Loop amplitude integrands from unitarity cuts

Jacob L. Bourjaily[1] and Simon Caron-Huot[2]

[1]*Institute for Gravitation and the Cosmos, Department of Physics, Pennsylvania State University, University Park, Pennsylvania 16802, USA*
[2]*Department of Physics, McGill University, 3600 Rue University, Montréal, H3A 2T8, Québec, Canada*

We derive novel recursion relations for all loop amplitude integrands of planar, maximally supersymmetric Yang-Mills theory in terms of unitarity-like "cuts" obtained via sequences of BCFW deformations in momentum-twistor space.

I. INTRODUCTION

Much of the recent progress in our understanding of scattering amplitudes in perturbative quantum field theories has stemmed from powerful new techniques to efficiently compute amplitudes and loop-amplitude integrands at large multiplicity and/or loop order. These techniques have provided a rich source of theoretical "data" from which a number of deep and important insights have been gleamed. For example, building on analytic insights into tree amplitudes (see e.g. [1]), the all-loop recursion relations for loop integrands discovered in [2] led directly to a wealth of theoretical data, against which new ideas were rapidly developed and tested.

The recursion relations described in [2] generated loop amplitude integrands from the "forward limits" of lower-loop amplitude integrands; although such forward limits are believed to be well defined for amplitudes in any supersymmetric field theory [3], they are known to involve considerable subtlety for less-than-maximal supersymmetry (especially for amplitudes represented via recursion) [4,5]; even for maximally supersymmetric Yang-Mills theory in the planar limit ("sYM"), forward limits prove an enormous source of computational inefficiency, greatly limiting their effective implementation. To date, they have been exploited for amplitudes only through two-loop order [6–8].

In this paper, we overcome this barrier by showing how sequences of BCFW recursions can be used to represent any L-loop, n-particle amplitude integrand of planar sYM, $\mathcal{A}_n^L$, in a form reminiscent of unitarity cuts:

$$\mathcal{A}_n^L = \sum \quad \widehat{j} \quad + \quad \widehat{\ell}_1 \quad \widehat{j} + \cdots + \widehat{\ell}_1 \ \widehat{\ell}_2 \cdots \widehat{\ell}_L \ \widehat{j}. \tag{1}$$

Here, the first term corresponds to the ordinary "BCFW bridge," and the second the one-loop "kermit" terms described in [7]. At two loops, a closed formula similar to (1) was given in the appendix of [8], but it involved two sums over triple products of amplitudes. Despite the similarity between (1) and unitarity cuts, the representation we derive features surprisingly novel types of cuts at three loops and higher.

Attached to this note, we have prepared an implementation of the representation (1) in *Mathematica*, which has been checked both internally and against a number of nontrivial examples and has been used to compute a number of previously inaccessible loop amplitudes including the 10-particle N^3MHV amplitude at 3 loops, the 8-particle N^2MHV amplitude at 4 loops, and the 5-particle MHV amplitude at 5 loops.

II. BCFW RECURSION IN MOMENTUM-TWISTOR SPACE

Our starting point will be the all-loop recursion of [2]. Similar to the tree-level BCFW recursion [9,10], it may be derived by shifting a pair of external momenta by a complex parameter while maintaining on-shell and momentum conservation constraints. In sYM (and for suitable helicity choices in nonsupersymmetric theories), the amplitude vanishes at infinite momentum shift [11] and therefore is fully determined by its poles at finite shifts. At tree level, these poles represent factorization channels, the

residues of which can be determined recursively as products of lower-point, on-shell amplitudes. For loop amplitude integrands, there are also poles corresponding to locations where internal propagators go on shell, which can be described (if not easily computed) as a lower-loop amplitude with two additional momenta taken in the forward limit.

The mechanics of on-shell recursion are simplified in momentum-twistor variables [12] which simultaneously trivialize both momentum conservation and the on-shell condition for massless particles. Thus, an arbitrary set of momentum twistors $z_a \in \mathbb{C}^4/\text{GL}(1)$ will encode a set of momentum-conserving, massless four-momenta. This is achieved using the twistor map [13] which associates the (projective) line $(a-1\,a) := \text{span}\{z_{a-1}, z_a\}$ (cyclic labeling understood) in twistor space with a point x_a in spacetime. Two points in spacetime will be lightlike separated if their corresponding lines in twistor space intersect; as such, the points x_a and x_{a+1} are manifestly lightlike separated because the corresponding lines in twistor space $(a-1\,a)$ and $(a\,a+1)$ intersect at z_a. The collection of pairwise, null-separated points $\{x_a\}$ can be used to define a set of null momenta $p_a := (x_{a+1} - x_a)$, for which momentum conservation is manifest. For supersymmetric Yang-Mills theory each twistor z_a can be upgraded to a projective vector in $\mathbb{C}^{4|\mathcal{N}}$ which captures the helicity dependence of scattering amplitudes. These variables make sYM formulas particularly compact as they manifest its dual conformal symmetry [14–16].

