Cylindrical black bounces and their field sources

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We apply the Simpson-Visser phenomenological regularization method to a cylindrically symmetric solution of the Einstein-Maxwell equations known as an inverted black hole. In addition to analyzing some properties of thus regularized space-time, including the Carter-Penrose diagrams, we show that this solution can be obtained from the Einstein equations with a source combining a phantom scalar field with a nonzero self-interaction potential and a nonlinear magnetic field. A similar kind of source is obtained for the cylindrical black bounce solution proposed by Lima *et al.* as a regularized version of Lemos's black string solution. Such sources are shown to be possible for a certain class of cylindrically, planarly, and toroidally symmetric metrics that includes the regularized solutions under consideration.

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I. INTRODUCTION

Space-time singularities are known to be an undesired though common feature of general relativity (GR) and other classical theories of gravity. A natural hope that all singularities will be inevitably suppressed by quantum gravity effects is made somewhat uncertain by the fact that different models and approaches of quantum gravity, being translated to the classical language, generally lead to quite different results, see, e.g., [1–9], see also a discussion in [10]. One can find among them black-hole–white-hole transitions [2–6], scenarios where the spherical radius beyond a black-hole horizon tends to a constant positive value [7], configurations having no horizons at all [9], etc. Such a diversity evidently indicates a still unfinished situation in developing quantum gravity by now.

It is therefore natural that attempts to simulate the possible effects of quantum gravity in the framework of classical space-time, leaving aside the details of quantization, cause rather large interest. The recent proposal of Simpson and Visser (SV) [11] can be regarded as one of such attempts: they regularized the Schwarzschild metric by replacing the spherical radius r with the nonzero expression $r(u) = \sqrt{u^2 + b^2}$ ($b > 0$ is a regularization parameter), thus avoiding the Schwarzschild singularity at $r = 0$. With different values of b, the resulting geometry can represent a wormhole (if $b > 2m$, where m is the Schwarzschild mass), a black hole with two horizons if $b < 2m$, and an intermediate case of an extremal black hole with a single horizon if $b = 2m$. In the black-hole case $b < 2m$, the subhorizon region is a Kantowski-Sachs type cosmology in which u is a temporal coordinate, and the hypersurface $u = 0$, where the radius $r(u)$ has a regular minimum, is actually a bounce of one of the two scale factors. This phenomenon was termed a *black bounce* [11]. One can also recall that such black bounces are a common feature of one more class of space-time models, called *black universes*, that is, black holes in which the internal region is a Kantowski-Sachs cosmology with late-time isotropization [12–14]. Such solutions have been obtained in GR with a phantom scalar field as well as in some scalar-tensor theories [15–17]. In the case $b = 2m$, the hypersurface $u = 0$ is null, being simultaneously a black-hole horizon and a throat, thus it may be called a *black throat* [18].

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Later on, a similar regularization was considered for the Reissner-Nordström space-time [19]. Lobo *et al.* [20] constructed a large class of regular black-hole and wormhole space-times. The diversity and richness of geometries obtained in this manner have attracted much attention, and their rotating extensions and those with a Newman-Unti-Tamburino (NUT) charge were also obtained and studied [21–25]. A further analysis concerned gravitational wave echoes at possible black-hole/wormhole transitions, quasinormal modes, and gravitational lensing in such space-times [26–36].

A question of interest emerging in this connection is whether these new regular geometries can be presented as solutions of GR with some reasonable sources, such as, for example, scalar and/or electromagnetic fields. As shown in [18] and [37,38], a large class of SV-like regular metrics, including the regularized Schwarzschild and Reissner-Nordström metrics, can be viewed as exact solutions to the Einstein equations with a sum of two stress-energy tensors (SETs): that of a minimally coupled self-interacting phantom scalar field and that of a nonlinear electromagnetic field in the framework of nonlinear electrodynamics (NED) with a Lagrangian as a function of the invariant $F = F_{\mu\nu}F^{\mu\nu}$. A scalar field or NED taken separately cannot be such a source due to algebraic properties of their SETs. For regularized Schwarzschild and Reissner-Nordström solutions, the explicit forms of scalar and NED constructed were found as well as their global structure diagrams for the cases of three and four horizons. Similar regularizations for some cosmological models were considered in [39].

In [40], similar regularizations were obtained for two other families of singular static, spherically symmetric solutions of GR: Fisher's solution with a massless scalar field [41] and a special subset of dilatonic black-hole solutions whose source consists of interacting scalar and electromagnetic fields [42–45]. In both cases, the SV trick ($x \mapsto \sqrt{u^2+b^2}$) is applied in the simplest way to a factor x producing a space-time singularity at its zero value. Scalar-NED sources have also been found in [40] for regularized versions of these space-times, and it turned out that the necessary scalar fields cannot be only canonical (possessing positive kinetic energy) or only phantom (having negative kinetic energy), but change their nature from one region to another in a regular manner. A situation where a scalar is phantom in a strong-field region and canonical elsewhere has been termed a "trapped ghost" [46], and some globally regular black-hole and wormhole solutions with such fields were obtained [46–48].

As a by-product, it has been shown in [40] that a sum of NED and minimally coupled scalar SETs can provide a source for *any* static, spherically symmetric metric, but in general the scalar can somewhere change its nature from canonical to phantom. A way to describe an arbitrary static, spherically symmetric metric using a single field source was found in [49] in the framework of the general (Bergmann-Wagoner-Nordtvedt) scalar-tensor theory of gravity, but such a description turned out to be possible only piecewise. This was demonstrated using as examples the Reissner-Nordström metric and the regularized Schwarzschild metric according to [11].

One more family of black-hole space-times, those with cylindrical symmetry, called black strings [50,51], with the metric

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2(dz^2 + d\varphi^2),$$
$$A(r) = \alpha^2 r^2 - \frac{b}{\alpha r}, \qquad \alpha, b = \text{const} > 0, \tag{1}$$

singular at $r = 0$, was also regularized in the manner of [11], by replacing $r \to \sqrt{u^2+a^2}$, $a > 0$, with, however, $dr \to du$ [52]. (Note that r, u, a, α^{-1} have here the dimension of length while z and b are dimensionless.[1]) The resulting nonsingular metric has the form

$$ds^2 = A(u)dt^2 - \frac{du^2}{A(u)} - (u^2+a^2)(dz^2 + d\varphi^2),$$
$$A(u) = \alpha^2(u^2+a^2) - \frac{b}{\alpha\sqrt{u^2+a^2}}. \tag{2}$$

Both metrics (1) and (2) are asymptotically AdS at large r or u, with the cosmological constant $\Lambda = -\alpha^2/3$. A further study of the metric (2) in [52] concerned the energy conditions, the thermodynamic properties of regular black strings, and possible stable and unstable circular orbits of photons and massive particles.

In the present paper, we study a similar regularization of another static, cylindrically symmetric metric of interest, called "inverted black holes" and being a special solution to the Einstein-Maxwell equations [53,54]. The metric has the form

$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - \frac{q^2x^2}{4k^2}(dz^2 + d\varphi^2),$$
$$A(x) = \frac{16k^4(1-x)}{q^2x^2}, \tag{3}$$

where q (characterizing the electric or magnetic charge density) and k are positive constants. The name "inverted black holes" was proposed in [53] because, contrary to "normal" black holes, a static region $x < 1$ in (3) with a singularity at $x = 0$ occurs at smaller values of the circular radius $r(x) = qx/(2k)$, inside the horizon $x = 1$, while the region $x > 1$ where $A < 0$ is nonstatic and represents a special kind of Bianchi-type I cosmology.

