Israel-Wilson-Perjes metrics in a theory with a dilaton field

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We are interested in the charged dust solutions of the Einstein field equations in stationary and axially symmetric spacetimes and inquire if the naked singularities of the Israel-Wilson-Perjes (IWP) metrics can be removed. The answer is negative in four dimensions. We examine whether this negative result can be avoided by adding scalar or dilaton fields. We show that IWP metrics also arise as solutions of the Einstein-Maxwell system with a stealth dilaton field. We determine the IWP metrics completely in terms of one complex function satisfying the Laplace equation. With the inclusion of the stealth dilaton field, it is now possible to add a perfect fluid source. In this case, the field equations reduce to a complex cubic equation. Hence, this procedure provides interior solutions to each IWP metric, and it is possible to cover all naked singularities inside a compact surface where there is matter distribution.

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I. INTRODUCTION

In four dimensions, the conformastatic or conformastat metrics [1]

$$ds^2 = -\lambda^{-2}(\vec{x})dt^2 + \lambda^2(\vec{x})d\vec{x}\cdot d\vec{x} \tag{1}$$

solve the Einstein-Maxwell-dust field equations with the vector potential $A^\mu = (2\lambda(\vec{x}), 0, 0, 0)$ if the metric function $\lambda(\vec{x})$ satisfies

$$\nabla^2\lambda + \frac{1}{2}\rho_m\lambda^3 = 0, \tag{2}$$

where ∇^2 denotes the three-dimensional Laplace operator in flat Cartesian coordinates, and ρ_m is the mass density of the dust distribution with the four-velocity $u_\mu = \delta^t_\mu/\lambda(\vec{x})$ [2,3]. The electric charge density ρ_e and the mass density ρ_m are equal; hence, the system is "extremely" charged. The $\rho_m = 0$ case, i.e., the conformastatic solutions of the Einstein-Maxwell equations, the so-called Majumdar-Papapetrou (MP) metrics [1,4,5], represent gravitational fields of multiple extremely charged black holes [6] of electrovacuum. One can extend these solutions by adding a charged dust distribution ([2,3,7] and references therein) where several interesting thin shell charged dust solutions without the singularities of the MP metrics were obtained.

The extension of the static MP solutions to the stationary case was done in [8,9], and a relevant question is to generalize the conformastatic MP metrics with extremely charged dust to conformastationary Israel-Wilson-Perjes (IWP) metrics with charged dust. This problem, in a sense, asks one to search for dust sources for the IWP metrics. Here we show that conformastationary spacetimes do not support charged dust solutions [10]. The integrability conditions of the rotation velocity and the magnetic potential vectors reduce the problem either to the sourceless case, i.e., the IWP metrics [8,9], or to MP metrics with dust [2,3,7].

To summarize our physical motivations and our findings in this work, let us note the following: Hartle and Hawking wrote the influential paper [6], in which, among other things related to the stationary solutions, in their own words they claim the following: "We also analyse some of stationary solutions of the Einstein-Maxwell equations discovered by Israel and Wilson. If space is asymptotically Euclidean we find that all of these solutions have naked singularities." As the nature of the singularity in gravity is an extremely important issue, this work received a lot of attention. There are, in principle, solutions of Einstein-Maxwell theory in which naked singularities are unavoidable. What we show below is an answer to the following question: By adding a source to the Einstein-Maxwell system, can one avoid these naked singularities? There are two nontrivial facets to this problem: First of all, the full theory with the source must admit the Israel-Wilson-Perjes metrics as solutions, and second, the naked singularities must be avoided. Both of these questions have, by no means obvious, answers. In fact,

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our first attempt to add a perfect fluid source (with pressure and density) will yield field equations that do not support these solutions with a nonzero pressure. However, the field equations force one to consider the dust source (i.e., a perfect fluid with zero pressure). So, with the inclusion of a dust, the Einstein-Maxwell system still supports these solutions, which is an answer to the first question. But, unfortunately, the introduced dust does not remove the naked singularity. Thus, this is our theorem (see the theorem below): The IWP metrics with charged perfect source reduce to either Majumdar-Papapetrau metrics (the no rotation case) or to the IWP metrics without a source. This theorem by itself is an important contribution to general relativity. Hence, the Hartle-Hawking no-go result is extended to the case of Einstein-Maxwell-dust theory in our work. We then answer the second question, that is, the question of removing the singularities, with the addition of a scalar field. It really is remarkable that such a complicated system both allows the IWP metrics as solutions and removes their stubborn naked singularities.

