Effective-one-body formalism for leading-order radiative effects in the postlinear framework

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(Received 9 January 2023; accepted 26 June 2023; published 14 July 2023)

In recent years, significant progress has been made in the computation of conservative and dissipative scattering observables using the post-Minkowskian approach to gravitational dynamics. However, for accurate modeling of unbound orbits, an appropriate effective-one-body (EOB) resummation of the post-Minkowski results that also accounts for dissipative dynamics is desirable. As a step in this direction, we consider the electromagnetic analog of this problem here. We show that a six-parameter equation of motion encapsulates the effective-one-body dynamics for the electromagnetic scattering problem appropriate to third order in the coupling constant. Three of these six parameters describe the conservative part of the dynamics, while the rest correspond to the radiation-reaction effects. Here we show that only two radiation-reaction-related parameters are important at the desired order, making the effective number of parameters in our formalism to be 5. We compute the explicit forms of these five parameters by matching EOB scattering observables to that of the original two-body ones computed by Saketh *et al.* [Phys. Rev. Res. **4**, 013127 (2022)]. Interestingly, our formalism leads to a conjecture for the subleading angular momentum loss, for which no precise computations exist. In addition, we demonstrate that the bound-orbit observables computed using our method are in perfect agreement with those calculated using unbound-to-bound analytical continuation techniques. Finally, we qualitatively discuss the extension of our formalism to gravity.

DOI: 10.1103/PhysRevD.108.024033

I. INTRODUCTION

With the dawn of gravitational wave (GW) astronomy [1–5], a new avenue has opened up to explore the deeper aspects of astrophysics and fundamental physics. Additionally, the GW detectors' network is expanding and being upgraded at a promising pace [6–8]. Future gravitational wave detectors, like the Cosmic Explorer and Einstein telescope, will have a very high signal-to-noise ratio and potentially help test general relativity (GR) in strong-gravity regimes [9,10]. This, in turn, has led to an increasing demand for accurate theoretical modeling of binary systems in GR and beyond. Among the several complementary theoretical perspectives on the two-body problem in relativistic gravity, the study of scattering dynamics has recently attracted much attention.

The post-Minkowskian (PM) scheme [11–14], wherein one computes the classical scattering observables in a weak field expansion and without restriction on velocities, has benefited from the recent progress made in scattering amplitude computations and other quantum field theory techniques [15–41]. Although PM formalism is naturally adapted for two-body gravitational scattering, one can also extract information about bound binary systems from the scattering results by mapping to effective-one-body (EOB) [12–14], effective-field-theory (EFT) methods [16,17] and judicious analytic continuation prescriptions [22,42]. Therefore, in particular, the PM scheme, in conjunction with EOB, furnishes a promising theoretical framework to address the full binary dynamics in GR.

Initially formulated for the post-Newtonian (PN) framework, the EOB formalism in GR has been extended only recently for the PM approach [12]. A variant of the EOB formalism for PM that is more compatible with scattering-amplitude-based techniques has also been developed [43]. The vast majority of PM literature until now, however, has focused on the conservative sector of the binary gravitational dynamics. The same is true about the literature on EOB resummation of PM results. However, dissipation and radiation are key features of binary systems. Therefore, it is imperative to have a better theoretical understanding of the dissipative effects in such systems. However, there have been promising developments in this field in recent years [25,28,33,37,44–48] that are pretty promising. Important progress can be made by constructing an EOB formalism that systematically accounts for dissipative and conservative effects at the necessary order in PM approximation and naturally connects to scattering-amplitude-based methods.

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In light of this, it is instructive to look at a closely related but more easily tractable problem, namely, to set up the EOB formulation of binary scattering dynamics in electromagnetism (EM). The scattering of charged particles in EM and, more generally, Yang-Mills theories is of significant interest. Although the difficulties introduced by nonlinearities are absent, the classical scattering problem in the Abelian gauge theory shares some of the technical hurdles one runs into in GR [49]. Moreover, the more direct gauge theory results may also be used to build the corresponding gravity results by using an appropriate version of the double-copy relations [50–53]. Hence, the EOB formalism for EM theory, which is the focus of this paper, not only serves as a useful toy model but also potentially as a source of ingredients to construct the gravity results. Recently, an EOB formalism for conservative and dissipative PM dynamics was put forward in [54]. The approach we set out to describe here is different in spirit from the one described in [54] and, therefore, we hope that the same can, in turn, motivate a novel approach to EOB for dissipative PM dynamics.

The paper is organized as follows: We review the classical electromagnetic scattering in Sec. II. Then in Sec. III, we describe the effective-one-body formalism adapted to address the relativistic EM scattering dynamics. The formalism is further elaborated in Sec. IV and, concluded by solving the unknown parameters in Sec. V. In Sec. VI, we illustrate the application of our formalism for extracting results concerning bound orbits. Then, in Sec. VII, we also present a brief comparison of our approach with certain other ones. We conclude with a summary and outline the future outlook in Sec. VIII. In the Appendixes, we outline some of the mathematical details concerning our work that are not explicitly discussed in the main text.

Throughout the paper, we shall follow the $(+,-,-,-)$ signature, $c = 1$, and the notation $k \equiv 1/(4\pi\epsilon_0)$, where ϵ_0 is the vacuum permittivity. We also follow the notations: $A \bullet B \equiv A^\mu B_\mu$, where A^μ and B^μ are two four-vectors and $|V|^2 \equiv -V \bullet V$, for a spacelike vector V^μ.

II. REVIEW OF THIRD-ORDER POSTLINEAR RESULTS

The postlinear (PL) approach is analogous to the PM scheme, where one seeks a formal series solution to the classical scattering problem in gauge theory. Like the PM scheme, the series is expanded in the coupling constant without any restriction on the velocities of the scatterers. For example, in Ref. [55], the scattering up to second order in the coupling was performed for EM (and GR). Recently, Saketh *et al.* [49] extended Westpfahl's results for EM to the third-order scattering problem, which we shall refer to as 3PL. As mentioned in the Introduction, the key result of this work is a prescription for a convenient EOB resummation of the EM scattering results up to 3PL and including leading-order dissipative effects due to radiation reaction (RR). As we show, our approach is also efficient in providing a one-to-one correspondence between certain observables concerning bound and unbound orbits. Before discussing the EOB formalism, we briefly review 3PL results, closely following Ref. [49]. (Also, see Ref. [56] for a closely related discussion, but using modern amplitude methods).

A. Kinematics

We consider relativistic EM scattering of two particles of rest masses m_i and charges q_i, respectively, where $i = 1, 2$. We solve the scattering problem as a function of the asymptotic initial momenta p_i^μ and the asymptotic initial impact parameter vector b^μ. The initial momenta, in turn, can be written in terms of the initial velocities u_i^μ as $p_i^\mu = m_i u_i^\mu$. By definition, $b \bullet u_i = 0$, for $i = 1, 2$. While most of the results do not depend on the sign of $q_1 \times q_2$, for concreteness, we will assume $q_1 q_2 < 0$. This will allow us to connect some EM results to gravitational results.

It is often convenient to rephrase these initial data in terms of certain center-of-momentum (COM) and "relative" quantities. The standard definitions for the initial relative velocity v, the corresponding Lorentz factor γ, and the COM energy E are

$$\frac{1}{\sqrt{1-v^2}} = \gamma = \frac{p_1 \bullet p_2}{m_1 m_2}, \tag{1}$$

$$\begin{aligned} E^2 = (p_1 + p_2)^2 &= m_1^2 + m_2^2 + 2m_1 m_2 \gamma \\ &= M^2 + 2mM(\gamma - 1), \end{aligned} \tag{2}$$

where the total mass $M = m_1 + m_2$ and the reduced mass $m = m_1 m_2 / M$. In addition, the COM velocity U^μ and the "relative momentum" P^μ are defined as

$$U^\mu = \frac{(p_1^\mu + p_2^\mu)}{E}, \tag{3}$$

$$P^\mu = \frac{1}{E}[(p_2 \bullet U) p_1^\mu - (p_1 \bullet U) p_2^\mu], \tag{4}$$

$$|P| \equiv \left(\frac{M}{E}\right)(m\gamma v), \tag{5}$$

where $|P|$ is the magnitude of the spacelike vector P^μ. With the above definitions, we find that the magnitude of initial COM angular momentum J has the form

$$J = |P||b| = \left(\frac{M}{E}\right)(m\gamma v b). \tag{6}$$

B. PL scattering dynamics

The scattering dynamics is described in terms of the worldlines $z_i^\mu(\tau_i)$, parametrized by the proper time τ_i of the respective particle. Further, we can describe the electromagnetic field generated by the particles by the gauge potential A^μ. Assuming the Lorentz gauge condition $\partial_\mu A^\mu = 0$, the worldlines are solutions to the following set of coupled differential equations:

$$\Box A^\mu(x) = 4\pi k J^\mu(x) \equiv 4\pi\left[\sum_{i=1}^{2} q_i \int d\tau_i \dot{z}_i^\mu \delta^4(x - z_i)\right] \quad \text{(Maxwell's Eq.)}, \tag{7}$$

$$m_i \ddot{z}_i = q_i F^\mu{}_\nu(z_i)\dot{z}_i^\nu + \frac{2kq_i^2}{3}(\dddot{z}_i^\mu + \dot{z}_i^\mu \ddot{z}_i \bullet \ddot{z}_i) \quad \text{(Lorentz-Dirac Eq.)}, \tag{8}$$

where $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ denotes the external field acting on a given particle, and the effects of the self-field are encapsulated in the second term on the rhs of Eq. (8). The exact scattering worldline functions $z_i^\mu(\tau_i)$ self-consistently satisfy Eqs. (7) and (8), with the initial conditions $\dot{z}_i^\mu(-\infty) = u_i^\mu$, $\lim_{\tau_1\to-\infty} z_1(\tau_1)\bullet b = b^2$, and $\lim_{\tau_2\to-\infty} z_2(\tau_2)\bullet b = 0$.