When two dual-momentum points x_a and x_b become lightlike separated, their corresponding lines in momentum-twistor space $(a-1\,a)$ and $(b-1\,b)$ intersect; this corresponds to a situation where the four twistors $\{z_{a-1}, z_a, z_{b-1}, z_b\}$ span a space of rank less than four. Defining the "four-bracket" $\langle abcd \rangle := \det\{z_a, z_b, z_c, z_d\}$, two lines intersect when $\langle a-1\,a\,b-1\,b \rangle = 0$.

For planar theories described in dual-momentum coordinates, physical poles must involve the sums of consecutive momenta $(x_a - x_b)^2 = (p_a + p_{a+1} + \ldots + p_{b-1})^2$, or involve some internal propagator $(x_{\ell_i} - x_b)^2$ where x_{ℓ_i} represents an internal loop momentum—to be associated with the line $(\ell_i) := (A_i B_i)$ in momentum twistors spanned by z_{A_i}, z_{B_i}.

Consider the BCFW deformation of (super)-twistors,

$$z_n \mapsto \bar{z}_n(\alpha) := z_n + \alpha z_{n-1}; \tag{2}$$

this deformation trivially preserves the on-shell condition and momentum conservation. Since the only *line* deformed is $(z_n z_1)$—corresponding to the dual point x_1—this is equivalent to BCFW-deforming the pair of momenta p_n and p_1. Although for sYM there is no pole at infinite external momenta, there is a pole at $\alpha \to \infty$ corresponding to the factorization involving a three-particle $\overline{\text{MHV}}$ amplitude (an "inverse soft-factor" in the language of [17,18])—which, in momentum twistor space, is simply the lower-point amplitude involving twistors $\{z_1, \ldots, z_{n-1}\}$.

The poles at finite values of α must involve the vanishing of either $\langle \bar{n}1\,j-1\,j \rangle$, a factorization, or of $\langle \bar{n}1 A_i B_i \rangle$, a forward limit. The locations of these poles are given by

$$\hat{n} = (n-1\,n) \cap (1j-1\,j) \quad \text{or} \quad \hat{n} = (n-1\,n) \cap (1A_i B_i), \tag{3}$$

where $(ab) \cap (cde) := z_a \langle bcde \rangle + z_b \langle cdea \rangle$—or, equivalently, in the parametrization (2), at the locations

$$\alpha = \frac{\langle n1j-1\,j \rangle}{\langle 1j-1\,jn-1 \rangle} \quad \text{or} \quad \alpha = \frac{\langle n1A_iB_i \rangle}{\langle 1A_iB_in-1 \rangle}. \tag{4}$$

Residues involving "external" lines $(j-1\,j)$ correspond to factorization channels involving two lower-multiplicity amplitude integrands involving any distribution of the total L loop momenta between them, represented by the first term of (1). Concretely, these terms are given by

$$\mathcal{A}_{n_T}^{L_T}(1, \ldots, j-1, \hat{\jmath}) \mathcal{B}_0[1(j-1\,j), (n-1\,n)]$$
$$\times \mathcal{A}_{n_B}^{L_B}(\hat{\jmath}, j, \ldots, n-1, \hat{n}) \tag{5}$$

where $L = L_B + L_T$, $\hat{\jmath} := (j-1\,j) \cap (n-1\,n1)$. Taking (as in [2]) the amplitudes $\mathcal{A}$ to be the conventional amplitude (divided by the Parke-Taylor MHV superamplitude), the "bridge" $\mathcal{B}_0[1, (j-1, j), (n-1, n)] := \mathcal{R}[1, j-1, j, n-1, n]$ is given by the "R-invariant" [19] in momentum-twistor variables [12]

$$\mathcal{R}[abcde] := \frac{\delta^{0|4}(\eta_a \langle bcde \rangle + \eta_b \langle cdea \rangle + \eta_c \langle deab \rangle + \eta_d \langle eabc \rangle + \eta_e \langle abcd \rangle)}{\langle abcd \rangle \langle bcde \rangle \langle cdea \rangle \langle deab \rangle \langle eabc \rangle},$$

where the η's are Grassmann variables labeling states.