The layout of the paper is as follows: In Sec. II, we study the conformostationary spacetimes as solutions of Einstein-Maxwell field equations with a perfect fluid distribution and show that such a configuration is not possible. In Sec. III, we introduce the dilaton field to the previous system and we show that in four dimensions, IWP metrics with a dust source is possible if the dilaton field has a vanishing energy-momentum tensor (that is, the dilaton becomes a stealth field). In Sec. IV, we reduce the field equations to a complex potential equation. In Sec. V, we discuss the nonvacuum cases. In the Appendix, we give an action formulation of the field equations.

II. CONFORMOSTATIONARY SPACETIMES

Einstein-Maxwell-perfect fluid field equations are given as

$$G_{\mu\nu} = \frac{1}{2}T_{\mu\nu} + (\rho_m + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{3}$$

$$\nabla_\alpha F^{\alpha\mu} = \rho_e u^\mu, \tag{4}$$

where the Einstein tensor $G_{\mu\nu}$ is defined as $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, ρ_m and ρ_e are the mass and electric charge densities, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $A_\mu = (A_0, \vec{A})$, and the energy-momentum tensor of the Maxwell field reads

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F^2, \tag{5}$$

with the definition $F^2 = F_{\sigma\rho} F^{\sigma\rho}$. Taking the metric as

$$ds^2 = -f^2 (dt + \vec{\Omega} \cdot d\vec{x})^2 + \frac{1}{f^2} d\vec{x} \cdot d\vec{x}, \tag{6}$$

where the metric functions f and $\vec{\Omega}$ depend on the spatial coordinates $\vec{x}$, one has $u_\mu = f(1, \vec{\Omega})$. The spatial indices are raised and lowered with the Kronecker delta δ_{ij}. The field equations (3) and (4) reduce to

$$\partial_i \Omega_j - \partial_j \Omega_i = 2\epsilon_{ijk} \Im(\lambda \partial_k \bar{\lambda}), \tag{7}$$

$$\partial_i A_j - \partial_j A_i = \frac{1}{f^2} \epsilon_{ijk} \partial_k \chi + \Omega_j \partial_i A_0 - \Omega_i \partial_j A_0, \tag{8}$$

where λ is a complex function, with real and imaginary parts given as $\Re(\lambda)$, $\Im(\lambda)$, while $|\lambda|$ is its magnitude, and in terms of λ, the real functions f, A_0, and χ are given as

$$\chi = \frac{2\Im(\lambda)}{|\lambda|^2}, \qquad f = \frac{1}{|\lambda|}, \qquad A_0 = -\frac{2\Re(\lambda)}{|\lambda|^2}. \tag{9}$$

Then, the remaining field equations give vanishing pressure $p = 0$, and the mass ρ_m and charge ρ_e densities are determined via

$$\Re(\lambda \nabla^2 \bar{\lambda}) + \frac{1}{2} \rho_m |\lambda|^4 = 0, \tag{10}$$

$$\Re(\lambda^2 \nabla^2 \bar{\lambda}) + \frac{1}{2} \rho_e |\lambda|^5 = 0. \tag{11}$$

Defining the vector potential as $\vec{A} = \vec{B} + \vec{\Omega} A_0$, one finds

$$\partial_i B_j - \partial_j B_i = -2\epsilon_{ijk} \partial_k \Im(\lambda). \tag{12}$$

We have the following lemma.

