Selecting Horndeski theories without apparent symmetries and their black hole solutions

Eugeny Babichev,[1] Christos Charmousis,[1] Mokhtar Hassaine,[2] and Nicolas Lecoeur[1]

[1]*Université Paris-Saclay, CNRS/IN2P3, IJCLab, 91405 Orsay, France*
[2]*Instituto de Matemática, Universidad de Talca, Casilla 747, Talca, Chile*

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Starting from a generalized Kaluza-Klein action including arbitrary Horndeski potentials, we establish integrability and compatibility conditions that solve the generic field equations for spherical symmetry. The resulting theories can be identified as general Horndeski theories having no apparent symmetries in four dimensions or as effective string theory actions with an IR logarithmic running for the dilaton, higher-order corrections and generalized Liouville-type potentials. For such actions, we then find black holes with secondary hair parametrized by two coupling constants essentially characterizing the theories at hand. One is related to an action which is conformally coupled in five dimensions, while the second is related to a Kaluza-Klein reduction of Lovelock theory. We show that the full action can also be interpreted as a sum of conformally coupled actions in differing dimensions. Known solutions are mapped within the general chart of the found theories, and novel general black holes are discussed, focusing on their important properties and some of their observational constraints.

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I. INTRODUCTION

General relativity (GR) is quite successful in passing numerous strong field tests that recent experimental and observational advances have brought forward. Among them, we can mention the direct detection of gravitational waves emitted by binary inspirals (see, e.g., Ref. [1]), the spectroscopy of x rays produced by accreting black holes (see, e.g., Ref. [2]), the observation of star trajectories orbiting the supermassive black hole Sagittarius A^* at the center of the Milky Way [3], and the recent direct observation of black holes by the Event Horizon Telescope [4]. These international collaborative efforts consolidate GR in the strong gravity regime. It goes without saying that recent observations also raise questions. A typical example is the mass of the secondary in the event GW190814, which remains puzzling in the framework of GR. Its mass of $2.59^{+0.08}_{-0.09}$ solar masses is too light for a black hole, and too heavy for a neutron star with a usual equation of state or angular momentum (and in accordance with the multimessenger observation of Ref. [1] disfavoring stiff equations of state). This is even more intriguing, as it is for small masses that we may hope or expect UV effects beyond GR, since the horizon or surface of the compact object is strongly curved. With expected upcoming observations of increasing precision, it is paramount to find compact objects of modified gravity theories which satisfy the previously mentioned tight experimental constraints and can also point beyond GR. Scalar-tensor theories can, for example, modify the no-hair relation of GR and admit black hole solutions which differ from the Kerr spacetime (see, for example, Ref. [5]), providing a theoretical framework to quantify possible observational deviations from GR. Furthermore, within scalar-tensor theories, it was, for example, found in Ref. [6] that there exist neutron star solutions with no mass gap, permitting more massive neutron stars (of generic equation of state) or lighter black hole secondaries, as, for example, detected in GW190814. These results, among others, are not as yet conclusive but are interesting in their own right as a measurable departure from GR.

It is not surprising that most analytic results concerning compact objects in such theories were obtained by making some assumption(s) of symmetry for the scalar-tensor action at hand. Early on, stealth/self-tuning solutions of spherical symmetry were obtained [7] under parity $\phi \to -\phi$ and shift symmetry $\phi \to \phi + \mathrm{cst}$. They were extended for generic classes of scalar-tensor theories [8]. These results were even extended to stealth stationary metrics [9] upon realizing the intricate relation of the scalar with the Hamilton-Jacobi functional of the Kerr metric described in the classic works of B. Carter [10]. For theories with broken parity or/and broken shift symmetry, numerical studies were initially made demonstrating nonstealth configurations (see, for example, Refs. [11,12]), nonperturbative effects like scalarization [13], or scalar shells around black holes [14]. On the analytic side, using a generalized Kaluza-Klein dimensional reduction [15] of Lovelock theory, one could construct black hole solutions [16] with higher-order corrections and without shift or parity symmetry (see also more recently Ref. [17]). Such a Kaluza-Klein truncation involved all Horndeski scalars for given exponential-type potentials. These solutions were, from a string theory point of view, the leading α' correction of two derivative Einstein dilaton solutions with a Liouville potential (initially discussed in Ref. [18]). At the absence of mass, a naked singularity was present for the two derivative Einstein dilaton theories. When these were completed to higher-order theories however, the naked singularity was cloaked by an event horizon originating from the higher-order terms of the action [16]. However, the mass term of such α'-corrected solutions did not have a Newtonian falloff. This was in agreement with their higher-dimensional Lovelock theory origin. This drawback was remedied very recently. Indeed, analytic solutions can be found for a precise combination of all Horndeski scalars [19] upon taking an intriguing four-dimensional singular limit (initially considered in Ref. [20]). For the theory in question [19], symmetries are still present but are now somewhat hidden, the scalar field equation of motion being conformally invariant, while the scalar-tensor action is not (see also Refs. [21,22] and the review in Ref. [23]). The Gauss-Bonnet curvature scalar, a crucial term of Lovelock theory and in this construction, is given by $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. In four dimensions, it is a topological invariant (the generalized Euler density), unless coupled with a scalar (see, for example, the nice discussion in Ref. [11]). The above mentioned theories [16,19] are higher-dimensional Kaluza-Klein truncations of Lovelock gravity. They can also be interpreted as low-energy effective holographic theories [24] at large but finite couplings, or noncritical string theory at finite-temperature backgrounds (see, e.g., Ref. [25]) with leading string tension α'-UV corrections. The common denominator is the presence of the scalar or dilaton field driving the low-energy dynamics in the IR. In a nutshell, generalized stringy Kaluza-Klein compactifications lead to the appearance of generic Horndeski terms in four-dimensional actions with particular couplings for the scalar field. So, can one go any further in this promising direction?

All things considered, there is no straightforward physical reason to have some symmetric configuration for the scalar-tensor theory. Our aim in this paper will be to strive a step further and obtain black hole solutions for scalar-tensor theories without some obvious symmetry present, including a generic form of Horndeski theory with no shift, parity, or partial conformal symmetry (for a recent brief review, see Ref. [26]). A good starting point toward this aim is to try to generalize the potentials appearing in Kaluza-Klein compactifications originating from Lovelock theory [16]. Such actions include all Horndeski terms with the scalar couplings depending on the nature of the compactified extradimensional space. Here, we should pause in order to emphasize a key difference of Lovelock theory from higher-dimensional GR first noted in Ref. [27]. Whereas locally compact spaces, present as horizons, are Einstein spaces in GR; for Lovelock theory, this property is not sufficient. Horizons must in addition have constant Gauss-Bonnet curvature, a rather strong condition, as it turns out. This reduces the possible spaces upon which to compactify Lovelock black hole solutions. In other words, uniform black strings or black branes are not solutions in Lovelock theories. A notable nontrivial example remains: the product of an arbitrary number of two-spheres giving rise to a black hole solution in higher dimensions [28]. It is this solution that truncates to the Kaluza-Klein solution found early on in Ref. [16]. It is the singular limit of this solution to four dimensions which gives one of the black hole solutions found independently by Fernandes [19].

Given these considerations, let us consider the following action which includes Kaluza-Klein potentials and particular subcases with known solutions, see, e.g., Refs. [16,19]:

$$S = \int d^4x\sqrt{-g}\Big\{(1+W(\phi))R - \frac{1}{2}V_k(\phi)(\nabla\phi)^2 + Z(\phi) + V(\phi)\mathcal{G} + V_2(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V_3(\phi)(\nabla\phi)^4 + V_4(\phi)\Box\phi(\nabla\phi)^2\Big\}. \tag{1}$$

Note the presence of a canonical kinetic term for the scalar field, which is important in order to evade possible strong coupling issues. Then, note the potential $W(\phi)$, which determines if the scalar is minimally coupled to the Ricci scalar (to lowest order) or not. The potential $V(\phi)$ multiplying the Gauss-Bonnet invariant is defined up to an additive constant which would only yield a boundary term. The potential term $Z(\phi)$ includes the cosmological constant, Liouville exponential terms most commonly present in noncritical string theories [25], in self-tuning scenarios [29], and in holographic gravitational backgrounds (see Ref. [16] and references within). With the usual notations of Horndeski theory—see, e.g., Ref. [30], with $X = -(\nabla\phi)^2/2$—the Horndeski functions $G_k(\phi, X)$, $k = 2, 3, 4, 5$, are[1]

$$G_2 = Z + XV_k + 4X^2V_3 + 8X^2(3 - \ln|X|)V_{\phi\phi\phi\phi},$$
$$G_3 = 2XV_4 + 4X(7 - 3\ln|X|)V_{\phi\phi\phi}, \tag{2}$$

$$G_4 = 1 + W + 4X(2 - \ln|X|)V_{\phi\phi},$$
$$G_5 = -4V_\phi \ln|X| - \int V_2 d\phi. \tag{3}$$

The theory as a whole, therefore, includes string effective theories with higher-order α' corrections. It also includes

[1]The expression for the Horndeski functions [Eqs. (2) and (3)] enables us to probe the validity domain of the EFT description—see, e.g., Ref. [31].

the typical Kaluza-Klein compactification one would get from Einstein and Gauss-Bonnet higher-dimensional gravity [16]. To these considerations, we finally add the interesting action considered by Fernandes [19], which fits in the framework of the action in Eq. (1) with the following potentials:

$$W = -\beta e^{2\phi}, \qquad V_k = 12\beta e^{2\phi}, \qquad Z = -2\lambda e^{4\phi} - 2\Lambda,$$
$$V = -\alpha\phi, \qquad V_2 = 4\alpha = V_4, \qquad V_3 = 2\alpha, \tag{4}$$

with three coupling constants: α, β, and λ. In Fernandes's case, the scalar field equation was conformally invariant, and as such, it led to an interesting universal geometric constraint. This is no longer true for the general action under present consideration.

We are interested in static and spherically symmetric spacetimes. The following metric ansatz is considered throughout this paper for spherical symmetry:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \qquad \phi = \phi(r). \tag{5}$$

This is not the general spherically symmetric ansatz. Therefore, we will have to be cautious on the compatibility of the field equations for the theory at hand. We will now seek a way to filter out theories, reducing the above general action to a more tractable yet quite general theory to solve. The most interesting results of our analysis are obtained when $\phi(r)$ is not constant, which we assume from now on, while we treat the case of constant scalar field in Appendix A.

The paper is organized as follows: In Sec. II, we present conditions on the potentials which enable us to rewrite the field equations as three simple compatibility conditions, parametrized by a unique real function $\mu(r)$. Section III contains the main results—namely, two novel black hole solutions obtained for the case $\mu(r) \equiv 1$ and with Newtonian mass falloff. In Sec. IV, the more general case of $\mu(r) \equiv \mu = \text{cst}$ is treated, also leading to static black hole solutions, while Sec. V is devoted to our conclusions.