However, even in the EM case, the exact solutions $z_i^\mu(\tau_i)$ are not available. Therefore, in the PL approach, one resorts to an iteration scheme, wherein one starts by expanding the worldlines as a series in coupling constant k,[1]

$$z_1^\mu(\tau_1) = b^\mu + u_1^\mu \tau_1 + k z_1^{(1)\mu}(\tau_1) + k^2 z_1^{(2)\mu}(\tau_1) + k^3 z_1^{(3)\mu}(\tau_1) + \cdots, \tag{9}$$

$$z_2^\mu(\tau_2) = u_2^\mu \tau_2 + k z_2^{(1)\mu}(\tau_2) + k^2 z_2^{(2)\mu}(\tau_2) + k^3 z_2^{(3)\mu}(\tau_2) + \ldots. \tag{10}$$

Note that the zeroth-order solutions describe the free straight-line trajectories. The above series ansatz is then fed into Eqs. (7) and (8) to iteratively solve the corrections to the straight-line paths at progressively higher orders in k; hence, the name "postlinear formalism." In Ref. [49], the solution of the worldlines up to $z_i^{(2)}(\tau_i)$ and momenta up to $m_i\dot{z}_i^{(3)}(\tau_i)$ were explicitly computed and, from them, the relevant scattering observables were also extracted. Unfortunately, the precise forms of the 3PL worldlines are rather lengthy and not quite illuminating. Hence, in the next subsection, we shall briefly summarize the scattering observables presented in Ref. [49].

[1]Technically, the series expansion is about a dimensionless quantity $\tilde{k} \equiv kq_1q_2/J$. However, we can be sloppy about it if we are sufficiently careful about the regime of validity ($\tilde{k} \ll 1$) of the results.

C. Scattering observables

In this subsection, we concentrate on three scattering observables—scattering angle χ, magnitude of the radiated angular momentum δJ, and the radiated COM energy δE—computed in Ref. [49], which are relevant to the current work. The scattering angle χ is defined via

$$\sin\chi = \frac{-\Delta p_1 \bullet b}{|b||P(\infty)|} = \frac{-\Delta p_1 \bullet b}{|b||P|} + \mathcal{O}(k^4), \tag{11}$$

where $\Delta p_1^\mu = m_1\dot{z}_1(\infty) - m_1\dot{z}_1(-\infty)$ is the net impulse on particle 1 and $|P(\infty)|$ is the final relative momentum. The 3PL result for χ is [49]

$$\chi = \chi_{\text{cons}} + \chi_{\text{rad}}, \tag{12}$$

$$\chi_{\text{cons}} = \frac{2kq_1q_2}{m\gamma|b|v^2}h_\nu(\gamma) - \frac{\pi k^2 q_1^2 q_2^2}{2m^2\gamma^2|b|^2v^2}h_\nu(\gamma) + \frac{2k^3q_1^3q_2^3[(2\gamma^2-3)h_\nu^2(\gamma) - 6\nu v^4\gamma^3]}{3m^3|b|^3\gamma^5 v^6}h_\nu(\gamma) + \mathcal{O}(k^4), \tag{13}$$

$$\chi_{\text{rad}} = -\frac{4k^3q_1^2q_2^2}{3m|b|^3\gamma v^3}\left[\left(\frac{q_1^2}{m_1^2} + \frac{q_2^2}{m_2^2}\right) - \frac{3q_1q_2}{m_1m_2}\left(\frac{1}{\gamma v^2} - \frac{\tanh^{-1}(v)}{\gamma^3 v^3}\right)\right]h_\nu(\gamma) + \mathcal{O}(k^4), \tag{14}$$

where $\chi_{\text{cons}}(\chi_{\text{rad}})$ denotes the conservative (radiative) part of the scattering angle. The symmetric mass ratio (ν) and $h_\nu(\gamma)$ are defined as

$$\nu = \frac{m}{M}, \qquad h_\nu(\gamma) = \frac{E}{M} = \sqrt{1 + 2\nu(\gamma - 1)}. \tag{15}$$

The radiated angular momentum δJ is the negative of the binary system's net change in total angular momentum. Hence, $\delta J = J_i - J_f$, where J_i and J_f are the initial and final angular momenta, respectively. With this definition, the radiated angular momentum in the COM frame δJ reduces to

$$\delta J \equiv k^2\delta J_2 + k^3\delta J_3 + \cdots, \tag{16}$$

$$= -\frac{4k^2q_1q_2m\gamma}{3|b|h_\nu(\gamma)}\left[\left(\frac{q_1^2}{m_1^2} + \frac{q_2^2}{m_2^2}\right) - \frac{3q_1q_2}{m_1m_2}\left(\frac{1}{\gamma v^2} - \frac{\tanh^{-1}(v)}{\gamma^3v^3}\right)\right] + \mathcal{O}(k^3). \tag{17}$$

The radiated linear momentum K^μ is defined as the negative of the sum of impulses on both particles. Hence,

$K^\mu = -\Delta p_1^\mu - \Delta p_2^\mu$, where $\Delta p_i^\mu = m_i \dot{z}_i^\mu(\infty) - p_i^\mu$, for $i = 1, 2$. The radiated COM energy δE is then defined as $K^\mu U_\mu$ and reduces to

$$\delta E \equiv k^2 \delta E_2 + k^3 \delta E_3 + \cdots \tag{18}$$

$$= \frac{\pi k^3 q_1^2 q_2^2}{4|b|^3 h_\nu(\gamma)} \left[\frac{3\gamma^2+1}{3\gamma v} \left(\frac{q_1^2}{m_1^2} + \frac{q_2^2}{m_2^2} \right) + \frac{(\gamma-1)(3\gamma^2+1)}{3\gamma v} \left(\frac{q_1^2 m}{m_1^3} + \frac{q_2^2 m}{m_2^3} \right) - \frac{\mathcal{G}(\gamma)}{\gamma v} \frac{q_1 q_2}{m_1 m_2} \right] + \mathcal{O}(k^4), \tag{19}$$

$$\mathcal{G}(\gamma) = \frac{(3\gamma^2+1)}{(\gamma v)^2} \left(\gamma - \frac{\tanh^{-1}(v)}{\gamma v} \right) - \frac{4}{(\gamma v)^2} (\gamma - 1)^2. \tag{20}$$

As mentioned earlier, the main aim of this work is to devise an EOB formulation that, in addition to effectively capturing the perturbative information contained in the above two-body scattering observables, also encapsulates certain nonperturbative aspects. This latter feature, in turn, can potentially enable us to go beyond the scope of the weak field regime of PL and also gain insights into the bound binary dynamics.

III. EFFECTIVE-ONE-BODY FORMALISM: THE SETUP

The EOB formalism is the canonical transformation that maps the dynamics of a relativistically interacting two-body system to that of a single particle in an external background. More specifically, the EOB maps the compact binary dynamics of general relativity to the test particle motion in an external background metric. The EOB formalism, first introduced in the seminal paper [14] and further developed in several works [57–60], is now routinely used in the detection of gravitational waves in LIGO-VIRGO-KAGRA.

Incidentally, the original paper of Buonanno and Damour [14] cites an old work of Brezin *et al.* (BIZ) [61], which essentially discusses an EOB formalism for EM as one of the primary inspirations. The BIZ paper used an approximate summation of the "crossed-ladder" Feynman diagrams of $2 \to 2$ EM scattering amplitude, which essentially recovered the eikonal asymptotic behavior and arrived at the bound-state energy spectrum E_n by inspecting the poles of the corresponding Green's function. Although this approximation did not effectively capture the centrifugal effects, their expression for E_n correctly accounted for the recoil effects. The BIZ expression for the two-body bound-state energy E_n can be mapped to the relativistic *one-body* spectrum ϵ_n of a particle, with mass being the reduced mass, moving in a static Coulomb potential as $E_n^2 = m_1^2 + m_2^2 + 2(m_1 + m_2)\epsilon_n$. However, the BIZ paper does not account for the radiative effects.

In the present discussion, we shall formulate an EOB approach to EM that is naturally adapted for the *classical* PL scheme. In the rest of this section, we briefly discuss EOB basics for completeness. In the following sections, we explicitly show that our proposal efficiently accounts for the leading-order radiation-reaction effects. We also qualitatively discuss how our proposal can be extended to GR.

A. EOB kinematics

As mentioned above, the starting point of EOB formalism is establishing a map between relevant quantities of the original two-body problem and the effective-one-body system. Because of the choice of canonical transformations, this mapping has some freedom. Here, we consider a choice introduced in [43] in the context of EOB formalism for PM gravity. This choice allows the conservative EOB dynamics up to 3PM to be cast as the motion of a particle in an effective metric. In contrast, the conventional mapping used in Refs. [14,57] leads to modifying the standard mass-shell condition. Moreover, the new mapping introduced in [43] also connects more directly with the results of the original two-body scattering observables.

The three key ingredients of the EOB mapping are as follows: (i) energy map, (ii) momentum map, and (iii) angular momentum map. The total energy E, the relative momentum P^μ, and the angular momentum J for a system of free particles are defined in Eqs. (2), (4), and (6), respectively. We shall denote the energy, spatial momentum, and angular momentum of the reduced mass in the effective-one-body description by ϵ, p^μ, and j, respectively. The EOB mapping amounts to the following identification:

$$\epsilon = \frac{E^2 - m_1^2 - m_2^2}{2M}, \tag{21}$$

$$p^\mu = h_\nu(\gamma) P^\mu, \tag{22}$$

$$j = h_\nu(\gamma) J, \tag{23}$$

$$|b| \to |b|, \tag{24}$$

$$\chi \to \chi, \tag{25}$$

where $h_\nu(\gamma)$ is defined in Eq. (15). The last two equations above emphasize that the impact parameters and scattering angles of the original two-body system and the EOB problem are identified. It is easily seen that $j = m v\gamma |b|$, $|p| = m v \gamma$, and $\epsilon = m\gamma$, as is desired.