Lemma 1.—Integrability of (7) and (12) implies that

$$\Im(\lambda \nabla^2 \bar{\lambda}) = 0, \tag{13}$$

$$\nabla^2 \Im(\lambda) = 0. \tag{14}$$

Proof of the Lemma 1 is easier when we convert Eqs. (7) and (12) to the following forms:

$$\epsilon^{ijk} \partial_i \Omega_j = 2\Im(\lambda \partial_k \bar{\lambda}), \tag{15}$$

$$\epsilon^{ijk} \partial_i B_j = 2\partial_k \Im(\lambda). \tag{16}$$

Taking the divergence of both sides of these equations, we find (13) and (14). Integrability condition (13) implies, from (10),

$$\nabla^2 \lambda + \frac{1}{2} \rho_m \lambda^2 \bar{\lambda} = 0, \tag{17}$$

and the integrability condition (14) implies, from (11),

$$\Re(\lambda) \rho_m = |\lambda| \rho_e. \tag{18}$$

Then we have the following theorem.

Theorem 1.—Conformostationary metrics (9) do not support Einstein-Maxwell-dust field equations. They reduce to either Israel-Wilson-Perjes metrics, i.e., $\rho_m = \rho_e = 0$, or to Majumdar-Papapetrou metrics, i.e., $\vec{\Omega} = 0$.

Proof of the theorem comes from the integrability conditions (13) and (14), and hence from (17), one has

$$\nabla^2 \Im(\lambda) + \frac{1}{2}\rho_m |\lambda|^2 \Im(\lambda) = 0. \tag{19}$$

Since the first term on the left-hand side of (19) vanishes due to (14), then the second term gives $\rho_m \Im(\lambda) = 0$. Hence, either $\rho_m = 0$ (Israel-Wilson-Perjes spacetimes) or $\lambda = \bar{\lambda}$ leading to, without losing any generality, $\vec{\Omega} = 0$ which corresponds to the Majumdar-Papapetrou spacetimes and λ satisfies (2).

Remark: Our theorem is consistent with the results of [10]. In [10], the authors consider also a magnetic current. In the absence of the magnetic current density, their result reduces to our result.

III. INCLUSION OF A DILATON FIELD

Following [11,12], we take the metric as $g_{\mu\nu} = e^{\frac{2\phi}{2-D}}(-u_\mu u_\nu + h_{\mu\nu})$ in D dimensions, where $u_\mu = e^{\phi}(1, \vec{q})$, and $h_{\mu\nu}$ is a constant two tensor with $u^\mu h_{\mu\nu} = 0$. Taking $F_{\alpha\beta} = \partial_\alpha u_\beta - \partial_\beta u_\alpha$, one finds

$$G_{\mu\nu} = \frac{4-D}{2(2-D)} T^{\phi}_{\mu\nu} + \frac{1}{2} e^{\frac{2\phi}{2-D}} T^{M}_{\mu\nu} + (\rho_m + p) v_\mu v_\nu + p g_{\mu\nu}, \tag{20}$$

$$\nabla_\alpha\left(e^{\frac{2\phi}{2-D}} F^{\alpha}{}_{\nu}\right) - \frac{1}{4} e^{\frac{4\phi}{2-D}} F^2 u_\nu = (\rho_e + p)\alpha^2 u_\nu, \tag{21}$$

$$\Box\phi + \frac{1}{4} e^{\frac{2\phi}{2-D}} F^2 = \rho_e - p, \tag{22}$$

where $v_\mu = \alpha u_\mu$ with $\alpha = e^{\frac{\phi}{2-D}}$, we have $g^{\mu\nu} u_\mu u_\nu = -1/\alpha^2$ and $g^{\mu\nu} v_\mu v_\nu = -1$. In addition, the energy-momentum tensor of the dilaton field is defined as

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} \partial_\rho\phi\partial^\rho\phi, \tag{23}$$

while the energy-momentum tensor for the electromagnetic field $T^{M}_{\mu\nu}$ is given in (5).