II. INTEGRABILITY AND COMPATIBILITY

In this section, we will see that a specific truncation of Eq. (1) can accommodate analytic solutions for the ansatz in Eq. (5). For this purpose, let us denote by $\mathcal{E}_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}$ the field equations arising from the variation of the action with respect to the metric.[2] Given a homogenous ansatz [Eq. (5)], it is quite common to consider the combination $\mathcal{E}_t^t - \mathcal{E}_r^r = 0$ in order to determine the expression of the scalar field. We obtain

$$\frac{\phi''}{(\phi')^2}[r^2 W_\phi + 4(1-f)V_\phi + 2frV_2\phi' + fr^2V_4(\phi')^2]$$
$$= -\frac{r^2}{2}(V_k + 2W_{\phi\phi}) - (V_2 + 4V_{\phi\phi})(1-f)$$
$$- fr(V_{2\phi} - 2V_4)\phi' - fr^2(V_{4\phi} - 2V_3)(\phi')^2, \tag{6}$$

where the prime stands for a radial derivative, while a subscript ϕ denotes a derivation with respect to ϕ. Upon close inspection, we see that, choosing the following as potentials of the theory,

$$V_k + 2W_{\phi\phi} = \frac{2}{d(\phi)}W_\phi, \qquad V_2 + 4V_{\phi\phi} = \frac{4}{d(\phi)}V_\phi,$$
$$V_{2\phi} - 2V_4 = \frac{2}{d(\phi)}V_2, \qquad V_{4\phi} - 2V_3 = \frac{1}{d(\phi)}V_4, \tag{7}$$

Equation (6) is factorized in a simple and elegant way, providing *a priori* two branches of solutions:

$$\left[\frac{\phi''}{(\phi')^2} + \frac{1}{d(\phi)}\right][r^2W_\phi + 4(1-f)V_\phi + 2frV_2\phi'$$
$$+ fr^2V_4(\phi')^2] = 0. \tag{8}$$

It is important that the conditions of Eq. (7) involve only the potentials of the theory—they cannot involve the characteristics of our ansatz [Eq. (5)] or the sought-for solutions. Here, $d = d(\phi)$ is some nonvanishing function of the scalar, and we will see below that, for the first branch of solutions in Eq. (8), it can be chosen to be constant without any loss of generality. It is interesting to note that, under the conditions of Eq. (7), the potentials V_k and V_i for $i = 2$, 3, and 4 can be parametrized in terms of the Einstein-Hilbert and Gauss-Bonnet potentials W and V as

$$V_k = \frac{2}{d}W_\phi - 2W_{\phi\phi}, \tag{9}$$

$$V_2 = \frac{4}{d}V_\phi - 4V_{\phi\phi}, \tag{10}$$

$$V_4 = -\frac{2}{d^2}(2 + d_\phi)V_\phi + \frac{6}{d}V_{\phi\phi} - 2V_{\phi\phi\phi}, \tag{11}$$

$$V_3 = \frac{1}{d^2}\left[\frac{1}{d}(1 + 2d_\phi)(2 + d_\phi) - d_{\phi\phi}\right]V_\phi$$
$$- \frac{1}{d^2}(5 + 4d_\phi)V_{\phi\phi} + \frac{4}{d}V_{\phi\phi\phi} - V_{\phi\phi\phi\phi}. \tag{12}$$

In other words, the factorization in Eq. (8) is made possible with any action [Eq. (1)] parametrized by three independent

[2] The metric field equations $\mathcal{E}_{\mu\nu} = 0$ and the scalar field equation $\delta S/\delta\phi = \mathcal{E}_\phi = 0$ are related by $\nabla^\nu\mathcal{E}_{\mu\nu} = -\mathcal{E}_\phi\nabla_\mu\phi$; see, e.g., Ref. [30]. Therefore, when the scalar field is not constant, the scalar field equation $\mathcal{E}_\phi = 0$ is implied by the metric field equations $\mathcal{E}_{\mu\nu} = 0$.

potentials—namely, W, Z, and V—provided that the remaining potentials are fixed by the above equations.

Now, let us take a closer look at the factorization in Eq. (8), and at its possible consequences for our purpose. First of all, the potentials Eq. (4) of Ref. [19] correspond to a constant function d given by $d(\phi) = -1$. As also occurs in this latter reference, Eq. (8) offers the possibility of two branches of solutions for the scalar field. We note that the first branch, corresponding to the first bracket in Eq. (8), does not involve coupling functions of the theory or the metric function. We have a simple ODE giving the scalar field independently of the geometry.[3] The second branch is much more involved, because the equation involves explicitly the coupling potentials of the theory and the metric function. We discuss this latter case in Appendix B. We here focus on the first branch, for which the scalar field satisfies the equation

$$\phi'' = -\frac{(\phi')^2}{d(\phi)}. \tag{13}$$

To go further, we now need to show the compatibility of the remaining equations with Eq. (13) and our ansatz for the metric [Eq. (5)]. This requires fixing the potentials W, Z, and V in such a way that the two remaining independent equations admit the same metric function f as a solution. Again, the potentials, which are functions of ϕ—or, equivalently, of r via Eq. (13)—must be independent of the metric solution f. It is quite remarkable that taking into account the expression of ϕ'' from Eq. (13), the two independent equations $\mathcal{E}_{rr} = 0$ and $\mathcal{E}_{\theta\theta} = 0$ can be integrated once and twice, respectively, to give

$$\mathcal{E}_{rr} \propto I_1'(r), \qquad \mathcal{E}_{\theta\theta} \propto I_2''(r), \tag{14}$$

with

$$I_1(r) = f^2(r^2V)''' - f(2r(1+\mathcal{W}') + 4V' + r^2\mathcal{W}'') + 2r + 2\mathcal{W} + r\mathcal{Z}' - \mathcal{Z}, \tag{15}$$

$$I_2(r) = f^2(rV)'' - fr(1+\mathcal{W}') + \mathcal{Z}, \tag{16}$$

and, where we have introduced for clarity two auxiliary functions $\mathcal{W}$ and $\mathcal{Z}$ determined by

$$W = \mathcal{W}', \qquad rZ = \mathcal{Z}''. \tag{17}$$

The integration of the equations given in Eq. (14) implies the existence of three integration constants: d_1, c_1, and c_2, such that

$$I_1 - d_1 = 0, \qquad I_2 - c_2 + c_1 r = 0. \tag{18}$$

As the following calculations show, the integration constants c_1, c_2, d_1 are not independent and are either gauged away or related to the mass of the black hole. Compatibility of the field equations is ensured once the two quadratic equations I_1 and I_2, defining the metric function f, are proportional. Denoting by $2\mu(r)$ this proportionality factor, which is *a priori* an arbitrary nonvanishing function of r, one obtains the following system of equations:

$$(r^2V)''' = 2\mu(rV)'', \tag{19}$$

$$4V' = 2(\mu-1)r(\mathcal{W}'+1) - r^2\mathcal{W}'', \tag{20}$$

$$2r + 2\mathcal{W} = d_1 - 2\mu c_2 + 2\mu c_1 r + (2\mu+1)\mathcal{Z} - r\mathcal{Z}', \tag{21}$$

where we assume that the factors in front of different powers of f should be proportional independently. For a given proportionality factor $\mu(r)$, these equations will determine the unfixed potentials W, Z, and V as functions of r—or, equivalently, of ϕ—while the quadratic equations [Eq. (18)] will give the metric function $f(r)$. Note that the parameters of the solution—in our case the mass—must not affect the relations of the theory potentials, as then the theory would be fine-tuned. We will see that this is not the case here.

As one may notice, the above conditions for compatibility of the equations are independent of the choice of $d(\phi)$, indicating that changing $d(\phi)$ does not change the physical results. Indeed, for any scalar field satisfying Eq. (13), the redefined scalar $\phi \to \int H(\phi)\phi' dr$ satisfies Eq. (13) with $d(\phi) = -1$, provided that H solves the ordinary differential equation $H_\phi - H^2 - \frac{H}{d} = 0$. One can therefore, without any loss of generality, fix $d(\phi) = -1$. Then, the general solution of Eq. (13) is

$$\phi(r) = \ln\left(\frac{c}{r+\tilde{c}}\right), \tag{22}$$

where c and $\tilde{c}$ are two integration constants. Note that the constant $\tilde{c}$ can be further fixed to have a specific value for convenience.[4] In particular, one can choose $\tilde{c} \propto c$, as in the examples just below, or $\tilde{c} = 0$ for the solutions presented in the following sections.

Let us first demonstrate how one can reproduce certain known solutions by using our formalism. If the

[3]This is typical in stringy black hole solutions with an IR logarithmic running for the dilaton (see, for example, Refs. [18,24]).

[4]Clearly, a different choice of $\tilde{c}$ in Eq. (22) amounts to a redefinition of the scalar field. We can use the residual freedom to redefine the scalar by choosing the constant of integration $\tilde{c}$. Indeed, the function $H(\phi)$ that provides $d(\phi) = -1$ is defined up to an integration constant, since it satisfies $H_\phi - H^2 - \frac{H}{d} = 0$. One can show that by adjusting this integration constant, one can change $\tilde{c}$.

Gauss-Bonnet potential $V = 0$, then Eq. (19) is satisfied automatically, and Eqs. (9)–(12) show that the action only has the Einstein-Hilbert potential W, the kinetic potential V_k, and the self-interaction and cosmological constant in Z. In this case, therefore, all higher-order terms in the action [Eq. (1)] are missing, and we are left with an action with at most two derivatives. This encompasses the Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) [32] and the Martinez-Troncoso-Zanelli (MTZ) [33] black holes. Indeed, consider the following potentials:

$$V = 0, \qquad W = -\beta e^{2\phi}, \qquad Z = -2\lambda e^{4\phi} - 2\Lambda,$$
$$V_k = 12\beta e^{2\phi}, \tag{23}$$

where V_k is determined by W according to Eq. (9). Then, we take into account the scalar field profile in Eq. (22) and solve the compatibility conditions [Eqs. (20) and (21)]. Finally, finding the metric function f from Eq. (18), we get the solution

$$\phi = \ln\left(\frac{M}{\sqrt{\beta}(r-M)}\right), \qquad f = \left(1 - \frac{M}{r}\right)^2 - \frac{\Lambda r^2}{3}, \tag{24}$$

provided the relation $\lambda = -\Lambda\beta^2$ holds. Equation (24) gives either the BBMB black hole (for $\Lambda = 0$) or the MTZ black hole (for $\Lambda \neq 0$), with a unique integration constant M playing the role of the black hole mass. The value of the function $\mu(r)$ in both these cases is $\mu(r) = 1 + M^2/(2M^2 - 3Mr + r^2)$.

A known solution for nonzero V is given in Ref. [19] for the theory with the potentials in Eq. (4) with $\lambda = \beta^2/(4\alpha)$. In our formalism, this solution corresponds to the choice $c = \sqrt{-2\alpha/\beta}$ and $\tilde{c} = 0$ in the solution for ϕ in Eq. (22), and the constant value[5] of $\mu(r) = 1$.