B. EOB dynamics

Let $x^\mu(\tau) = (t(\tau), \vec{x}(\tau))$ denote the effective worldline that describes the relative dynamics of the two-body such that $m\dot{x}^i = p^i$, where the dot $(\cdot)$ denotes derivative with respect to the proper time of the reduced mass. The final ingredient of the EOB formalism for PL is a prescription for effective dynamics for x^μ, which encapsulates all the information contained in the two-body scattering observables up to the desired order.

The success and utility of the conventional EOB formalism for GR stem from the fact that, once the relevant parameters of the effective dynamics are fixed by matching the appropriate observables to a given order of an approximation scheme (like, for instance, PM, post-Newtonian), the EOB formalism can make sensible predictions even a bit beyond the regime of validity of the original approximation scheme. This feature can be attributed to the fact that the EOB formalism implicitly resums the approximate series expansion (i.e., either PM or PN expansions) and effectively translates the same to a systematic deformation about the test-particle limit (i.e., $\nu \to 0$). For instance, by construction, the effective metric in the PN EOB formalism can be viewed as a ν-deformed Schwarzschild metric with the mass parameter being $M = m_1 + m_2$ (see, for instance, [14,57,60]).

Motivated by this, we seek an EOB dynamics for the PL formalism that can also be viewed as a deformation of the test-particle limit in EM. The equation of motion in the test-particle limit $m_1/m_2 \to 0$ is given simply by the Lorentz-Dirac equation,

$$m_1\ddot{x}^\mu = q_1 F^{(c)\mu}{}_\nu \dot{x}^\nu + \frac{2kq_1^2}{3}(\dddot{x}^\mu + \dot{x}^\mu \ddot{x}\bullet\ddot{x}), \tag{26}$$

where $F^{(c)}{}_{\mu\nu}$ is the static Coulomb field generated by the charge q_2. It is instructive to define the Coulomb force-field tensor $\mathcal{F}^{(c)}{}_{\mu\nu}$ via $\mathcal{F}^{(c)}{}_{\mu\nu} \equiv q_1 F^{(c)}{}_{\mu\nu}$. The Coulomb force-field tensor can be written in terms of the vector potential $\mathcal{A}^{(c)}{}_\mu$, which takes the form

$$\mathcal{A}^{(c)}{}_\mu = \left(\frac{kq_1q_2}{r}, 0, 0, 0\right). \tag{27}$$

The Lorentz-Dirac equation, recast in terms of the Coulomb force-field tensor, is given by

$$\begin{aligned} m_1\ddot{x}^\mu &= \mathcal{F}^{(c)\mu}{}_\nu \dot{x}^\nu \\ &+ \frac{2km_1}{3}\Bigg[\left(\frac{q_1^2}{m_1^2}\right)\mathcal{F}^{(c)\mu}{}_{\nu,\alpha}\dot{x}^\nu\dot{x}^\alpha + \left(\frac{q_1^2}{m_1^3}\right)(\mathcal{F}^{(c)\mu}{}_\nu \mathcal{F}^{(c)\nu}{}_\alpha \dot{x}^\alpha \\ &- \mathcal{F}^{(c)\alpha}{}_\nu \mathcal{F}^{(c)\nu}{}_\beta \dot{x}^\beta \dot{x}_\alpha \dot{x}^\mu) \\ &+ \frac{2k}{3}\left(\frac{q_1^4}{m_1^3}\right)\mathcal{F}^{(c)\mu}{}_{\nu,\alpha\sigma}\dot{x}^\nu\dot{x}^\alpha\dot{x}^\sigma\Bigg] + \mathcal{O}(k^4). \end{aligned} \tag{28}$$

We neglect the $O(k^4)$ terms since they will not be relevant for the 3PL results. Now, motivated by the form of the test-particle limit given by Eq. (28), we propose that the EOB dynamics is described by the following deformed Lorentz-Dirac equation:

$$\begin{aligned} m\ddot{x}^\mu &= \mathcal{F}^\mu{}_\nu \dot{x}^\nu \\ &+ \frac{2km}{3}\Bigg[A\mathcal{F}^\mu{}_{\nu,\alpha}\dot{x}^\nu\dot{x}^\alpha + B(\mathcal{F}^\mu{}_\nu\mathcal{F}^\nu{}_\alpha\dot{x}^\alpha - \mathcal{F}^\alpha{}_\nu\mathcal{F}^\nu{}_\beta\dot{x}^\beta\dot{x}_\alpha\dot{x}^\mu) \\ &+ \frac{2k}{3}C\mathcal{F}^\mu{}_{\nu,\alpha\sigma}\dot{x}^\nu\dot{x}^\alpha\dot{x}^\sigma\Bigg] + O(k^4), \end{aligned} \tag{29}$$

where A, B, and C are three parameters of dimensions q_1^2/m_1^2, q_1^2/m_1^3, and q_1^4/m_1^3, respectively. Although we have motivated the above EOB equation of motion as a natural deformation of the Lorentz-Dirac equation, in the spirit of conventional EOB formalism in the gravitational case, the same deformed dynamics can be shown to follow from general principles, like Lorentz invariance, gauge invariance, etc., and some minimal further assumptions, as we discuss in Sec. VII A. This alternative perspective, akin to the effective-field-theory approach, signifies that Eq. (29) is not entirely *ad hoc*.

The proposed deformation of the dynamics has two distinct aspects: (i) the deformation of the *conservative* force field (or, equivalently, the potential); $\mathcal{F}^{(c)}{}_{\mu\nu} \to \mathcal{F}_{\mu\nu}$ (or, equivalently, $\mathcal{A}^{(c)}{}_\mu \to \mathcal{A}_\mu$), and (ii) the deformation of the *radiation-reaction* force; accomplished by replacement of the coefficients q_1^2/m_1^2, q_1^2/m_1^3, and q_1^4/m_1^3 in the leading radiation-reaction term in square brackets on the right-hand side of Eq. (28) by, respectively, A, B, and C. While the former captures the conservative part of the two-body dynamics, the latter encodes the radiative effects. As in gravity, we consider a static and radial external force field. Further, we can assume the following form for the deformed potential:

$$\mathcal{A}_\mu = (\phi, 0, 0, 0), \tag{30}$$

$$\phi(r) = E\sum_{n=1}^{\infty}\left(\frac{kq_1q_2}{rM}\right)^n \phi_n(\nu,\gamma). \tag{31}$$

In the limit $\nu \to 0$, the scalar potential ϕ must reduce to kq_1q_2/r. We shall explicitly see that this is indeed the case. Moreover, it is also easy to see that the conservative dynamics is effectively described by the following Hamiltonian:

$$\mathcal{H}(\vec{p}, \vec{x}) = \sqrt{\vec{p}\bullet\vec{p} + m^2} + \phi(|\vec{x}|), \tag{32}$$

where $\vec{x}$ and $\vec{p}$ are the effective-one-body coordinate and momenta, respectively. In particular, Hamilton's equations of motion that follow from $\mathcal{H}$, namely, $dx^i/dt = \partial_{p^i}\mathcal{H}$ and $dp^i/dt = -\partial_{x^i}\mathcal{H}$, can be shown to be equivalent to the conservative limit (i.e., $A, B, C \to 0$) of Eq. (29).

It is well known that Hamiltonian formalism cannot describe generic dissipative systems. For the special case of dissipative forces that are linear in velocity, one can employ the Rayleigh dissipation function [62]. However, this approach fails to describe a more general class of dissipative features. For a discussion of this issue and a modified Hamiltonian formalism that can accommodate generic dissipative systems, see Ref. [63]. Nevertheless, we can study the classical evolution using the effective deformed Lorentz-Dirac equation. The parameters A and B, in the radiation-reaction terms therein, can be assumed in the following form:

$$A = \alpha_1(\nu,\gamma)\left(\frac{q_1^2}{m_1^2}+\frac{q_2^2}{m_2^2}\right)+\alpha_2(\nu,\gamma)\frac{q_1q_2}{m_1m_2}, \tag{33}$$

$$B = \beta_1(\nu,\gamma)\left(\frac{q_1^2}{m_1^2 m}+\frac{q_2^2}{m_2^2 m}\right)+\beta_2(\nu,\gamma)\left(\frac{q_1^2}{m_1^3}+\frac{q_2^2}{m_2^3}\right) + \beta_3(\nu,\gamma)\frac{q_1q_2}{mm_1m_2}. \tag{34}$$

Here, we expect $\alpha_1(0,\gamma)=\beta_1(0,\gamma)+\beta_2(0,\gamma)=1$, so that the Lorentz-Dirac equation is retained in the test-particle limit. To accommodate all potential terms of the required dimension that can be generated from q_1, q_2, m_1, m_2, one would have naively assumed an infinite series expansion for both A and B based on the dimension analysis. For instance, one could have added terms proportional to $q_1^3/(q_2m_2^2)$, $q_1^{100}/(q_2^{98}m_2^2)$, etc., to the ansatz for A. However, from the form of dissipative scattering observables δJ and δE, given by Eqs. (16) and (18), respectively, one can guess that the finite expansions in Eqs. (33) and (34) would suffice. On the other hand, the form of C can only be fixed when the dissipative variables are known to $\mathcal{O}(k^4)$, which is currently unavailable. However, fortunately, the parameter C does not contribute to either χ or δE, up to $\mathcal{O}(k^3)$. Hence, we shall ignore this term for the most part, barring some comments at a few junctures.

In summary, up to third order, we have three sets of unknown dimensionless parameters in the EOB dynamics: $\{\phi_1,\phi_2,\phi_3\}$, $\{\alpha_1,\alpha_2\}$, and $\{\beta_1,\beta_2,\beta_3\}$. The next step is to determine the explicit forms of these parameters by matching the observables of the EOB problem to those of the original two-body problem, as discussed in Sec. III A.

IV. EFFECTIVE-ONE-BODY FORMALISM: THE OBSERVABLES

The effective potential $\phi(r)$, as defined in Eq. (31), is characterized by an infinite set of coefficients $\{\phi_n|n=1,2,3,\ldots\}$. However, as mentioned earlier, for results up to 3PL, the subset $\{\phi_1,\phi_2,\phi_3\}$ would suffice. We can fix $\{\phi_1,\phi_2,\phi_3\}$ by matching the conservative part of the scattering angle of the EOB problem to that of the original two-body problem; whereas, to fix A and B, we shall do a similar exercise with the leading-order radiated angular momentum and radiated energy. Therefore, our next step is to solve the scattering problem of the EOB system to 3PL.