For $D = 4$, we have

$$g_{\mu\nu} = e^{-\phi}(-u_\mu u_\nu + h_{\mu\nu}),$$

and the gravitational field equation takes a special form for which the backreaction of the scalar field disappears as the energy-momentum tensor of the dilaton field drops from the gravitational field equation, and ϕ becomes a stealth dilaton field

$$G_{\mu\nu} = \frac{1}{2} e^{-\phi} T^{M}_{\mu\nu} + (\rho + p) v_\mu v_\nu + p g_{\mu\nu}, \tag{24}$$

$$\nabla_\alpha(e^{-\phi} F^{\alpha}{}_{\nu}) - \frac{1}{4} e^{-2\phi} F^2 u_\nu = (\rho + p) e^{-\phi} u_\nu, \tag{25}$$

$$\Box\phi + \frac{1}{4} e^{-\phi} F^2 = \rho - p. \tag{26}$$

We find that $p = 0$, then

$$G_{\mu\nu} = \frac{1}{2} e^{-\phi} T^{M}_{\mu\nu} + \rho v_\mu v_\nu, \tag{27}$$

$$\nabla_\alpha(e^{-\phi} F^{\alpha}{}_{\nu}) - \frac{1}{4} e^{-2\phi} F^2 u_\nu = \rho e^{-\phi} u_\nu, \tag{28}$$

$$\Box\phi + \frac{1}{4} e^{-\phi} F^2 = \rho. \tag{29}$$

The last equation is also the zeroth component of the first Maxwell equation. The spatial components of Maxwell equation (28) reduce to

$$\partial_i(e^{-2\phi} F_{ij}) = 0. \tag{30}$$

The dilaton equation (29) is not independent and can be obtained by contracting u^ν with Maxwell equation (28). The matter density ρ_m needs to be equal to the charge density ρ_e as $\rho = \rho_m = \rho_e$.

IV. SIMPLIFIED FIELD EQUATIONS

The Maxwell and dilaton field equations take the form

$$\partial_k(e^{\phi} f_{ik}) = 0, \tag{31}$$

$$\nabla^2\phi - \frac{1}{2}\vec{\nabla}\phi \cdot \vec{\nabla}\phi + \frac{1}{2} e^{2\phi} f_{ik} f_{ik} = \rho e^{-\phi}, \tag{32}$$

where $f_{ij} = \partial_i q_j - \partial_j q_i$. Maxwell equation (31) implies that

$$f_{ij} = e^{-\phi} \varepsilon_{ijk} \partial_k \chi, \tag{33}$$

where χ is a function satisfying $\vec{\nabla} \cdot (e^{-\phi} \vec{\nabla}\chi) = 0$. Then,

$$\nabla^2\phi - \frac{1}{2}\vec{\nabla}\phi \cdot \vec{\nabla}\phi + \frac{1}{2}\vec{\nabla}\chi \cdot \vec{\nabla}\chi = e^{-\phi}\rho. \tag{34}$$

Electrovacuum case: For $\rho = 0$, we have

$$f_{ij} = e^{-\phi} \varepsilon_{ijk} \partial_k \chi, \tag{35}$$

where χ is a function satisfying the equation $\vec{\nabla}\cdot(e^{-\phi}\vec{\nabla}\chi)=0$. Then,

$$\nabla^2\phi - \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}\vec{\nabla}\chi\cdot\vec{\nabla}\chi = 0. \tag{36}$$

We obtain the solutions of the field equation by solving the coupled nonlinear differential equations:

$$\vec{\nabla}\cdot(e^{-\phi}\vec{\nabla}\chi) = 0, \tag{37}$$

$$\nabla^2\phi - \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + \frac{1}{2}\vec{\nabla}\chi\cdot\vec{\nabla}\chi = 0. \tag{38}$$

The metric is

$$ds^2 = -e^{\phi}(dt + \vec{q}\cdot d\vec{x})^2 + e^{-\phi}d\vec{x}\cdot d\vec{x}. \tag{39}$$