As we mentioned above, we have a choice to fix the constant of integration $\tilde{c}$ in Eq. (22). For the BBMB and MTZ solutions, we took $\tilde{c} \neq 0$ in order to stick with the standard form of these solutions, while to demonstrate the solution of Ref. [19] we chose $\tilde{c} = 0$, again to be in accord with the presentation of the original paper. From now on, we will set $\tilde{c} = 0$, and thus we consider

$$\phi(r) = \ln\left(\frac{c}{r}\right), \tag{25}$$

where $c > 0$ is a constant with dimension 1. As we will see below, the constant c of the scalar field solution in Eq. (25) is related to the coupling constants of the theory once the compatibility conditions of Eqs. (19)–(21) are solved.

[5] It is interesting to note that in this case, the couplings c, $\tilde{c}$, and $\mu(r)$ are independent of the mass integration constant M. This is unlike the BBMB and MTZ cases, where it is important that the action functions do not end up depending on the parameter of the solution M, as is indeed the case in Eq. (23).

It turns out that compatibility conditions (19)–(21) can be solved for any constant μ, the most interesting case being $\mu = 1$. Indeed, while we have seen above that the solution of Ref. [19] corresponds to $\mu = 1$, it appears conversely that solving the compatibility conditions with $\mu = 1$ leads to a more general action than the one considered in Ref. [19], acquiring new black hole solutions with faraway Newtonian asymptotics. The following section is dedicated to this $\mu = 1$ case, while the more general case of constant μ is explained in Sec. IV.

III. CASE OF $\mu = 1$: BLACK HOLES WITH A NEWTONIAN FALLOFF

For a constant proportionality factor, $\mu(r) = \mu = \text{cst}$, the compatibility conditions in Eqs. (19)–(21) are integrable, and new explicit solutions can be found. This will be the core of the next two sections. Different choices of constant μ yield solutions of differing faraway asymptotics, and only for $\mu = 1$ do the metric solutions have a standard four-dimensional Newtonian behavior at infinity—i.e., $f \sim 1–2M/r - (\Lambda r^2/3)$. Here, the optional Λ term in parentheses stands for the cosmological constant if present in the action, while M is the mass of the solution.

Let us thus focus in the current section on $\mu = 1$, and present the results in a way which enables us to interpret them easily. We consider the following potentials for W, Z, and V:

$$W = -\beta_4 e^{2\phi} - \beta_5 e^{3\phi}, \qquad Z = -2\lambda_4 e^{4\phi} - 2\lambda_5 e^{5\phi} - 2\Lambda,$$
$$V = -\alpha_4\phi - \alpha_5 e^{\phi}, \tag{26}$$

where β_4, β_5, λ_4, λ_5, α_4, and α_5 are six coupling constants, and Λ is the usual cosmological constant. The choice of subscripts "4" and "5" will become clear momentarily. The remaining potentials are given by the compatibility conditions (9)–(12) with $d = -1$, giving the following action:

$$\begin{aligned} S = \int d^4x\sqrt{-g}\{&R - 2\Lambda - 2\lambda_4 e^{4\phi} - 2\lambda_5 e^{5\phi} \\ &- \beta_4 e^{2\phi}(R + 6(\nabla\phi)^2) - \beta_5 e^{3\phi}(R + 12(\nabla\phi)^2) \\ &- \alpha_4(\phi\mathcal{G} - 4G^{\mu\nu}\phi_\mu\phi_\nu - 4\Box\phi(\nabla\phi)^2 - 2(\nabla\phi)^4) \\ &- \alpha_5 e^{\phi}(\mathcal{G} - 8G^{\mu\nu}\phi_\mu\phi_\nu - 12\Box\phi(\nabla\phi)^2 - 12(\nabla\phi)^4)\}, \end{aligned} \tag{27}$$

where for short we have defined $\phi_\mu \equiv \partial_\mu\phi$. One can see that the resulting action for $\lambda_5 = \beta_5 = \alpha_5 = 0$ coincides with the theory [Eq. (4)] presented in Ref. [19]. We recall that for this theory, the scalar field equation is conformally invariant in four dimensions, although the α_4-dependent part of the action *is not*. As regards the parts of the action depending on λ_5, β_5, and α_5, they correspond to the most general densities which are *conformally invariant in five* *dimensions*—see, for example, Refs. [34,35]. This motivates our use of the subscripts 4 and 5 for the parametrization of the full action under consideration [Eq. (27)]. The distant relation of our full action to conformal invariance is rather surprising, as at no point in our construction did we advocate conformal symmetry of any sort. In the Horndeski vocabulary [Eqs. (2) and (3)], this action corresponds to

$$\begin{aligned} G_2 &= -2\Lambda + 8X^2(\alpha_4 + 3\alpha_5 e^{\phi}) + 12X(\beta_4 e^{2\phi} + 2\beta_5 e^{3\phi}) \\ &\quad - 2(\lambda_4 e^{4\phi} + \lambda_5 e^{5\phi}) + 8\alpha_5 e^{\phi} X^2 \ln|X|, \\ G_3 &= 8\alpha_4 X + 4\alpha_5 e^{\phi} X(3\ln|X| - 1), \\ G_4 &= 1 - \beta_4 e^{2\phi} - \beta_5 e^{3\phi} + 4\alpha_4 X + 4\alpha_5 e^{\phi} X(\ln|X| - 2), \\ G_5 &= -8\alpha_5 e^{\phi} + 4(\alpha_4 + \alpha_5 e^{\phi})\ln|X|. \end{aligned}$$

As announced, the potentials [Eq. (26)], along with the scalar field [Eq. (25)], solve the compatibility conditions [Eqs. (19)–(21)] for $\mu = 1$, for two distinct sets of relations holding between the coupling constants and the constant appearing in the scalar field, thus yielding two distinct metric solutions of the form in Eq. (5). The first solution exists with all coupling constants switched on—namely,

$$\lambda_4 = \frac{\beta_4^2}{4\alpha_4}, \qquad \lambda_5 = \frac{9\beta_5^2}{20\alpha_5}, \qquad \frac{\beta_5}{\beta_4} = \frac{2\alpha_5}{3\alpha_4}. \tag{28}$$

The solution reads

$$\begin{aligned} &\phi = \ln\left(\frac{\eta}{r}\right), \qquad \eta = \sqrt{\frac{-2\alpha_4}{\beta_4}}, \\ &f(r) = 1 + \frac{2\alpha_5\eta}{3\alpha_4 r} + \frac{r^2}{2\alpha_4}\left[1 \pm \sqrt{\left(1 + \frac{4\alpha_5\eta}{3r^3}\right)^2 + 4\alpha_4\left(\frac{\Lambda}{3} + \frac{2M}{r^3} + \frac{2\alpha_4}{r^4} + \frac{8\alpha_5\eta}{5r^5}\right)}\right], \end{aligned} \tag{29}$$

where M is a free integration constant. A number of comments can be made concerning this solution, which is of the Boulware-Deser type [36] typical of higher-order metric theories, admitting two branches. The "plus" branch is asymptotically of de Sitter or anti–de Sitter type, much like Ref. [36]. In Lovelock theory, this upper branch is perturbatively unstable [37], although no direct analogy can be made with the case here. We will not consider this branch any further, as we are mostly interested in asymptotically flat spacetimes. So, for simplicity, let us set $\Lambda = 0$ and consider the "negative" branch in order to discuss some properties of the solution. For the negative branch, we have a Schwarzschild limit as the coupling constants α_4, α_5 tend to zero. Also, for $\alpha_5 = 0$, which automatically implies $\beta_5 = \lambda_5 = 0$, the first class of solutions of Ref. [19] is recovered. The asymptotics $r \to \infty$ of the full solution are given by

$$f(r) = 1 - \frac{2M}{r} - \frac{2\alpha_4}{r^2} - \frac{8\alpha_5\eta}{5r^3} + \mathcal{O}\left(\frac{1}{r^4}\right), \tag{30}$$

and M is therefore the ADM mass. The function f has the same behavior as the Schwarzschild metric up to first order, while the next order is controlled by the coupling α_4, and the other couplings α_5 and β_4 appear via η in the higher corrections. When we have a horizon, the black hole therefore has secondary hair, as the only integration constant appearing in the metric is M, while all other constants are fixed by the theory. Note that even if $M = 0$, the spacetime is not trivial, and is in fact a black hole or naked singularity. On the other hand, as $r \to 0$, the metric function [Eq. (29)] behaves as

$$f(r) = \begin{cases} -\frac{1}{5} - \frac{21\alpha_4 r}{50\alpha_5\eta} + \mathcal{O}(r^2) & \text{if } \alpha_5 > 0 \\ \frac{4\alpha_5\eta}{3\alpha_4 r} + \frac{11}{5} + \mathcal{O}(r) & \text{if } \alpha_5 < 0, \end{cases} \tag{31}$$

and while $f(0)$ is finite for $\alpha_5 > 0$, spacetime curvature is infinite at $r = 0$, since $f(r)$ does not possess a regular de Sitter core $f(r) = 1 + \delta r^2 + o(r^2)$ (with δ a constant). In fact, the spacetime might not be defined in the whole $r \in (0, \infty)$, but only on (r_S, ∞) where $r_S > 0$ is such that the square root ceases to be well defined below r_S. This branch singularity is quite typical for Lovelock spacetimes [38]. Before discussing the horizon structure of the solution in Eq. (29), let us present the second $\mu = 1$ solution arising from action (27).