A. Perturbative approach to the EOB dynamics

As described in Eq. (29), we now solve the hyperbolic-like orbits of the reduced mass, interacting with an effective conservative force $\mathcal{F}(r)$ and a radiation-reaction force. Because of the system's symmetry, the entire scattering orbit will be constrained on a plane, where we introduce the polar coordinate system (r,θ). As is standard in the study of orbital dynamics, for convenience, we introduce the variable $u\equiv 1/|\vec{x}|\equiv 1/r$. Now, we want to obtain the function $u(\theta)$ that mathematically represents the orbit. In the spirit of the PL scheme, we shall solve the scattering orbit $u(\theta)$ as a series in the coupling constant. To this end, we expand as follows:

$$\begin{aligned} u(\theta) &= \frac{m\gamma}{j}\sin\theta+\left(\frac{kq_1q_2}{j}\right)\tilde{u}_1(\theta)+\left(\frac{kq_1q_2}{j}\right)^2\tilde{u}_2(\theta) \\ &\quad+\left(\frac{kq_1q_2}{j}\right)^3\tilde{u}_3(\theta)+\cdots \\ &\equiv u_0(\theta)+ku_1(\theta)+k^2u_2(\theta)+k^3u_3(\theta)+\cdots, \end{aligned} \tag{35}$$

where j is the *initial* angular momentum. Let us denote the angular momentum along the orbit of the EOB particle by $\mathcal{J}(\theta)$. Because of the radiation reaction, $\mathcal{J}(\theta)$ is not a constant along the orbit. It is also convenient to expand the varying angular momentum $\mathcal{J}(\theta)$ in powers of k as follows:

$$\begin{aligned} \mathcal{J}(\theta) &\equiv mr^2\dot{\theta} \\ &= j+\left(\frac{kq_1q_2}{j}\right)^2\tilde{\mathcal{J}}_2(\theta)+\left(\frac{kq_1q_2}{j}\right)^3\tilde{\mathcal{J}}_3(\theta)+\cdots \\ &\equiv j+k^2\mathcal{J}_2(\theta)+k^3\mathcal{J}_3(\theta)+\cdots. \end{aligned} \tag{36}$$

Note that θ-dependent corrections start only at $\mathcal{O}(k^2)$, the reason for which will be clear shortly. Similarly, we can also expand the energy function $\mathcal{E}(\theta)$ as

$$\begin{aligned} \mathcal{E}(\theta) &\equiv \varepsilon+\phi \\ &= \epsilon+\left(\frac{kq_1q_2}{j}\right)^2\tilde{\mathcal{E}}_2(\theta)+\left(\frac{kq_1q_2}{j}\right)^3\tilde{\mathcal{E}}_3(\theta)+\cdots \\ &\equiv \epsilon+k^2\mathcal{E}_2(\theta)+k^3\mathcal{E}_3(\theta)+\cdots, \end{aligned} \tag{37}$$

where $\varepsilon\equiv m\dot{t}$. The evolution equations of $u(\theta)$, $\mathcal{J}(\theta)$, and $\mathcal{E}$ can be derived from Eq. (29) and the mass-shell condition written in polar coordinates. The orbit is a solution to the following differential equation:

$$u''+u-\frac{\varepsilon\varepsilon'}{\mathcal{J}^2u'}+\frac{\varepsilon^2\mathcal{J}'}{\mathcal{J}^3u'}=0, \tag{38}$$

where the prime ($'$) denotes derivative with respect to θ. The rates of change of angular momentum and energy function take the forms

$$\mathcal{J}'(\theta) = \frac{2km}{3}\left[A\left(\frac{\varepsilon\mathcal{F}}{m^2 u}\right) - B\left(\frac{\mathcal{F}^2}{u^2} + \frac{\mathcal{F}^2\mathcal{J}^2}{m^2}\right) + \frac{2k}{3}C\left(\frac{\mathcal{J}^2 u' u^3 \mathcal{F}}{m^2} + \frac{\mathcal{J}^2 u' u^4 \partial_u \mathcal{F}}{m^2}\right)\right], \tag{39}$$

$$\mathcal{E}'(\theta) = \frac{2km}{3}\left[A\left(\frac{\mathcal{J}u\mathcal{F}}{m} - \frac{\mathcal{J}(u')^2\partial_u\mathcal{F}}{m}\right) - B\left(\frac{\mathcal{J}\varepsilon\mathcal{F}^2}{m^2}\right) + \frac{2k}{3m^2}C(\mathcal{J}^2 uu'(3u\mathcal{F} + u\partial_u\mathcal{F}) - (u')^2(2\partial_u\mathcal{F} + \partial_u^2\mathcal{F}u)))\right]. \tag{40}$$

Now, we can perturbatively solve the system of the three equations (38)–(40) in powers of k. To this end, we need to substitute Eq. (35) into the three equations and iteratively solve the system. In particular, the equation of motion of the orbit at the nth order can be cast in the following form:

$$u_n'' + u_n + \mathcal{U}_n(u_{n-1}, u_{n-2}, \ldots, u_0) = 0. \tag{41}$$

On the other hand, the equation describing the dissipation of energy and angular momentum simplifies to

$$\mathcal{J}_n' = \mathcal{T}_n(u_{n-2}, u_{n-3}, \ldots, u_0), \tag{42}$$

$$\mathcal{E}_n' = \mathcal{P}_n(u_{n-2}, u_{n-3}, \ldots, u_0). \tag{43}$$

1. Comments on the C-dependent RR term

Although we have displayed the C-dependent RR term in Eq. (29) for generality, as we show in this subsection, the observables at $\mathcal{O}(k^3)$ are independent of C. To see this, we first rewrite the C-dependent RR term as a total derivative, modulo $\mathcal{O}(k^4)$ terms,

$$\frac{4k^2 mC}{9}\mathcal{F}^{\mu}{}_{\nu,\alpha\sigma}\dot{x}^{\nu}\dot{x}^{\alpha}\dot{x}^{\sigma} = \frac{4k^2 mC}{9}\frac{d}{d\tau}[\mathcal{F}^{\mu}{}_{\nu,\alpha}\dot{x}^{\nu}\dot{x}^{\alpha}] + \mathcal{O}(k^4). \tag{44}$$

Substituting the rhs of the above expression in Eq. (29), we see that the contribution to the energy radiated from this term goes at most as $\lim_{\tau\to\infty}[u^3(\tau) - u^3(-\tau)]$ and, hence, vanishes at $\mathcal{O}(k^3)$. Similarly, the C-dependent piece of the rate of change of angular momentum, Eq. (39),

$$\frac{4k^2 C}{9}\left(\frac{\mathcal{J}^2 u' u^3 \mathcal{F}}{m^2} + \frac{\mathcal{J}^2 u' u^4 \partial_u \mathcal{F}}{m^2}\right) \propto k^3 \frac{d}{d\theta}u^6 + \mathcal{O}(k^4), \tag{45}$$

is also a total derivative. Since $u \to 0$ at the asymptotes, it follows that the contribution of the above term to $\delta\mathcal{J}$ vanishes at $\mathcal{O}(k^3)$. Thus, from the above two observations, one can naively conclude that C cannot be fixed with the knowledge of $\mathcal{O}(k^3)$ observables. On the other hand, the same observations also indicate that we may predict the subleading contribution to δJ from only knowing the explicit forms of the coefficients A and B, whose explicit forms are not available in the literature. In Sec. V A, we discuss this in detail.

B. Scattering observables: The conservative parts

We define the conservative scattering angle χ_{cons} as the part of the total scattering angle χ that can be attributed to the dynamics generated by the conservative potential while neglecting the radiation-reaction terms. The standard formula below gives the conservative scattering angle to be

$$\chi_{\text{cons}} = \pi - 2j\int_{r_{\min(c)}}^{\infty} \frac{dr}{r\sqrt{[(\epsilon-\phi)^2 - m^2]r^2 - j^2}}, \tag{46}$$

where $r_{\min(c)}$ is the conservative piece of the value of radial coordinate at the closest approach and is given by

$$r_{\min(c)} = |b| + \frac{h_\nu q_1 q_2 \phi_1}{\gamma m v^2}k + \frac{h_\nu q_1^2 q_2^2(h_\nu\phi_1^2 + 2\gamma^2\nu v^2\phi_2)}{2|b|\gamma^4 m^2 v^4}k^2 + \frac{h_\nu \nu q_1^3 q_2^3(\gamma\nu\phi_3 - h_\nu\phi_1\phi_2)}{|b|^2\gamma^2 m^3 v^2}k^3 + \mathcal{O}(k^4) \tag{47}$$

$$\equiv |b| + b_1 k + b_2 k^3 + b_3 k^3 + \mathcal{O}(k^4), \tag{48}$$

where $b_1, b_2, b_3, ..$ are constant coefficients defined by the above series. A straightforward computation yields

$$\chi_{\text{cons}} = \frac{2kq_1q_2\phi_1 h_\nu}{|b|\gamma m v^2} + \frac{\pi k^2 q_1^2 q_2^2 h_\nu(2\gamma\nu\phi_2 - h_\nu\phi_1^2)}{2|b|^2\gamma^2 m^2 v^2} + \frac{2k^3 q_1^3 q_2^3 h_\nu}{3|b|^3\gamma^6 m^3 v^6}[\phi_1 h_\nu(\gamma(2\gamma^2-3)(\phi_1^2 h_\nu - 6\gamma\nu\phi_2) - 6\nu\phi_2) + 6\gamma^5\nu^2 v^4\phi_3] + \mathcal{O}(k^4). \tag{49}$$