Equations (37) and (38) can be combined as one complex equation by defining $\xi = \phi \mp i\chi$,

$$\nabla^2\xi - \frac{1}{2}\vec{\nabla}\xi\cdot\vec{\nabla}\xi = 0, \tag{40}$$

or $\nabla^2 e^{-\frac{1}{2}\xi} = 0$. Hence, $e^{-\frac{1}{2}\xi} = \psi_1 + i\psi_2$ where both ψ_1 and ψ_2 satisfy the Laplace equation. We find that

$$e^{-\phi} = \psi_1^2 + \psi_2^2, \qquad \chi = \pm 2\tan^{-1}\left(\frac{\psi_2}{\psi_1}\right), \tag{41}$$

and

$$f_{ij} = \partial_i q_j - \partial_j q_i = \varepsilon_{ijk}(\psi_1\partial_k\psi_2 - \psi_2\partial_k\psi_1). \tag{42}$$

The solution of the above equation for $\vec{q}$ is given by

$$\vec{q} = \vec{x}\times\vec{E}, \tag{43}$$

where [13]

$$\vec{E} = \int_0^1 [\psi_1(s\vec{x})\vec{\nabla}\psi_2(s\vec{x}) - \psi_2(s\vec{x})\vec{\nabla}\psi_1(s\vec{x})]s ds. \tag{44}$$

Hence, the metric is completely determined:

$$ds^2 = -\frac{1}{\psi_1^2+\psi_2^2}(dt - \vec{E}\cdot(\vec{x}\times d\vec{x}))^2 + (\psi_1^2+\psi_2^2)d\vec{x}\cdot d\vec{x}. \tag{45}$$

V. NONVACUUM CASE

The metric (45) may have singularities without a horizon enclosing them. To avoid such situations, consider the case with $\rho \neq 0$. In this case, the field equations become

$$\nabla^2\xi - \frac{1}{2}\vec{\nabla}\xi\cdot\vec{\nabla}\xi = e^{-\phi}\rho, \tag{46}$$

where $\xi = \phi \pm i\chi$. The above equation can be converted to

$$\nabla^2\Lambda + \frac{1}{2}\rho(\Lambda\bar{\Lambda})\Lambda = 0, \tag{47}$$

where $\Lambda = e^{-\frac{1}{2}\xi}$. The vector $\vec{q}$ remains the same as in the vacuum case (43). For the thin shell model and other nonvacuum cases, it is better to have a different and useful parametrization other than $\Lambda = e^{-\frac{1}{2}\xi}$ used in Sec. IV. Let $\Lambda = Re^{i\theta}$, and then one has

$$R = e^{-\phi/2}, \qquad \theta = \pm\frac{\chi}{2}.$$

With this parametrization, (47) reduces to two real equations as

$$\nabla^2 R - \vec{\nabla}\theta\cdot\vec{\nabla}\theta + \frac{1}{2}\rho R^3 = 0, \qquad \vec{\nabla}\cdot(R^2\vec{\nabla}\theta) = 0. \tag{48}$$

We have the following interesting cases:

(i) When $\theta =$ constant, then the IWP metrics reduce to the MP metrics, but when $R =$ constant, the source density is $\rho = \frac{2}{R^2}\vec{\nabla}\theta\cdot\vec{\nabla}\theta$, and the resulting metric is given by

$$ds^2 = -\frac{1}{R^2}(dt - \vec{E}\cdot(\vec{x}\times d\vec{x}))^2 + R^2 d\vec{x}\cdot d\vec{x}, \tag{49}$$

with

$$\vec{E} = R^2\vec{\nabla}\int_0^1 \theta(s\vec{x})s ds, \tag{50}$$

where $\nabla^2\theta = 0$. The spacetime obtained is a dust filled rotating universe.

(ii) When $\rho = \rho_0\theta(F)$, where ρ_0 is the density function in a region $F > 0$, $F = 0$ defines a compact surface outside of which is the vacuum (IWP) metric (45). Hiding the singularities of the IWP metrics inside the compact surface $F = 0$, we get rotating regular charged solutions of Einstein field equations with a dilaton field.