Indeed, a second, quite distinctive class of solutions exists, provided the couplings of the generalized conformal action of Ref. [19] are switched off:

$$\lambda_4 = \beta_4 = \alpha_4 = 0, \qquad \lambda_5 = \frac{9\beta_5^2}{20\alpha_5}, \tag{32}$$

with a scalar field and a metric function given by

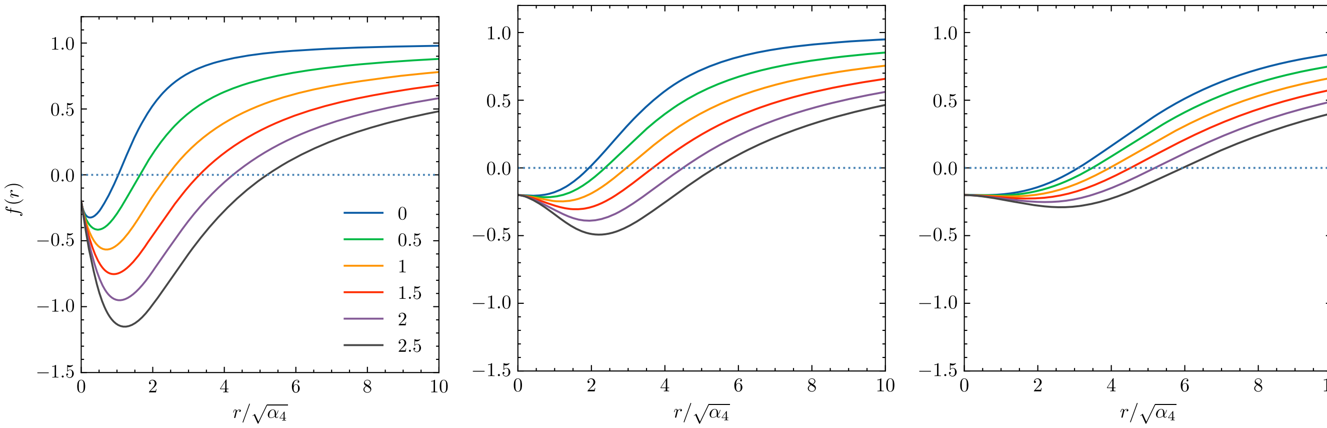

FIG. 1. Metric profile $f(r)$ of Eq. (29) for $\alpha_4 > 0$ and different values of the mass (in units of $\sqrt{\alpha_4}$, indicated by the colors) and different positive values of the product $\alpha_5\eta$ (in units of $\alpha_4^{3/2}$). Left: $\alpha_5\eta = 0.25$. Middle: $\alpha_5\eta = 20$. Right: $\alpha_5\eta = 100$. The spacetime is a black hole for any mass, with a hidden singularity at $r = 0$.

$$\phi = \ln\left(\frac{\eta}{r}\right), \qquad \eta = 2\sqrt{\frac{-\alpha_5}{3\beta_5}},$$
$$f(r) = \frac{1}{1+\frac{4\alpha_5\eta}{3r^3}}\left[1-\frac{\Lambda r^2}{3}-\frac{2M}{r}-\frac{4\alpha_5\eta}{15r^3}\right], \qquad (33)$$

where M is a mass integration constant and Λ is the cosmological constant, which we again set to zero for simplicity. Note that although the action includes higher-order terms, these do not yield branching solutions as is common in Lovelock theories [36]. In fact, the solution is very much "GR-like." Indeed, asymptotically as $r \to \infty$, the metric function behaves as

$$f(r) = 1 - \frac{2M}{r} - \frac{8\alpha_5\eta}{5r^3} + \mathcal{O}\left(\frac{1}{r^4}\right). \qquad (34)$$

On the other hand, close to the origin, the metric function does not blow up and behaves as

$$f(r) = -\frac{1}{5} - \frac{3Mr^2}{2\alpha_5\eta} + \mathcal{O}(r^3). \qquad (35)$$

Note that although the r^2 term is what is needed for a regular black hole (see, for example, Ref. [39]), regularity is spoiled by $f(0) = -1/5$. Hence, at $r = 0$ we have a curvature singularity. If $\alpha_5 < 0$, the spacetime becomes singular at $0 < r_S = (-4\alpha_5\eta/3)^{1/3}$, unless the numerator of Eq. (33) also vanishes at r_S, which occurs for a mass $M = M_S$ with

$$M_S = \frac{6^{2/3}(-\alpha_5\eta)^{1/3}}{5}. \qquad (36)$$

If $\alpha_5 > 0$, on the other hand, we always have a black hole horizon even for the case $M = 0$. In fact, one can see that for $\alpha_5 > 0$ and constant M, the horizon position is shifted at larger r, in comparison to GR—i.e., $r_h > 2M$. One can also show that there is an unstable light ring at $r_L > 3M$ again at larger r in comparison to GR (see, for example, Ref. [4]).

More generally, as regards the horizon of both spacetimes [Eqs. (29) and (33)], they turn out to be given by a cubic polynomial equation,

$$15r_h^3 - 30Mr_h^2 - 15\alpha_4 r_h - 4\alpha_5\eta = 0, \qquad (37)$$

where $r = r_h$ is the location of the event horizon, and of course $\alpha_4 = 0$ in the case of Eq. (33). This condition is necessary, but not sufficient. It is sufficient in the case of Eq. (29) if $\alpha_4 > 0$ and $\alpha_5 > 0$, and in the case of Eq. (33) if $\alpha_5 > 0$, or if $\alpha_5 < 0$ and $M > M_S$. In order to sketch the general aspect of the spacetimes and their horizons, we present various plots of the functions $f(r)$ of Eqs. (29) and (33). Regarding the solution in Eq. (29), note that the subcase $\alpha_5 = \beta_5 = \lambda_5 = 0$ was studied in Ref. [22], where it was found that there is a bias towards negative values[6] of α_4. The plots in Figs. 1–4 present the cases ($\alpha_4 > 0$, $\alpha_5 > 0$), ($\alpha_4 > 0$, $\alpha_5 < 0$), ($\alpha_4 < 0$, $\alpha_5 > 0$), and ($\alpha_4 < 0$, $\alpha_5 < 0$), respectively. The obtained spacetimes have at most one horizon. It is only when α_4 and α_5 are positive that there is always a horizon (even for $M = 0$). This is due to the fact that the square root is never zero and no branch singularity is possible. For all the other cases, however, we may have naked singularities for certain values of the coupling constants. An exotic result, in the case of the left and middle plots of Fig. 3, is a mass gap and horizon gap between light black holes and heavy black holes: masses $M \in (M_1, M_2)$ give rise to naked singularities, while $M < M_1$ or $M > M_2$ give black holes, the black holes

[6]This is based on the observation of atomic nuclei while assuming a Birkhoff-type argument for this theory. It is also based on the observation that the canonical kinetic term has the usual sign for $\alpha_4 < 0$.

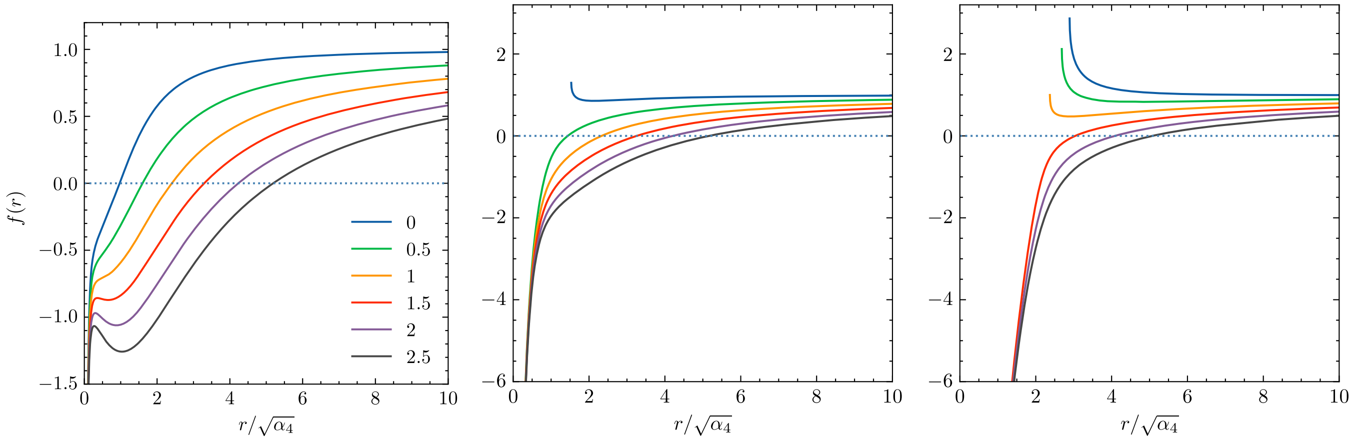

FIG. 2. Metric profile $f(r)$ of Eq. (29) for $\alpha_4 > 0$ and different values of the mass (in units of $\sqrt{\alpha_4}$, indicated by the colors) and different negative values of the product $\alpha_5\eta$ (in units of $\alpha_4^{3/2}$). Left: $\alpha_5\eta = -0.25$. Middle: $\alpha_5\eta = -2$. Right: $\alpha_5\eta = -10$. On the left, the spacetime is a black hole for any mass. When $|\alpha_5\eta|$ increases (middle and right plots), the light spacetimes acquire a naked singularity at a radius $r_S > 0$, while the heavier spacetimes remain black holes with a hidden singularity at $r = 0$.

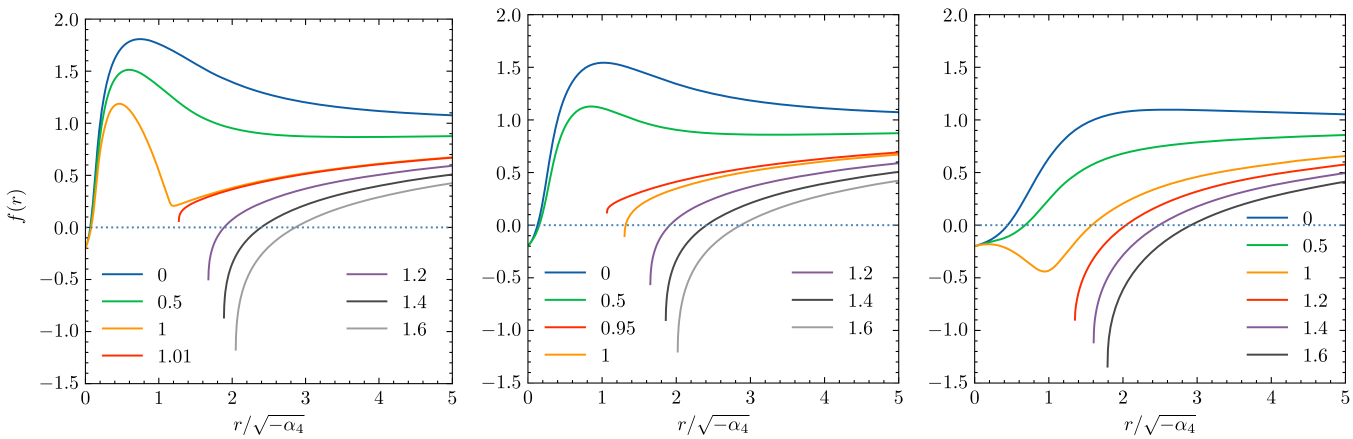

FIG. 3. Metric profile $f(r)$ of Eq. (29) for $\alpha_4 < 0$ and different values of the mass (in units of $\sqrt{-\alpha_4}$, indicated by the colors) and different positive values of the product $\alpha_5\eta$ [in units of $(-\alpha_4)^{3/2}$]. Left: $\alpha_5\eta = 0.25$. Middle: $\alpha_5\eta = 0.5$. Right: $\alpha_5\eta = 2$. For small $\alpha_5\eta$ (left and middle plots), light and heavy masses give black holes, with a hidden singularity at $r = 0$ or at $r_S > 0$, respectively, while intermediate masses give a naked singularity at $r_S > 0$ (see the red curves in the left and middle plots). For sufficiently large $\alpha_5\eta$ (right plot), all spacetimes are black holes, with a hidden singularity at $r = 0$ for light masses or at $r_S > 0$ for large masses.

with $M < M_1$ having very tiny horizons. It is interesting to note, in comparison with Ref. [22], that for $\alpha_4 < 0$ we may do away with cases of naked singularities by having sufficiently positive α_5 (right plot of Fig. 3). Concerning Eq. (33), its profile is presented in Fig. 5: if $\alpha_5 > 0$, it is a black hole for any mass, while for $\alpha_5 < 0$, it is a black hole only for $M \geq M_S$, where M_S is given by Eq. (36).

