} \tag{4.22}$$

B. Penrose limit of single-trace $T\bar{T}$ deformation of $AdS_3 \times S^3 \times T^4$

As a warm-up exercise, we first do the Penrose limit on the single-trace $T\bar{T}$ deformation of $AdS_3\times S^3\times T^4$, where we know for sure that the deformation is given by the TsT transformation.

The metric is (reintroducing the factor of R^2 which was needed for the pp wave limit, as it is taken to infinity, now corresponding to $R^2_{\mathrm{AdS}} = kl_s^2$, and writing the metric on S^3 as above, in terms of a ψ and an S^2)

$$R^{-2}ds^2 = \frac{e^{2\rho}(-dt^2+dx^2)}{1+2\gamma e^{2\rho}} + d\rho^2 + \frac{1}{4}(d\psi^2 + \sin^2\psi d\Omega_2^2) + ds^2(T^4). \tag{4.23}$$

Note that, strictly speaking, $R = R_{\mathrm{AdS}} \to \infty$ can be obtained only for $k\to\infty$, in which case we cannot observe any potential phase transition in k.

Then (1) the effective Lagrangian for motion in (t,ρ,ψ) is (ignoring the overall R^2)

$$L = -\frac{e^{2\rho}}{1+2\gamma e^{2\rho}}\dot{t}^2 + \dot{\rho}^2 + \frac{1}{4}\dot{\psi}^2. \tag{4.24}$$

It is independent (explicitly) on t and ψ, so the Lagrange equations of motion are integrated with integrals of motion,

$$\frac{\partial L}{\partial \dot{t}} = -2\frac{\dot{t}e^{2\rho}}{1+2\gamma e^{2\rho}} = -2E = \text{constant},$$
$$\frac{\partial L}{\partial \dot{\psi}} = \frac{1}{2}\dot{\psi} = 2\mu = \text{constant}, \tag{4.25}$$

giving

$$\dot{t} = e^{-2\rho}(1+2\gamma e^{2\rho})E, \qquad \dot{\psi} = 4\mu. \tag{4.26}$$

The constraint $L=0$ (from $ds^2=0$, null geodesic) gives

$$\dot{\rho}^2 = \frac{e^{2\rho}}{1+2\gamma e^{2\rho}}\dot{t}^2 - \frac{1}{4}\dot{\psi}^2 \Rightarrow \dot{\rho} = \pm e^{-\rho}\sqrt{E^2(1+2\gamma e^{2\rho}) - 4\mu^2 e^{2\rho}}, \tag{4.27}$$

so that we obtain

$$\lambda = \pm\int \frac{e^\rho d\rho}{\sqrt{E^2(1+2\gamma e^{2\rho}) - 4\mu^2 e^{2\rho}}} = \int\frac{d(e^\rho)}{\sqrt{E^2 - e^{2\rho}(4\mu^2 - 2\gamma)}}, = \frac{1}{4\mu^2 - 2\gamma}\arcsin\left(\frac{e^\rho\sqrt{4\mu^2-2\gamma}}{E}\right), \quad \text{if } 4\mu^2 - 2\gamma > 0,$$
$$\frac{d\psi}{d\rho} = \pm 4\mu\frac{e^\rho}{\sqrt{E^2(1+2\gamma e^{2\rho}) - 4\mu^2 e^{2\rho}}},$$
$$\frac{dt}{d\rho} = E\frac{e^{-\rho}(1+2\gamma e^{2\rho})}{\sqrt{E^2(1+2\gamma e^{2\rho}) - 4\mu^2 e^{2\rho}}}. \tag{4.28}$$

Then the coordinate transformation (t,ρ,ψ) to (λ,β,ϕ) that puts $g_{\lambda\beta}=1, g_{\lambda\phi}=0$ is (we fix the coefficients of β,ϕ in this way)

$$d\rho = \frac{d\rho}{d\lambda}d\lambda = e^{-\rho}\sqrt{E^2(1+2\gamma e^{2\rho}) - 4\mu^2 e^{2\rho}}d\lambda,$$
$$d\psi = \frac{d\psi}{d\lambda}d\lambda + Ed\phi = 4\mu d\lambda + Ed\phi,$$
$$dt = \frac{dt}{d\lambda}d\lambda - \frac{d\beta}{E} + \mu d\phi = e^{-2\rho}E(1+2\gamma e^{2\rho})d\lambda - \frac{d\beta}{E} + \mu d\phi. \tag{4.29}$$

Indeed, (2) substituting the above transformation in the metric (4.23) we obtain

$$R^{-2}ds^2 = 2d\lambda d\beta - \frac{d\beta^2}{E^2}\frac{e^{2\rho}}{1+2\gamma e^{2\rho}} + \frac{2\mu}{E}d\beta d\phi\frac{e^{2\rho}}{1+2\gamma e^{2\rho}} + d\phi^2\left(\frac{E^2}{4} - \frac{\mu^2 e^{2\rho}}{1+2\gamma e^{2\rho}}\right) + \frac{dx^2 e^{2\rho}}{1+2\gamma e^{2\rho}} + \sin^2(\psi_\lambda + E\phi)d\Omega_2^2 + ds^2(T^4). \tag{4.30}$$

Taking the (3) Penrose limit as usual by $R\to\infty$, with

$$U = \lambda = u, \qquad V = \beta = \frac{v}{R^2}, \qquad Y^i = \frac{y^i}{R}, \tag{4.31}$$

where Y^i stands in for ϕ, Ω_2, x, T^4, so

$$\phi = \frac{\varphi}{R}, \qquad \vec{\Omega}_2 = \frac{\vec{y}_2}{R}, \qquad x = \frac{x'}{R}, \qquad T_4 = \frac{T'_4}{R}, \tag{4.32}$$

and where $\rho = \rho(\lambda = u), \psi_\lambda = \psi_\lambda(\lambda = u)$, we obtain the pp wave metric in Rosen coordinates,

$$ds^2 = 2dudv + d\varphi^2\left[\frac{E^2}{4} - \frac{\mu^2 e^{2\rho(u)}}{1+2\gamma e^{2\rho(u)}}\right] + \frac{(dx')^2 e^{2\rho(u)}}{1+2\gamma e^{2\rho(u)}} + \sin^2\psi_\lambda(u)d\vec{y}_2^2 + ds^2(T'_4). \tag{4.33}$$

The dilaton is unchanged through the coordinate changes, and is

$$e^{2\Phi} = \frac{e^{2\Phi_0}}{1-2\gamma e^{2\rho(u)}}. \tag{4.34}$$

The B field with the correct power of R in front is

$$B_{01}dt\wedge dx = -R\frac{e^{2\rho(u)}}{1+2\gamma e^{2\rho(u)}}dt\wedge dx, \\ \to R\frac{e^{2\rho(u)}}{1+2\gamma e^{2\rho(u)}}e^{-2\rho(u)}(1+2\gamma e^{2\rho(u)})Edu\wedge dx, \\ = -Edu\wedge dx'. \tag{4.35}$$

On the other hand, on the S^3, we have

$$H_{S^3} = R^2\frac{1}{4}\sigma_1\wedge\sigma_2\wedge\sigma_3 \Rightarrow B_{S^3}\sim R^2\psi d\Omega_1\wedge d\Omega_2, \tag{4.36}$$

so in the Penrose limit, we get ($\psi = 4\mu x^+$ in the Penrose limit)

$$B_{S^3}\sim 4\mu x^+\sin^2 4\mu x^+ dy_1\wedge dy_2. \tag{4.37}$$

To go to (4) the Brinkmann coordinates, we first define the vielbeins,

$$e^{\varphi}_{\varphi} = \sqrt{\frac{E^2}{4}-\mu^2\frac{e^{2\rho}}{1+2\gamma e^{2\rho}}}, \qquad e^{x'}_{x'} = \frac{e^{\rho}}{\sqrt{1+2\gamma e^{2\rho}}}, \\ e^y_y = \sin\psi_\lambda(u), \tag{4.38}$$

so the coordinate change is

$$\lambda = u = x^+, \\ \tilde{\varphi} = \varphi\sqrt{\frac{E^2}{4}-\mu^2\frac{e^{2\rho}}{1+2\gamma e^{2\rho}}}, \\ \tilde{x} = x'\frac{e^{\rho}}{\sqrt{1+2\gamma e^{2\rho}}}, \\ \tilde{y} = y\sin\psi_\lambda(u), \\ z = z(\lambda = u = x^+), \tag{4.39}$$

and x^- does not need to be written.