(iii) For $\rho = \rho_0\delta(F)$ where ρ_0 is the density function on the thin shell $F = 0$, we have a shell model. In both sides of the thin shell, the spacetime is described by the IWP metric (45) with $\rho = 0$. As an example of such a model, let $\rho(x,y,z) = \rho_0(x,y)\delta(z)$. Above the thin shell ($z > 0$), the metric is an IWP metric with the metric functions (R_1,θ_1) and below the thin shell ($z < 0$), the metric is also an IWP metric with the metric functions (R_2,θ_2) (see Fig. 1). These metric functions are continuous at $z = 0$, but $\frac{\partial R}{\partial z}$ is

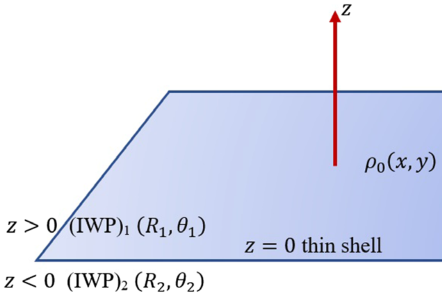

FIG. 1. The thin shell is located at $z=0$ as an infinite plane on which the charge density is $\rho_0(x,y)$. The thin shell separates two different IWP spacetimes described by the metric functions (R_1,θ_1) and (R_2,θ_2).

not continuous and satisfies the following jump condition:

$$\frac{\partial R_1}{\partial z}-\frac{\partial R_2}{\partial z}+\frac{1}{2}R^3\rho_0(x,y)=0,\quad \text{at } z=0. \tag{51}$$

VI. CONCLUSIONS AND DISCUSSION

We have considered the IWP metrics in four dimensions and have shown that these metrics have many interesting properties which can be summarized as follows:

(i) We have shown that the IWP metrics, with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ where $A_\mu=(A_0,\vec{A})$, solve the Einstein-Maxwell field equations but do not admit charged perfect fluids as sources. Furthermore, these metrics contain naked singularities; hence, they do not represent the gravitational fields of black holes [6].

(ii) We have shown that the IWP metrics in four dimensions $g_{\mu\nu}=e^{-\phi}(-u_\mu u_\nu+h_{\mu\nu})$ or

$$ds^2=-e^{\phi}(dt+\vec{q}\cdot d\vec{x})^2+e^{-\phi}d\vec{x}\cdot d\vec{x}, \tag{52}$$

for $F_{\mu\nu}=\partial_\mu u_\nu-\partial_\nu u_\mu$ admit charged dust distributions as sources with a stealth (having a vanishing energy-momentum tensor) dilaton field ϕ.

(iii) For $\rho=0$, we have given the complete solution in terms of two harmonic functions ψ_1 and ψ_2. For $\rho\neq 0$, the field equations reduce to a cubic equation for a complex function Λ. It is now possible to cover all naked singularities inside a compact surface $F=0$ where the matter density $\rho\neq 0$. Outside of the compact surface $F=0$ where we allow matter distribution, the geometry is described by the IWP metrics with $\rho=0$.

(iv) It is possible to obtain exact solutions for multi-IWP universes separated by thin shells represented with the matter density

$$\rho=\sum_{i=1}^{N}\rho_{0,i}(x)\delta(F_i), \tag{53}$$

where $\rho_{0,i}$, $(i=1,2,\ldots,N)$ are functions defined on layers $F_i=0$, $(i=1,2,\ldots,N)$. Here the layers are parallel 2-surfaces in $\mathbb{R}^3$ having the same normal vectors $\hat{n}$. Examples are (a) planar multilayers $F_i=z-a_i=0$, (b) spherical cocentrical layers $F_i=r-a_i$, (c) cylindrical cocentrical layers $F_i=r-a_i=0$, etc. In all of these cases, the jump conditions across the surfaces are given by

$$\sqrt{\gamma}\left(\frac{\partial R_{i+1}}{\partial n}-\frac{\partial R_i}{\partial n}\right)+\frac{1}{2}R_i^3\rho_{0,i}=0\quad \text{at } F_i=0,\quad i=1,2,\ldots,N, \tag{54}$$

where γ is the determinant of the metric on the two surfaces, and $\frac{\partial R}{\partial n}$ is the derivative along the normal direction.