The second type of pole—that involving an internal line $(A_i B_i)$—corresponds to a one-lower-loop amplitude involving two additional particles taken in the forward limit as described in [2]; graphically, it corresponds to

$$\equiv \underset{A_1 \to B_1}{\text{FL}}\left(1, \ldots, n-1, \hat{n}, A_1, B_1\right) \tag{6}$$
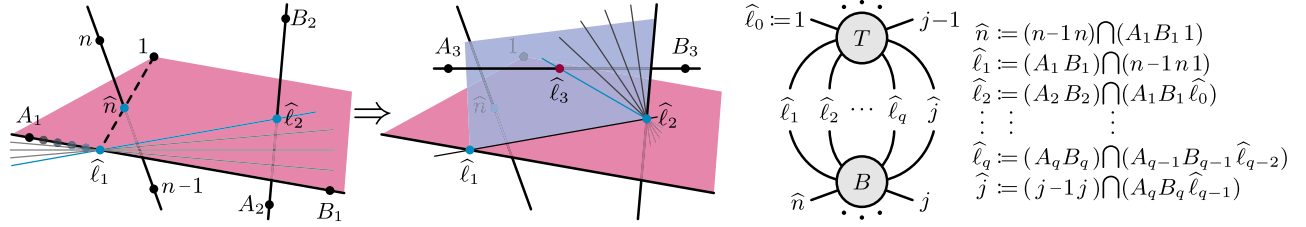
FIG. 1. Illustration of iterated BCFW deformations and resulting locations in momentum twistor space. In the first deformation, z_n is translated by z_{n-1} until it intersects the plane $(\ell_1 1)$ at the point $\hat{n} := (n-1n) \cap (A_1 B_1 1)$ (cutting the local pole $\langle \ell_1 n1 \rangle$); in this configuration, we may translate A_1 to $\hat{\ell}_1 := (A_1 B_1) \cap (n-1n1)$. After this, we perform a second BCFW deformation by translating B_1 in the direction of $\hat{n}$ [so as to remain in the plane $(A_1 B_1 1)$] until it intersects the line $(A_2 B_2)$ representing ℓ_2 at the point $\hat{\ell}_2 := (A_2 B_2) \cap (A_1 B_1 1)$—which cuts the internal propagator $\langle \ell_1 \ell_2 \rangle$. In the second figure, this sequence is continued, by deforming ℓ_2 in the plane $(A_2 B_2 \hat{\ell}_1)$ until it intersects the line $(A_3 B_3)$ at the point $\hat{\ell}_3 := (A_3 B_3) \cap (A_2 B_2 \hat{\ell}_1)$, and so on.

However, the computational complexities of analytically taking lower-loop amplitudes in the forward limit have continued to stymie implementation or wide applications.

The loop integrand obtained in this fashion, after symmetrizing in the loop variables x_{ℓ_i}, is a well-defined and gauge-invariant function (as opposed to an equivalence class modulo shifts, which it normally is in a nonplanar theory). It is conjectured to be the canonical volume form on the amplituhedron [20,21] and can be alternatively computed using a Wilson loop in twistor space [22]. By exploiting the duality between Wilson loops and scattering amplitudes in the sYM theory (see [23–25]), it is also equal to a correlator of null polygonal Wilson loops with Lagrangian insertions [26] and can be obtained from the null limit of stress-tensor multiplet correlators [27–29]. The latter has been discussed recently for massive amplitudes along the Coulomb branch [30], but we focus here on the massless case where we benefit from the massless momentum twistor formalism.

III. CUTS FROM SEQUENCES OF BCFW DEFORMATIONS

We would like to simplify the forward-limit term by recursing it in a way that will allow us to compute it analytically. Let us choose to label the *first* loop momentum cut to be "ℓ_1." As the amplitude depends only on the *line* $(A_1 B_1)$, we may freely fix A_1 to be a special point on this line: $A_1 \mapsto \hat{\ell}_1 := (A_1 B_1) \cap (n-1n1)$—that is, $(A_1 B_1) \simeq (\hat{\ell}_1 B_1)$. Now consider a BCFW deformation which translates B_1 in the direction of $\hat{n}$—preserving the plane $(A_1 B_1 1) \simeq (A_1 B_1 \hat{n})$.

With this deformation, physical poles will correspond to places where the deformed line $(\hat{\ell}_1 B_1)$ intersects either an external line $(j-1j)$ or an internal line corresponding to another loop momentum, say $\ell_2 := (A_2 B_2)$. The former results in the second term of (1), and the latter corresponds to a diagram with two particles taken in the forward limit (two internal lines on shell). To resolve the latter, we repeat this procedure by translating $A_2 \mapsto \hat{\ell}_2$ and deforming B_2, and so on, until all loop momenta have been exhausted—at which point the final ℓ_L can be deformed to expose one final "factorization" involving either an external or already-localized internal line $(\hat{\ell}_{a-1}\hat{\ell}_a)$.