Then, for the Brinkmann metric, we find

$$A_{\tilde{y}\tilde{y}} = \frac{1}{\sin\psi_\lambda}\frac{d^2}{d\lambda^2}\sin\psi_\lambda = \tan^{-1}\psi_\lambda\frac{d^2\psi_\lambda}{d\lambda^2} - \left(\frac{d\psi_\lambda}{d\lambda}\right)^2 \\ = 0 - 16\mu^2. \tag{4.40}$$

Moreover, using that

$$\frac{d}{d\lambda}\left(\frac{e^{\rho}}{\sqrt{1+2\gamma e^{2\rho}}}\right) = \frac{\sqrt{E^2(1+2\gamma e^{2\rho})-4\mu^2e^{2\rho}}}{(1+2\gamma e^{2\rho})^{3/2}}, \tag{4.41}$$

we find

$$A_{\tilde{x}\tilde{x}} = e^{-\rho}\sqrt{1+2\gamma e^{2\rho}}\frac{d^2}{d\lambda^2}\left(\frac{e^{\rho}}{\sqrt{1+2\gamma e^{2\rho}}}\right), \\ = e^{-\rho}\sqrt{1+2\gamma e^{2\rho}}\frac{d}{d\lambda}\left[\frac{\sqrt{E^2(1+2\gamma e^{2\rho})-4\mu^2e^{2\rho}}}{(1+2\gamma e^{2\rho})^{3/2}}\right], \\ = -\frac{4}{(1+2\gamma e^{2\rho})^2}[(1+2\gamma e^{2\rho})(\gamma E^2+\mu^2)-6\gamma\mu^2e^{2\rho}], \tag{4.42}$$

and also

$$A_{\tilde{\varphi}\tilde{\varphi}} = \frac{1}{\sqrt{\frac{E^2}{4}-\mu^2\frac{e^{2\rho}}{1+2\gamma e^{2\rho}}}}\frac{d^2}{d\lambda^2}\sqrt{\frac{E^2}{4}-\mu^2\frac{e^{2\rho}}{1+2\gamma e^{2\rho}}}, \\ = \frac{1}{\sqrt{\frac{E^2}{4}-\mu^2\frac{e^{2\rho}}{1+2\gamma e^{2\rho}}}}\frac{d}{d\lambda}\left[-4\mu^2\frac{e^{\rho}}{(1+2\gamma e^{2\rho})^{3/2}}\right], \\ = -8\mu^2\frac{1-4\gamma e^{2\rho}}{(1+2\gamma e^{2\rho})^2}. \tag{4.43}$$

The pp wave metric is then

$$ds^2 = 2dx^+dx^- + H(x^+)(dx^+)^2 + d\tilde{\varphi}^2 + d\tilde{x}^2 \\ + d\tilde{\vec{y}}_2^2 + ds^2(T^4), \tag{4.44}$$

and

$$H(x^+) = A_{\tilde{\varphi}\tilde{\varphi}}\tilde{\varphi}^2 + A_{\tilde{x}\tilde{x}}\tilde{x}_2^2 + A_{\tilde{y}\tilde{y}}\tilde{\vec{y}}_2^2. \tag{4.45}$$

Also, we have found $\lambda(\rho)$, and we saw that $\lambda = x^+$, so inverting it, we get

$$e^{\rho(x^+)} = \frac{E}{\sqrt{4\mu^2-2\gamma}}\sin\left(x^+\sqrt{4\mu^2-2\gamma}\right), \quad \text{if } 4\mu^2-2\gamma>0. \tag{4.46}$$

The string action is written, as usual, in the gauge $x^+ = \tau$, and the conformal (unit) gauge $\sqrt{h}h^{ab} = \eta^{ab}$, and with $\epsilon^{01} = +1$, we get

$$\begin{aligned} S_{\text{string}} &= -\frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+} d\sigma \int d\tau[\eta^{ab}G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \epsilon^{ab}B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + 2\pi\alpha'\Phi\mathcal{R}^{(2)}], \\ &= -\frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+} d\sigma \int d\tau\Bigg[\eta^{ab}\sum_{i\neq\pm}\partial_a X^i \partial_b X^i - 8\mu^2\tilde{\varphi}^2\frac{1-4\gamma e^{2\rho(\tau)}}{1+2\gamma e^{2\rho(\tau)}} - 16\mu^2\tilde{\vec{y}}_2^2 \\ &\quad -4\tilde{x}^2\frac{(1+2\gamma e^{2\rho(\tau)})(\mu^2+\gamma E^2) - 6\gamma\mu^2 e^{2\rho}}{(1+2\gamma e^{2\rho(\tau)})^2} - E\partial_1 x' + 4\mu x^+ \sin^2(4\mu x^+)(\partial_0 y_1 \partial_1 y_2 - \partial_1 y_1 \partial_0 y_2)\Bigg]. \end{aligned} \tag{4.47}$$

Note that $x' = xe^{-\rho}\sqrt{1+2\gamma e^{2\rho}}$ and $y = \tilde{y}/\sin(4\mu x^+) = \tilde{y}/\sin(4\mu\tau)$ in the above (for the B field).

C. Penrose limit of proposed single-trace $T\bar{T}$ deformation of $AdS_5 \times S^5$

We move on to the most interesting case, of the proposed single-trace $T\bar{T}$ deformation of $AdS_5 \times S^5$, and we repeat the procedure.

For the solution

$$\begin{aligned} R^{-2}ds^2 &= \frac{e^{2\rho}(-dt^2 + d\vec{x}_3^2)}{1+\gamma^2 e^{4\rho}} + d\rho^2 + d\psi^2 + \sin^2\psi d\Omega_4^2, \\ B_{01} &= B_{23} = \frac{\gamma e^{2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}}, \\ e^{2\Phi} &= e^{2\Phi_0}\left(\frac{e^{-2\rho}}{e^{-2\rho}+\gamma^2 e^{2\rho}}\right)^2, \end{aligned} \tag{4.48}$$

we consider motion in (t,ρ,ψ), with the effective Lagrangian

$$R^{-2}L = -\frac{e^{2\rho}\dot{t}^2}{1+\gamma^2 e^{4\rho}} + \dot{\rho}^2 + \dot{\psi}^2, \tag{4.49}$$

and equations of motion

$$\begin{aligned} \frac{\partial L}{\partial \dot{t}} &= -\frac{2e^{2\rho}\dot{t}}{1+\gamma^2 e^{4\rho}} = -2E, \\ \frac{\partial L}{\partial \dot{\psi}} &= 2\dot{\psi} = 2\mu = \text{const.}, \end{aligned} \tag{4.50}$$

so

$$\begin{aligned} \dot{t} &= e^{-2\rho}E(1+\gamma^2 e^{4\rho}), \\ \dot{\psi} &= \mu, \end{aligned} \tag{4.51}$$

plus the condition of null motion, so $L = 0$, i.e.,

$$\dot{\rho}^2 = \frac{e^{2\rho}\dot{t}^2}{1+\gamma^2 e^{4\rho}} - \dot{\psi}^2 \Rightarrow \dot{\rho} = \pm e^{-\rho}\sqrt{E^2(1+\gamma^2 e^{4\rho}) - \mu^2 e^{2\rho}}, \tag{4.52}$$

integrated to

$$\begin{aligned} \lambda &= \pm\int \frac{e^\rho d\rho}{\sqrt{E^2(1+\gamma^2 e^{4\rho}) - \mu^2 e^{2\rho}}} \\ &= \frac{1}{E\gamma}\int \frac{d(e^\rho)}{\sqrt{\frac{1}{\gamma^2} - \frac{\mu^2}{4E^4\gamma^4} + \left(e^{2\rho} - \frac{\mu^2}{2E^2\gamma^2}\right)^2}}. \end{aligned} \tag{4.53}$$

We obtain

$$\begin{aligned} \frac{d\psi}{d\rho} &= \pm\mu\frac{e^\rho}{\sqrt{E^2(1+\gamma^2 e^{4\rho}) - \mu^2 e^{2\rho}}}, \\ \frac{dt}{d\rho} &= \pm E\frac{e^{-\rho}(1+\gamma^2 e^{4\rho})}{\sqrt{E^2(1+\gamma^2 e^{4\rho}) - \mu^2 e^{2\rho}}}, \end{aligned} \tag{4.54}$$

leading to the change of coordinates

$$\begin{aligned} d\rho &= \frac{d\rho}{d\lambda}d\lambda = d\lambda e^{-\rho}\sqrt{E^2(1+\gamma^2 e^{4\rho}) - \mu^2 e^{2\rho}}, \\ d\psi &= \frac{d\psi}{d\lambda}d\lambda + Ed\phi = \mu d\lambda + Ed\phi \Rightarrow \\ \psi &= \psi_\lambda + E\phi, \\ dt &= \frac{dt}{d\lambda}d\lambda - \frac{d\beta}{E} + \mu d\phi, \\ &= e^{-2\rho}E(1+\gamma^2 e^{4\rho})d\lambda - \frac{d\beta}{E} + \mu d\phi, \end{aligned} \tag{4.55}$$

which when substituted in the metric leads to (we obtain $g_{\lambda\phi} = g_{\lambda\lambda} = 0$, $g_{\lambda\beta} = 1$)

$$\begin{aligned} R^{-2}ds^2 &= 2d\lambda d\beta - \frac{d\beta^2}{E^2}\frac{e^{2\rho}}{1+\gamma^2 e^{4\rho}} + 2d\beta d\phi\frac{\mu}{E}\frac{e^{2\rho}}{1+\gamma^2 e^{4\rho}} \\ &\quad + d\phi^2\left(E^2 - \mu^2\frac{e^{2\rho}}{1+\gamma^2 e^{4\rho}}\right). \end{aligned} \tag{4.56}$$

Take the Penrose limit,

$$U = \lambda = u, \qquad V = \beta = \frac{v}{R^2}, \qquad Y^i = \frac{y^i}{R}, \tag{4.57}$$

so specifically

$$\phi = \frac{\varphi}{R}, \qquad \vec{\Omega}_4 = \frac{\vec{y}_4}{R}, \qquad \vec{x}_3 = \frac{\vec{x}'_3}{R}, \tag{4.58}$$

and $\rho=\rho(\lambda=u)$, $\psi_\lambda=\psi_\lambda(\lambda=u)$. We obtain the pp wave metric in Rosen coordinates,

$$ds^2_{\rm Rosen}=2dudv+d\varphi^2\left[E^2-\mu^2\frac{e^{2\rho(u)}}{1+\gamma^2e^{4\rho(u)}}\right]+\sin^2(\psi_\lambda(u))d\vec{y}_4^{\,2}+(d\vec{x}_3')^2\frac{e^{2\rho(u)}}{(1+\gamma^2e^{4\rho(u)})^2}. \quad (4.59)$$