Finally, let us comment on the scalar field sector of the full theory we have used here. From the vantage point of theory, scalar fields show up in many different settings, yet from the experimental point of view besides the Higgs field, no fundamental scalar field has yet been detected. This, of course, should not deter one to consider gravity theories that have scalar fields, as they could be relevant in the early Universe or in the strong field regime of gravity pertaining to compact objects. Including the inflaton field that is employed to explain in the initial inflation phase of the Universe, there are several we can mention which are consistent with observations. For example, string-loop modification of the low-energy couplings of the dilaton may provide a mechanism for fixing the vacuum expectation value of a massless dilaton in a way which is naturally compatible with the existing experimental data [14]. The string expansion involves massless fields other than gravitation, the most relevant being the dilaton [15]. In a string model, there have been some discussions in which the dilaton field changes the dynamical properties of the system drastically [16]. A wide class of scalar-tensor theories can pass the present Solar System tests [17] and still exhibit large, strong field-induced observable deviations in systems involving neutron stars [18].

APPENDIX: FIELD EQUATIONS FROM AN ACTION

Let us obtain the field equations (27)–(29) from an action. For this, first consider the Einstein-Maxwell-dilaton action of the form

$$S=\int d^4x\sqrt{-g}\left[R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{\gamma\phi}F^2\right], \tag{A1}$$

which is widely studied in the literature; see, for example, [19,20] or [21] where the Lagrangian density is augmented with the Liouville potential $V(\phi)=2\Lambda e^{-\delta\phi}$. The field equations of this action are

$$G_{\mu\nu} = \frac{1}{2} T^{\phi}_{\mu\nu} + \frac{1}{2} e^{\gamma\phi} T^{M}_{\mu\nu}, \tag{A2}$$

$$\nabla_\mu (e^{\gamma\phi} F^{\mu\nu}) = 0, \tag{A3}$$

$$\Box\phi = \frac{\gamma}{4} e^{\gamma\phi} F^2. \tag{A4}$$

On the other hand, the Einstein-Maxwell action with a constraint on the norm of the electromagnetic four-potential vector A^μ of the form

$$S = \int d^4x \sqrt{-g} \left[R - 2\Lambda - \frac{c_1}{2} F^2 + \zeta (u^\mu u_\mu + 1) \right] \tag{A5}$$

was studied in the literature in the context of Einstein aether theories (see, for example, [22] and the references therein) with the field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = c_1 T^{M}_{\mu\nu} - \zeta u_\mu u_\nu, \tag{A6}$$

$$\nabla_\mu F^{\mu\nu} = -\frac{\zeta}{c_1} u^\nu, \tag{A7}$$

$$u^\mu u_\mu = -1. \tag{A8}$$

The action

$$S = \int d^4x \sqrt{-g} \left[R - \frac{1}{4} e^{-\phi} F^2 + \zeta (u_\mu u^\mu + e^{\phi}) \right], \tag{A9}$$

where the kinetic term for the scalar field is removed, yields the field equations

$$G_{\mu\nu} = \frac{1}{2} e^{-\phi} T^{\mathrm{M}}_{\mu\nu} - \zeta u_\mu u_\nu, \tag{A10}$$

$$\nabla_\mu (e^{-\phi} F^{\mu\nu}) = -2\zeta u^\nu, \tag{A11}$$

$$\frac{1}{4} e^{-\phi} F^2 + \zeta e^{\phi} = 0, \tag{A12}$$

$$u_\mu u^\mu = -e^{\phi}. \tag{A13}$$

The equation coming from the variation of the scalar field yields the Lagrange multiplier ζ as

$$\zeta = -\frac{1}{4} e^{-2\phi} F^2. \tag{A14}$$

Then, the electromagnetic field equation becomes

$$\nabla_\mu (e^{-\phi} F^{\mu\nu}) - \frac{1}{2} e^{-2\phi} F^2 u^\nu = 0. \tag{A15}$$

At this point, to have a match with (28), ρ needs to be

$$\rho = \frac{1}{4} e^{-\phi} F^2, \tag{A16}$$

which reduces (29) to

$$\Box\phi = 0. \tag{A17}$$

At this point, we must emphasize that the variational principle yields a special case of the equations of motion that we provided in (27)–(29).

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