This sequence of BCFW deformations and cut solutions is illustrated in Fig. 1. Letting the ath loop momentum be encoded by the line $(A_a B_a)$ in twistor space, the cut points appearing in the q-loop bridge take the form summarized on the right-hand side of Fig. 1.

A subtlety which will be clarified below is that the final line cut through this process need not involve any external momenta: the line "$(j-1j)$" may be chosen from any pair of adjacent elements of the list $(\hat{\ell}_{q-2}, \hat{\ell}_{q-3}, \ldots, \hat{\ell}_1, 1, \ldots, n-1, \hat{n}, \hat{\ell}_1, \hat{\ell}_2, \ldots, \hat{\ell}_{q-2})$.

A. EXPLICIT FORM OF THE "BRIDGED" AMPLITUDES RESULTING FROM RECURSION

When q internal loop momenta are cut via the sequence of BCFW deformations described above, the resulting residue of the loop integrand takes the form

$$:= \int \prod_{a=2}^{q} d^4 \eta_{\hat{\ell}_a} \left\{ \mathcal{A}_T^{L_T}(\hat{\ell}_q \hat{\ell}_{q-1} \hat{\ell}_{q-2} \cdots \hat{j}) \right.$$
$$\mathcal{B}_q[1, (j-1j), (n-1n)]$$
$$\left. \mathcal{A}_B^{L_B}(\hat{j} \cdots \hat{\ell}_{q-2} \hat{\ell}_{q-1} \hat{\ell}_q) \right\} \tag{7}$$

where $L_T + L_B = L - q$ and the bridge factors "$\mathcal{B}_q$" are derived by performing the GL_2 integrals required for the forward limit (see [2]) within products of R invariants. The 0-loop bridge $\mathcal{B}_0$ was given above in (5); for $q = 1$ it is

$$\mathcal{B}_1[1, (j-1\,j), (n-1\,n)] := \frac{\langle \ell_1(j-1\,j1) \cap (n-1\,n1) \rangle^2}{\langle \ell_1 j-1\,j \rangle \langle \ell_1 j1 \rangle \langle \ell_1 1\,j-1 \rangle \langle \ell_1 n-1\,n \rangle \langle \ell_1 n1 \rangle \langle \ell_1 1\,n-1 \rangle}, \tag{8}$$

which appears as the "kermit" of Refs. [7,31]; and for $q \geq 2$ we obtain the new result

$$\mathcal{B}_q\limits_{(q\geq2)}[1, (j-1\,j), (n-1\,n)] := \frac{\langle (n-1\,n1)\hat{\ell}_2 \rangle (\langle \hat{\ell}_0 \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 \rangle \langle \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 \hat{\ell}_4 \rangle \cdots \langle \hat{\ell}_{q-3} \hat{\ell}_{q-2} \hat{\ell}_{q-1} \hat{\ell}_q \rangle \langle \hat{\ell}_{q-2} \hat{\ell}_{q-1} \hat{\ell}_q \hat{j} \rangle) \langle \hat{\ell}_{q-1} \hat{\ell}_q (j-1\,j) \rangle}{\langle \ell_1 n-1\,n \rangle \langle \ell_1 n1 \rangle \langle \ell_1 1\,n-1 \rangle (\langle \ell_1 \ell_2 \rangle \cdots \langle \ell_{q-1} \ell_q \rangle) \langle \ell_q j-1\,j \rangle \langle \ell_q j \hat{\ell}_{q-1} \rangle \langle \ell_q \hat{\ell}_{q-1} j-1 \rangle}. \tag{9}$$

Notice that the lower-loop amplitudes being bridged have an excess R-charge of $4(1-q)$, which is compensated by the $(q-1)$ $d^4 \eta_{\hat{\ell}_a}$ Grassmann integrations in (7). It is worth emphasizing that the expression (7) is exactly dual-conformal [although not manifest in the expression given in (9)]: the factors of $\mathcal{B}_q$ carry net weight $+4$ in $\hat{\ell}_2, \ldots, \hat{\ell}_q$ (which is compensated by the $\eta_{\hat{\ell}_a}$-integrations) and weight -4 in $\ell_1, \ldots, \ell_q$.

Finally, all results must be symmetrized over the L lines $\{(A_1 B_1), \ldots (A_L B_L)\}$.