[1]We adopt the metric signature $(+,-,-,-)$ and use the geometric units where $8\pi G = c = 1$.

We regularize the metric (3) as before, replacing $x \to \sqrt{u^2+a^2}$, to obtain

$$ds^2 = A(u)dt^2 - \frac{du^2}{A(u)} - \frac{q^2(u^2+a^2)}{4k^2}(dz^2+d\varphi^2),$$
$$A(u) = \frac{16k^4(1-\sqrt{u^2+a^2})}{q^2(u^2+a^2)}, \qquad (4)$$

and study its properties in a manner similar to [52].

In addition to such regularizations, we construct the Carter-Penrose global structure diagrams and determine possible field sources for both regularized metrics (2) and (4) in the framework of GR similarly to [18] in terms of a combination of NED and a self-interacting scalar field.

It is necessary to mention that all the metrics (1)–(4) can be interpreted not only as cylindrically symmetric ones [such that $z \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$] but also in terms of two other kinds of symmetry: planar, if both $z \in \mathbb{R}$ and $\varphi \in \mathbb{R}$, and toroidal, if both z and φ range on finite segments with identified ends. All local quantities discussed in this paper do not depend on these topological assumptions. We will adhere to interpretations in terms of cylindrical symmetry that seems to be of greater interest than the other two.

The paper is organized as follows. We begin the next Sec. II with giving some general relations for cylindrically symmetric metrics and then consider some properties of the regularized metrics (2) and (4). Section III is devoted to finding and discussing field sources for these metrics, and Sec. IV is a conclusion.

II. REGULARIZED METRICS

A. General relations

In general, the line element that describes static space-times with cylindrical symmetry is written as

$$ds^2 = A(x)dt^2 - B(x)dx^2 - C(x)dz^2 - D(x)d\varphi^2. \qquad (5)$$

However, in this paper we are dealing with space-times possessing locally flat orbits of the spatial isometry group, such that $C(x) = D(x)$, and using the coordinate condition $B(x) = 1/A(x)$, the metric can be written in the form[2]

$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)(dz^2 + d\varphi^2), \qquad (6)$$

where we assume $z \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$ according to cylindrical symmetry. The nonzero Riemann tensor components are

$$K_1 = R^{01}{}_{01} = -\frac{1}{2}A'', \qquad K_2 = R^{02}{}_{02} = R^{03}{}_{03} = -\frac{A'r'}{2r},$$
$$K_3 = R^{12}{}_{12} = R^{13}{}_{13} = -\frac{2Ar'' + A'r'}{2r},$$
$$K_4 = R^{23}{}_{23} = -\frac{Ar'^2}{r^2}, \qquad (7)$$

where the coordinates are numbered as $(t, x, z, \varphi) = (0, 1, 2, 3)$, and primes denote d/dx. Since the Kretschmann invariant $\mathcal{K} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4K_1^2 + 8K_2^2 + 8K_3^2 + 4K_4^2$ is a sum of squares, it is clear that the finiteness of all K_i from (7) is a *necessary and sufficient condition* for finiteness of all algebraic invariants of the Riemann tensor.

Also, in full similarity with the more familiar case of spherical symmetry, regular zeros of $A(x)$ (provided $r \neq 0$) correspond to Killing horizons at which the Killing vector ∂_t that is timelike at $A > 0$, becomes null. At such horizons all K_i are finite and well behaved, thus illustrating the well-known fact that Killing horizons are regular surfaces.

Let us also present the expressions for nonzero components of the Einstein tensor $G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R$, to be used in the Einstein equations $G_\mu^\nu = -T_\mu^\nu$, where T_μ^ν is the stress-energy tensor of matter:

$$G_0^0 = \frac{1}{r^2}[A(2rr'' + r'^2) + A'rr'],$$
$$G_1^1 = \frac{1}{r^2}[Ar'^2 + A'rr'],$$
$$G_2^2 = G_3^3 = \frac{1}{r}\left[Ar'' + A'r' + \frac{1}{2}A''r\right]. \qquad (8)$$

B. The regularized black string

Let us begin with the metrics (1) and (2), already considered in [52]. In terms of (6), in the metric (1) we have

$$r(x) = x, \qquad A(x) = \alpha^2x^2 - b/(\alpha x). \qquad (9)$$

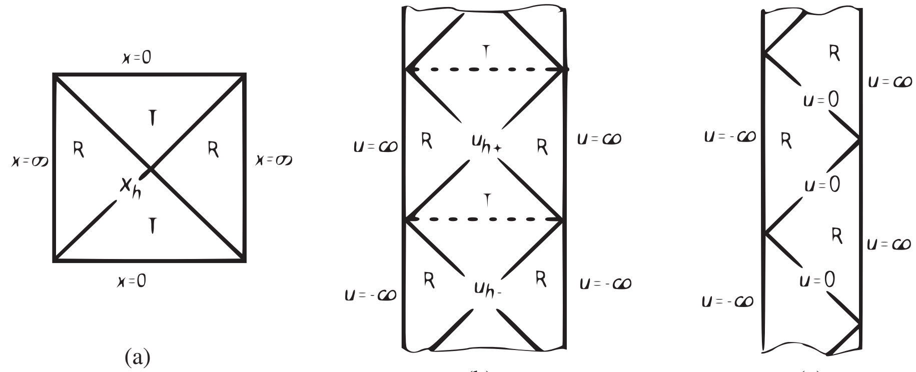
Outside the horizon, at $x > x_h = b^{1/3}/\alpha$, there is a static (R) region with an AdS-like asymptotic behavior at large x. Inside the horizon, at $0 < x < x_h$, there is a dynamic (T) region with a special Bianchi-type I geometry and a singularity at $x = 0$.

The regularized metric (2), with $u \in \mathbb{R}$, has the same AdS-like behavior at large r as the original one, but now this happens at both limits $u \to \pm\infty$ since $r = \sqrt{u^2 + a^2}$.

The global properties of space-time (2) depend on the value of the regularization parameter a:

(i) If $a < x_h$, it is a regular black hole, but now, unlike (1) [see Fig. 1(a)], it has two horizons at $u = u_{h\pm} = \pm\sqrt{x_h^2 - a^2}$ [Fig. 1(b)]. At $u = 0$ there

[2] In the whole paper, to avoid confusion, we denote by r the quantity $\sqrt{-g_{22}} = \sqrt{-g_{33}}$ that has a clear geometric meaning of a scale factor of 2-surfaces (cylinders) parametrized by $x^2 = z$ and $x^3 = \varphi$ (similar to the spherical radius r in the case of spherical symmetry). For the radial coordinate x^1 we use other letters, x or u.



(a) (b) (c)

FIG. 1. Carter-Penrose diagrams for the black string metric (1) [panel (a)] and its regularized version (2): panel (b) for $a < x_h$, and panel (c) for $a = x_h$. All inner lines in the diagrams depict horizons, the letters R and T mark static (R) and dynamic (T) regions, respectively. The dashed line in panel (b) shows the black bounce instant $u = 0$. Diagrams (b) and (c) are infinitely continued upward and downward.

is a minimum of $r(u)$ in a T region, in other words, a black bounce.