For the B field, we find (since $dt\wedge dx_1+dx_2\wedge dx_2\to\frac{1}{R}Ee^{-2\rho(u)}(1-\gamma^2e^{4\rho(u)})du\wedge dx_1')$

$$B=R\frac{\gamma}{1+\gamma^2e^{4\rho}}(dt\wedge dx_1+dx_2\wedge dx_3),\\ =\gamma Ee^{2\rho(u)}du\wedge dx_1', \quad (4.60)$$

and the dilaton remains the same,

$$e^{2\Phi}=\frac{e^{2\Phi_0}}{(1+\gamma^2e^{4\rho(u)})^2}. \quad (4.61)$$

To go to the Brinkmann coordinates, we note that

$$e^\varphi_\varphi=\sqrt{E^2-\mu^2\frac{e^{2\rho}}{1+\gamma^2e^{4\rho}}},\qquad e^{x_j'}_{x_i'}=\frac{e^\rho}{\sqrt{1+\gamma^2e^{4\rho}}}\delta_i^j,\\ e^{u_j}_{y_i}=\sin\psi_\lambda(u)\delta_i^j, \quad (4.62)$$

so the coordinate transformation is

$$\lambda=u=x^+,\qquad \tilde{y}=u\sin\psi_\lambda(u),\\ \tilde{\varphi}=\varphi\sqrt{E^2-\mu^2\frac{e^{2\rho}}{1+\gamma^2e^{4\rho}}},\qquad z=z(\lambda=u=x^+),\\ \tilde{\vec{x}}=\vec{x}'\frac{e^\rho}{\sqrt{1+\gamma^2e^{4\rho}}}. \quad (4.63)$$

Then, we find

$$A_{\tilde{y}\tilde{y}}=\frac{1}{\sin\psi_\lambda}\frac{d^2}{d\lambda^2}\sin\psi_\lambda=\tan^{-1}\psi_\lambda\frac{d^2\psi_\lambda}{d\lambda^2}-\left(\frac{d\psi_\lambda}{d\lambda}\right)^2\\ =0-\mu^2. \quad (4.64)$$

Moreover, using that

$$\frac{d}{d\lambda}\left(\frac{e^\rho}{\sqrt{1+\gamma^2e^{4\rho}}}\right)=\sqrt{E^2(1+\gamma^2e^{4\rho})-\mu^2e^{2\rho}}\frac{1-\gamma^2e^{4\rho}}{(1+\gamma^2e^{4\rho})^{3/2}}, \quad (4.65)$$

we find

$$A_{\tilde{x}\tilde{x}}=e^{-\rho}\sqrt{1+\gamma^2e^{4\rho}}\frac{d^2}{d\lambda^2}\frac{e^\rho}{\sqrt{1+\gamma^2e^{4\rho}}},\\ =e^{-\rho}\sqrt{1+\gamma^2e^{4\rho}}\frac{d}{d\lambda}\left(\sqrt{E^2(1+\gamma^2e^{4\rho})-\mu^2e^{2\rho}}\times\frac{1-\gamma^2e^{4\rho}}{(1+\gamma^2e^{4\rho})^{3/2}}\right),\\ =-\frac{E^2(1+\gamma^2e^{4\rho})-\mu^2e^{2\rho}}{(1+\gamma^2e^{4\rho})^2}2\gamma^2e^{2\rho}(3-\gamma^2e^{4\rho})\\ -(-2E^2\gamma^2e^{2\rho}+\mu^2)\frac{1-\gamma^2e^{4\rho}}{1+\gamma^2e^{4\rho}}, \quad (4.66)$$

and also

$$A_{\varphi\varphi}=\frac{1}{\sqrt{E^2-\mu^2\frac{e^{2\rho}}{1+\gamma^2e^{4\rho}}}}\frac{d^2}{d\lambda^2}\sqrt{E^2-\mu^2\frac{e^{2\rho}}{1+\gamma^2e^{4\rho}}},\\ =\frac{1}{\sqrt{E^2-\mu^2\frac{e^{2\rho}}{1+\gamma^2e^{4\rho}}}}\frac{d}{d\lambda}\left[-2\mu^2\frac{e^\rho(1-\gamma^2e^{4\rho}}{(1+\gamma^2e^{4\rho})^{3/2}}\right],\\ =-2\mu^2\frac{1-8\gamma^2e^{4\rho}-\gamma^4e^{8\rho}}{(1+\gamma^2e^{4\rho})^2}. \quad (4.67)$$

Then the pp wave metric is

$$ds^2=2dx^+dx^-+\left[A_{\varphi\varphi}\varphi^2+A_{\tilde{x}\tilde{x}}\vec{\tilde{x}}_3^2+A_{\tilde{y}\tilde{y}}\vec{\tilde{y}}_4^2\right](dx^+)^2\\ +d\tilde{\varphi}^2+(d\vec{\tilde{x}}_3)^2+(d\vec{\tilde{y}}_4)^2, \quad (4.68)$$

and the string action, in the light-cone gauge with $x^+=\tau$, is

$$S_{\rm string}=-\frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+}d\sigma\int d\tau\left[\eta^{ab}G_{\mu\nu}\partial_aX^\mu\partial_bX^\nu+\epsilon^{ab}B_{\mu\nu}\partial_aX^\mu\partial_bX^\nu+2\pi\alpha'\Phi\mathcal{R}^{(2)}\right],\\ =-\frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+}d\sigma\int d\tau\left\{\eta^{ab}\sum_{i\neq\pm}\partial_aX^i\partial_bX^i-2\mu^2\tilde{\varphi}^2\frac{1-8\gamma^2e^{4\rho(\tau)}-\gamma^4e^{8\rho(\tau)}}{(1+\gamma^2e^{4\rho(\tau)})^2}\right.\\ \left.-\mu^2\vec{\tilde{y}}_4^2-\vec{\tilde{x}}_3^2\left[(-2E^2\gamma^2e^{2\rho(\tau)}+\mu^2)\frac{1-\gamma^2e^{4\rho(\tau)}}{1+\gamma^2e^{4\rho(\tau)}}+\frac{E^2(1+\gamma^2e^{4\rho(\tau)})-\mu^2e^{2\rho(\tau)}}{(1+\gamma^2e^{4\rho(\tau)})^2}2\gamma^2e^{2\rho(\tau)}(3-\gamma^2e^{4\rho(\tau)})\right]\right\}+\gamma e^{2\rho(\tau)}\partial_\sigma x_1', \quad (4.69)$$

and note that in the above, $x' = \tilde{x}e^{-\rho}\sqrt{1+\gamma^2 e^{4\rho}}$ (for the term coming from the B field).

As a simple check, note that the $\gamma \to 0$ limit of the above reduces to the string on the maximally supersymmetric pp wave, with parameter μ (the bosonic part of the action).

D. Interpretation in $\mathcal{N} = 4$ SYM of the deformation

1. Symmetries

To understand the $T\bar{T}$ deformation, and more specifically its Penrose limit, consider first the symmetries of the gravitational background, and match them against the symmetries in field theory.

In the case of the pp wave of $\mathrm{AdS}_5 \times S^5$, the initial symmetry of $PSU(2,2|4)$, with bosonic subgroup $SO(4,2)\times SO(6)$, gets changed in the Penrose limit to the pp wave algebra. In particular, $SO(4,2)$ breaks to $SO(4)_1 \times SO(2)_1$, where $SO(2)_1$ corresponds to x^+ translations, and $SO(4)_1$ rotates the $\vec{\tilde{x}}_3$ and φ, and $SO(6)$ breaks to another $SO(4)_2\times SO(2)_2$, where $SO(2)_2$ corresponds to x^- translations, and $SO(4)_2$ rotates the $\vec{\tilde{y}}_4$.

Of course, at least as far as the bosonic action goes, $SO(4)_1\times SO(4)_2$ is actually enhanced to $SO(8)$, but that is irrelevant, since the 5-form F_5 (the coupling to fermions) breaks $SO(8)$ back to $SO(4)_1 \times SO(4)_2$.

Now, in the $T\bar{T}$ deformed case, for the metric, $x^+ = \tau$ translation is not a symmetry anymore, so $SO(2)_1$ is gone, and $SO(4)_1$ is further broken to $SO(3)$, that rotates $\vec{x}_3$ only (no φ). However, the B field breaks further $SO(3)$ to $SO(2)'_1$, rotating only x_2, x_3. So, we have the symmetry $SO(2)'_1 \times SO(4)\times SO(2)_2$. But, *like in the undeformed case*, there must also be some translation-type generators e_i and e_i^*, or a_i and $a_i^\dagger$, that supplement the breaking of the rotational symmetry during the Penrose limit, giving a total number of generators equal to the one before the limit (since the number of generators cannot decrease in the Penrose limit).

Indeed, before the Penrose limit we have $SO(2)_1\times SO(2)_2\times SO(6)$, for x_0, x_1 and x_2, x_3 rotations, and for S^5 rotations, for a total of $1+1+15 = 17$ bosonic generators. But after the Penrose limit we find only $1+1+6=8$, so we miss 9 generators. Since the we have the same $-\mu^2 \vec{\tilde{y}}_4^2 (dx^+)^2$ term as in the undeformed metric, we have the same $4+4$ extra Killing spinors (see [30,31]; $\partial_i = \partial/\partial\tilde{y}^i$)

$$\xi_{e_i} = -\cos(\mu x^+)\partial_i - \mu\sin(\mu x^+)\tilde{y}^i\partial_-,$$
$$\xi_{e_i^*} = -\mu\sin(\mu x^+)\partial_i + \mu^2\cos(\mu x^+)\tilde{y}^i\partial_-. \tag{4.70}$$

Then we can write (as in [16]) $a_i \sim e_i + e_i^*$ and $M_{ij} = x^i\partial_j - x^j\partial_i = i(a_i^\dagger a_j - a_j^\dagger a_i)$, with $[a_i, a_j^\dagger] = \delta_{ij}$ (harmonic oscillators) and $e = -p_-$ commuting with everything. We also have $h = -p_+ = \mu\sum_i a_i^\dagger a_i$ and this gives the bosonic algebra. We are still missing one generator, but this is just $M_{23} = x_2\partial_3 - x_3\partial_2$.