C. Scattering observables: The dissipative parts

From Eqs. (39) and (40), and noting that $\mathcal{F}$ is $\mathcal{O}(k)$, we see that $\mathcal{E}'$ and $\mathcal{J}'$ are $\mathcal{O}(k^2)$. Moreover, to solve $\mathcal{E}(\theta)$ and $\mathcal{J}(\theta)$ to $\mathcal{O}(k^3)$, we require only the orbital equation $u(\theta)$ to $\mathcal{O}(k)$. Solving Eq. (38) iteratively, we get

$$u(\theta) = \frac{mv\gamma}{j}\sin\theta + \left(\frac{kq_1q_2}{j}\right)\left(\frac{h_\nu\phi_1 m\gamma}{j}\right)(\cos\theta - 1) + \mathcal{O}(k^2). \tag{50}$$

Substituting the above in Eqs. (39) and (40), and integrating, we get

$$\mathcal{J}(\theta) = j - \left(\frac{kq_1q_2}{j}\right)^2 \left(\frac{2jAm^2\gamma^2 v\phi_1 h_\nu}{3q_1q_2}\right)(\cos\theta - 1) + \mathcal{O}(k^3), \tag{51}$$

$$\mathcal{E}(\theta) = \epsilon - \left(\frac{kq_1q_2}{j}\right)^2 \left(\frac{2\gamma^3 Am^3 v^3\phi_1 h_\nu}{3q_1q_2}\right)\sin^2\theta\,\cos\theta + \mathcal{O}(k^3). \tag{52}$$

We have delegated the rather lengthy expressions for $\mathcal{J}_3$ and $\mathcal{E}_3$ to Appendix A. An interesting thing to note is that, although the perturbations of both energy and angular momentum start at $\mathcal{O}(k^2)$, the radiated angular momentum $\delta\mathcal{J}$ is $\mathcal{O}(k^2)$, while the radiated energy $\delta\mathcal{E}$ starts only at $\mathcal{O}(k^3)$. More precisely,

$$\delta\mathcal{J} \equiv -[\mathcal{J}(\pi-\chi) - \mathcal{J}(0)] \tag{53}$$

$$= -\left(\frac{kq_1q_2}{j}\right)^2 \frac{4Aj\gamma^2m^2v\phi_1h_\nu}{3q_1q_2} + \left(\frac{kq_1q_2}{j}\right)^3 \frac{\pi\gamma^2 jm^2[\phi_1^2 h_\nu(4A(v^2+2) + B(3\gamma^2+1)mv^2) - 8A\gamma\nu v^2\phi_2]}{12q_1q_2} + \mathcal{O}(k^4), \tag{54}$$

$$\delta\mathcal{E} \equiv -[\mathcal{E}(\pi-\chi) - \mathcal{E}(0)] \tag{55}$$

$$= \left(\frac{kq_1q_2}{j}\right)^3 \frac{\pi\gamma^3k^3m^3v^2h_\nu^2}{12q_1^2q_2^2}(4A + 3B\gamma^2 mv^2) + \mathcal{O}(k^4). \tag{56}$$

As mentioned in Sec. IV A 1, the third-order contribution to δJ is indeed independent of C.

V. DETERMINING THE EOB PARAMETERS

In the previous section, we perturbatively obtained the expressions for $u(\theta), \mathcal{E}(\theta)$, and $\mathcal{J}(\theta)$, including the radiation-reaction terms. In this section, we determine the hitherto unknown coefficients $\{\phi_i\}, \{\alpha_i\}$, and $\{\beta_i\}$ in the EOB formulation. We also discuss how the EOB formalism can provide a way to obtain subleading contribution to the loss in angular momentum.

To determine the first set of parameters $\{\phi_1, \phi_2, \phi_3\}$, we equate the conservative piece of the scattering angle of the original problem to that of the EOB one. This yields

$$\phi_1 = 1, \tag{57}$$

$$\phi_2 = \frac{h_\nu - 1}{2\gamma\nu}, \tag{58}$$

$$\phi_3 = \frac{(2\gamma^4 - 3\gamma^2 + 1)(h_\nu - 1)h_\nu - 2\gamma^5\nu v^4}{2\gamma^6\nu^2v^4}. \tag{59}$$

Similarly, we obtain the radiative coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ using the matching conditions derived from the EOB mapping given in Sec. III A,

$$\frac{\delta\mathcal{E}}{h_\nu} = \delta E + \frac{(\delta E)^2}{2E} = \delta E + \mathcal{O}(k^4), \tag{60}$$

$$\frac{\delta\mathcal{J}}{h_\nu} = \delta J + \frac{J}{E}\delta E + \frac{\delta E\delta J}{E} = \delta J + \mathcal{O}(k^3). \tag{61}$$

The above mapping yields

$$\alpha_1 = \frac{1}{h_\nu}; \qquad \alpha_2 = \frac{3}{\gamma^3v^3h_\nu}(\tanh^{-1}(v) - \gamma^2 v), \tag{62}$$

$$\beta_1 = \frac{3\gamma^2 - 4\gamma h_\nu + 1}{3\gamma^3v^2h_\nu^2}, \qquad \beta_2 = \frac{(\gamma-1)(3\gamma^2+1)}{3\gamma^3v^2h_\nu^2},$$
$$\beta_3 = \frac{4h_\nu[\gamma^3 v - \gamma\tanh^{-1}(v)] + (3\gamma^2+1)\tanh^{-1}(v) + \gamma v[(4-3\gamma)\gamma^2 - 9\gamma + 4]}{\gamma^6v^5h_\nu^2}. \tag{63}$$

This concludes our EOB formalism for the electromagnetic scattering problem, which accounts for the leading-order radiative effects. Appendix C contains the expansion of the EOB potential in the test-particle limit.

A. Determining subleading radiated angular momentum

We now argue how the EOB formalism can be used to derive the subleading contribution to the angular momentum loss. As mentioned earlier, the direct computation of this $\mathcal{O}(k^3)$ term to δJ is currently unavailable. However, once the explicit forms of A and B are known, one can use Eq. (54) along with the EOB mapping from $\delta\mathcal{J} \to \delta J$ to "predict" the third-order contribution to δJ. We find that this contribution δJ_3 is

$$\delta J_3 = \left[\frac{J}{3m}\left(h_\nu\left(\frac{3}{\gamma}+\frac{4}{\gamma^3 v^2}\right)-3\nu\right)\right]\delta E_3 + \left[\frac{\pi q_1 q_2 (h_\nu(3\gamma^2(v^2-2)+4)-3\gamma^2 v^2)}{12\gamma^2 J v h_\nu}\right]\delta J_2, \quad (64)$$

where δJ_2 and δE_3 correspond to the leading-order radiated angular momentum [Eq. (16)] and energy [Eq. (18)].

VI. RESULTS FOR BOUND ORBITS

In the gravitational case, one of the most important applications of classical scattering results has been accurately forecasting bound-orbit dynamics. Several promising methods have been proposed to achieve this, including EFT methods [16,17] and analytic continuations [22,42]. Here, we explore using EOB formalism to extract binary-orbit results from the unbound-orbit ones.

For the EM case, which is the focus of this work, binary orbits exist for $q_1 q_2 < 0$. These binary orbits satisfy $E < M$, which in the EOB language translates to $\epsilon < m$. The method initiated in Refs. [22,42] corresponds to obtaining certain bound-orbit results by the analytical continuation of appropriate scattering observables into the domain $\gamma < 1$. As a consistency check of our formalism, we now verify whether we can reproduce the bound orbits' results obtained using analytical continuation methods. In this section, we explicitly evaluate (1) the periastron shift $\Delta\Phi$ and (2) the energy radiated per orbital period (ΔE), and (3) the angular momentum loss per orbital period (ΔJ) using the EOB formalism and compare them against the expressions evaluated using analytic continuation. For completeness, we have exhibited the expressions from the analytical continuation method in Appendix D.

To compare bound-orbit results, it is convenient to employ certain useful variables. We start with the mass-subtracted energy (or "the nonrelativistic energy") $E_{nr} \equiv E - M$. The bound orbits correspond to those with $E_{nr} < 0$. Recalling that $E_{nr} < 0$ corresponds to $\epsilon/m = \gamma < 1$, the function $h_\nu = E/M = \sqrt{1+2\nu(\gamma-1)}$ for the bound orbits satisfies $h_\nu < 1$. Since h_ν and γ were originally defined for the scattering problem (for which $h_\nu, \gamma \geq 1$), we introduce a subscript (b) for these variables whenever they specifically refer to bound orbits; i.e., $h_{\nu(b)}$ and $\gamma_{(b)}$ for $E_{nr} < 0$. Moreover, for bound orbits $v = i v_b$, where $v_b \in \mathbb{R}$.

A. Periastron shift

The leading-order periastron shift is expected at $\mathcal{O}(k^2)$. So, we shall next find the conservative bound orbit at this order. The (conservative) equation of motion for the orbit takes the form

$$u'' + \omega^2 u + \left(\frac{k q_1 q_2}{j^2}\right)\gamma_{(b)} m \phi_1 h_{\nu(b)} + \mathcal{O}(k^3) = 0, \quad (65)$$

$$\omega^2 = 1 - \left(\frac{k q_1 q_2}{j^2}\right)^2 h_{\nu(b)}\,(\phi_1^2 h_{\nu(b)} - 2\gamma_{(b)}\nu\phi_2), \quad (66)$$

where the subscript (b) with γ and h_ν refers to bound orbits (i.e., $\gamma_{(b)}, h_{\nu(b)} < 1$). The orbital equation is

$$u_b(\theta) = \bar{u}\,\cos[\omega(\theta-\bar{\theta})] - \frac{\gamma_{(b)} k m q_1 q_2 \phi_1 h_{\nu(b)}}{j^2}, \quad (67)$$

where the constant $\bar{u}$ can be found by demanding the mass-shell condition and $\bar{\theta}$ just captures the initial condition. Therefore, the periastron shift at $\mathcal{O}(k^2)$ can be found from

$$\Delta\Phi = \frac{2\pi}{\omega} - 2\pi + \mathcal{O}(k^4) \quad (68)$$

$$= \frac{\pi k^2 q_1^2 q_2^2 h_{\nu(b)}(\phi_1^2 h_{\nu(b)} - 2\gamma_{(b)}\nu\phi_2)}{j^2} + \mathcal{O}(k^4) \quad (69)$$

$$= \frac{\pi k^2 q_1^2 q_2^2}{J^2 h_{\nu(b)}} + \mathcal{O}(k^4), \quad (70)$$

where in Eq. (70) we have used Eqs. (57) and (58), along with the EOB map $J = h_\nu j$. Note that the above expression matches exactly with (D2).