(ii) If $a = x_h$, it is a regular black hole with a single extremal horizon at $u = 0$, it separates two R regions [Fig. 1(c)].

(iii) If $a > x_h$, it is a cylindrically symmetric wormhole with a throat at $u = 0$.

The physical properties of space-time (2) are discussed in detail in [52].

C. The regularized inverted black hole

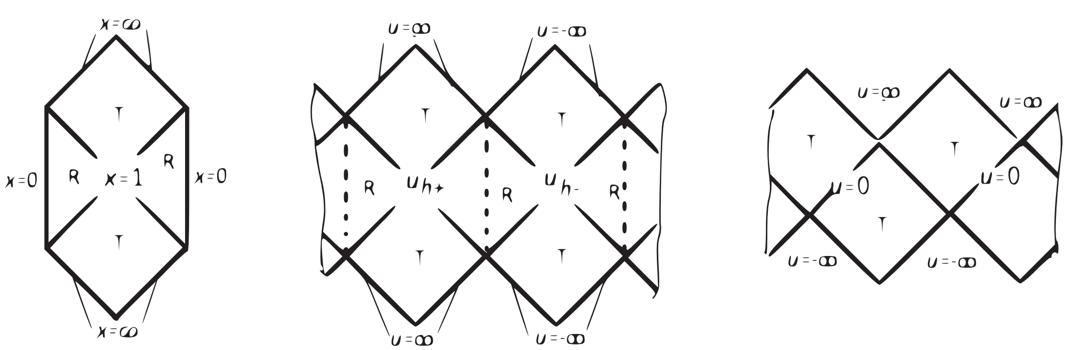
The inverted black-hole line element (3) contains the metric functions in terms of (6)

$$A(x) = \frac{16(1-x)k^4}{q^2x^2}, \qquad r^2(x) = \frac{q^2x^2}{4k^2} > 0. \quad (10)$$

This space-time contains a horizon at $x = 1$ and a singularity at $x = 0$, where $r \to 0$ [a singular axis in cylindrically symmetric space-time that may be interpreted as a charged thread repelling neutral particles since $g_{00} = A(x) \to \infty$ there]. The horizon at $x = 1$ has a common feature with a de Sitter horizon in that the metric has a cosmological, time-dependent nature at larger values of r. The "far end," $x \to \pm\infty$, corresponds to an infinitely remote past or future in which the universe is highly anisotropic (x is a time coordinate there, the scale factor in the t direction, now spatial, is $a_t \sim |x|^{-1/2} \to 0$, while the other two scale factors are $a_z = a_\varphi \sim |x| \to \infty$). The corresponding global causal structure diagram is presented in Fig. 2(a).

In the regularized metric (4), we have, in terms of the new coordinate u,

$$A(u) = \frac{16k^4\left(1-\sqrt{u^2+a^2}\right)}{q^2(u^2+a^2)}, \qquad r^2(u) = \frac{q^2(u^2+a^2)}{4k^2}, \qquad u \in \mathbb{R}. \quad (11)$$



(a) (b) (c)

FIG. 2. Carter-Penrose diagrams for the inverted black-hole metric (3) [panel (a)] and its regularized version (4): panel (b) for $a < 1$, and panel (c) for $a = 1$. All inner solid lines in the diagrams depict horizons, the letters R and T mark static (R) and dynamic (T) regions, respectively. The dashed line in panel (b) shows the wormhole throat $u = 0$. Diagrams (b) and (c) are infinitely continued to the left and to the right.

As was the case with the black string metric, the global properties of this space-time depend on the value of the parameter a:

(i) If $a < 1$, we obtain a cylindrically symmetric wormhole with a throat at $u = 0$, surrounded by two cosmological-type horizons at $u = u_{h\pm} = \pm\sqrt{1-a^2}$ [see Fig. 2(b)].

(ii) If $a = 1$, it is a regular Bianchi-type I cosmological model with a single extremal horizon at $u = 0$ that separates two T regions [Fig. 2(c)].

(iii) If $a > 1$, it is a regular Bianchi-type I cosmological model with a bounce (minimum of r) at $u = 0$.

By substituting (11) to the expressions (7) for K_i it is straightforward to verify that this metric is globally regular. Still we present, for completeness, the corresponding expression for the Kretschmann scalar:

$$\mathcal{K} = \frac{256k^8[-3a^4(u^2-4)-40a^2u^2+9a^6+4u^4(3u^2+14)]}{q^4(a^2+u^2)^6} - \frac{1024k^8(-11a^2u^2+5a^4+12u^4)}{q^4(a^2+u^2)^{11/2}}, \tag{12}$$

where it is evident, in particular, that $\mathcal{K}$ is finite at $u = 0$ and decays as u^{-6} at large $|u|$.

Substituting the quantities (11) to (8), we find the components of the Einstein tensor:

$$G_0^0 = -\frac{16k^4[2a^2(\sqrt{a^2+u^2}-1)+u^2]}{q^2(a^2+u^2)^3}, \tag{13}$$

$$G_1^1 = -\frac{16k^4u^2}{q^2(a^2+u^2)^3}, \tag{14}$$

$$G_2^2 = \frac{8k^4(2u^2-a^2\sqrt{a^2+u^2})}{q^2(a^2+u^2)^3}. \tag{15}$$

Using the Einstein equations, we can find the effective stress-energy tensor that can be a material source of the metric, and analyze the fulfillment or violation of various energy conditions by a source of metric under consideration.

It is, however, necessary to bear in mind that only in a static region ($A > 0$) we deal with the usual relations