IV. NOVELTY RELATIVE TO ORDINARY "UNITARITY CUTS"

Although a propagator involving each loop momentum is cut sequentially in the recursion, poles resulting from the final BCFW deformation [of ℓ_q within the plane $(A_{q-1} B_{q-1} \hat{\ell}_{q-2})$] need not involve an "external" propagator: in addition to the external poles of the form $\langle A_q B_q j-1\,j \rangle$, there can also be those involving the lines $(\hat{\ell}_1 1)$ or $(\hat{n} \hat{\ell}_1)$, or even the purely "internal" poles of the form $\langle A_q B_q \hat{\ell}_{a-1} \hat{\ell}_a \rangle$ for any $a \leq (q-2)$.

Examples of the first type of novelty are illustrated in the first two figures in Fig. 2, respectively. These yield a product of two amplitudes in which one of the factors only contains internal lines. This is possible since, in contrast to the Cutkosky rules [32], there is no requirement that all energies in our complexified cuts be positive. These first appear at three loops as all amplitudes being bridged must involve at least four legs.

The second type of novelty is illustrated in the last two examples of Fig. 2, where an internal propagator is cut at the final stage of the recursion and one finds a completely internal island on one side of the bridge. This "cut" is unquestionably unusual and in fact involves one amplitude with at least one pair of (necessarily nonadjacent) legs taken in the forward limit. This may seem problematic, since the complexity involved in evaluating forward limits required by the recursion formula of [2] was a primary technical obstruction motivating our present work. Nevertheless, we find that nonadjacent forward limits are rather benign and that any term that would require delicacy vanishes directly upon $d^4 \eta_{\hat{\ell}_a}$ integration. Thus, all terms with the requisite $\eta_{\hat{\ell}_a}$ support may be evaluated naively in the forward limit.

It is interesting to count the number of terms $N(n, k, L)$ generated by applying the recursion (1) for the n-point N^kMHV amplitude at L loops. Defining a generating function $\mathcal{G}(\nu, \kappa, \lambda) =: \sum_{n,k,L} N(n, k, L) \nu^n \kappa^k \lambda^L$ with $n \geq 4$, the recursion implies that

$$\mathcal{G} = \nu^4 + \nu(1 + \kappa)\mathcal{G} + \left(\frac{\kappa^2 \mathcal{G}^2}{\nu^2 \kappa - \lambda} \right)\Big|_{\text{reg}}, \tag{10}$$

$$\mathcal{A}_T(\hat{\ell}_4, \hat{\ell}_3, \hat{\ell}_2, \hat{\ell}_1, \hat{j})$$
$$\mathcal{A}_B(\hat{j}, 1, \ldots, \hat{n}, \hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3, \hat{\ell}_4)$$
$$\text{with } \hat{j} := (\hat{\ell}_1 1) \bigcap (\hat{\ell}_4 \hat{\ell}_3)$$

$$\mathcal{A}_T(\hat{\ell}_4, \hat{\ell}_3, \hat{\ell}_2, \hat{\ell}_1, 1, \ldots, \hat{n}, \hat{j})$$
$$\mathcal{A}_B(\hat{j}, \hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3, \hat{\ell}_4)$$
$$\text{with } \hat{j} := (\hat{n} \hat{\ell}_1) \bigcap (\hat{\ell}_4 \hat{\ell}_3)$$

$$\mathcal{A}_T(\hat{\ell}_4, \hat{\ell}_3, \hat{\ell}_2, \hat{j})$$
$$\mathcal{A}_B(\hat{j}, \hat{\ell}_1, 1, \ldots, \hat{n}, \hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3, \hat{\ell}_4)$$
$$\text{with } \hat{j} := (\hat{\ell}_2 \hat{\ell}_1) \bigcap (\hat{\ell}_4 \hat{\ell}_3)$$