The two cases where a horizon exists for any mass—that is to say, Fig. 1, i.e., solution (29) with $\alpha_4 > 0$, $\alpha_5 > 0$, and the left plot of Fig. 5, i.e., solution (33) with $\alpha_5 > 0$—are in fact strongly constrained by the following argument, which was formerly developed in Ref. [6] and assumes that the considered solutions verify a Birkhoff-type argument. More precisely, it assumes that they are the unique static, spherically symmetric solutions of their respective theories. In this case, these solutions must represent the gravitational field created by an atomic nucleus of radius $R \sim 10^{-15}$ m and mass $M \sim 10^{-54}$ m. Since these nuclei can be experimentally probed, they are not covered by a horizon, and therefore $r_h < R$, where r_h is a root of Eq. (37). It is easy to show that, for the considered two cases, this leads to the constraints

$$0 < \alpha_4 < R(R - 2M) \sim 10^{-30}\ \mathrm{m}^2,$$
$$0 < \alpha_5\eta < \frac{15}{4}R^2(R - 2M) \sim 10^{-45}\ \mathrm{m}^3, \tag{38}$$

where of course the first inequality does not concern the left plot of Fig. 5, which already has $\alpha_4 = 0$. Such stringent

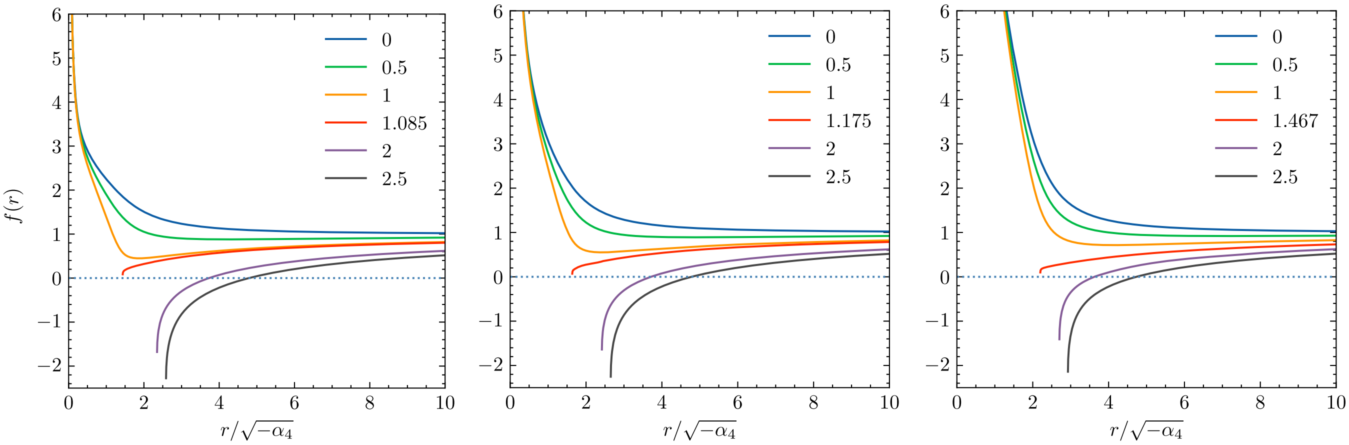

FIG. 4. Metric profile $f(r)$ of Eq. (29) for $\alpha_4 < 0$ and different values of the mass (in units of $\sqrt{-\alpha_4}$, indicated by the colors) and different negative values of the product $\alpha_5\eta$ [in units of $(-\alpha_4)^{3/2}$]. Left: $\alpha_5\eta = -0.25$. Middle: $\alpha_5\eta = -1$. Right: $\alpha_5\eta = -5$. The spacetime is a naked singularity at $r = 0$ for light masses, a naked singularity at $r_S > 0$ for intermediate masses (see the red curves), and a black hole with a hidden singularity at $r_S > 0$ for large masses.

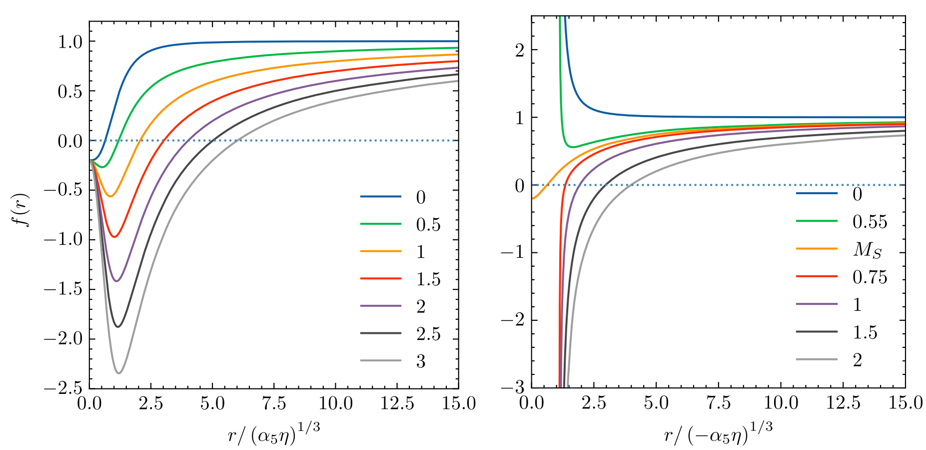

FIG. 5. Metric profile $f(r)$ of Eq. (33) for $\alpha_5 > 0$ (left plot) or $\alpha_5 < 0$ (right plot) and different values of the mass (in units of $|\alpha_5\eta|^{1/3}$, indicated by the colors). For $\alpha_5 > 0$, the spacetime is a black hole for any mass. For $\alpha_5 < 0$, the only mass giving a singularity at $r = 0$ is M_S [see Eq. (36)], and this singularity is hidden by a horizon. For other masses, there is a singularity at $r_S = (-4\alpha_5\eta/3)^{1/3}$, either naked for $M < M_S$ or hidden by a horizon if $M > M_S$.

bounds make the associated gravitational effects probably undetectable.

In the case of Eq. (33), there remains most likely a unique case, $\alpha_5 < 0$ (Fig. 5, right plot), where black holes have a minimal mass M_S given by Eq. (36). Another argument from Ref. [6] can then be used to constrain the value of $|\alpha_5\eta|$ for this theory. Indeed, the minimal mass M_S must be lower than the mass of experimentally detected black holes. In GW200115, the second object is a black hole of mass $M = 5.7^{+1.8}_{-2.1} M_\odot$ at a 90% credible interval, giving

$$|\alpha_5\eta| \lesssim 2070^{+565}_{-659}\ \mathrm{km}^3. \qquad (39)$$

If we take into account other events for which the second object is lighter, but whose black hole nature is not certain—namely, GW170817 and GW190814—we rather get $|\alpha_5\eta| \lesssim 230\ \mathrm{km}^3$ and $|\alpha_5\eta| \lesssim 194\ \mathrm{km}^3$. Finding such a simple constraint would not be possible in the other cases (Figs. 2–4), since in these cases, the minimal mass depends nontrivially on both α_4 and $\alpha_5\eta$.

This completes our study of solutions of the form in Eq. (5) to action (27) with a nontrivial scalar field that has a typical logarithmic running. From a string-theoretical point of view, this is to be corrected in the UV from higher-order corrections, but one can also question the existence of solutions with a constant scalar field $\phi = \phi_0$. A more general analysis of such solutions is displayed in Appendix A. It is easy to see that, for the considered action [Eq. (27)], a constant scalar field solution exists provided the couplings satisfy

$$0 = \alpha_4^2[\alpha_4^3\alpha_5(\beta_4\lambda_5 - 2\beta_5\lambda_4) + \alpha_4^4\beta_5\lambda_5 - 5\alpha_4\alpha_5^3\lambda_5 + 4\alpha_5^4\lambda_4] + 2\alpha_5^5(2\alpha_5\beta_4 - 3\alpha_4\beta_5)\Lambda, \quad (40)$$

$$0 \neq \alpha_5^3 - \alpha_4^2\alpha_5\beta_4 + \alpha_4^3\beta_5, \quad (41)$$

and the solution is a Schwarzschild-(A)dS black hole:

$$\phi_0 = \ln\left(-\frac{\alpha_4}{\alpha_5}\right),$$
$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha_4^4(\alpha_4\lambda_5 - \alpha_5\lambda_4) - \Lambda\alpha_5^5}{3\alpha_5^2(\alpha_5^3 - \alpha_4^2\alpha_5\beta_4 + \alpha_4^3\beta_5)}r^2. \quad (42)$$

Interestingly enough, this solution is valid for the theory with coupling constants given by Eq. (28), where we also have the black hole solution [Eq. (29)]. However, while Eq. (29) is asymptotically flat if $\Lambda = 0$, this is not the case with Eq. (42), which has an effective cosmological constant.

IV. CASE OF CONSTANT $\mu \neq 1$

In the previous section, we have analyzed in some detail the action and solutions for $\mu = 1$ obtained from integrating Eqs. (19)–(21). Using the method of Sec. II, we now proceed to solve the potentials of our theory V, W, Z from Eqs. (19)–(21) for μ, an arbitrary constant. Without loss of generality, we have set $d = -1$ and $\tilde{c} = 0$, and by taking into account the scalar field profile [Eq. (25)], we can find the potentials as functions of ϕ:

$$V = \frac{\alpha}{c^{2(1-\mu)}}(e^{2(1-\mu)\phi} - 1) - \frac{\gamma}{c}(2\mu+1)(\mu-1)e^{\phi},$$
$$W + 1 = 2(\mu-1)c^{2(\mu-1)}e^{2(1-\mu)\phi}\left(1 + \frac{2\alpha}{c^2}e^{2\phi} + \frac{2\gamma}{c^{2\mu+1}}e^{(2\mu+1)\phi}\right),$$
$$Z = 2(\mu-1)c^{2(\mu-1)}e^{2(1-\mu)\phi}\left(\frac{-2\Lambda\mu(2\mu+1)}{3(2\mu-1)} + \frac{2(\mu-1)}{c^2}e^{2\phi} + \frac{2\alpha(\mu-2)}{c^4}e^{4\phi} - \frac{12\gamma}{(2\mu+3)c^{2\mu+3}}e^{(2\mu+3)\phi}\right). \quad (43)$$

Putting aside nonrelevant constants, the potentials end up depending on four parameters—α, γ, c, and Λ—or, rather, coupling constants of the theory. In Appendix C, the cases $\mu = \pm\frac{1}{2}$ and $\mu = \pm\frac{3}{2}$ are treated separately, since they correspond to some degenerate characteristic equations with metric solutions involving a logarithmic radial dependence.[7]

Before proceeding to the discussion of the solutions, let us write down the resulting action and provide an interesting geometrical interpretation. To this end, we introduce the following three Lagrangian densities:

$$\mathcal{L}_1(n) = 2e^{n\phi},$$
$$\mathcal{L}_2(n) = e^{(n-2)\phi}[R + (n-1)(n-2)(\nabla\phi)^2],$$
$$\mathcal{L}_3(n) = e^{(n-4)\phi}[\mathcal{G} - 4(n-3)(n-4)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - 2(n-2)(n-3)(n-4)\Box\phi(\nabla\phi)^2 - (n-2)(n-3)^2(n-4)(\nabla\phi)^4]. \quad (44)$$

The actions corresponding to the first two Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ are conformally invariant in dimension $n > 2$, while for $\mathcal{L}_3$ the conformal invariance holds for dimension $n > 4$.