$$\rho = -G_0^0, \qquad p_r = G_1^1, \qquad p_\perp = G_2^2, \tag{16}$$

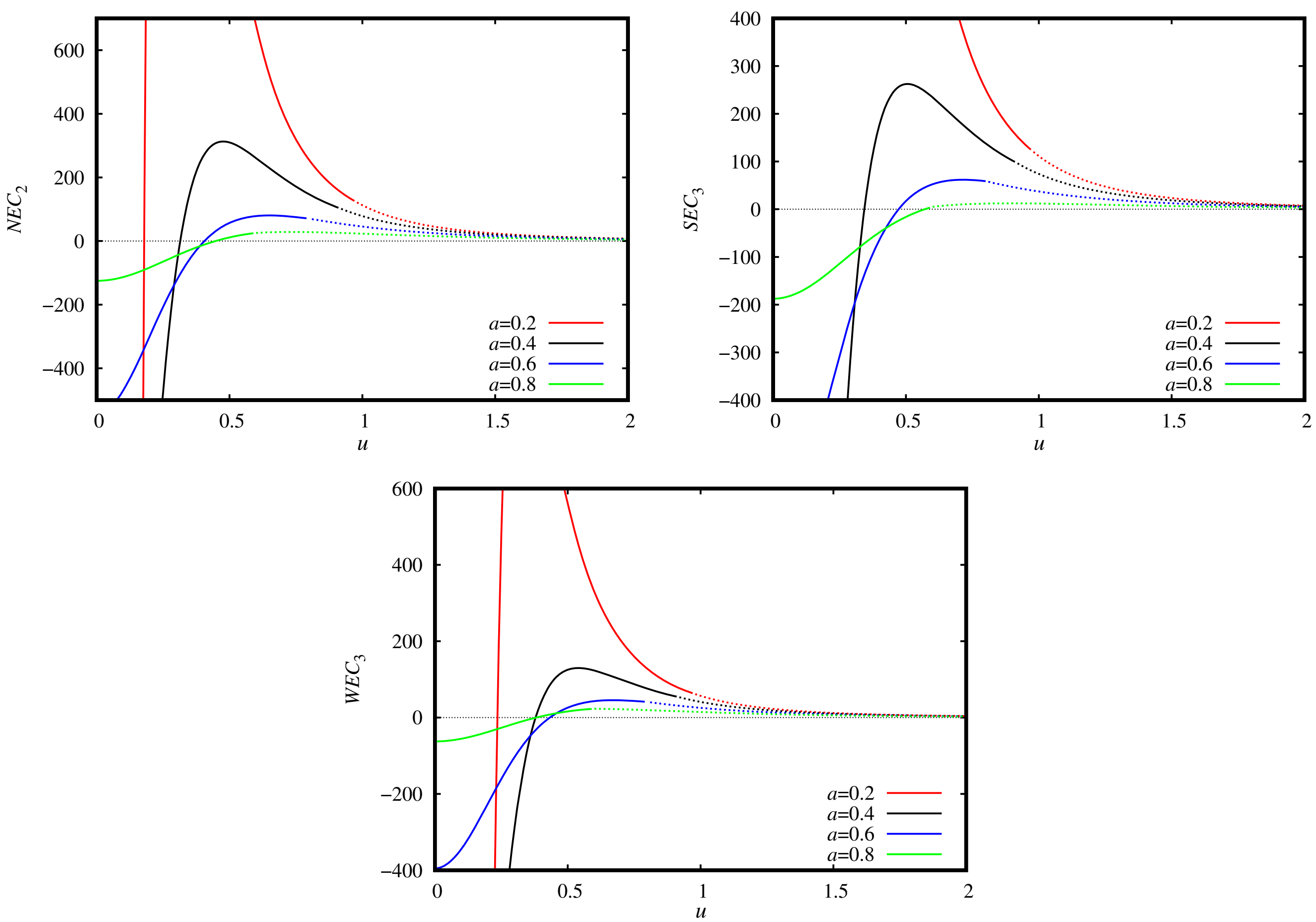


FIG. 3. Plots characterizing the fulfillment of energy conditions at $k = 1$, $q = 0.5$, and different values of a. Solid lines present the functions inside the horizon (at $A > 0$), and dashed lines outside the horizon (at $A < 0$).

where ρ is the density, p_r the radial pressure, and $p_\perp$ the tangential pressure. In T regions, where $A < 0$, we must identify

$$\rho = -G^1_1, \qquad p_r = G^0_0, \qquad p_\perp = G^2_2. \tag{17}$$

As we are working with solutions containing throats or bounces that require exotic sources, it is of interest to analyze the fulfillment of the null (NEC), weak (WEC), strong (SEC), and dominant (DEC) energy conditions: They can be presented as follows:

$$\mathrm{NEC}_{1,2} = \mathrm{WEC}_{1,2} = \mathrm{SEC}_{1,2} \Leftrightarrow \rho + p_{r,\perp} \geq 0, \tag{18}$$

$$\mathrm{SEC}_3 \Leftrightarrow \rho + p_r + 2p_\perp \geq 0, \tag{19}$$

$$\mathrm{DEC}_{1,2} \Rightarrow \rho - p_{r,\perp} \geq 0 \quad \text{and} \quad \rho + p_{r,\perp} \geq 0, \tag{20}$$

$$\mathrm{DEC}_3 = \mathrm{WEC}_3 \Leftrightarrow \rho \geq 0. \tag{21}$$

In essence, all energy conditions are violated once the NEC is violated, and in our case the condition NEC_1 is everywhere violated once $r''/r > 0$. The condition DEC_1 is also violated since it includes NEC_1 as its component. In Fig. 3 we illustrate the behavior of the other energy conditions. At small values of a, these conditions are satisfied outside the horizon (i.e., where $A < 0$).

In these and all subsequent figures we restrict ourselves to positive values of u; for $u < 0$ the plots are unnecessary due to the symmetry $u \leftrightarrow -u$.

III. FIELD SOURCES FOR THE REGULARIZED METRICS

A. General consideration

By analogy with [18,37], let us find field sources for the cylindrically symmetric metrics (2) and (4) in GR as NED plus a self-interacting scalar field, with the total action

$$S = \int \sqrt{-g} d^4x [R + 2h(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) - L(F)], \tag{22}$$

where g is the determinant of the metric $g_{\mu\nu}$, ϕ is the scalar field, $V(\phi)$ is its potential, $L(F)$ is the NED Lagrangian, $F = F^{\mu\nu} F_{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. The function $h(\phi)$ is included here not only for generality (expressing the freedom of scalar field parametrization), but also to include possible cases where ϕ changes its nature from canonical ($h > 0$) to phantom ($h < 0$), exhibiting a "trapped ghost" behavior [40,46].

Varying the action (22) with respect to ϕ, A_μ, and $g^{\mu\nu}$, we obtain the field equations

$$\nabla_\mu [L_F F^{\mu\nu}] = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} L_F F^{\mu\nu}] = 0, \tag{23}$$

$$2h(\phi) \nabla_\mu \nabla^\mu \phi + \frac{dh}{d\phi} \partial^\mu \phi \partial_\mu \phi + \frac{dV(\phi)}{d\phi} = 0, \tag{24}$$

$$G^\nu_\mu = -T^\nu_\mu[\phi] - T^\nu_\mu[F], \tag{25}$$

where $L_F = dL/dF$, $T^\nu_\mu[\phi]$ and $T^\nu_\mu[F]$ are the stress-energy tensors of the scalar and electromagnetic fields, respectively:

$$T^\nu_\mu[F] = \frac{1}{2} \delta^\mu_\nu L(F) - 2 L_F F^{\nu\alpha} F_{\mu\alpha}, \tag{26}$$

$$T^\nu_\mu[\phi] = 2h(\phi) \partial^\nu \phi \partial_\mu \phi - \delta^\nu_\mu (h(\phi) \partial^\alpha \phi \partial_\alpha \phi - V(\phi)). \tag{27}$$

Let us show that any static, cylindrically symmetric metric of the form (6) can be presented as a solution to Eqs. (23)–(25) with a proper choice of ϕ and $F_{\mu\nu}$, quite similarly to the static, spherically symmetric case studied in [40]. The symmetry of the metric (6) makes us assume $\phi = \phi(x)$ and single out among $F_{\mu\nu}$ either a "radial" electric field with the only nonzero component $F_{01} = -F_{10}$ or a "radial" magnetic field with $F_{23} = -F_{32} \neq 0$. For certainty, let us consider a magnetic field, so that $F_{\theta\varphi} = -F_{\varphi\theta} = Q = \text{const}$, and Q may be interpreted as a magnetic charge, while Eq. (23) is trivially satisfied. Then the electromagnetic invariant F is given by $F = 2Q^2/r^4$, and the stress-energy tensors (27) and (26) take the form

$$T^\nu_\mu[\phi] = h(\phi) A(x) \phi'^2 \mathrm{diag}(1, -1, 1, 1) + \delta^\nu_\mu V(\phi), \tag{28}$$

$$T^\nu_\mu[F] = \frac{1}{2} \mathrm{diag}\left(L, L, L - \frac{4Q^2}{r^4} L_F, L - \frac{4Q^2}{r^4} L_F \right). \tag{29}$$

For an arbitrary metric (6), all three components (8) of the Einstein tensor are different; therefore, taken separately, a scalar or electromagnetic field cannot solve the problem because of the equalities $T^0_0[F] = T^1_1[F]$ for the electromagnetic field and $T^0_0[\phi] = T^2_2[\phi]$ for the scalar field. So we will consider them together.