Since the $SO(4)_2$ symmetry is maintained, and that corresponded to a R-symmetry rotation of the four positive J fermions, it seems to suggest that the supersymmetry is still $\mathcal{N} = 4$.

2. Quantization and eigenvalues vs. SYM anomalous dimensions

The equations of motion of the modes are

$$(-\partial_\tau^2 + \partial_\sigma^2 - \mu^2)y^i = 0, \quad i = 1,\ldots,4$$
$$\left[-\partial_\tau^2 + \partial_\sigma^2 - 2\mu^2\frac{1-8\gamma^2e^{4\rho(\tau)} - \gamma^4 e^{8\rho(\tau)}}{(1+\gamma^2 e^{4\rho(\tau)})^2}\right]\varphi = 0,$$
$$\left\{-\partial_\tau^2 + \partial_\sigma^2 - \left[(-2E^2\gamma^2e^{2\rho(\tau)} + \mu^2)\frac{1-\gamma^2e^{4\rho(\tau)}}{1+\gamma^2e^{4\rho(\tau)}} + \frac{E^2(1+\gamma^2e^{4\rho(\tau)}) - \mu^2e^{2\rho(\tau)}}{(1+\gamma^2e^{4\rho(\tau)})^2}2\gamma^2e^{2\rho(\tau)}(3-\gamma^2e^{4\rho(\tau)})\right]\right\}\tilde{x}_a = 0, \quad a = 2,3$$
$$\left\{-\partial_\tau^2 + \partial_\sigma^2 - \left[(-2E^2\gamma^2e^{2\rho(\tau)} + \mu^2)\frac{1-\gamma^2e^{4\rho(\tau)}}{1+\gamma^2e^{4\rho(\tau)}} + \frac{E^2(1+\gamma^2e^{4\rho(\tau)}) - \mu^2e^{2\rho(\tau)}}{(1+\gamma^2e^{4\rho(\tau)})^2}2\gamma^2e^{2\rho(\tau)}(3-\gamma^2e^{4\rho(\tau)})\right]\right\}\tilde{x}_1$$
$$+\gamma\partial_\sigma\left[e^{2\rho(\tau)}e^{-\rho(\tau)}\sqrt{1+\gamma^2e^{4\rho(\tau)}}\right] = 0. \tag{4.71}$$

We choose as usual for all the modes

$$X^i = X_0^i\exp[-i\omega\tau + ik_i\sigma], \tag{4.72}$$

and with the usual rescaling by p^+ for the gauge condition, we get for the quantization of the momenta around the σ circle

$$k_{i,n} = \frac{n_i}{\alpha' p^+}. \tag{4.73}$$

Then the only simple modes are the $\tilde{y}^i$s, for which we get the usual

$$(\omega_y^2 - k_i^2 - \mu^2)\tilde{y}_0^i = 0 \Rightarrow \omega_y^2 = \mu^2 + k_i^2 \Rightarrow \frac{\omega_y}{\mu} = \sqrt{1 + \frac{n_i^2}{(\mu\alpha' p^+)^2}}. \tag{4.74}$$

3. Discretization and spin chain

Consider, as usual, a $Z = X^1 + iX^2$ field that is charged under the $J = i\partial_\psi$ in the gravity dual, and the rest we call Φ^i, $i = 1, 2, 3, 4$, and their insertions into $\mathrm{Tr}[Z^J]$ correspond to string modes.

The undeformed case ($\mathcal{N} = 4$ SYM) has a pp wave with $SO(4)_1 \times SO(2)_1 \times SO(4)_2 \times SO(2)_2$ symmetry, with the 1 index corresponding to the $\mathcal{N} = 4$ SYM directions, so the insertions of $D_i Z = \partial_i Z + [A_i, Z]$, whereas the 2 index corresponding to the transverse scalar directions, so the insertions of the Φ^is.

On the other hand, in the $T\bar{T}$ deformed case, we have a pp wave with $SO(2)'_1 \times SO(4)_2 \times SO(2)_2$ symmetry, and again the 2 index corresponds to transverse scalar directions, so insertions of the Φ^is, and it is unchanged. Moreover, the interactions coming from the (transverse part of the) pp wave are the same, so the interaction Hamiltonian (the potential) in field theory is unchanged, namely $[\Phi^i, \Phi^j]^2$.

But the $D_i Z$ insertions must change, since now we have only $SO(2) \times SO(2)$ symmetry, instead of $SO(4)_1$. That should mean a *dipole* theory change in the kinetic term, breaking Lorentz invariance, (01) and (23) being singled out. That happens for the case of noncommutative theories, for instance (which are examples of dipolelike theories). Also, $SO(4)_2 \simeq SU(2) \times SU(2)$, which means that there is one $\mathcal{N} = 2$ supersymmetry acting on half of the fermions [fermions are in some complex representation, so of $SU(2)$], and another $\mathcal{N} = 2$ supersymmetry on the other half of fermions (since Lorentz symmetry is broken).

Because the TsT deformation was argued in [21–25] to be dual to the noncommutative (star product) deformation of the field theory, it is very likely, though the Penrose limit could involve extra subtleties, that the dipole theory is noncommutative. In fact, in [32] the (null) dipole deformed theory of [33] was related to some TsT deformation, and the full spin chain [in the $SL(2)$ sector, away from the BMN limit] was described.[3]

In conclusion, we have the $T\bar{T}$ deformation, followed by Penrose limit gives a deformation of the $\mathcal{N} = 4$ SYM in the kinetic term, for instance through the change of the usual product with the star product, giving a noncommutative theory, but other possibilities for the dipole theory could also happen.

V. $T\bar{T}$ DEFORMATION OF STRING WORLD SHEET ON $\mathrm{AdS}_5 \times S^5$ pp WAVE VS. $\mathcal{N} = 4$ SYM SPIN CHAIN DEFORMATION

Next, we consider the opposite order: first Penrose limit, then $T\bar{T}$ deformation. That is, we would like to find the $T\bar{T}$ deformation of the BMN sector of $\mathcal{N} = 4$ SYM. There are several ways that could be defined, however.

A. First try: Discretization of two-dimensional $T\bar{T}$ deformed string world sheet

Since the BMN spin chain Hamiltonian, describing $\mathcal{N} = 4$ SYM interactions in the BMN limit, is obtained from a discretization of the string Hamiltonian on the pp wave, written in terms of $\phi_i = \frac{a_i + a_i^\dagger}{\sqrt{2}}$, just that a_i are Cuntz oscillators at a site, we can try first to discretize the $T\bar{T}$ deformed string worldsheet Hamiltonian.

Then, by $T\bar{T}$ deforming the string world sheet Hamiltonian on the pp wave and discretizing, we should, almost by definition, obtain the $T\bar{T}$ deformed spin chain in terms of Cuntz oscillators: we could, in fact, *define* the deformed spin chain like that. The question then is whether this would be useful, meaning whether the resulting Hamiltonian can be derived from SYM, or from a deformed SYM, using the same procedure [16] did for the undeformed case.

In [4], a $T\bar{T}$ deformed Lagrangian density for a two-dimensional scalar with a potential V was given. Specializing for the case of just a mass term, $V = \mu^2 X^2/2$, it is

$$\mathcal{L} = -\frac{\sqrt{1 + 2\lambda(\partial_\mu X)^2(1 - \lambda\mu^2 X^2/2)} - (1 - \lambda\mu^2 X^2)}{2\lambda(1 - \lambda\mu^2 X^2/2)}. \tag{5.1}$$

If we consider several scalars X_i, we can consider the *independent $T\bar{T}$ deformation* for each of them, and sum the corresponding Lagrangians, which is what we will do here.