The resulting action for $\mu \neq 1$ can be conveniently written as a linear combination of the densities in Eq. (44) as[8]

$$S_{\mu\neq 1} = \frac{1}{2}\int d^4x\sqrt{-g}\Big\{\alpha\Big[2(\mu-2)c^{2(\mu-3)}\mathcal{L}_1(6-2\mu) + 4c^{2(\mu-2)}\mathcal{L}_2(6-2\mu) + \frac{c^{2(\mu-1)}}{(\mu-1)}\mathcal{L}_3(6-2\mu)\Big] - \gamma\Big[\frac{12}{(2\mu+3)c^5}\mathcal{L}_1(5) - \frac{4}{c^3}\mathcal{L}_2(5) + \frac{(2\mu+1)}{c}\mathcal{L}_3(5)\Big] + 2(\mu-1)c^{2(\mu-2)}\mathcal{L}_1(4-2\mu) + 2c^{2(\mu-1)}\mathcal{L}_2(4-2\mu)\Big\}. \quad (45)$$

As one can see, the "integrable" four-dimensional action we have obtained for constant $\mu \neq 1$ turns out to be a linear combination of densities that *are conformally invariant* in dimensions $D = 5$, $D = 6 - 2\mu$, and $D = 4 - 2\mu$. Strictly speaking, and given the current restriction $\mu \notin \{\pm 1/2, \pm 3/2\}$, this interpretation holds true if $\mu = -\frac{k}{2}$, with $k \in \mathbb{N}\backslash\{1, 3\}$. Case $\mu = 1$ can also be recovered from Eq. (45) in a nontrivial way. Indeed, all terms of Eq. (45) but one have a well-defined limit as $\mu \to 1$, and notably, the last term gives a contribution $\mathcal{L}_2(4-2\mu) \to \mathcal{L}_2(2)$,

[7] A convenient overall rescaling of the potentials V, $W+1$, and Z by $\tilde{V} = \frac{V}{2(\mu-1)c^{2(\mu-1)}e^{2(1-\mu)\phi}}$, etc., gives back the $\mu \to 1$ potentials of the previous section [Eq. (26)] with couplings related by Eq. (28), up to redefinitions of the coupling constants.

[8] For simplicity, we have set the cosmological constant $\Lambda = 0$, and we have normalized the full action by a global factor for latter convenience.

yielding the standard Einstein-Hilbert piece. As regards the *a priori* singular term $\frac{c^{2(\mu-1)}}{(\mu-1)}\mathcal{L}_3(6-2\mu)$, a singular limiting procedure as described in Ref. [40] yields precisely the α_4 part of Eq. (27). Another particular value is $\mu=0$, which corresponds in Eq. (45) to densities which are conformally invariant in dimensions $D=4$, $D=5$, and $D=6$. However, even with the proportionality factor null, $\mu=0$, in Eqs. (19)–(21), the below presented metric solutions, Eqs. (47) and (48), can be verified to be solutions to the corresponding $\mu=0$ theory, and therefore, all comments made on these solutions are also valid for this degenerate case.

Note that for the special case $\gamma=0$, the theory, and its solution presented below [see Eq. (47)], correspond to the Galileon theory and black hole found in Ref. [16]. This Galileon black hole has a higher-dimensional (i.e., non-Newtonian) falloff, as it originates from a higher-dimensional Lovelock solution with a horizon given as a product of two-spheres. More precisely, it comes from a diagonal Kaluza-Klein reduction along an internal space which is a product of $s=\mu-1$ two-spheres. This higher-dimensional interpretation holds, strictly speaking, for any integer $\mu\geq 2$, while the other values of μ are just an analytic continuation of this result. In particular, the identified favorable case, $\mu=1$, corresponds formally to an empty product of two-spheres, which explains that it can be captured by the Kaluza-Klein reduction only through a singular limit. In a nutshell, in action (45), the parts which have no γ factor have two possible, complementary interpretations: in terms of Lagrangians which have conformal invariance in dimensions $6-2\mu$ and $4-2\mu$, or in terms of Kaluza-Klein reduction. Interestingly, the former interpretation seems more relevant when $\mu\leq 0$, while the latter interpretation makes more sense for $\mu\geq 1$. As regards the γ parts, they are just the Lagrangians with conformal invariance in five dimensions, and their interpretation is the same for any μ.

Having identified the geometric nature of the action, we now concentrate on the solutions of the theory [Eq. (45)]. The quadratic equation satisfied by f can be written generically for any $\mu\notin\{\pm 3/2,\pm 1/2\}$ as

$$\alpha f^2-\frac{f}{2\mu-1}(r^2+2\gamma r^{1-2\mu}+2\alpha)+\frac{1}{(2\mu-1)^2}\left(r^2-\frac{\Lambda r^4}{3}-2Mr^{3-2\mu}-\frac{2\gamma(2\mu-1)}{2\mu+3}r^{1-2\mu}+\frac{\alpha(2\mu-1)}{2\mu-3}\right)=0, \quad (46)$$

and its solution is given by

$$f(r)=\frac{1}{2\mu-1}\left[1+\frac{\gamma r^{1-2\mu}}{\alpha}+\frac{r^2}{2\alpha}\left(1\pm\sqrt{\left(1+\frac{2\gamma}{r^{2\mu+1}}\right)^2+\frac{4\alpha\Lambda}{3}+\frac{8\alpha M}{r^{2\mu+1}}+\frac{16\alpha\gamma(2\mu+1)}{(2\mu+3)r^{2\mu+3}}-\frac{8\alpha^2}{(2\mu-3)r^4}}\right)\right]. \quad (47)$$

It is easy to see that in the limiting case $\mu=1$, the expression of f reduces to the $\mu=1$ solution previously found in Eq. (29) with $\alpha=\alpha_4$ and $\gamma=2\alpha_5\eta/3$. For brevity, we discuss these solutions when we switch off the potential term Λ (which played the role of the cosmological constant for the $\mu=1$ solution). The asymptotic behavior at $r\to\infty$ of the metric function depends on the range of the parameter μ, and also on the sign of γ (>0, <0, or $=0$) with respect to the sign in front of the square root. If $\mu>-1/2$, the metric potential f grows at infinity with a power exponent that never exceeds 2. Indeed, in this case, for the lower branch we have asymptotically either $f\sim(2\mu-1)^{-1}$ (for $\mu>1/2$) or $f\sim r^{1-2\mu}$ (for $\mu<1/2$), while for the upper branch $f\sim r^2$. On the other hand, for $\mu<-1/2$, there are three cases. If γ has a sign opposite to the sign in front of the square root in Eq. (47), then $f\sim r^2$. In the case when those signs coincide (or when $\gamma=0$), f has an asymptotic dependence on r with the power which exceeds that of (A)dS—namely, $f\sim r^{1-2\mu}$ (or $f\sim r^{3/2-\mu}$).

This is a fairly general case of a black hole solution where the solution has a unique integration constant, the mass M, and two coupling constants α and γ appearing in the metric. It is a black hole with secondary hair. We can note that the metric solution does not depend on the coupling c, while the parameter Λ loosely parametrizes the Liouville potential, which is the higher-dimensional cosmological constant for noncritical string theories [25]. The theory can be perceived as a higher-order Liouville theory with higher-order corrections parametrized here by α and γ. It is also clear that depending on the ranges of the constant $\mu\neq 1$, the metric function will display different asymptotic behaviors, and none of them will reproduce the standard falloff $f\sim 1–2M/r$. The "mass" contribution as $r\to\infty$ is, for instance, $\sim Mr^{1-2\mu}$ for $\mu>-1/2$. Because of that, one can wonder whether these solutions have a finite mass. In fact, a quick thermodynamic analysis of the solution through the Euclidean method on the action in Eq. (45) reveals that the mass $\mathcal{M}$ is finite for both branches, and it is parametrized in terms of the proportionality factor μ as

$$\mathcal{M}=\frac{\mu}{2\mu-1}M,$$

or in terms of the horizon r_h as

$$\mathcal{M} = \frac{\mu}{2}\left[\frac{\alpha}{(2\mu-3)} r_h^{2\mu-3} + \frac{1}{(2\mu-1)} r_h^{2\mu-1} - \frac{2\gamma}{(2\mu+3) r_h^2}\right].$$

Similarly to the $\mu = 1$ case, for $\alpha = 0$, the quadratic equation [Eq. (46)] reduces to a linear equation for f whose solution is given by

$$f(r) = \frac{1}{(2\mu-1)(1+\frac{2\gamma}{r^{2\mu+1}})}\left[1 - \frac{\Lambda r^2}{3} - \frac{2M}{r^{2\mu-1}} - \frac{2\gamma(2\mu-1)}{(2\mu+3) r^{2\mu+1}}\right]. \quad (48)$$

The corresponding theory is given by the linear combination of the conformal densities of the dimensions $D = 5$ and $D = 4 - 2\mu$, and for $\mu = 1$, one recovers the previous solution [Eq. (33)], along with its theory [Eq. (27)] with couplings [Eq. (32)]. Note that the solution [Eq. (48)] and its associated action both remain valid for $\gamma = 0$. Also, as in the quadratic case, the asymptotic behavior of the metric solution (48) depends on the parameter μ.

V. CONCLUSIONS

In this paper, starting from a quite general version of Horndeski theory inspired by Kaluza-Klein theories, we have selected specific theories admitting interesting and explicit black hole solutions with no apparent four-dimensional symmetries. Our starting point was a subclass of Horndeski theory with arbitrary ϕ-dependent potentials and higher-order operators; the latter are related to Kaluza-Klein reduction of Lovelock theory (see, for example, Ref. [41] and references within). We then, for spherical symmetry, established integrability [Eq. (8)] and compatibility relations [Eqs. (19)–(21)] that permitted the selection of interesting and quite general subsets of Horndeski theories without apparent symmetries. There is no particular bias in the Horndeski functionals for the filtered-out theory; for example, there is no assumption of shift or parity symmetry—and in four dimensions, no conformal invariance of the scalar field equation. The compatibility conditions we found, by fully integrating the field equations, involved an arbitrary function $\mu(r)$. We established the repertory of some previously known solutions in terms of our potentials and this function μ. The BBMB [32] or MTZ [33] solutions are found to yield quite complex functions $\mu(r)$, whereas those of Fernandes [19] are tailored to $\mu = 1$. We then chose μ to be constant, for simplicity, in the principal part of our study. Our filtered theories are at the end parametrized in terms of this constant parameter and four coupling constants. The solutions we have found are rare examples of unbiased Horndeski functionals along with a canonical kinetic term for the scalar. They have a higher-dimensional geometric interpretation either through conformally coupled actions (of various dimensions) or Kaluza-Klein reduction of Lovelock theories. The physical properties of the solutions depend on the value of μ, which essentially fixes the asymptotic behavior.