Assuming that $A(x)$ and $r(x)$ are known and substituting them to Eq. (25), we find

$$G^2_2 - G^0_0 = T^0_0 - T^2_2 = \frac{2Q^2}{r^4} L_F \tag{30}$$

[the scalar field does not contribute to this combination of Eq. (25)]. Hence we know L_F as a function of x, and recalling that $F(x) = 2Q^2/r^4(x)$ is also known, we can write from (30)

$$L' = \frac{F'}{F} (G^2_2 - G^0_0) = -\frac{4r'}{r} (G^2_2 - G^0_0) \tag{31}$$

and integrate. We thus know $L(x)$, $F(x)$, and finally $L(F)$, at least in a range where $r(x)$ is monotonic.

Furthermore, from (25) we have

$$T_0^0 - T_1^1 = 2h(\phi)A(x)\phi'^2 = G_1^1 - G_0^0 = -2A(x)\frac{r''}{r}, \quad (32)$$

which gives us $h(\phi)\phi'^2$ as a function of x. It is easy to notice that we have $h(\phi) \geq 0$ (the scalar field is canonical) as long as $r'' \leq 0$ and $h(\phi) \leq 0$ (the scalar field is phantom) if $r'' \geq 0$. In the general case, $r''(x)$ can change its sign and then the sign of $h(\phi)$ also varies. In all cases we can freely choose the scalar field parametrization and conclude that we know the functions $\phi(x)$ and $h(x)$. The only still unknown quantity, the potential $V(\phi)$, can now be found from any components of Eq. (25), for example, $G_t^t = -T_t^t$. The scalar field equation (24) is known to follow from the Einstein equations (25) [actually, as long as $\phi \neq$ const, Eq. (24) directly follows from the "conservation" law $\nabla_\nu T_\mu^\nu$, which is in turn a consequence of the Einstein equations]. So we can assert that the whole set of equations is fulfilled.

Another algorithm consists in using again (32) to determine ϕ and $h(\phi)$ and then finding $V(\phi(x))$ from the scalar field equation (24). With known $V(x)$, the function $L(F(x))$ is found from one more combination of the Einstein equations,

$$L(x) = -2V(x) - G_0^0 - G_1^1, \quad (33)$$

and with known $F(x)$ it is then easy to find $L(F)$. Needless to say that both algorithms must lead to the same result, up to the choice of integration constants when finding $L(x)$ in the first algorithm and $V(x)$ in the second one.

For our particular regular metrics (2) and (4), $r'' > 0$, hence we will inevitably deal with a phantom scalar field and can safely put $h(\phi) = -1$.

We can note that a representation of any metric (6) using an electric instead of magnetic field is also possible but is slightly more complicated.

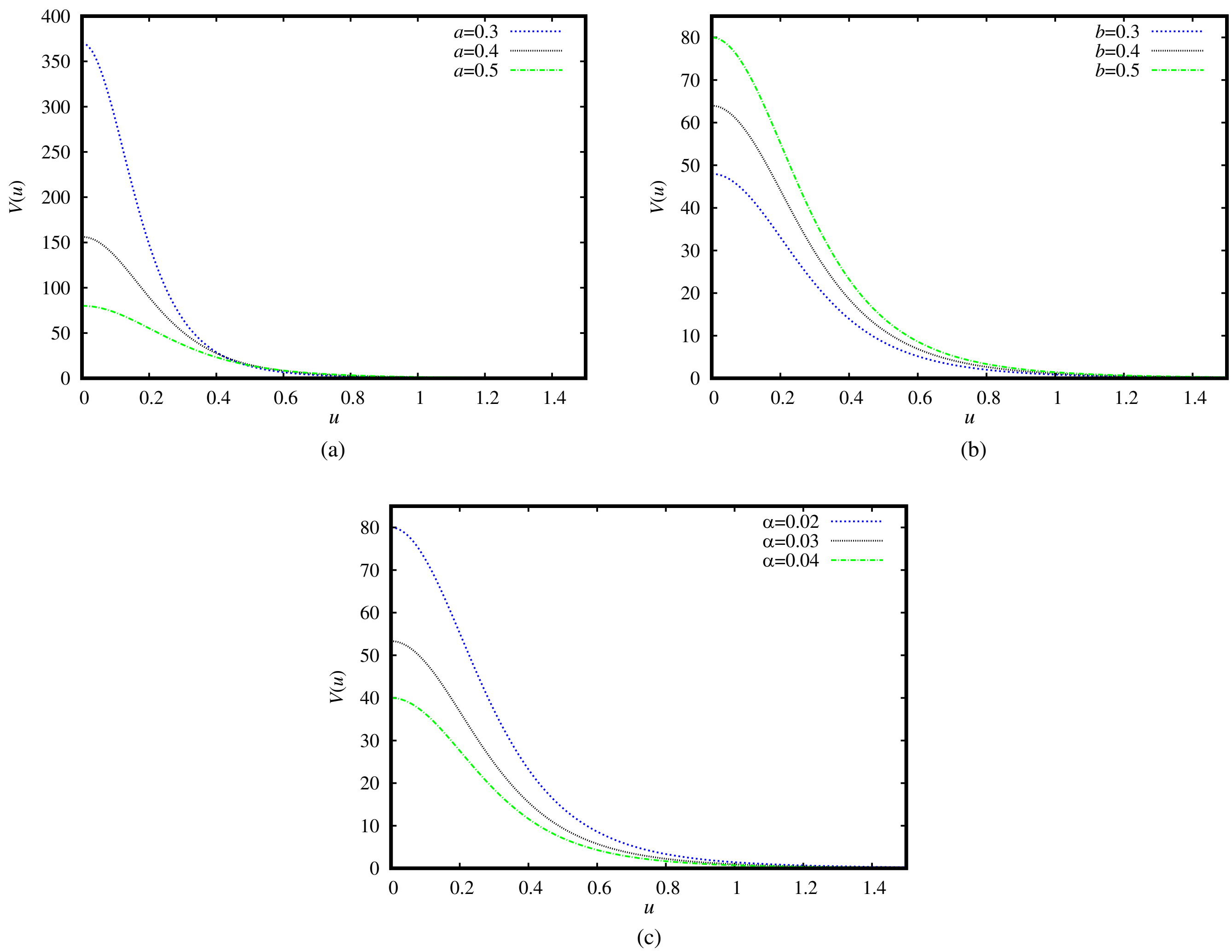


FIG. 4. The potential $V(\phi(u))$, Eq. (36), as a function of the coordinate u. In (a) we fixed $b = 0.5$ and $\alpha = 0.02$. In (b) we fixed $a = 0.5$ and $\alpha = 0.02$. In (c) we fixed $a = 0.5$ and $b = 0.5$.