Therefore when discretizing, and when considering several coordinates (scalars) X^I, we obtain the $T\bar{T}$ deformed Lagrangian

$$L = \int dx \mathcal{L} \to \sum_{I,i} L_i^I, \tag{5.2}$$

where, since $(\partial_\mu X)^2 = -\dot{X}^2 + (X')^2$, and $(X')^2$ discretizes as $(X_i - X_{i+1})^2/a^2$ (a is the length of a step on the chain), we have (no sum over i and I)

[3] We thank Fedor Levkovich-Maslyuk for mentioning his work to us after the first version of our paper appeared on the arXiv.

$$L_i^I = -\frac{\sqrt{1+2\lambda(-(\dot{X}_i^I)^2+(X_i^I-X_{i+1}^I)^2/a^2)(1-\lambda\mu_I^2(X_i^I)^2/2)}-(1-\lambda\mu_I^2(X_i^I)^2)}{2\lambda(1-\lambda\mu_I^2(X_i^I)^2/2)}. \tag{5.3}$$

Expanding in λ, we get

$$L_i^I = \frac{1}{2}[(\dot{X}_i^I)^2-(X_i^I-X_{i+1}^I)^2/a^2-\mu_I^2(X_i^I)^2] + \lambda\left[\frac{1}{4}((\dot{X}_i^I)^2-(X_i^I-X_{i+1}^I)^2/a^2)^2-\frac{1}{4}\mu_I^4(X_i^I)^4\right] + \cdots. \tag{5.4}$$

The canonical momentum is, in the λ expansion,

$$p_i^I = \frac{\partial L_i^I}{\partial \dot{X}_i^I} = \dot{X}_i^I + \lambda\dot{X}_i^I((\dot{X}_i^I)^2-(X_i^I-X_{i+1}^I)^2/a^2)+\cdots, \tag{5.5}$$

and the Hamiltonian, in the same expansion,

$$\begin{aligned} H_i^I &= p_i^I\dot{X}_i^I - L_i^I \\ &= \frac{1}{2}[(\dot{X}_i^I)^2+(X_i^I-X_{i+1}^I)^2/a^2+\mu_I^2(X_i^I)^2] \\ &\quad + \frac{\lambda}{4}\left[3(\dot{X}_i^I)^4-2(\dot{X}_i^I)^2(X_i^I-X_{i+1}^I)^2/a^2 \right.\\ &\quad \left. -(X_i^I-X_{i+1}^I)^4/a^4\mu_I^4(X_i^I)^4\right]+\cdots. \end{aligned} \tag{5.6}$$

Next, in order to find the spin chain Hamiltonian (in terms of Cuntz oscillators at a site a_i^I), we need to write

$$X_i^I = \frac{a_i^I e^{-i\mu_I t}+(a_i^I)^\dagger e^{i\mu t}}{\sqrt{2}}, \tag{5.7}$$

and *only at the end of the calculation, after taking the time derivatives*, put $t=0$. In the leading term, we obtain

$$H_i^I = \frac{a_i^I(a_i^I)^\dagger+(a_i^I)^\dagger a_i^I}{2}+\frac{1}{a^2}\left(\frac{a_i^I+(a_i^I)^\dagger}{\sqrt{2}}-\frac{a_{i+1}^I+(a_{i+1}^I)^\dagger}{\sqrt{2}}\right)^2 \tag{5.8}$$

(this was what was obtained in [16]), but the next term looks more complicated.

The total Hamiltonian H needs to be diagonalized in order to find the spectrum. For that, we must define (at least in the leading term, not clear if we need to modify this for the other terms) first a Fourier transform,

$$a_j = \frac{1}{\sqrt{L}}\sum_{n=1}^{L} e^{\frac{2\pi i j n}{L}} b_n, \tag{5.9}$$

then the mixing of forward and backward (left and right) waves,

$$b_n = \frac{c_{n,1}+c_{n,2}}{\sqrt{2}}, \qquad b_{L-n} = \frac{c_{n,1}-c_{n,2}}{\sqrt{2}}, \tag{5.10}$$

for $n \le L/2$, and finally a Bogoliubov transformation mixing cs and $c^\dagger$s,

$$d_{n,i} = \alpha_{n,i}c_{n,i}+\beta_{n,i}c_{n,i}^\dagger, \quad i=1,2, \tag{5.11}$$

to obtain a diagonal Hamiltonian. Moreover, one can check that, *in the dilute gas approximation $d_{n,i}$ approximately* satisfy the usual commutation relations ($[d,d^\dagger]=1$, $[d,d]=0, [d^\dagger,d^\dagger]=0$), and not anymore the Cuntz ones.

We should check that/if the transformation above still holds, and the commutation relations in the dilute gas approximation still hold.

However, it does not hold, as we now show.

The Hamiltonian is found as follows. For the one-scalar Lagrangian, we find

$$\mathcal{H}(\lambda) = p\dot{\phi}-\mathcal{L} = \frac{1}{2\bar{\lambda}}\frac{1+2\bar{\lambda}\phi'^2}{\sqrt{1+2\bar{\lambda}(-\dot{\phi}^2+\phi'^2)}}+\tilde{V}, \tag{5.12}$$

where

$$\bar{\lambda} = \lambda(1-\lambda V), \qquad \tilde{V} = -\frac{1}{2\bar{\lambda}}(1-2\lambda V). \tag{5.13}$$

Then, in our case, with X_i^I, we have

$$p_i^I = \frac{\partial L_i^I}{\partial \dot{X}_i^I} = \frac{4\dot{X}_i^I}{\sqrt{1+2\lambda\left(1-\lambda\frac{\mu^2(X_i^I)^2}{2}\right)(-(\dot{X}_i^I)^2+(X_i'^I)^2)}}, \tag{5.14}$$

and so the Hamiltonian is

$$\mathcal{H} = \frac{1}{2\lambda\left(1-\frac{\lambda\mu^2(X_i^I)^2}{2}\right)} \times \frac{1+2\lambda(1-\frac{\lambda\mu^2(X_i^I)^2}{2})(X_i'^I)^2}{\sqrt{1+2\lambda\left(1-\lambda\frac{\mu^2(X_i^I)^2}{2}\right)(-(\dot{X}_i^I)^2+(X_i'^I)^2)}} - \frac{1-\lambda\mu^2(X_i^I)^2}{2\lambda\left(1-\frac{\lambda\mu^2(X_i^I)^2}{2}\right)}. \tag{5.15}$$

In order to get the spin chain Hamiltonian, we substitute in it the fields in terms of creation and annihilation operators, so

$$(\dot{X}_i^I)^2(t=0) = \frac{\mu^2}{2}\left[-(a_i^I)^2-(a_i^{\dagger I})^2+a_i^I a_i^{\dagger I}+a_i^{\dagger I}a_i^I\right],$$
$$(X_i'^I)^2 = (X_i^I - X_{i+1}^I)^2/a^2,$$
$$\mu^2(X_i^I)^2 = \frac{\mu^2}{2}\left[(a_i^I)^2+(a_i^{\dagger I})^2+a_i^I a_i^{\dagger I}+a_i^{\dagger I}a_i^I\right]. \tag{5.16}$$

The explicit formula is somewhat long, so we do not write it here.

Then, in terms of the Fourier modes of the a_j^I, namely $a_j^I = \frac{1}{\sqrt{L}}\sum_{n=1}^L e^{\frac{2\pi ijn}{L}} b_n^I$, where if (approximately, for Cuntz oscillators at a site acting on states in the dilute gas approximation) $[a_k, a_j^\dagger] = \delta_{jk}$, we obtain $[b_n, b_m^\dagger] = \delta_{nm}$, and

$$\sum_{j=0}^{L-1} a_j^{\dagger I} a_j^I = \sum_{n=0}^{L-1} b_n^{\dagger I} b_n^I, \tag{5.17}$$

and

$$A_j^I \equiv \left[(a_j^I + a_j^{\dagger I}) - (a_{j+1}^I + a_{j+1}^{\dagger I})\right],$$
$$= \sum_{n=0}^{L-1}\frac{1}{\sqrt{L}}\left[e^{\frac{2\pi inj}{L}}(e^{\frac{2\pi in}{L}}-1)b_n^I + e^{-\frac{2\pi inj}{L}}(e^{-\frac{2\pi in}{L}}-1)b_n^{\dagger I}\right]. \tag{5.18}$$

Further, defining the forward and backward (left and right) waves, $b_n^I = \frac{c_{n,1}^I + c_{n,2}^I}{\sqrt{2}}$, $b_{L-n}^I = \frac{c_{n,1}^I - c_{n,2}^I}{\sqrt{2}}$, the commutation relations are once again respected, and moreover

$$b_n^{\dagger I} b_n^I + b_{L-n}^{\dagger I} b_{L-n}^I = c_{n,1}^{\dagger I} c_{n,1}^I + c_{n,2}^{\dagger I} c_{n,2}^I, \tag{5.19}$$

and

$$A_j^I = \sum_{n=0}^{L/2}\frac{1}{\sqrt{L}}\left[e^{\frac{2\pi inj}{L}}(e^{\frac{2\pi in}{L}}-1)b_n^I + e^{-\frac{2\pi inj}{L}}(e^{-\frac{2\pi in}{L}}-1)b_n^{\dagger I} + e^{-\frac{2\pi inj}{L}}(e^{-\frac{2\pi in}{L}}-1)b_{L-n}^I + e^{\frac{2\pi inj}{L}}(e^{\frac{2\pi in}{L}}-1)b_{L-n}^{\dagger I}\right]. \tag{5.20}$$

Until now, the steps are useful and continue to work in the same way in our Lagrangian.

However, the essential next step does not, since it relies on making the sum $\sum_j (A_j^I)^2$, and in our case, $(A_j)^2$ appears inside a complicated expression with square root, and only then it is summed over.

With

$$\sum_{j=0}^{L_1}(A_j^I)^2 = \sum_{n=0}^{L/2}\left\{-\left(1-\cos\frac{2\pi n}{L}\right)\times\left(c_{n,1}^{\dagger I}c_{n,1}^I + c_{n,1}^I c_{n,1}^{\dagger I} + c_{n,2}^{\dagger I}c_{n,2}^I + c_{n,2}^I c_{n,2}^{\dagger I}\right) + 2\left(1-\cos\frac{2\pi n}{L}\right)\times\left[(c_{n,1}^I + c_{n,1}^{\dagger I})^2 - (c_{n,2}^I - c_{n,2}^{\dagger I})^2\right]\right\}, \tag{5.21}$$

we would have the Hamiltonian finally in a form in which we could use a Bogoliubov transformation.