$$\mathcal{A}_T(\hat{\ell}_8, \hat{\ell}_7, \hat{\ell}_6, \hat{\ell}_5, \ldots, \hat{\ell}_2, \hat{\ell}_1, 1, \ldots, \hat{n}, \hat{\ell}_1, \hat{\ell}_2, \ldots, \hat{\ell}_5, \hat{j})$$
$$\mathcal{A}_B(\hat{j}, \hat{\ell}_6, \hat{\ell}_7, \hat{\ell}_8) \text{ with } \hat{j} := (\hat{\ell}_5 \hat{\ell}_6) \bigcap (\hat{\ell}_8 \hat{\ell}_7)$$
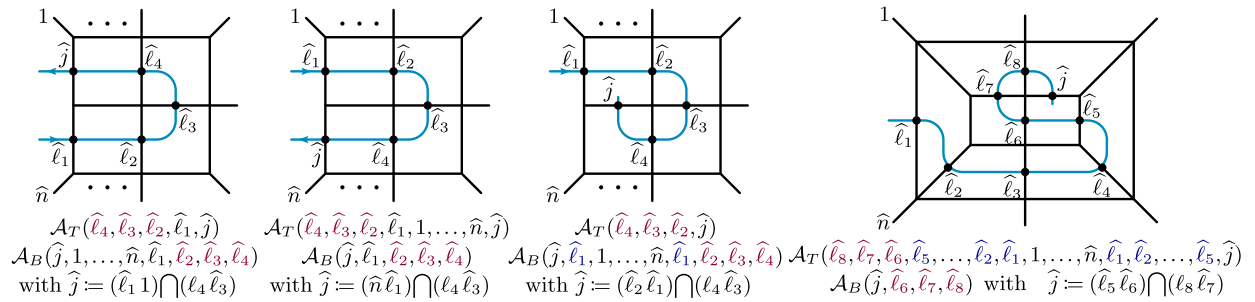
FIG. 2.   Examples of novel cut sequences appearing in the recursion. The first two figures illustrate diagrams where one amplitude involves no external momenta while the third and fourth figures illustrate terms where one side of the bridge is entirely contained within the other.

where the second term accounts for tree-level bridges involving a three-point vertex of either parity, and the quadratic term with $\frac{\kappa^2}{\nu^2\kappa-\lambda} = \frac{\kappa}{\nu^2} + \frac{\lambda}{\nu^4} + \frac{\lambda^2}{\nu^6\kappa} + \cdots$ accounts for all the other bridges. The "reg." operation removes any terms with negative or too-large powers of κ so as to maintain the N^kMHV degree within the range $k \in [0, n-4]$—terms outside this range vanish trivially upon Grassmann integration in (7). The solution for $\lambda = 0$ gives the Catalan numbers familiar from tree-level recursion (see e.g. [18]), while the one-loop counting agrees with [18]. We do not know a closed form for $N(n, k, L)$ for $L \geq 2$, but these numbers agree with the number of terms produced by the function `preAmp[]` in the attached package.

We found that a significant (and recursion-scheme-dependent) fraction of these terms vanish upon Grassmann integration. At three loops, for example, we have found expressions for the 4-particle amplitude involving anywhere between 88 and 98 nonvanishing terms after Grassmann integration (significantly fewer than the 146 terms prior to Grassmann integration). In the included *Mathematica* package, we default to a recursion scheme wherein non-cut-loop variables are preferentially chosen for subsequent recursion, but we have also included a function `superAmpRandom[]` which randomly selects the legs to be BCFW deformed at each step of the recursion. The agreement between the different resulting expressions is a powerful consistency check on our implementation of the all-loop formula (1).

V. CONCLUSIONS AND DISCUSSION

In this paper, we have used a sequence of BCFW deformations to arrive at a novel, recursive representation of any L-loop, n-point N^kMHV amplitude in planar, maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory in terms of bridges between lower-loop amplitudes in the theory. By avoiding any reference to problematic forward limits, the formula given in (1) can be directly used to evaluate amplitude integrands well beyond the reach of existing, general algorithms. We have implemented these tools in a relatively simple *Mathematica* package, included among the Supplemental Material [33]. (These tools have been built using those included in the works of [6–8].) We have explicitly checked that the results for our integrands match the local expressions for four points through five loops (see e.g. [34–38]).

It has been conjectured that all multiplicity, lower-loop amplitudes can be determined via an n-point lightlike limit of the 4-point correlation function at sufficiently high loops [27–29,39–42]; in the case of 5 particles, this encoding is explicit, giving the complete L-loop integrand in terms of the $(L + 2)$-loop 4-point correlator [this includes an extra loop needed to tease out the parity-odd part of the integrand (see e.g. [28,42,43])]. We have checked this result explicitly through 5 loops, which can be viewed as both a check on the correctness of the n-point projection from the 4-point correlator and on the correlator of stress tensor itself, as taken from [44–46].

Although transparent and relatively efficient, it is worth noting that—unlike BCFW applied to amplitudes at tree level—the representations that result from (1) are *far* from compact, dramatically exceeding the number of terms required in "local" integrand formulas such as those of [7,8], for example. The four-particle integrand at 5-loop order (before loop label symmetrization) is given by 23,072 terms by our package (using default options for how bridged amplitudes are recursed); this is far greater than representations of this amplitude in terms of local integrands. For example, even including the dihedral images of the 34 archetype terms of the representation of [38], the local loop integrand representation involves a mere 193 terms.