B. A field source for a regularized black string

According to the above, since $r'' > 0$ in the metric (2), we put $h(\phi) = -1$. Let us try to model the solution with a magnetically charged source and a phantom scalar field. The magnetic field and the invariant F are given by

$$F_{23} = Q, \qquad F(u) = \frac{2Q^2}{(u^2+a^2)^2}. \tag{34}$$

Let us use the second algorithm described above. From (32) we find

$$\phi(u) = \arctan(u/a), \tag{35}$$

and from the scalar equation (24) we determine, choosing the integration constant so that V vanishes at large $|u|$,

$$V(u) = 2a^2\left(\frac{b}{5\alpha(a^2+u^2)^{5/2}} + \frac{\alpha^2}{a^2+u^2}\right). \tag{36}$$

Next, Eq. (33) yields

$$L(u) = \frac{6a^2 b}{5\alpha(u^2+a^2)^{5/2}} - 6\alpha^2, \tag{37}$$

while the combination (30) leads to

$$L_F(u) = \frac{3a^2 b}{4\alpha Q^2\sqrt{u^2+a^2}}. \tag{38}$$

These quantities satisfy the condition

$$L_F - \frac{dL}{du}\left(\frac{dF}{du}\right)^{-1} = 0, \tag{39}$$

confirming the correctness of our calculations. We can now write $L(F)$ and $V(\phi)$ as

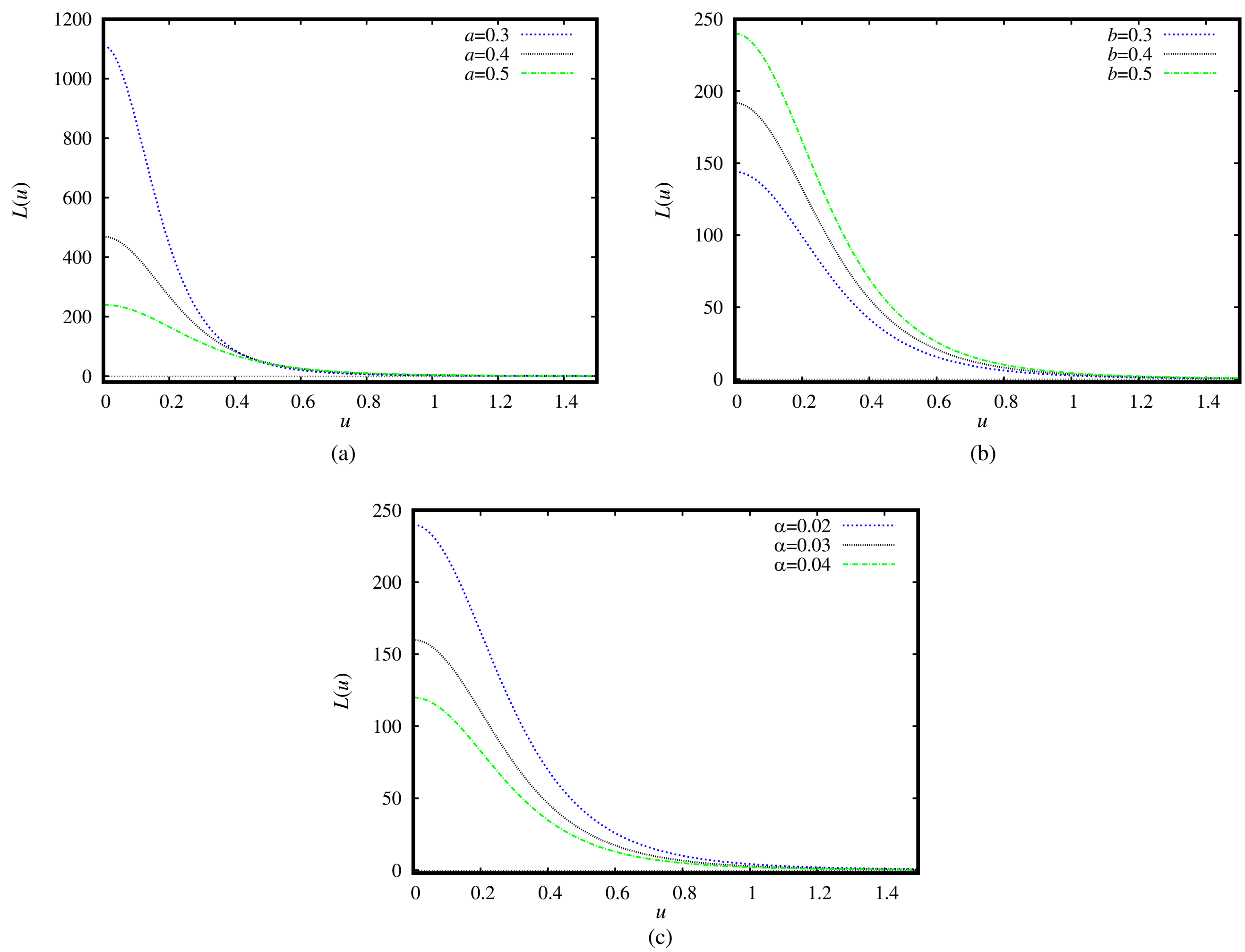


FIG. 5. The function $L(F(u))$, Eq. (37), as a function of the coordinate u. In (a) we fixed $b = 0.5$ and $\alpha = 0.02$. In (b) we fixed $a = 0.5$ and $\alpha = 0.02$. In (c) we fixed $a = 0.5$ and $b = 0.5$. At large $|u|$, the function $L(F(u))$ tends to $-6\alpha^2$ instead of zero, according to Eq. (37), but this contribution is too small to be visible in the figure.

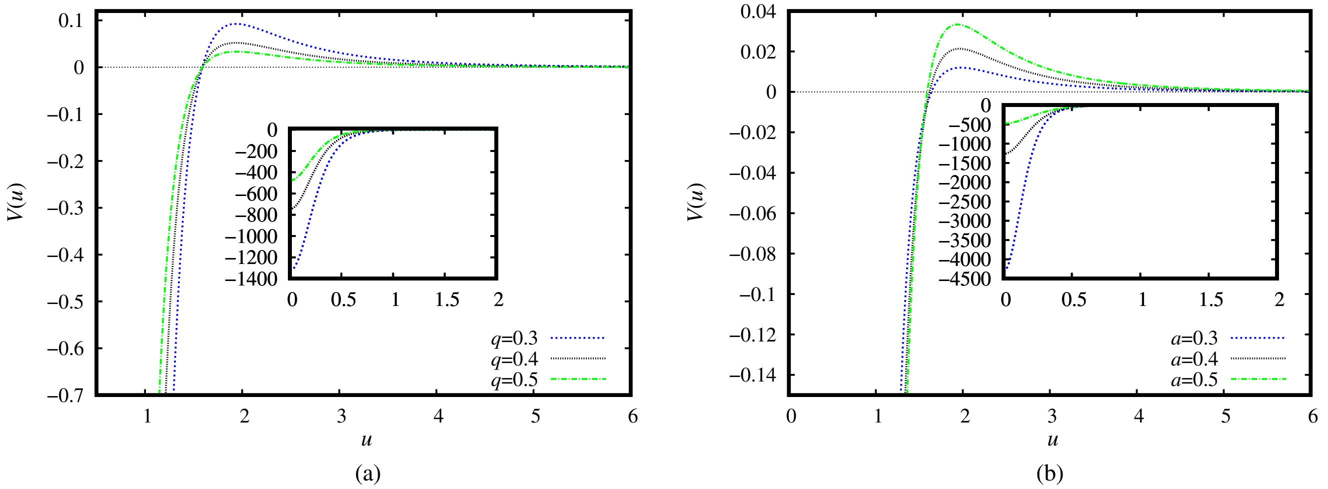


FIG. 6. The potential $V(\phi(u))$, Eq. (44), with $k=1$, as a function of the coordinate u. In (a) we fixed $a=0.5$ and in (b) we fixed $q=0.5$.