Indeed, for a general Hamiltonian

$$H = \beta\frac{aa^\dagger + a^\dagger a}{2} \pm \alpha\frac{(a\pm a^\dagger)^2}{2} = \beta\left[\left(1+\frac{\alpha}{\beta}\right)\frac{aa^\dagger + a^\dagger a}{2} \pm \frac{\alpha}{2\beta}(a^2 + (a^\dagger)^2)\right], \tag{5.22}$$

the Bogoliubov transformation is

$$b = \tilde{\alpha}a + \tilde{\beta}a^\dagger, \tag{5.23}$$

and if we impose that $[b, b^\dagger] = 1$ (like $[a, a^\dagger] = 1$), we get $|\alpha|^2 - |\beta|^2 = 1$. Imposing diagonalization, so no b^2 or $(b^\dagger)^2$ terms in H, we obtain the condition

$$\left(1+\frac{\alpha}{\beta}\right)\tilde{\alpha}^+\tilde{b}^* = \pm\frac{\alpha}{\beta}[(\tilde{\alpha}^*)^2 + (\tilde{\beta}^*)^2]. \tag{5.24}$$

If we have $\alpha, \beta \in \mathbb{R}$, then we can define $\tilde{\alpha} - \tilde{\beta} \equiv \frac{1}{\sqrt{\omega}}$, $\tilde{\alpha} + \tilde{\beta} = \sqrt{\omega}$, and obtain

$$\omega_1 = \sqrt{\frac{1-\alpha/\beta}{1+3\alpha/\beta}}, \qquad \omega_2 = \sqrt{\frac{1+3\alpha/\beta}{1-\alpha/\beta}}, \tag{5.25}$$

and for both values of ω the diagonal Hamiltonian

$$H = \omega\frac{bb^\dagger + b^\dagger b}{2}. \tag{5.26}$$

B. $T\bar{T}$ deformation of quantum mechanical spin chain in $\mathcal{N}=4$ SYM

If the first try was a deformation of the two-dimensional system, followed by a discretization, and it was not very successful (very useful), in that we could not diagonalize the Hamiltonian to find the spectrum, we can next try to deform directly the one-dimensional spin chain.

One approach is defined specifically for spin chains. In [34,35] it was argued that the previous integrability-preserving deformation of spin chains defined by Bargheer, Beisert, and Loebbert in [36,37] is actually a $T\bar{T}$ deformation.

They claim that, first, the seminal paper [3] defining explicit $T\bar{T}$ deformations of 2 dimensional QFTs already defines them via a Bethe ansatz, so it is worth following the same way to define the $T\bar{T}$ deformations of spin chains [35].

They also say that the deformations of the Bethe-Yang equations

$$e^{ip_jR}\prod_{k\neq j}^{N} S(p_k,p_j)=1, \tag{5.27}$$

are via a deformation of the Castillejo-Dalitz-Dyson (CDD) factor appearing in the S matrix. Specifically, one can keep the S matrix fixed and modify the equations as

$$e^{ip_jR+i\alpha(X_jY-Y_jX)}\prod_{k\neq j}^{N} S(p_j,p_k)=1, \tag{5.28}$$

or equivalently, modify the S matrix by multiplication with a phase (CDD factor),

$$S(p_j,p_k)\to e^{i\alpha(X_kY_k-X_kY_j)}S(p_j,p_k). \tag{5.29}$$

These were the integrable deformations of spin chains found by [36,37].

In the case of the $T\bar{T}$ deformation, the claim is that one can take $X_j=p_j$ and $Y_j=H(p_j)$, but the arguments are somewhat indirect (in particular, the fact that integrability must be preserved).

One should also mention the works [38–40] where the $T\bar{T}$ deformation of the *world sheet* string in gravity duals, in particular on pp waves, is considered through an analysis of the world sheet Hamiltonian arising in uniform light-cone gauge, and is found to equal to the TsT transformation in time t and one compact direction ϕ in *spacetime*. For the $\mathrm{AdS}_3\times S^3$ case [38], in the massless case one indeed finds the usual [3,4] two-dimensional deformation of massless bosons (on the world sheet), though in the massive case and in the $\mathrm{AdS}_5\times S^5$ case [39,40] one finds a difference, which is hard to understand physically. We do not understand this method and the discrepancy well enough to comment intelligently on it. Instead, in the method used below, such discrepancy is not possible by construction.

However, there is a parallel line of inquiry, one that is also cited by [35] as previous work, by David Gross *et al.* [17,41], in which the $T\bar{T}$ deformation of quantum mechanics is proposed, based on the holographic proposal of McGough *et al.* [12] and AdS_2, and a dimensional reduction from AdS_3, and a corresponding one on the boundary.[4,5] This is the approach we will follow here. Note that there is some controversy about the applicability of this result to the case with a potential: In [45] it was argued that the original two-dimensional theory must be conformal invariant, so no potential can be present. But the controversy is mostly about semantics: In the Gross *et al.* prescription, the *definition* of the one-dimensional analog of $T\bar{T}$ deformation was such that it coincides with the holographic AdS_2 one of [12]. Otherwise, the reduction prescription from two dimensions might not work for nonconformal seed theories, as [45] argued. We will, however, continue applying the original procedure as it was developed.

In Secs. 3 and 4 of [17], the deformation of quantum mechanics is defined as[6]

$$H=\frac{1-\sqrt{1-8\lambda\left(\sum_i\frac{p_i^2}{2}+V(\{q_i\})\right)}}{4\lambda}, \tag{5.32}$$

which leads to

[4]Note that there is also a third deformation, in terms of a bilinear operator, in the papers of Cardy and Doyon [42], and Yunfeng Jiang [43], though it was not developed further, so we will not describe it.

[5]See also [44] for an alternative derivation of this proposal.

[6]Note some signs are different with respect to [17]. It was easy to check that the signs were wrong, since the $\lambda\to 0$ limit does not work. Instead, [17] has the formula

$$L=\frac{1-\sqrt{(1-4\lambda\sum_i\dot{q}_i^2)(1-8\lambda V)}}{4\lambda}, \tag{5.30}$$

which would correspond to taking $\lambda\to-\lambda$ AND $V\to-V$ in (5.33). However, in the Hamiltonian, this was not what was considered, so their formula has incorrect signs.
One observation is that the *two-dimensional* $T\bar{T}$ deformed Lagrangian density is

$$\mathcal{L}=\frac{1-2\lambda V}{2\lambda(1-\lambda V)}-\frac{\sqrt{[1+2\lambda\sum_i(-\dot{q}_i^2+q_i'^2)](1-\lambda V)}}{2\lambda(1-\lambda V)}, \tag{5.31}$$

and if we naively put $q'=0$ in (5.30), we get a similar (but not quite! the formula above in two dimensions has the correct sign for V in the $\lambda\to 0$ limit; it only matches with their Lagrangian for $V=0$, and rescaling λ by 2) formula to the above, which may be the reason for the confusion.

$$L = -\frac{1-\sqrt{(1+4\lambda\sum_i \dot{q}_i^2)(1-8\lambda V)}}{4\lambda}, \qquad (5.33)$$

via

$$p_i = \dot{q}_i\sqrt{\frac{1-8\lambda V}{1+4\lambda\sum_i \dot{q}_i^2}}. \qquad (5.34)$$

Further, in [41] it was shown that this $T\bar{T}$ deformation of the quantum mechanics replaces the Hamiltonian H with a function of it, $f(H)$, which means that the eigenfunctions do not change.

Note that, in general, if

$$H(q_i, p_i) = f(H_0(q_i, p_i)), \qquad (5.35)$$

and H_0 has conserved quantities, $\dot{p}_i = 0$, so $\frac{\partial H_0}{\partial q_i} = 0$, then also

$$\frac{\partial H}{\partial q_i} = f'(H_0)\frac{\partial H_0}{\partial q_i} = 0, \qquad (5.36)$$

so all classical integrals of motion remain integrals of motion, and therefore a classically integrable system remains integrable after the ($T\bar{T}$, in this case) deformation.

We see that the Hamiltonian deformation (A12) is very easy to work with. Anything that we did with H_0 and E_0 we can do with $H(\lambda)$ and $E(\lambda)$. In particular, we can write the explicit form in $a, a^\dagger$ oscillators, and diagonalize it.

So the discretized pp wave Hamiltonian (dual to the original spin chain in the dilute gas approximation) (5.8) can be $T\bar{T}$ deformed,

$$H(\lambda) = \mu\frac{1}{4\lambda\mu}\left[1-\sqrt{1-8\lambda\mu\sum_{i,I}\left(\frac{a_i^I(a_i^I)^\dagger + (a_i^I)^\dagger a_i^I}{2} + \frac{1}{a^2}\left(\frac{a_i^I + (a_i^I)^\dagger}{\sqrt{2}} - \frac{a_{i+1}^I + (a_{i+1}^I)^\dagger}{\sqrt{2}}\right)^2\right)}\right], \qquad (5.37)$$

and the deformation can be diagonalized, obtaining (in the $AdS_5 \times S^5$ case)

$$E(\lambda, g^2N, n/J) = \mu\frac{1}{4\lambda\mu}\left(1-\sqrt{1-8\lambda\mu\sqrt{1+\frac{g^2N}{\pi^2}\sin^2\frac{\pi n}{J^2}}}\right). \qquad (5.38)$$

Note that the discretization and $T\bar{T}$ deformation are not commutative. In the previous subsection we $T\bar{T}$ deformed first, and then discretized, now we do the opposite. This is besides the noncommutativity of the $T\bar{T}$ deformation and Penrose limit, which is also present: Here we take the Penrose limit first, then deform, whereas in the previous section, we first deformed, then took the Penrose limit.

C. Deformation of $\mathcal{N} = 4$ SYM

We now try to interpret the quantum mechanical $T\bar{T}$ deformation from the point of view of $\mathcal{N} = 4$ SYM.

1. Symmetries and symmetry algebra

We start with an analysis of the symmetries and their algebra.