In the case $\mu = 1$, we have a four-dimensional Newtonian falloff, and the theory and solutions present the most phenomenological interest. In this case, the overall theory has distinct subparts with four- and five-dimensional (conformal) symmetries associated with the coupling of the scalar field, while no symmetry is associated with the full action in four dimensions. The first of these subactions, $\mathcal{L}_4(g,\phi)$, is a singular limit giving a conformally coupled scalar field equation in four dimensions [19]. The second is a Lagrangian density $\mathcal{L}_5(g,\phi)$ which is conformally invariant in $D = 5$ dimensions. Summing up, the full action in the case $\mu = 1$ is given by the Einstein-Hilbert plus the linear combination of the aforementioned Lagrangians, $\mathcal{L}_4(g,\phi)$ and $\mathcal{L}_5(g,\phi)$. The solutions emanating from this action are asymptotically flat [or (A)dS in the presence of a cosmological constant in the action]. The black holes are asymptotically very similar to Schwarzschild, depending on a mass parameter and two coupling constants associated with the aforementioned four- and five-dimensional parts in the action. Each of these two parts of the action accompanied by the Einstein-Hilbert term possesses black hole solutions—and, somewhat surprisingly, their combination also admits a black hole which, in a sense, is a superposition of the latter two. Usually, because of nonlinearity, solutions of different actions cannot be superposed; nevertheless, the black holes we presented in this case can be viewed as a nonlinear superposition of subsolutions. If we switch off the $\mathcal{L}_4(g,\phi)$ part of the action, we find a solution very much like a Schwarzschild type, although the action itself is of higher order. The vacua of such theories are not trivial, and we have made additional analysis in this direction—see Appendix D.

In fact, for all remaining constant μ, the action we consider involves again two subparts: the five-dimensional piece $\mathcal{L}_5(g,\phi)$ is a common denominator for all μ. The second piece can be identified in two ways: as a Kaluza-Klein reduction originating from a higher-dimensional Lovelock theory or interestingly, as a combination of conformally coupled actions in differing dimensions. The black holes for the general action will have—not surprisingly, given their higher-order nature—a non-Newtonian falloff for the mass of the solution. The solution is again a superposition of two differing solutions for the two subset actions. A relevant question is whether there exists an oxidation of the full solution we have found into vacuum Lovelock metric theory. Or, are these solutions only scalar-tensor? Furthermore, could it be that the five-dimensional piece is some form of higher-dimensional matter [42] for a Lovelock black hole [28]? This is the most probable outcome, as in Lovelock theory there are strong constraints on the possible horizon metrics which are permissible [27].

The solutions we have found in the $\mu = 1$ case bifurcate known no-hair theorems, either due to the absence of symmetry in the action or due to the presence of higher-order corrections. Let us comment on a no-hair theorem that concerns a two-derivative scalar-tensor action of the form

$$S = \int d^4x\sqrt{-g}[\kappa R - \beta\mathcal{L}_2(n) - U(\phi)], \qquad (49)$$

where $\mathcal{L}_2(n)$, defined in Eq. (44), corresponds to the nonminimally coupled scalar. Here, κ and β are constants, and U is a function that only depends on the scalar field. As mentioned before, in four dimensions, the density $\mathcal{L}_2(n)$ is conformally invariant only for $n = 4$, and in this case, a black hole solution for $U \equiv 0$ is known [32]. Generalizing this solution for a nonminimal coupling parameter $n \neq 4$ remains an open problem. Regarding this question, a no-hair theorem was established in Ref. [43] stipulating the nonexistence of asymptotically flat black hole solutions with a positive semidefinite potential for $n \in [0, 2[-\{1\}$. In the present case, our solution [Eq. (33)] arises from an action of the form in Eq. (49) with $n = 5$ and with $U \propto e^{5\phi}$, but supplemented by the Lagrangian $\mathcal{L}_3(5)$ as defined in Eq. (44). Our theory violates the hypotheses of the no-hair theorem due to the presence of the higher-order terms in $\mathcal{L}_3(5)$. Nevertheless, the value of n, for our solution $n = 5$, is not forbidden by the no-hair theorem. Therefore, one may wonder if one could extend the theorem of Ref. [43] to include higher-order corrections.

There are several ways to extend our present analysis. For a start, we chose to take $\mu = \text{cst.}$ to facilitate the integration of the compatibility equations in Eqs. (19)–(21). Other choices, however, are possible, with the BBMB and MTZ solutions recovered for nonconstant μ, as we mentioned earlier. As these examples show, it is actually intriguing that once μ is not constant, the function invariably depends on the mass parameter of the solution. This renders the solution quite different: for example, it is known that the BBMB solution explodes at the horizon of the black hole, while for mass equal to zero the scalar is trivial. Furthermore, although the mass parameter enters in the coupling constants of the scalar field, these are in turn accommodated in the theory, so that the mass never appears in the action. Note that in the contrary case, this would have meant a fine-tuning of the theory for a given mass black hole. Are these facts accidents or more general observations? Furthermore, our integrability condition was adapted to a homogenous spherically symmetric ansatz for which $-g_{tt} = g^{rr}$. In fact, it is not so clear how to extend the integrability condition for a general spherically symmetric ansatz. Is it possible to go beyond this and establish again more general integrability conditions?

The integrability condition we have found admits a second branch of possible solutions which we did not manage to solve for more general theories than those found previously [19]—see Appendix B. In this second branch, the scalar field has explicit dependence on the metric function as well, as it incorporates the flat vacuum. Furthermore, it incorporates an additional constant independent of the mass which one may associate with the symmetry of the conformally coupled scalar in Ref. [19]. It is then normal to question if the absence of extensions of the second branch is due to the absence of conformal symmetry for the scalar: the scalar field equation arising from the variation of the action in Eq. (45) is conformally invariant only for $\mu = 1$ and $\gamma = 0$. This intuition conforms to our results in higher dimensions, where we have shown that this second branch indeed exists for a conformally coupled scalar field [40]. Another interesting question concerns the possible presence of multiple solutions and the issue of scalarization [13], for a *given specific* theory. Our unbiased Horndeski theories possibly allow for scalarization, and obtaining explicit—rather than numerical—solutions is an intriguing possibility. This is an interesting subject, as well as undertaking issues of cosmological stability in such scalarized cases [44]. This and some of the points mentioned above deserve further study, which we hope will be successfully undertaken in the near future.

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APPENDIX A: CONSTANT SCALAR FIELD

In this appendix, we solve the field equations obtained from the action in Eq. (1) for a constant scalar field $\phi = \phi_0 = \text{cst}$ and a static, spherically symmetric metric ansatz as given in Eq. (5). For such a constant scalar field, two cases must be distinguished: either $1 + W(\phi_0)$ vanishes, or not. Given the form of Eq. (1), the vanishing of $1 + W(\phi_0)$ makes the Einstein-Hilbert term disappear, which is equivalent to having all potentials (but W) in the action (1) displaying infinite coupling constants. In other words, $1 + W(\phi_0) = 0$ amounts to a strongly coupled constant scalar field.

Let us start with the most relevant case, where the scalar field is not strongly coupled: $W(\phi_0) \neq -1$. Then, the metric field equations $\mathcal{E}_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}$ are solved by a Schwarzschild-(A)dS metric profile, with the cosmological constant determined by the constant scalar field,

$$f(r) = 1 - \frac{2M}{r} + \frac{Z(\phi_0)r^2}{6(1+W(\phi_0))}, \tag{A1}$$

where M is a mass integration constant. Entering this into the scalar field equation $\mathcal{E}_\phi = \frac{\delta S}{\delta \phi}$ gives

$$\left[Z_\phi + \frac{2Z}{1+W}\left(\frac{ZV_\phi}{3(1+W)} - W_\phi\right) + \frac{48M^2V_\phi}{r^6}\right]_{\phi=\phi_0} = 0, \tag{A2}$$

and hence, this will be a solution only if $V_\phi(\phi_0) = 0$ and

$$\left[Z_\phi - \frac{2ZW_\phi}{1+W}\right]_{\phi=\phi_0} = 0. \tag{A3}$$

We now turn for completeness to the strongly coupled case, $W(\phi_0) = -1$. Then the equations $\mathcal{E}_{\mu\nu}$ are degenerate and are solved if $Z(\phi_0) = 0$, while $\mathcal{E}_\phi$ then fully determines the metric, giving the solution[9]

$$f(r) = \begin{cases} 1 + \frac{r^2 W_\phi}{4V_\phi}\left(1 \pm \sqrt{1 + \frac{8V_\phi}{W_\phi}\left(\frac{2M}{r^3} - \frac{q}{r^4} - \frac{Z_\phi}{12W_\phi}\right)}\right) & \text{if } V_\phi(\phi_0) \neq 0 \quad \text{and} \quad W_\phi(\phi_0) \neq 0 \\ 1 \pm \sqrt{1 + 2Mr - q - \frac{Z_\phi r^4}{24V_\phi}} & \text{if } V_\phi(\phi_0) \neq 0 \quad \text{and} \quad W_\phi(\phi_0) = 0 \\ 1 - \frac{2M}{r} + \frac{q}{r^2} + \frac{r^2 Z_\phi}{12W_\phi} & \text{if } V_\phi(\phi_0) = 0 \quad \text{and} \quad W_\phi(\phi_0) \neq 0, \end{cases} \tag{A4}$$

where it is implicit that the potentials and their derivatives are evaluated at ϕ_0. These profiles display two integration constants, a mass M and a kind of "charge" q.

We can, for instance, apply the previous analysis to the case of Ref. [19], with the potentials given by Eq. (4). Then $V_\phi = -\alpha \neq 0$; therefore, solutions are obtained only in the strongly coupled case $W(\phi_0) = -1$, giving $\phi_0 = -\ln(\beta)/2$, while the condition $Z(\phi_0) = 0$ gives $\lambda = -\Lambda\beta^2$, and the metric is given by the first line in the above equation, in agreement with the results presented in Ref. [19].