$$L(F)=\frac{6a^2bF}{5\alpha}\left(\frac{F}{2Q^2}\right)^{5/4}-6\alpha^2, \tag{40}$$

$$V(\phi)=2\alpha^2\cos^2\phi+\frac{2b}{5\alpha a^3}\cos^5\phi. \tag{41}$$

Their behavior as functions of u is illustrated in Figs. 4 and 5. The magnitude of $V(u)$ and $L(u)$ decreases as a and α increase, and increases as b increases.

C. A field source for a regularized inverted black hole

Again, for a magnetically charged source, the magnetic field is given by

$$F_{23}=Q, \qquad F(u)=\frac{2Q^2}{r^4}=\frac{32Q^2k^4}{q^4(u^2+a^2)^2}. \tag{42}$$

Let us note that the magnetic charge Q of our anticipated source has nothing to do with the initial charge q belonging to the "seed" solution with the metric (3) and involved in the regularized metric (4). However, as a special case, they can coincide.

As before, with $h(\phi)=-1$, for the scalar field we obtain from (32)

$$\phi(u)=\arctan(u/a). \tag{43}$$

Then the scalar field equation (24) yields

$$V(u)=\frac{32a^2k^4}{15q^2(u^2+a^2)^3}\left(-5+3\sqrt{u^2+a^2}\right). \tag{44}$$

From the Einstein equations we then obtain

$$L(u)=\frac{32k^4}{15q^2(u^2+a^2)^3}\times\left[15(u^2+a^2)+9a^2\sqrt{u^2+a^2}-20a^2\right], \tag{45}$$

$$L_F(u)=\frac{q^2}{Q^2}\left(\frac{3a^2}{4\sqrt{a^2+u^2}}+\frac{u^2-a^2}{u^2+a^2}\right). \tag{46}$$

As in the previous case, the correctness of calculations is verified by Eq. (39). With our magnetic source, the knowledge of $u(F)$ and $u(\phi)$ enables us to write the Lagrangian $L(F)$ and the potential $V(\phi)$ as follows:

$$L(F)=\frac{q^2F}{Q^2}\left(1+\frac{3a^2}{5x}-\frac{4a^2}{3x^2}\right), \qquad x=\frac{2k}{q}\left(\frac{2Q^2}{F}\right)^{1/4}, \tag{47}$$

$$V(\phi)=\frac{32k^4}{15q^2a^4}\cos^5\phi(3a-5\cos\phi). \tag{48}$$

There is good reason to suppose the equality $Q=q$ since it provides $L\approx F$ as $F\to 0$, the asymptotically Maxwell behavior of $L(F)$ at small magnetic fields. So in what follows we put $Q=q$.

The behavior of $V(u)$ is shown in Fig. 6. With other values of the parameters, the potential behaves qualitatively in the same way. As the charge q increases, the peaks of the potential become lower, while they grow at increasing values of the parameter a. The function $L(u)$ is plotted in Fig. 7. The peaks in the electromagnetic Lagrangian become lower as a or q increase.

The behavior of $L(F)$ and $V(\phi)$ is presented in Figs. 8 and 9. At small F, the dominating term in the electromagnetic Lagrangian is $L(F)\approx F$. The first nonlinear term that appears is of the order $F^{5/4}$. The potential looks like a

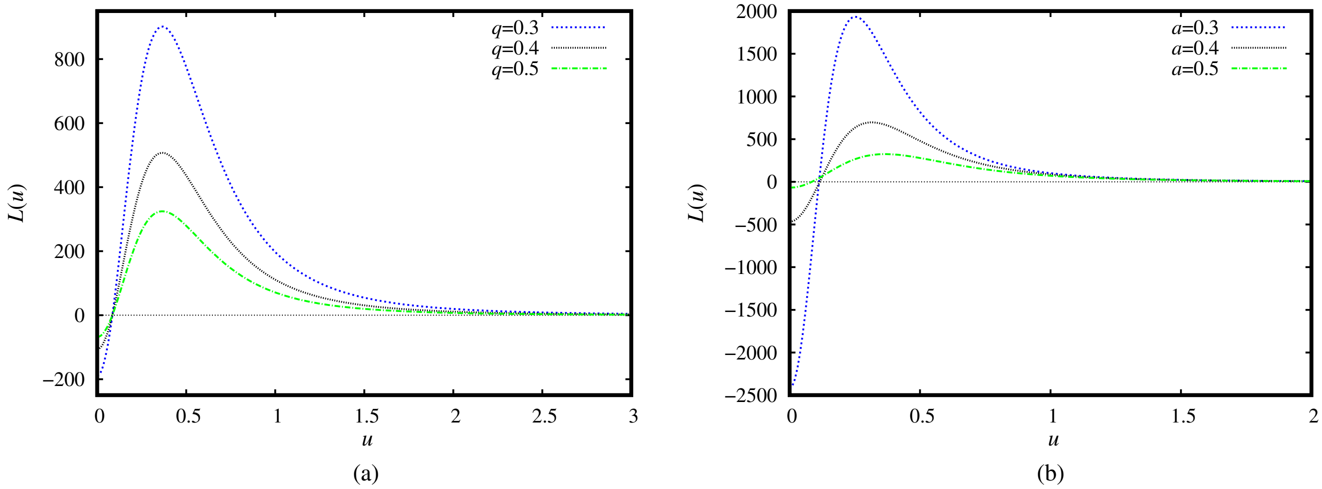


FIG. 7. The function $L(F(u))$, Eq. (45), with $k = 1$, as a function of the coordinate u. In (a) we fixed $a = 0.5$ and in (b) we fixed $q = 0.5$.