Since $H(\lambda) = f(H_0)$, the global symmetries of the Hamiltonian continue to be symmetries of the deformed one. In particular, the $SO(2)_1 \times SO(4)_1 \times SO(2)_2 \times SO(4)_2$ with, in the bosonic case, $SO(4)_1 \times SO(4)_2$ extended to $SO(8)$, continues to hold for the deformed pp wave Hamiltonian, as it was for the usual pp wave Hamiltonian. Also true for the $\mathcal{N} = 4$ supersymmetry (or, more precisely, to the 16 supercharges).

This means that in this case we are dealing with a *deformed sector within* $\mathcal{N} = 4$ *SYM*.

The bosonic symmetry algebra *of the string* pp *wave Hamiltonian on* $AdS_5 \times S^5$, matching the one of $\mathcal{N} = 4$ SYM, is defined in terms of Killing spinors (see [31]) for the pp wave variables (with respect to that paper, we have defined $\mu = 2\lambda$),

$$\begin{aligned} h &= \xi_{e^+} = -\partial_+, \xi_{e^-} = -\partial_-,\\ \xi_{e_i} &= -\cos(\mu x^+)\partial_i - \mu\sin(\mu x^+)\tilde{y}^i\partial_-, \quad i = 1,\ldots,8\\ \xi_{e_i^*} &= -\mu\sin(\mu x^+)\partial_i + \mu^2\cos(\mu x^+)\tilde{y}^i\partial_-,\\ \xi_{M_{ij}} &= x_i\partial_j - x_j\partial_i, \quad i,j = 1,\ldots,4 \quad \text{or } 5,\ldots,8. \end{aligned} \qquad (5.39)$$

The algebra is obtained by defining harmonic oscillators $a_i = (e_i + ie_i^*)/\sqrt{2}$, so $[a_i, a_j^\dagger] = \delta_{ij}$, then $M_{ij} = i(a_i^\dagger a_j - a_j^\dagger a_i)$ and $H = H_0 = -p_+ = \mu\sum_i a_i^\dagger a_i$, while $e = -p_-$ commutes with everything.

More precisely, we want to obtain the commutation relations

$$\begin{aligned} &[a_i, a_j^\dagger] = e\delta_{ij},\\ &[ih, a_i^\dagger] = \mu a_i^\dagger, [-ih, a_i^\dagger] = -\mu a_i,\\ &[M_{ij}, a_k] = -\delta_{ik}a_j + \delta_{jk}a_i, [M_{ij}, a_k^\dagger] = -\delta_{ik}a_j^\dagger + \delta_{jk}a_i^\dagger. \end{aligned} \qquad (5.40)$$

But the symmetry algebra of the ξs (which we will now call just e_i, e_i^*, M_{ij}, h) is, *redefining* $e_i^* \to \mu e_i^*$,

$$
\begin{aligned}
&[e_i, e_j^*] = (\mu e)\delta_{ij},\\
&[h, e_i] = \mu e_i^*, \qquad [h, e_i^*] = -\mu e_i,\\
&[M_{ij}, e_k] = -\delta_{ik}e_j + \delta_{jk}e_i, \qquad [M_{ij}, e_k^*] = -\delta_{ik}e_j^* + \delta_{jk}e_i^*.
\end{aligned} \tag{5.41}
$$

Then, defining

$$
a_i = ae_i + ibe_i^*, \qquad a_i^\dagger = a^*e_i^* - ib^*e_i, \tag{5.42}
$$

we obtain

$$
[a_i, a_j^\dagger] = (|a|^2 + |b|^2)(\mu e)\delta_{ij}, \tag{5.43}
$$

so

$$
|a|^2 + |b|^2 = \frac{1}{\mu}. \tag{5.44}
$$

Moreover,

$$
\begin{aligned}
[-ih, a_i] &= -\mu(be_i + iae_i^*),\\
&\equiv -\mu(ae_i + iae_i^*) = -\mu a_i \Rightarrow a = b,\\
[ih, a_i^\dagger] &= +\mu(b^*e_i^* - ia^*e_i),\\
&\equiv +\mu(a^*e_i^* - ib^*e_i) = +\mu a_i^\dagger \Rightarrow a = b.
\end{aligned} \tag{5.45}
$$

So we finally have

$$
a = b = \frac{1}{\sqrt{2\mu}} \Rightarrow a_i = \frac{e_i + ie_i^*}{\sqrt{2\mu}}, \qquad a_i^\dagger = \frac{e_i^* - ie_i}{\sqrt{2\mu}}. \tag{5.46}
$$

Then we can represent

$$
M_{ij} = i(a_i^\dagger a_j - a_j^\dagger a_i), \tag{5.47}
$$

and

$$
ih = \frac{\mu}{e}\sum_i a_i^\dagger a_i. \tag{5.48}
$$

We can redefine $a_i = \sqrt{e}\tilde{a}_i$ (since we can treat e as a number, since it is a central charge: it commutes with everything, it is $= -p_- = -p^+$), and then we have (we write H instead of h, to underline the fact that we now talk about the $T\bar{T}$ deformed one-dimensional Hamiltonian)

$$
\begin{aligned}
(\pm i)H &= \mu\sum_i \tilde{a}_i^\dagger\tilde{a}_i,\\
&= \sum_{n\geq 1} c_n\frac{1}{\lambda}(\lambda H_0)^n \equiv \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda H_0}\right),\\
&= \sum_{n\geq 1} c_n\frac{1}{\lambda}\left[\lambda\mu_0\sum_i \tilde{a}_{0,i}^\dagger\tilde{a}_{0,i}\right]^n,\\
&= \mu_0\sum_{n\geq 1}(\lambda\mu_0)^{n-1}\left[\sum_i \tilde{a}_{0,i}^\dagger\tilde{a}_{0,i}\right]^n.
\end{aligned} \tag{5.49}
$$

Here $e_0, \tilde{a}_{0,i}, \tilde{a}_{0,i}^\dagger$ are the undeformed symmetry generators (in some abstract sense, equal to $e, \tilde{a}_i, \tilde{a}_i^\dagger$, namely they *satisfy the same algebra, though one with* μ *and another with* μ_0), namely their expression in terms of fields are for the undeformed sector. The algebra of deformed generators *must be the same as of the undeformed generators*. The above relation must be solved for $\tilde{a}_i$ in terms of $\tilde{a}_{0,i}$. One obvious solution is (there does not seem to be a solution involving only as, no $a^\dagger$s, since the noncommutation of the two makes it unlikely to disentangle)

$$
\tilde{a}_i = \sum_{n\geq 0}\tilde{c}_n\left[\lambda\mu\sum_j \tilde{a}_{0,j}^\dagger\tilde{a}_{0,j}\right]^n\tilde{a}_{0,i}, \tag{5.50}
$$

which follows from (the following defines the constants c_n and $\tilde{c}_n$)

$$
\begin{aligned}
&\frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda x}\right)\\
&\equiv \sum_{n\geq 1} c_n\lambda^{n-1}x^n \Rightarrow \sqrt{\frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda x}\right)}\\
&\equiv \sqrt{x}\sum_{n\geq 0}\tilde{c}_n\lambda^n x^n.
\end{aligned} \tag{5.51}
$$

Indeed, we can check that then the deformation of the Hamiltonian is obtained, if we consider that

$$
e = e_0 \tag{5.52}
$$

(the central charge is undeformed, and it refers to the same p^+), but μ is identified with the energy, so $\mu = E(\lambda)$ and $\mu_0 = E_0$.

Then the fact that the deformed and undeformed symmetry algebras are the same is obtained from the fact that both are written in the same way in terms of the creation and annihilation operators [we just have the relation (5.50) between the as and a_0s].

Thus we have (5.49) relating H and H_0, (5.50) relating a_i with $a_{0,i}$ (and its dagger for $a_i^\dagger$ with $a_{i,0}^\dagger$), $e = e_0$, and

$$M_{ij} = \sum_{n\geq 0}\Big[\lambda\mu_0\sum_j \tilde{a}^\dagger_{0,j}\tilde{a}_{0,j}\Big]^n M_{0,ij}. \qquad (5.53)$$

The relation between as and a_0s must be one of *equivalence* of the two creation/annihilation operator sets, and consequently of their Fock spaces.

This symmetry algebra must also be obtained from $\mathcal{N}=4$ SYM, on the large J charge sector, though it was not written explicitly in terms of fields in [16], so we must first do that. Next we must define what are the *deformed* generators in terms of the $\mathcal{N}=4$ SYM fields.

When acting on (undeformed) BMN operators (states in the BMN sector), we can represent

$$(\tilde{a}_{0,i})^\alpha{}_\beta = \frac{\delta}{\delta(\Phi_i)^\alpha{}_\beta}, \qquad (\tilde{a}^\dagger_{0,i})^\alpha{}_\beta = (\Phi_i)^\alpha{}_\beta \mathbf{In}, \qquad (5.54)$$

where Φ_i, $i=1,\ldots,4$ are the 4 scalars that are inserted inside $\mathrm{Tr}[Z^J]$, with their matrix indices, and $\mathbf{In}$ refers to insertion of the matrix element inside the trace.