This "inefficiency" is easy to understand: each term appearing in (1) exposes a very specific sequence of physical cuts whereas a single local integrand can capture the contributions from many individual cuts simultaneously. This inefficiency may reflect the eventual tension between the singularity structure of particular amplitudes and the locality of individual Feynman integrals: our representation should represent a triangulation of the amplituhedron, related to a particular "$d\log$" form [47] in the Grassmannian; as such, it might do a better job of exposing the analytic structure of amplitudes (or even perhaps be easier to integrate—once IR regularization and loop integration are understood for these expressions [48]) than expressions involving Feynman-diagram-like master integrands.

It would be interesting to know if a formula such as (1) could be applied to less than maximally supersymmetric theories, or to maximally supersymmetric theories beyond the planar limit.

[1] H. Elvang and Y.-t. Huang, Scattering amplitudes, arXiv:1308.1697.

[2] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot, and J. Trnka, The all-loop integrand for scattering amplitudes in planar $\mathcal{N} = 4$ SYM, J. High Energy Phys. 01 (2011) 041.

[3] S. Caron-Huot, Loops and trees, J. High Energy Phys. 05 (2011) 080.

[4] P. Benincasa, On-shell diagrammatics and the perturbative structure of planar gauge theories, arXiv:1510.03642.

[5] P. Benincasa and D. Gordo, On-shell diagrams and the geometry of planar $\mathcal{N} < 4$ SYM theories, J. High Energy Phys. 11 (2017) 192.

[6] J. L. Bourjaily, Efficient tree-amplitudes in $\mathcal{N} = 4$: Automatic BCFW recursion in *Mathematica*, arXiv:1011.2447.

[7] J. L. Bourjaily, S. Caron-Huot, and J. Trnka, Dual-conformal regularization of infrared loop divergences and the chiral box expansion, J. High Energy Phys. 01 (2015) 001.

[8] J. L. Bourjaily and J. Trnka, Local integrand representations of all two-loop amplitudes in planar SYM, J. High Energy Phys. 08 (2015) 119.

[9] R. Britto, F. Cachazo, and B. Feng, New recursion relations for tree amplitudes of gluons, Nucl. Phys. **B715**, 499 (2005).

[10] R. Britto, F. Cachazo, B. Feng, and E. Witten, Direct Proof of Tree-Level Recursion Relation in Yang-Mills Theory, Phys. Rev. Lett. **94**, 181602 (2005).

[11] N. Arkani-Hamed and J. Kaplan, On tree amplitudes in gauge theory and gravity, J. High Energy Phys. 04 (2008) 076.

[12] A. Hodges, Eliminating spurious poles from gauge-theoretic amplitudes, J. High Energy Phys. 05 (2013) 135.

[13] R. Penrose, Twistor algebra, J. Math. Phys. (N.Y.) **8**, 345 (1967).

[14] J. Drummond, J. Henn, V. Smirnov, and E. Sokatchev, Magic identities for conformal four-point integrals, J. High Energy Phys. 01 (2007) 064.

[15] J. M. Drummond, J. M. Henn, and J. Plefka, Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory, J. High Energy Phys. 05 (2009) 046.

[16] L. Mason and D. Skinner, Dual superconformal invariance, momentum twistors and Grassmannians, J. High Energy Phys. 11 (2009) 045.

[17] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, and J. Trnka, Unification of residues and Grassmannian dualities, J. High Energy Phys. 01 (2011) 049.

[18] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, and J. Trnka, *Grassmannian Geometry of Scattering Amplitudes* (Cambridge University Press, Cambridge, England, 2016).

[19] J. Drummond, J. Henn, G. Korchemsky, and E. Sokatchev, Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory, Nucl. Phys. **B828**, 317 (2010).

[20] N. Arkani-Hamed and J. Trnka, The amplituhedron, J. High Energy Phys. 10 (2014) 030.

[21] N. Arkani-Hamed and J. Trnka, Into the amplituhedron, J. High Energy Phys. 12 (2014) 182.

[22] L. Mason and D. Skinner, The complete planar S-matrix of $\mathcal{N} = 4$ SYM as a Wilson loop in twistor space, J. High Energy Phys. 12 (2010) 018.

[23] L. F. Alday and J. M. Maldacena, Gluon scattering amplitudes at strong coupling, J. High Energy Phys. 06 (2007) 064.

[24] N. Berkovits and J. Maldacena, Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection, J. High Energy Phys. 09 (2008) 062.