B. Energy loss per orbit

Equation (40) gives the general expression for the rate of change of energy. Hence, the total energy *loss* per orbital period of the EOB particle $\Delta\mathcal{E}$ is

$$\Delta\mathcal{E} = -\oint \mathcal{E}' d\theta. \quad (71)$$

Substituting the bound-orbit solution $u_b(\theta)$ from Eq. (67) into the above equation yields

$$\Delta\mathcal{E} = \frac{\pi\gamma_{(b)}(\gamma_{(b)}^2-1)k^3 m^3 q_1^2 q_2^2 \phi_1^2 h_{\nu(b)}^2 (4A + 3B(\gamma_{(b)}^2-1)m)}{6 j^3} + \mathcal{O}(k^4). \quad (72)$$

Using the EOB mapping in Eq. (60), the above equation translates to

$$\Delta E = \frac{\pi m^3 k^3 q_1^2 q_2^2 (\gamma_{(b)}^2 - 1)}{2J^3 h_{\nu(b)}^4}\left[\frac{(3\gamma_{(b)}^2+1)}{3}\left(\frac{q_1^2}{m_1^2}+\frac{q_2^2}{m_2^2}\right) + \frac{(\gamma_{(b)}-1)(3\gamma_{(b)}^2+1)}{3}\left(\frac{q_1^2 m}{m_1^3}+\frac{q_2^2 m}{m_2^3}\right) - \mathcal{G}(\gamma_{(b)})\frac{q_1 q_2}{m_1 m_2}\right] + \mathcal{O}(k^4), \tag{73}$$

where we have also used the explicit forms of A and B as found in Sec. V to arrive at the final expression. The above expression matches exactly with that in Eq. (D3).

As a further consistency check, it is instructive to consider the nonrelativistic limit of the above expression. To this end, recall the definition of nonrelativistic energy $E_{nr} \equiv E - M$, so that $h_{\nu(b)} = E_{nr}/M + 1$. Therefore, the nonrelativistic limit ΔE_{nr} of the energy loss takes the form

$$\Delta E = \Delta E_{nr} + \mathcal{O}\left(\frac{E_{nr}^2}{M^2}\right), \tag{74}$$

$$\Delta E_{nr} = \frac{4\pi m^2 k^3 q_1^2 q_2^2 E_{nr}}{3J^3}\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)^2 + \mathcal{O}(k^4). \tag{75}$$

The above expression matches perfectly with the leading-order nonrelativistic limit of energy loss computed in Appendix B.

C. Angular momentum loss per orbit

Equation (39) describes the rate of change of angular momentum. The angular momentum lost per orbit of the EOB particle is

$$\Delta\mathcal{J} = -\oint \mathcal{J}' d\theta. \tag{76}$$

Substituting the bound orbit $u_b(\theta)$ from Eq. (67) into the above equation, we get

$$\Delta\mathcal{J} = \frac{2\pi A k^3 m^2 q_1^2 q_2^2 h_{\nu(b)}(\gamma_{(b)}^2 + 2\gamma_{(b)}^2 h_{\nu(b)} - 1)}{3j^2} + \frac{\pi B(3\gamma^4 - 2\gamma_{(b)}^2 - 1)k^3 m^3 q_1^2 q_2^2 h_{\nu(b)}^2}{6j^2} + \mathcal{O}(k^4). \tag{77}$$

Now, using the EOB mapping in Eq. (61), we can evaluate the angular momentum loss per orbit of the original binary system,

$$\Delta J = \frac{2\pi A k^3 m^2 q_1^2 q_2^2 h_{\nu(b)}(-\nu\gamma_{(b)}^3 + \gamma_{(b)}\nu + h_{\nu(b)}(\gamma_{(b)}^2 + 2\gamma_{(b)}^2 h_{\nu(b)} - 1))}{3J^2} - \frac{\pi B(\gamma_{(b)}^2 - 1)k^3 m^3 q_1^2 q_2^2 h_{\nu(b)}(3\gamma_{(b)}(\gamma_{(b)}^2 - 1)\nu - (3\gamma_{(b)}^2 + 1)h_{\nu(b)}^2)}{6J^2} + \mathcal{O}(k^4). \tag{78}$$

Substitution of A and B in the above equation produces a lengthy expression, which is not illuminating and will not be given here. However, the final expression thus obtained matches exactly the one obtained from Eq. (D5). In other words, ΔJ from our EOB formalism is precisely $2k^3\delta J_3 + \mathcal{O}(k^4)$.

As in the case of energy loss, the nonrelativistic limit of ΔJ yields

$$\Delta J = \Delta J_{nr} + \mathcal{O}\left(\frac{E_{nr}}{M}\right), \tag{79}$$

$$\Delta J_{nr} = \frac{4\pi k^3 m^2 q_1^2 q_2^2}{3J^2}\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)^2. \tag{80}$$

The above expression for ΔJ_{nr} matches precisely with the explicit nonrelativistic computation of angular momentum loss given in Appendix B.

VII. COMPARISON TO OTHER APPROACHES

In the previous section, we demonstrated how to use the EOB formalism to derive physical quantities for the binary orbit from the unbound-orbit ones. We also showed that the EOB formalism gives identical results to the other approaches. In this section, we compare and contrast our approach to certain other ones, although mainly concerning gravitational dynamics, already available in the literature.

A. Effective-field-theory-inspired approach

The EFT-based approach for gravitational binaries of nonrotating compact objects was pioneered by Goldberger and Rothstein [64,65]. The extension of the approach that accounts for the spin was also later developed [66,67]. For a modern review of the EFT approach to gravitational dynamics, consult [68]. EFT methods have recently been used to study PM dynamics [69–74], even including the radiation effects [45,75–81].

In Sec. III B, we invoked Eq. (29) as a deformation of the standard Lorentz-Dirac equation to account for the recoil effects. However, one can also motivate the same deformed Lorentz-Dirac equation from EFT-inspired reasoning. To this end, instead of referring to the Lorentz-Dirac equation, we can seek the most general form of the radiation-reaction force that is allowed by symmetries of the system. More specifically, we look for additions to the Lorentz force equation that are (1) Lorentz invariant, (2) gauge invariant, (3) orthogonal to $\dot{x}^\nu$. The above assumptions, along with the fact that $\mathcal{F}^\mu{}_\nu = \mathcal{O}(k)$ and kq^2/m has dimensions of length, imply that the allowed terms, up to $\mathcal{O}(k^3)$, are proportional to

$$\begin{aligned}&\{A\mathcal{F}^\mu{}_{\nu,\alpha}\dot{x}^\nu\dot{x}^\alpha, B(\mathcal{F}^\mu{}_\nu\mathcal{F}^\nu{}_\alpha\dot{x}^\alpha - \mathcal{F}^\alpha{}_\nu\mathcal{F}^\nu{}_\beta\dot{x}^\beta\dot{x}_\alpha\dot{x}^\mu),\\ &\quad kC\mathcal{F}^\mu{}_{\nu,\alpha\sigma}\dot{x}^\nu\dot{x}^\alpha\dot{x}^\sigma\},\end{aligned} \tag{81}$$

where A, B, and C are three parameters of dimensions q_1^2/m_1^2, q_1^2/m_1^3, and q_1^4/m_1^3, respectively. The idea is to generate terms by either taking multiple derivatives of $\mathcal{F}$ and contraction with an appropriate number of velocity vectors or taking products of $\mathcal{F}$'s and contracting by an appropriate number of velocity vectors or a combination of these two. Of course, while generating these terms, one should be mindful of the order in k of the same as well as introduce appropriate coefficients so that the terms have the desired dimension. The simplest term obtained in this manner, therefore, has the schematic form $\partial_x\mathcal{F}\dot{x}\dot{x}$, followed by $\mathcal{F}\mathcal{F}\dot{x}$ and, finally, $\partial x\partial x\mathcal{F}\dot{x}\dot{x}\dot{x}$, exhausting terms up to $\mathcal{O}(k^3)$. By extending this logic, one can write down the radiation-reaction terms to any desired order and introduce appropriate parameters, which can be fixed by matching observables of the EOB problem to that of the original two-body one.

B. Conventional approaches to EOB for dissipative dynamics

The dissipative effects due to radiation can be accounted for by adding an appropriate radiation-reaction term to the equation of motion. Retaining our notation for the polar coordinates for the relative coordinate, the radiation-reaction $\mathbf{f}$ force can be generically written as $\mathbf{f} = f_r\hat{r} + (f_\theta/r)\hat{\theta}$. The rate of change of total angular momentum and energy takes the following form [33,82]:

$$\frac{d\mathcal{E}}{dt} = \frac{dr}{dt}f_r + \frac{d\theta}{dt}f_\theta, \tag{82}$$

$$\frac{d\mathcal{J}}{dt} = f_\theta. \tag{83}$$

Finding the components $\{f_r, f_\theta\}$ of the radiation-reaction force requires balancing the energy and angular momentum emitted to infinity by radiation with that dissipated by the EOB system. In the gravity case, this PM-inspired approach was used to determine the radiation response force as a series in G. To facilitate this, in Ref. [44], the RR force components were written in terms of two functions c_r and c_p, such that

$$\mathbf{f} \equiv (c_r + c_p)p_r\hat{r} + \frac{c_p\mathcal{J}}{r}\hat{\theta}, \tag{84}$$

where p_r is the radial component of the relative momentum. The idea is to find the functions $\{c_r, c_p\}$ by an appropriate matching of the energy and angular momentum radiated as gravitational waves to, respectively, the energy and angular momentum lost by the binary system. In the spirit of PM and further assuming $\{c_r, c_p\}$ to be functions of only r and $|p|^2$, Ref. [44] describes the general procedure to find these functions as a series in G and gives explicit results up to $\mathcal{O}(G^2)$.