Next we must define the set of operators of the deformed subsector. The vacua must be the same, since moreover $e=e_0$ and this corresponds to having an undeformed p^+, so undeformed J. Thus consider the same vacuum $\mathrm{Tr}[Z^J]$, and insert $\sum_{n\geq 0}\tilde{c}_n(\lambda\mu_0\sum_j\Phi_j\frac{\delta}{\delta\Phi_j})^n\Phi_i$, and $\tilde{c}_n$ solved from c_n as suggested above, so

$$a^{\dagger i}_m|0\rangle \sim \sum_l \mathrm{Tr}\left[Z^l\left(\sum_{n\geq 0}\tilde{c}_n\left[\lambda\mu_0\sum_j\Phi^j\frac{\delta}{\delta\Phi_j}\right]^n\right)\Phi_i Z^{J-l}\right]e^{\frac{2\pi m l}{J}}. \qquad (5.55)$$

In this way, we find that, at the same time we have a *deformed* BMN sector, and it is *equivalent to the original one*. Thus $H(\lambda)$ has both the same eigenstates (the undeformed BMN sector), and the new eigenstates (the deformed BMN sector), since they are supposed to be equivalent. Then $H(\lambda)$ in the undeformed eigenstates gives $E(\lambda)$, whereas $H(\lambda)$ in the deformed eigenstates gives E_0.

2. Comments on the properties of $H_\lambda = f(\lambda, H_0)$

As was described in detail in [41], once we have $H_\lambda = f(\lambda, H_0)$, with the extra condition of analyticity at $\lambda=0$, so $\exists H_0$ limit, we can calculate anything in the deformed theory from the undeformed theory, in particular the correlations functions are found by a general formula. Moreover, obviously (with the extra analyticity assumption, needed for the case of an infinite dimensional Hilbert space) we have the same spectrum of eigenstates for the two Hamiltonians.

But note that this does not mean that the Lagrangians are also related, $L_\lambda \neq g(\lambda, L_0)$. We can easily see this in the formula for L (5.33), in the nontrivial case of $V\neq 0$. Reversely, if $L_\lambda = f(\lambda, L_0)$, it does not mean that $H_\lambda = f(\lambda, H_0)$.

For the DBI example = $T\bar{T}$ deformation of the $V=0$ case in two dimensions,

$$\mathcal{L} = \frac{1}{\lambda}\left[1-\sqrt{1+\lambda(-\dot{\phi}^2+\phi'^2)}\right], \qquad (5.56)$$

so

$$p = \frac{\dot{\phi}}{\sqrt{1+\lambda(-\dot{\phi}^2+\phi'^2)}}, \qquad (5.57)$$

so

$$H = \frac{1}{\lambda}\left[-1+\sqrt{(1+\lambda p^2)(1+\lambda\phi'^2)}\right], \qquad (5.58)$$

so it is not a function of $H_0 = \frac{1}{2}(p^2+\phi'^2)$.

VI. CONCLUSIONS AND DISCUSSION

In this paper we have considered $T\bar{T}$ deformations in the context of holography, and more specifically in the context of the pp wave correspondence.

We have applied the TsT procedure of obtaining the gravity dual to a single trace $T\bar{T}$ deformation in the $\mathrm{AdS}_3\times S^3\times T^4$ case to the $\mathrm{AdS}_5\times S^5$ case, and used the Penrose limit to understand it, proposing that the deformation corresponds to some dipole theory, probably a noncommutative one.

Reversely, we have considered the $T\bar{T}$ deformation of the Penrose limit of $\mathrm{AdS}_5\times S^5$, in two ways. We have discretized the $T\bar{T}$ deformation of the world sheet string on the pp wave, though the corresponding spin chain looks complicated. We have instead considered the $T\bar{T}$ deformation of the quantum mechanical model obtained from the discretization of the string Hamiltonian on the pp wave, known to correspond to the spin chain one in $\mathcal{N}=4$ SYM. Based on the corresponding symmetry algebra of the deformed pp wave theory, we have argued that the deformation can be understood within $\mathcal{N}=4$ SYM as deformation of the (BMN) sector within it, but not of the theory, obtaining moreover an equivalent sector.

There are many open questions left for further work. The exact nature of the deformation to $\mathcal{N}=4$ SYM dual to the TsT transformed $\mathrm{AdS}_5\times S^5$ is not yet clear. The spin chain obtained by discretizing the $T\bar{T}$ deformed world sheet string for the pp wave has no good interpretation yet. One can try to use the same methods considered here in other holographic cases for dimensions $d>2$, for instance the $d=3$ case of the Aharony-Bergman-Jafferis-Maldacena (ABJM) model, as well as more realistic cases like the Klebanov-Witten, Klebanov-Strassler, Maldacena-Nunez, and Maldacena-Nastase ones. In particular holographic pp wave theories associated with confining gauge dynamics analyzed in [46]. Finally, we have only considered the bosonic part of the string world sheet theory on the pp wave; it would be interesting to see what happens when we add the fermions, in all the cases considered. This is in particular interesting, since one can construct supersymmetric $T\bar{T}$ deformations, and the deformation appears as a supersymmetric descendant [47,48].

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APPENDIX: REVIEW OF DERIVATION OF $T\bar{T}$ DEFORMATION OF GENERAL QUANTUM MECHANICAL MODEL

Here we review the derivation in [17] of the $T\bar{T}$ deformation of general quantum mechanics systems, via dimensional reduction, assuming the $\mathrm{AdS}_3/\mathrm{CFT}_2$.

In the two-dimensional CFT, it is assumed that the flow equation is

$$\frac{\partial S_E(\lambda)}{\partial\lambda} = \int d^2x\sqrt{\gamma}8T\bar{T}, \tag{A1}$$

and that, along the flow, we have the equation (following from holography of the $T\bar{T}$ deformation following [12])

$$T^\mu_\mu = -16\lambda T\bar{T} = -2T(T_{ij}T^{ij} - (T^i_i)^2). \tag{A2}$$

The $\mathrm{AdS}_3/\mathrm{CFT}_2$ gravity dual of Bañados-Teitelboim-Zanelli-black hole (BTZ-BH) type is

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{dr^2}{r^2 - r_+^2} + r^2 d\phi^2 \to_{bd.} r^2(d\tau^2 + d\phi^2). \tag{A3}$$

Then the condition on the flow is

$$T^\mu_\mu = T^\phi_\phi + T^\tau_\tau = -2\lambda(T_{ij}T^{ij} - (T^i_i)^2), \tag{A4}$$

and

$$T_{ij}T^{ij} - (T^i_i)^2 = T_{\tau\tau}T^{\tau\tau} + T_{\phi\phi}T^{\phi\phi} + 2T_{\tau\phi}T^{\tau\phi} - (T^\tau_\tau + T^\phi_\phi)^2, \\ = 2T_{\tau\phi}T^{\tau\phi} + 2T^\tau_\tau T^\phi_\phi, \tag{A5}$$

so the condition along the flow becomes

$$T^\phi_\phi = \frac{T^{\tau_\tau} + 4\lambda T_{\tau\phi}T^{\tau\phi}}{4\lambda T^\tau_\tau - 1}, \tag{A6}$$

and, given that $T^\mu_\mu = T^\phi_\phi + T^\tau_\tau = -16\lambda T\bar{T}$ on the flow, we have

$$8T\bar{T} = \frac{T^\phi_\phi + T^\tau_\tau}{-2\lambda} = \frac{(T^\tau_\tau)^2 + T_{\tau\phi}T^{\tau\phi}}{1/2 - 2\lambda T^\tau_\tau} \Rightarrow \frac{\partial S_E(\lambda)}{\partial\lambda} = \int d^2x\sqrt{\gamma}\frac{(T^\tau_\tau)^2 + T_{\tau\phi}T^{\tau\phi}}{1/2 - 2\lambda T^\tau_\tau}. \tag{A7}$$

But, since $\langle T_{\tau\phi}\rangle = \langle T^{\tau\phi}\rangle = iJ$ and $\langle T^\tau_\tau\rangle = E$, by which we mean more precisely integral over a circle of radius 1, $\int T_{\tau\phi} = \int T^{\tau\phi} = iJ$ and $\int T^\tau_\tau = E$ (and implicit factorization of the square) besides the Vacuum Expectation Value (VEV), putting $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$, peeling off the time integral in S_E, we obtain the equation for the energy eigenstates,

$$\frac{\partial E(\lambda)}{\partial\lambda} = \frac{E^2 - J^2}{1/2 - 2\lambda E} \Rightarrow E(\lambda) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda E_0 + 16\lambda^2 J^2}\right), \tag{A8}$$

the usual solution.

Dimensional reduction proceeds as follows: Put $T_{\phi\tau} = T^{\tau\phi} = 0$ and $T^\tau_\tau \equiv T$, to obtain

$$\frac{\partial S_E}{\partial\lambda} = \int d\tau\frac{T^2}{1/2 - 2\lambda T}. \tag{A9}$$

Then, again peeling of $d\tau$ and using $\langle T\rangle = E$ (VEV) and factorization of the square, we obtain the equation for the energy eigenstates,

$$\frac{\partial E(\lambda)}{\partial\lambda} = \frac{E^2}{1/2 - 2\lambda E} \Rightarrow E(\lambda) = \frac{1 - \sqrt{1 - 8\lambda E_0}}{4\lambda}, \tag{A10}$$

or, equivalently as we see, just put $J = 0$ in the two-dimensional result.

Note that if we take $\lambda \to -\lambda$, we obtain

$$\frac{\partial E(\lambda)}{\partial\lambda} = -\frac{E^2}{1/2 + 2\lambda E} \Rightarrow E(\lambda) = -\frac{1 - \sqrt{1 + 8\lambda E_0}}{4\lambda}. \tag{A11}$$

See the discussion in the main text with respect to the signs in H and L.

Now, we assume then that the same relation is true for the Hamiltonians (since it is valid for all its eigenstates, and the Hamiltonian is a Hermitian operator), so

$$H(\lambda) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda H_0}\right) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda\left(\sum_i \frac{p_i^2}{2} + V(\{q_i\})\right)}\right). \quad \text{(A12)}$$

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