[25] N. Beisert, R. Ricci, A. A. Tseytlin, and M. Wolf, Dual superconformal symmetry from AdS$_5 \times S^5$ superstring integrability, Phys. Rev. D **78**, 126004 (2008).

[26] S. Caron-Huot, Notes on the scattering amplitude/Wilson loop duality, J. High Energy Phys. 07 (2011) 058.

[27] B. Eden, P. Heslop, G. P. Korchemsky, and E. Sokatchev, The super-correlator/super-amplitude duality: Part I, Nucl. Phys. **B869**, 329 (2013).

[28] B. Eden, P. Heslop, G. P. Korchemsky, and E. Sokatchev, The super-correlator/super-amplitude duality: Part II, Nucl. Phys. **B869**, 378 (2013).

[29] T. Adamo, M. Bullimore, L. Mason, and D. Skinner, A proof of the supersymmetric correlation function/Wilson loop correspondence, J. High Energy Phys. 08 (2011) 076.

[30] S. Caron-Huot and F. Coronado, Ten dimensional symmetry of $\mathcal{N} = 4$ SYM correlators, J. High Energy Phys. 03 (2022) 151.

[31] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. Hodges, and J. Trnka, A note on polytopes for scattering amplitudes, J. High Energy Phys. 04 (2012) 081.

[32] R. E. Cutkosky, Singularities and discontinuities of Feynman amplitudes, J. Math. Phys. (N.Y.) **1**, 429 (1960).

[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.108.025008, for the included Mathematica notebook and codebase implement our main results, allowing the user to generate closed-form expressions for the L-loop integrand for the n-particle N^{k}MHV amplitude, including those obtained via randomized recursion schemata (to internally verify our results).

[34] Z. Bern, J. Rozowsky, and B. Yan, Two-loop four-gluon amplitudes in $\mathcal{N} = 4$ super-Yang-Mills, Phys. Lett. B **401**, 273 (1997).

[35] Z. Bern, L. J. Dixon, and D. A. Kosower, Two-loop splitting amplitudes, Nucl. Phys. B, Proc. Suppl. **135**, 147 (2004).

[36] Z. Bern, L. J. Dixon, and V. A. Smirnov, Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond, Phys. Rev. D **72**, 085001 (2005).

[37] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang-Mills theory, Phys. Rev. D **75**, 085010 (2007).

[38] Z. Bern, J. Carrasco, H. Johansson, and D. Kosower, Maximally supersymmetric planar Yang-Mills amplitudes at five loops, Phys. Rev. D **76**, 125020 (2007).

[39] L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena, and E. Sokatchev, From correlation functions to Wilson loops, J. High Energy Phys. 09 (2011) 123.

[40] B. Eden, G. P. Korchemsky, and E. Sokatchev, From correlation functions to scattering amplitudes, J. High Energy Phys. 12 (2011) 002.

[41] B. Eden, G. P. Korchemsky, and E. Sokatchev, More on the duality correlators/amplitudes, Phys. Lett. B **709**, 247 (2012).

[42] P. Heslop and V.-V. Tran, Multi-particle amplitudes from the four-point correlator in planar $\mathcal{N} = 4$ SYM, J. High Energy Phys. 07 (2018) 068.

[43] R. G. Ambrosio, B. Eden, T. Goddard, P. Heslop, and C. Taylor, Local integrands for the five-point amplitude in planar $\mathcal{N} = 4$ SYM up to five loops, J. High Energy Phys. 01 (2015) 116.

[44] J. L. Bourjaily, A. DiRe, A. Shaikh, M. Spradlin, and A. Volovich, The soft-collinear bootstrap: $\mathcal{N} = 4$ Yang-Mills amplitudes at six and seven loops, J. High Energy Phys. 03 (2012) 032.

[45] J. L. Bourjaily, P. Heslop, and V.-V. Tran, Perturbation Theory at Eight Loops: Novel Structures and the Breakdown of Manifest Conformality in $\mathcal{N} = 4$ Super-symmetric Yang-Mills Theory, Phys. Rev. Lett. **116,** 191602 (2016).

[46] J. L. Bourjaily, P. Heslop, and V.-V. Tran, Amplitudes and correlators to ten loops using simple, graphical bootstraps, J. High Energy Phys. 11 (2016) 125.

[47] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, and J. Trnka, Singularity Structure of Maximally Supersymmetric Scattering Amplitudes, Phys. Rev. Lett. **113,** 261603 (2014).

[48] A. E. Lipstein and L. Mason, From $d$logs to dilogs; the super Yang-Mills MHV amplitude revisited, J. High Energy Phys. 01 (2014) 169.