One can employ a similar approach for the EM case as well. As opposed to the top-down approach for RR force in Ref. [44], our method begins with the most general expression for the RR force at the desired order, so the series expansions of the corresponding functions $\{c_r, c_p\}$ are straightforward and follow directly from expanding the appropriate components of Eq. (29). To this end, we first assume the functions c_r and c_p to have the following general expansion:

$$c_r = \frac{1}{r}\left[\left(\frac{kq_1q_2}{Mr}\right)^2 c_{(2)r} + \left(\frac{kq_1q_2}{Mr}\right)^3 c_{(3)r} + \cdots\right], \tag{85}$$

$$c_p = \frac{1}{r}\left[\left(\frac{kq_1q_2}{Mr}\right)^2 c_{(3)p} + \left(\frac{kq_1q_2}{Mr}\right)^3 c_{(3)p} + \cdots\right]. \tag{86}$$

On the other hand, the radial and angular components of the RR force in Eq. (29) that, ignoring the C-dependent term, take the forms

$$f_r = \left[\frac{2kA}{3}\partial_r\mathcal{F} - \frac{2kB\mathcal{J}^2}{3mr^2\varepsilon}\mathcal{F}^2\right]p_r+, \tag{87}$$

$$f_\theta = \left[\frac{2kA\mathcal{F}}{3r} - \frac{2kmB\mathcal{F}^2}{\varepsilon}\left(1 + \frac{\mathcal{J}^2}{m^2r^2}\right)\right]\mathcal{J}, \tag{88}$$

where $\mathcal{F} = -\partial_r\phi$ and $\varepsilon = m\dot{t}$. In the above equation, we can use the definition of $\{c_r, c_p\}$ from Eq. (84) and the series expansion for the ϕ obtained in Sec. V to arrive at

$$\begin{aligned}c_r = \frac{1}{r}\Bigg[&-\frac{2Ak^2q_1q_2\phi_1h_\nu}{r^2} + \Bigg(-\frac{16Ak^3\nu q_1^2q_2^2\phi_2h_\nu}{3mr^3}\\ &+ \frac{2Bk^3q_1^2q_2^2\phi_1^2h_\nu^2}{3\gamma r^3}\Bigg) + \mathcal{O}(k^4)\Bigg],\end{aligned} \tag{89}$$

$$\begin{aligned}c_p = \frac{1}{r}\Bigg[&\frac{2Ak^2q_1q_2\phi_1h_\nu}{3r^3} + \Bigg(\frac{4Ak^3\nu q_1^2q_2^2\phi_2h_\nu}{3mr^3}\\ &- \frac{2Bk^3q_1^2q_2^2\phi_1^2h_\nu^2(j^2+m^2r^2)}{3\gamma m^2r^5}\Bigg) + \mathcal{O}(k^4)\Bigg],\end{aligned} \tag{90}$$

where A and B are also as found in Sec. V.

We can now compare our findings to those related to gravity [44]: First, in Ref. [44], it was found that demanding $\delta\mathcal{E}$ and $\delta\mathcal{J}$ at $\mathcal{O}(G^2)$ in the gravitational case fixes the RR force entirely at that order. For instance, the fact that $\delta\mathcal{E}$ vanishes at $\mathcal{O}(k^2)$ implies that $c_p = -3c_r + \mathcal{O}(G^4)$. Using Eqs. (89) and (90), it is easy to see that the analogous condition, namely, $c_p = -3c_r + \mathcal{O}(k^4)$, is automatically satisfied in our formalism.

VIII. SUMMARY AND FUTURE OUTLOOK

We have described the effective-one-body formalism for conservative and radiative dynamics of a relativistic electromagnetic binary system. As a concrete illustration of the formalism, we discussed the details of EOB dynamics at $\mathcal{O}(k^3)$. At this order, the system's symmetry requires that the EOB dynamics be parametrized by three conservative parameters $\{\phi_1, \phi_2, \phi_3\}$ and three dissipative parameters $\{A, B, C\}$. While the former set describes the conservative potential of the EOB dynamics, the latter captures the radiation-reaction force. However, physical arguments show that the parameter C is irrelevant for observables at $\mathcal{O}(k^3)$. By matching the conservative part of the $\mathcal{O}(k^3)$ scattering angle of the original two-body problem and that of the EOB system, we found the explicit forms of the parameters $\{\phi_1, \phi_2, \phi_3\}$. Similarly, by comparing the $\mathcal{O}(k^3)$ radiated energy and $\mathcal{O}(k^2)$ angular momentum on both sides, following the EOB mapping reviewed in Sec. III A, we found the explicit forms of the dissipative parameters A, B. With the exact values of $\{\phi_1, \phi_2, \phi_3\}$ and A, B, our formalism describes the full dynamics of an electromagnetically charged binary system at $\mathcal{O}(k^3)$, including certain nonperturbative aspects. As a further application and cross-check, we have studied the bound-orbit dynamics of the system at $\mathcal{O}(k^3)$ using our formalism. To this end, we focused on calculating three observables: the periastron shift ($\Delta\phi$), radiated energy per orbit (ΔE), and radiated angular momentum per orbit (ΔJ). Our results for these observables match perfectly with the ones expected from the method of unbound-to-bound analytical continuation.

Interestingly, our formalism leads to a conjecture for the subleading contribution to the net angular momentum loss for the unbound orbits (δJ_3), whose explicit computations are unavailable yet. The significance of the subleading angular momentum loss in a scattering encounter is that via an analytic continuation argument [see Eq. (D5) and, for more details, [49]]; one can show that δJ_3 is directly related to the *leading-order* radiated angular momentum per period in a bound orbit. Using our EOB formalism, we have also independently shown that this is, in fact, the case. As a corollary, our formalism leads to the leading-order PL expression for radiated angular momentum per period for a bound orbit, valid to all orders in velocity. The nonrelativistic limit of the angular momentum loss can be independently derived from standard results on dipole radiation (see Appendix B). We verified that the nonrelativistic limit of our expression for ΔJ matches precisely with that obtained by explicit nonrelativistic calculations, potentially indicating that our conjecture is accurate. It must be emphasized that a mere analytic continuation of δJ up to the order available in [49] gives $\Delta J = 0 + \mathcal{O}(k^3)$, rendering the approach essentially futile in making predictions of leading-order angular momentum loss in bound orbits. Moreover, as noted in Ref. [49], the explicit computation of δJ_3 (and, by analytical continuation, ΔJ) requires first solving the exact forms of the worldlines of the original scattering particles at $\mathcal{O}(k^3)$. The solving of the worldlines at 3PL involves rather cumbersome integrals. However, if correct, the EOB approach offers an extremely economical way to derive δJ_3 and ΔJ. In summary, our conjecture for the leading-order ΔJ shows that our EOB formalism is not only a convenient resummation of the already available 3PL results, but also has predictive power.

In Sec. VII A, we have briefly discussed a plausible alternate interpretation of the EOB dynamics in terms of the EFT framework. However, it is desirable to formalize the reasoning therein, which will be the subject of a forthcoming publication. Here, we will briefly outline how one could achieve this objective. The standard Lagrangian-based EFT approach cannot efficiently account for dissipative effects. An elegant Lagrangian formulation of classical dissipative systems, inspired by the Schwinger-Keldysh formalism for nonequilibrium quantum systems, was proposed in [63]. For the EOB particle considered in this work, the application of this formalism starts with the doubling $x^\mu \to \{x_1^\mu, x_2^\mu\}$, while the Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_{\mathrm{EM}}(x_1^\mu, \dot{x}_1^\mu) - \mathcal{L}_{\mathrm{EM}}(x_1^\mu, \dot{x}_1^\mu) + K(x_i^\mu, \dot{x}_i^\mu), \quad (91)$$

where $i = 1, 2$, $\mathcal{L}_{Em}$ is the standard quadratic Lagrangian for a particle in an external electromagnetic field, and K is a function that cannot be written as a difference of the form $f(x_1^\mu, \dot{x}_1^\mu) - f(x_2^\mu, \dot{x}_2^\mu)$. The radiation-reaction terms can be accounted for by an appropriate choice of $K(x_i^\mu, \dot{x}_i^\mu)$. This choice, in turn, may be further constrained by the symmetries of the systems and expanded in powers of coupling k in a manner analogous to the standard EFT approach. Presumably, such a systematic procedure would lead to an alternate, more rigorous justification for the deformed Lorentz-Dirac equation given in Eq. (29). Further, it would be interesting to see if the RR coefficients (i.e., A, B, C, etc.) can be directly related to the scattering amplitude analogous to the conservative potential extracted from amplitudes.

We are confident that the formalism outlined here for the EM case can be extended to gravity. However, one immediate hurdle to a naive extension of our approach to the GR case is that, even in the test-particle limit, the leading-order RR force, as, for instance, described by the MiSaTaQuWa equation [83,84], is not a simple local function of the worldline of the particle. Rather, it depends on the history of the particle as well. Nevertheless, one might seek a suitable deformation of the MiSaTaQuWa equation that can serve as the RR term in the EOB equation of motion.

Alternatively, one might employ the series expansion approach described in [44] (and briefly revised in Sec. VII B). Regarding this, in Ref. [44], it was assumed that c_r and c_p are functions of r and $|\vec{p}|^2$. It appears that the EM analog of this assumption is inconsistent with the expressions for c_r and c_p given by Eqs. (89) and (90). It seems more reasonable to regard $\{c_{(i)r}, c_{(i)p}\}$ as functions of both $|\vec{p}|^2$ and $(\mathcal{J}^2/m^2r^2)$. Consequently, the assumption that the RR force can be written in radial gauge appears to be not true, at least in the EM case. In GR, this assumption (of dependence on only r and $|\vec{p}|^2$) may hold, while the same is violated in EM. If this is the case, it merits additional investigation and will be the subject of subsequent studies.