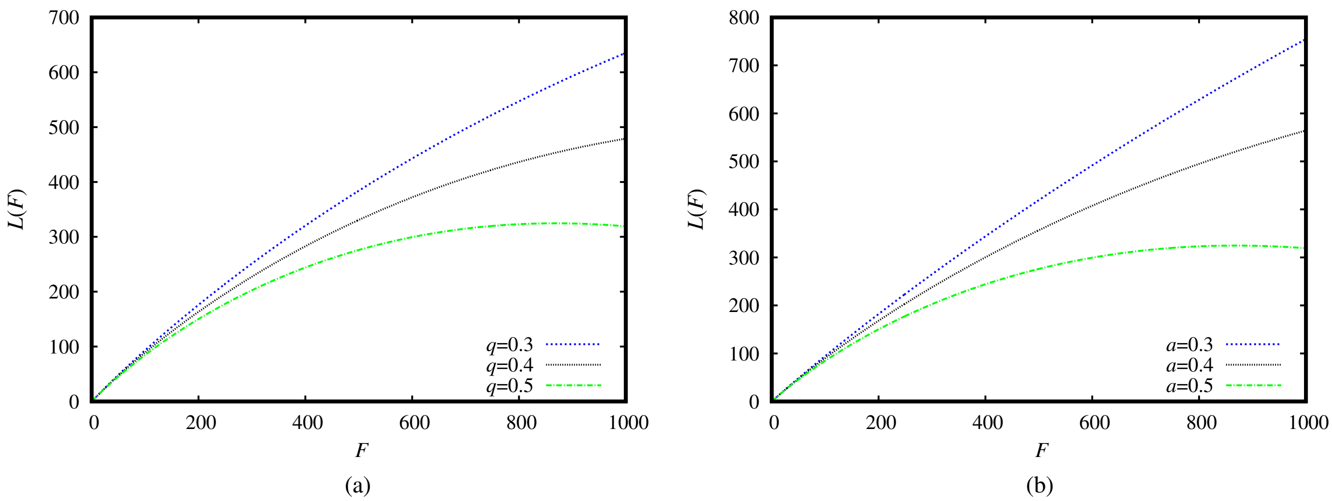


FIG. 8. The function $L(F)$, Eq. (47), with $k = 1$. In (a) we fixed $a = 0.5$, and in (b) we fixed $q = 0.5$.

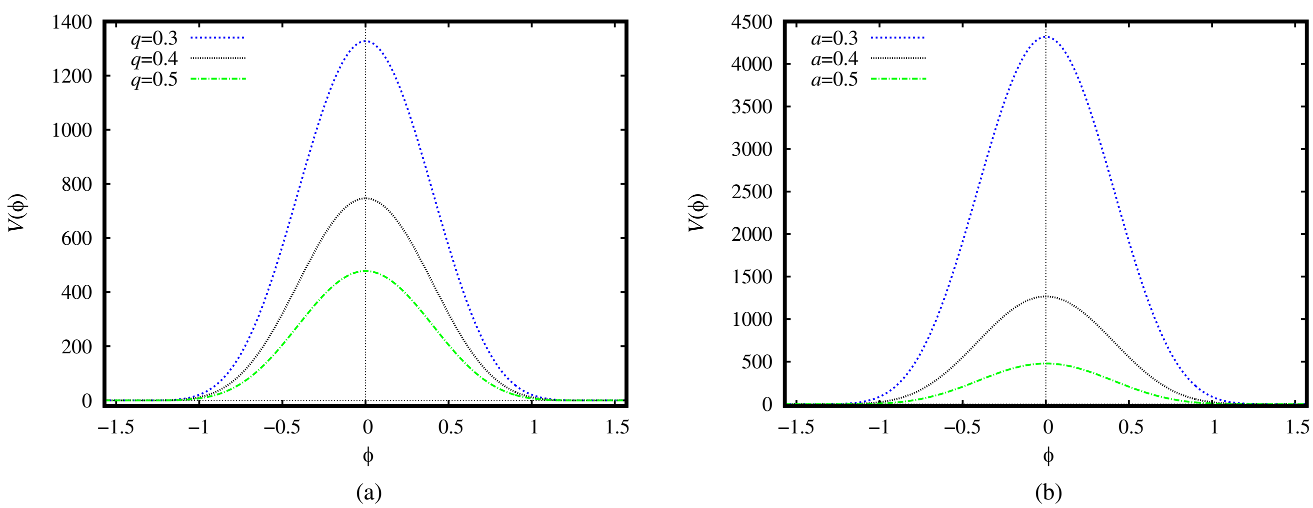


FIG. 9. The potential $V(\phi)$, Eq. (48), with $k = 1$. In (a) we fixed $a = 0.5$ and in (b) we fixed $q = 0.5$.

barrier as the scalar field changes. At zero scalar, the potential tends to a constant, and the first correction term is proportional to ϕ^2, simulating a scalar field mass such that $m_\phi^2 = 8(2-a)/(a^2q^2)$.

IV. CONCLUSION

In this paper, we have considered SV-type regularizations for two cylindrically symmetric space-times with Killing horizons: (i) the black string proposed by Lemos [50] (previously analyzed by Lima *et al.* [52]), and (ii) the so-called inverted black hole [51, 52]. For both families of regularized space-times, we have studied the global causal structures, constructing the appropriate Carter-Penrose diagrams, and found possible field sources in the framework of GR, consisting of a phantom minimally coupled scalar field with a nonzero potential and a nonlinear magnetic field. The global regularity of these regularized metrics (2) and (4) is verified by finiteness of the Riemann tensor components (7) or the Kretschmann invariant.

Next, we proposed a way to find, in the framework of GR, field sources for a class of cylindrically, planarly, or toroidally symmetric metrics of the form (6), by combining a minimally coupled scalar field ϕ with a self-interaction potential $V(\phi)$ and a nonlinear magnetic field described by a certain NED theory with a Lagrangian $L(F)$. As both regularized metrics (2) and (4) belong to this class, we have obtained explicit forms of field sources for both of them.

In particular, the regularized black string solution, which may be called a black-bounce version of the black string proposed by Lemos [50], has different structures, depending on the value of the regularization parameter a (at small a we have a regular black hole with two horizons and a black bounce at $u=0$, and at larger a we obtain a regular black hole with a single extremal horizon at $u=0$) or a cylindrically symmetric wormhole with a throat at $u=0$. (A similar study was recently carried out by Lima *et al.* [55] for the charged version of black strings [56].)

In the regularized inverted black-hole solution, at $a<1$ there is a throat at $u=0$ surrounded by two cosmological-type horizons. At $a=1$ a static region vanishes, and we have a regular Bianchi-type I cosmology with an extremal horizon at $u=0$. At $a>1$, we have a regular Bianchi-type I cosmology with a bounce at $u=0$.

In both regularized solutions, the null energy condition is violated everywhere due to $r''/r>0$, and the scalar part of the source is necessarily phantom; still some of the standard energy conditions are respected at least in a certain part of space-time, as can be seen from Fig. 3.

The scalar field in both solutions is of kink type and varies between two finite limits, while the potential $V(\phi)$, being everywhere finite, rapidly tends to zero as $|u|\to\infty$, although these infinities are of drastically different nature in the two solutions. Meanwhile, the NED sources of the solutions are quite different in nature. In the case of a regularized black string, $L(F)$ has no Maxwell weak field limit and has a constant contribution $-6\alpha^2$ [see Eq. (37)] corresponding to the AdS asymptotic behavior of the metric. Unlike that, in the regularized inverted black-hole solution, both $V(u)$ and $L(u)$ tend to zero as $|u|\to\infty$, and $L(F)$ has a Maxwell weak field limit.

A common feature of these regular solutions is that the magnetic fields exist without their own source (which could be imagined as some distributed current or monopole charge density), their lines of force stretching from one infinity to the other.

We can conclude that SV-type regularization of known solutions of GR with horizons and singularities leads to a number of geometries of interest, and, in turn, their possible field sources shed a certain new light on their properties and can be useful, in particular, for studying their stability under various kinds of perturbations.

It is clear that the SV regularization trick can be applied to many other singular metrics, including general cylindrically symmetric ones, described by Eq. (5). Since they contain one more degree of freedom as compared to (6), they will require more general field sources than those described here, and this can be a subject of a future study.

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