ACKNOWLEDGMENTS

The work is supported by a SERB-MATRICS grant.

APPENDIX A: THIRD-ORDER ENERGY AND ANGULAR MOMENTUM

The third-order angular momentum is given by

$$\mathcal{J}_3(\theta) = A\mathcal{J}_{3(A)}(\theta) + B\mathcal{J}_{3(B)}(\theta) + C\mathcal{J}_{3(C)}(\theta), \tag{A1}$$

$$\mathcal{J}_{3(A)} = \frac{\gamma^2 m^2 q_1^2 q_2^2 h_\nu (\phi_1^2 h_\nu (4\sin(\theta) - 2\theta(v^2+2) + v^2\sin(2\theta)) + 2\gamma\nu v^2 \phi_2 (2\theta - \sin(2\theta)))}{6j^2}, \tag{A2}$$

$$\mathcal{J}_{3(B)} = -\frac{\gamma^2 m^3 q_1^2 q_2^2 v^2 \phi_1^2 h_\nu^2 (4\theta(3\gamma^2 v^2 + 4) + \gamma^2 v^2 \sin(4\theta) - 8\sin(2\theta)(\gamma^2 v^2 + 1))}{48j^2}, \tag{A3}$$

$$\mathcal{J}_{3(C)} = \frac{4\gamma^6 m^5 q_1 q_2 v^6 \phi_1 h_\nu \sin^6(\theta)}{9j^4}. \tag{A4}$$

The third-order energy turns out to be

$$\mathcal{E}_3 = A\mathcal{E}_{3(A)}(\theta) + B\mathcal{E}_{3(B)}(\theta) + C\mathcal{E}_{3(C)}(\theta), \tag{A5}$$

$$\mathcal{E}_{3(A)}(\theta) = \frac{\gamma^3 h m^3 q_1^2 q_2^2 v^2 (h\phi_1^2(-2\theta + \sin(\theta) + 5\sin(2\theta) - 3\sin(3\theta)) - 8\gamma\nu v^2 \phi_2 \sin^3(\theta)\cos(\theta))}{6j^3}, \tag{A6}$$

$$\mathcal{E}_{3(B)}(\theta) = -\frac{\gamma^5 h^2 m^4 q_1^2 q_2^2 v^4 \phi_1^2 (12\theta - 8\sin(2\theta) + \sin(4\theta))}{48j^3}, \tag{A7}$$

$$\mathcal{E}_{3(C)}(\theta) = -\frac{2\gamma^5 h m^4 q_1 q_2 v^5 \phi_1 \sin^3(\theta)(3\cos(2\theta) + 1)}{9j^3}. \tag{A8}$$

APPENDIX B: NONRELATIVISTIC LIMIT OF ANGULAR MOMENTUM AND ENERGY LOSS

The dipole formula for the rate of radiated angular momentum loss is [85]

$$\frac{d\mathbf{J}}{dt} = -\frac{2k}{3}\mathbf{D} \times \dddot{\mathbf{D}}, \tag{B1}$$

where $\mathbf{D}$ is the dipole moment vector. For a binary system of charges, we have

$$\mathbf{D} = \left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right) m\mathbf{r}, \tag{B2}$$

where $\mathbf{r}$ is the position vector of particle 2 with respect to particle 1. In the nonrelativistic limit, the force between the charges is the simple Coulomb force. Using this fact, the rate of angular momentum loss

$$\frac{dJ}{d\theta} = -\frac{2k^2 m q_1 q_3}{3r}\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)^2. \tag{B3}$$

The nonrelativistic orbit is given by $r = R/(1+\sigma\cos\theta)$, where $\sigma^2 = 1 + \frac{2E_{nr}J^2}{m(kq_1q_2)^2}$ and $R = J^2/(kq_1q_2m)$. Integrating the above equation over one orbital period, we find that the nonrelativistic limit of angular momentum loss per orbit is given by

$$\Delta J_{nr} = \frac{4\pi k^3 m^2 q_1^2 q_2^2}{3J^2}\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)^2. \tag{B4}$$

The dipole energy loss, on the other hand, has the following expression [85]:

$$\frac{dE}{dt} = -\frac{2k}{3}\dddot{\mathbf{D}} \bullet \dot{\mathbf{D}}. \tag{B5}$$

Again, using the fact that the force between the particles is Coulombic, the above equation simplifies to

$$\frac{dE}{d\theta} = \frac{2k^3 m q_1^2 q_2^2}{3Jr^2}\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)^2. \tag{B6}$$

Upon integrating the above equation for one orbit, we get

$$\Delta E_{nr} = \frac{4\pi m^2 k^3 q_1^2 q_2^2 E_{nr}}{3J^3}\left(\frac{q_1}{m_1} - \frac{q_2}{m_2}\right)^2 + \mathcal{O}(k^4). \tag{B7}$$

APPENDIX C: EXPANSION ABOUT THE TEST-PARTICLE LIMIT

It is instructive to look at expansions of the EOB parameters in the symmetric mass ratio ν. The effective potential takes the form

$$\begin{aligned}\phi(r) = {}& \frac{kq_1q_2}{r} \\ & + \nu\left[\frac{(\gamma-1)kq_1q_2}{r} + \frac{(\gamma-1)k^2q_1^2q_2^2}{2\gamma m r^2}\right. \\ & \left. - \frac{(2\gamma+1)k^3q_1^3q_2^3}{2\gamma^2(\gamma+1)m^2r^3} + \mathcal{O}(k^4)\right] + \mathcal{O}(\nu^2),\end{aligned} \tag{C1}$$

which has the leading-order term consistent with that expected from the test-particle limit. On the other hand, the radiative coefficients have the expansions

$$\alpha_1 = 1 + (1-\gamma)\nu + \mathcal{O}(\nu^2), \tag{C2}$$

$$\begin{aligned}\alpha_2 = {}& \frac{3(\tanh^{-1}(v) - v\gamma^2)}{v^3\gamma^3} + \frac{3(\gamma-1)(v\gamma^2 - \tanh^{-1}(v))\nu}{v^3\gamma^3} \\ & + \mathcal{O}(\nu^2),\end{aligned} \tag{C3}$$

$$\beta_1 + \beta_2 = 1 + \frac{(2-6\gamma^2)\nu}{3\gamma+3} + \mathcal{O}(\nu^2), \tag{C4}$$

$$\beta_3 = \frac{\nu((-6\gamma^2+4\gamma-2)\cosh^{-1}(\gamma) + 2\gamma(3\gamma((\gamma-2)\gamma+3)-4)v)}{\gamma^4(\gamma+1)v^3} + \frac{(3\gamma-1)\cosh^{-1}(\gamma) + \gamma((5-3\gamma)\gamma-4)v}{\gamma^4(\gamma+1)v^3} + \mathcal{O}(\nu^2). \tag{C5}$$

In particular, at the leading order, α_1 and $\beta_1 + \beta_2$ are both unity, which is also consistent with the expected test-particle limit.

APPENDIX D: RESULTS FOR BOUND ORBITS FROM OTHER METHODS

In this appendix, we give the results of the analytical continuation method for (1) the periastron shift $\Delta\Phi$ [22,42], (2) the energy radiated per orbital period (ΔE) [49,86], and (3) the angular momentum loss per orbital period (ΔJ) [49]. Following Refs. [22,42], the periastron shift can be found from the scattering angle by analytically continuing E_{nr} and J,

$$\Delta\Phi(E_{nr}, J) = -\chi(E_{nr}, J) - \chi(E_{nr}, -J). \tag{D1}$$

Note that the left-hand side is defined for $E_{nr} < 0$, while the right-hand side is written in terms of functions originally defined for $E_{nr} > 0$, but analytically continued to the bound-orbit domain $E_{nr} < 0$. Hence, from Eq. (13), the leading-order periastron shift takes the following form:

$$\Delta\Phi = \frac{\pi k^2 q_1^2 q_2^2}{J^2 h_{\nu(b)}} + \mathcal{O}(k^4). \tag{D2}$$

Similarly, following Refs. [49,86], the energy loss per orbit can be obtained by analytically continuing the energy, i.e.,

$$\Delta E(E_{nr}, J) = \delta E(E_{nr}, J) - \delta E(E_{nr}, -J). \tag{D3}$$

As in the expression for the periastron shift, on the left-hand side is a function naturally defined for bound orbits ($E_{nr} < 0$), while the right-hand side is constructed from analytic continuations of functions originally defined for scattering orbits ($E_{nr} > 0$). From Eq. (18), the energy loss per orbit is then given by

$$\begin{aligned}\Delta E = {}& \frac{\pi m^3 k^3 q_1^2 q_2^2 (\gamma_{(b)}^2 - 1)}{2J^3 h_{\nu(b)}^4}\left[\frac{(3\gamma_{(b)}^2+1)}{3}\left(\frac{q_1^2}{m_1^2} + \frac{q_2^2}{m_2^2}\right)\right. \\ & \left. + \frac{(\gamma_{(b)}-1)(3\gamma_{(b)}^2+1)}{3}\left(\frac{q_1^2 m}{m_1^3} + \frac{q_2^2 m}{m_2^3}\right) - \mathcal{G}(\gamma_{(b)})\frac{q_1q_2}{m_1m_2}\right] \\ & + \mathcal{O}(k^4).\end{aligned} \tag{D4}$$

Likewise, the loss of angular momentum per orbit is [49]

$$\Delta J(E_{nr}, J) = \delta J(E_{nr}, J) + \delta J(E_{nr}, -J). \tag{D5}$$

Since the second-order contribution to δJ is odd in J, the above equation implies that ΔJ is $\mathcal{O}(k^3)$. In particular,

$$\Delta J = 2k^3 \delta J_3 + \mathcal{O}(k^4), \tag{D6}$$

where δJ_3 on the right-hand side of the above equation is to be understood as the analytic continuation of the same [i.e., as given in Eq. (64)] to the domain $E_{nr} < 0$.

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