APPENDIX B: ANALYSIS OF THE SECOND BRANCH

We study in this appendix the second branch identified in Sec. II by requiring that the second factor of the factorization [Eq. (8)] vanish. Using Eqs. (9)–(12), this amounts to rewriting the equation as

$$r^2 W_\phi + 4V_\phi - 4fV_\phi\left[1 + 2r\phi' \frac{V_{\phi\phi} - \frac{V_\phi}{d}}{V_\phi} + r^2(\phi')^2 \frac{\frac{2+d_\phi}{d^2}V_\phi + V_{\phi\phi\phi} - \frac{3}{d}V_{\phi\phi}}{2V_\phi}\right] = 0. \tag{B1}$$

In this generic form, there is no possibility to solve this equation, and an option would be to look for separability of this equation, as happens in the particular case of Ref. [19]. For this purpose, let us consider $\mathcal{U}$ such that

$$\mathcal{U}_\phi = \frac{V_{\phi\phi}}{V_\phi} - \frac{1}{d}, \qquad \mathcal{U}_\phi^2 = \frac{2+d_\phi}{2d^2} + \frac{V_{\phi\phi\phi}}{2V_\phi} - \frac{3V_{\phi\phi}}{2dV_\phi}. \tag{B2}$$

Then, the latter equation takes the form

$$f(1 + r\mathcal{U}')^2 - \left(1 + \frac{r^2 W_\phi}{4V_\phi}\right) = 0. \tag{B3}$$

By introducing $u = \frac{1}{\alpha}\exp(\mathcal{U})$, with α as a coupling constant, one has $u' = u\mathcal{U}'$, so that

$$f(Y')^2 - \frac{Y^2}{r^2}\left(1 + \frac{W_\phi}{4V_\phi u^2}Y^2\right) = 0, \tag{B4}$$

where $Y = ru$. Therefore, the equation is separable if the last term in the parenthesis is a pure function of Y—that is to say, there exists a coupling constant β such that

$$\frac{W_\phi}{4V_\phi u^2} = \frac{\beta}{2\alpha}. \tag{B5}$$

Compiling all these conditions, one gets the following compatibility conditions:

$$\begin{aligned} V_{\phi\phi\phi} &= 2\frac{V_{\phi\phi}^2}{V_\phi} - \frac{d_\phi}{d^2}V_\phi - \frac{V_{\phi\phi}}{d}, \\ W_\phi &= \frac{2\beta}{\alpha}\gamma^2 V_\phi^3 \exp\left(-2\int\frac{\mathrm{d}\phi}{d}\right), \\ u &= \gamma V_\phi \exp\left(-\int\frac{\mathrm{d}\phi}{d}\right), \end{aligned} \tag{B6}$$

[9]If $V_\phi(\phi_0) = 0$ and $W_\phi(\phi_0) = 0$, the field equations are degenerate and reduce to $Z_\phi(\phi_0) = 0$ without any constraint on $f(r)$.

where γ is an additional coupling constant. These conditions enable us to solve the second branch for $Y = ru$, yielding the following expression for u:

$$u = \frac{\sqrt{-2\alpha/\beta}}{r\cosh\left(c \pm \int \frac{\mathrm{d}r}{r\sqrt{f}}\right)} \quad \text{if } \beta \neq 0,$$
$$u = \exp\left(c + \int \frac{\pm 1 - \sqrt{f}}{r\sqrt{f}}\mathrm{d}r\right) \quad \text{if } \beta = 0, \tag{B7}$$

where c is an integration constant. One can see that in the case of Ref. [19] with the coupling functions of Eq. (4), $u = \exp(\phi)$, and this reproduces the solution found in this reference. More generally, for $d(\phi) = \mathrm{cst}$, the conditions of Eq. (B6) can be integrated and, taking into account the remaining field equations, one gets the solution of Ref. [19], up to scalar field redefinition. Hence, one concludes that the second parenthesis of Eq. (8) can be made a separable equation if the conditions of Eq. (B6) are verified, and these conditions are then easily tractable for $d(\phi) = \mathrm{cst}$,[10] in which case one retrieves solutions of Ref. [19] up to scalar field redefinitions. We have not proven in all generality that this second branch cannot lead to new metric solutions, either in the separable case by a judicious choice of nonconstant $d(\phi)$, or even in the nonseparable case, but after long study, it has not seemed to us that an analytic solution could be obtained in these complicated configurations.

APPENDIX C: CASES $\mu = \pm 1/2$ AND $\mu = \pm 3/2$

We complete the integration of compatibility conditions [Eqs. (19)–(21)] for constant μ by turning in this appendix to the special values $\mu \in \{\pm 3/2, \pm 1/2\}$, not included in the analysis of Sec. IV. For such values, some factors or powers vanish in the previously found potentials [Eq. (43)] and/or in the corresponding metric solutions [Eqs. (47) and (48)]. This gives rise to additional terms in ϕ in the potentials, and to logarithmic terms $\ln(r/c)$ in the metric function. The potentials can again be written in terms of four parameters: α, γ, c, and Λ. For $\mu = 3/2$, the potentials all coincide with Eq. (43). For $\mu = -3/2$, V and W also coincide, while

$$Z = -\frac{5e^{5\phi}}{c^5}\left(\frac{\Lambda}{2} - \frac{5e^{2\phi}}{c^2} - \frac{7\alpha e^{4\phi}}{c^4} - 13\gamma - 12\gamma\phi\right). \tag{C1}$$

For $\mu = -1/2$, the potentials are given by

$$V = \frac{\alpha e^{3\phi}}{c^3} - \frac{3\gamma e^{\phi}}{2c},$$
$$W + 1 = -\frac{3e^{3\phi}}{c^3}\left(1 + \frac{2\alpha}{c^2}e^{2\phi} - 2\gamma\phi\right),$$
$$Z = -\frac{3e^{3\phi}}{c^3}\left(-\frac{\Lambda}{6} - \frac{3+\gamma}{c^2}e^{2\phi} - \frac{5\alpha e^{4\phi}}{c^4} + \frac{6\gamma\phi e^{2\phi}}{c^2}\right). \tag{C2}$$

For $\mu = 1/2$, they read

$$V = \frac{2\alpha\phi e^{\phi}}{c} + \frac{\gamma e^{\phi}}{c},$$
$$W + 1 = -\frac{e^{\phi}}{c}\left(1 + \frac{2(\alpha+\gamma)e^{2\phi}}{c^2} + \frac{4\alpha\phi e^{2\phi}}{c^2}\right),$$
$$Z = -\frac{e^{\phi}}{c}\left(\frac{\Lambda}{3} - \frac{e^{2\phi}}{c^2} - \frac{(6\gamma+7\alpha)e^{4\phi}}{2c^4} - \frac{6\alpha\phi e^{4\phi}}{c^4}\right). \tag{C3}$$

Once again, f is given by a polynomial equation of degree 2 if $\alpha \neq 0$, and degree 1 if $\alpha = 0$. When possible, we keep the following expressions as close to the generic solution in Eq. (47) as possible and thus do not always explicitly replace μ with its value. First, if $\alpha \neq 0$, $\mu = 3/2$ gives

$$f(r) = \frac{1}{2\mu - 1}\left[1 + \frac{\gamma r^{1-2\mu}}{\alpha} + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{\left(1 + \frac{2\gamma}{r^{2\mu+1}}\right)^2 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha M}{r^{2\mu+1}} + \frac{16\alpha\gamma(2\mu+1)}{(2\mu+3)r^{2\mu+3}} - \frac{8\alpha^2\ln(r/c)}{r^4}}\right)\right], \tag{C4}$$

$\mu = -3/2$ gives

$$f(r) = \frac{1}{2\mu - 1}\left[1 + \frac{\gamma r^{1-2\mu}}{\alpha} + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{\left(1 + \frac{2\gamma}{r^{2\mu+1}}\right)^2 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha M}{r^{2\mu+1}} + 32\alpha\gamma\ln\left(\frac{r}{c}\right) - \frac{8\alpha^2}{(2\mu-3)r^4}}\right)\right], \tag{C5}$$

$\mu = -1/2$ gives

$$f(r) = \frac{1}{2\mu - 1}\left[1 + \frac{\gamma r^2\ln(r/c)}{\alpha} + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{\left(1 + 2\gamma\ln\left(\frac{r}{c}\right)\right)^2 + \frac{4\alpha\Lambda\ln(r/c)}{3} + 8\alpha M - \frac{8\alpha\gamma}{r^2} - \frac{8\alpha^2}{(2\mu-3)r^4}}\right)\right], \tag{C6}$$

[10]We recall that only for the first branch of solutions, the function $d(\phi)$ can be taken as a constant without any loss of generality.

and $\mu = 1/2$ gives

$$f(r) = -\frac{1}{2}\left[1 + 2\left(\frac{\gamma}{2\alpha} - \ln\left(\frac{r}{c}\right)\right) + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{\left(1 + \frac{2(\gamma - 2\alpha\ln(r/c))}{r^2}\right)^2 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha M}{r^2} + \frac{8\alpha\ln(r/c)}{r^2} + \frac{2\alpha(3\alpha+2\gamma)}{r^4} - \frac{8\alpha^2\ln(r/c)}{r^4}}\right)\right]. \tag{C7}$$

On the other hand, let us consider $\alpha = 0$. Then, if $\mu = 3/2$, f coincides with the formula in Eq. (48) for generic μ. For $\mu = -3/2$,

$$f(r) = \frac{1}{(2\mu-1)\left(1+\frac{2\gamma}{r^{2\mu+1}}\right)}\left[1 - \frac{\Lambda r^2}{3} - \frac{2M}{r^{2\mu-1}} + 2\gamma r^2\left(1 - 4\ln\left(\frac{r}{c}\right)\right)\right], \tag{C8}$$

for $\mu = -1/2$,

$$f(r) = \frac{1}{(2\mu-1)(1+2\gamma\ln(\frac{r}{c}))}\left[1 + \left(2\gamma - \frac{\Lambda r^2}{3}\right)\ln\left(\frac{r}{c}\right) - \frac{2M}{r^{2\mu-1}} + 2\gamma\right], \tag{C9}$$

and for $\mu = 1/2$,

$$f(r) = -\frac{1}{2\left(1+\frac{2\gamma}{r^2}\right)}\left[1 - \frac{\Lambda r^2}{3} - 2M + \frac{\gamma}{r^2} - 2\ln\left(\frac{r}{c}\right)\right]. \tag{C10}$$

The behavior at $r \to \infty$ and $r \to 0$ is accordingly modified with logarithmic terms as compared to generic μ, but the qualitative picture is the same as for the generic μ case of Sec. IV: no Newtonian mass term is obtained.

APPENDIX D: STEALTH FLAT SPACETIME SOLUTIONS TO THE ACTION (27) WITH A NONTRIVIAL SCALAR FIELD $\phi = \phi(r)$

In Sec. III, we have presented two new classes of black hole solutions [Eqs. (29) and (33)], provided the coupling constants of action (27) satisfy Eq. (28) or Eq. (32), respectively, and these solutions turn out to be asymptotically flat for a vanishing bare cosmological constant $\Lambda = 0$. In this appendix, we would like to investigate whether these asymptotically flat black holes coexist with some stealth flat spacetimes. To this aim, we insert the ansatz in Eq. (5) with $f(r) = 1$ into the field equations of the action in Eq. (27), taking $\Lambda = 0$. The combination of field equations $\mathcal{E}_t^t - \mathcal{E}_r^r = 0$ admits as well a similar factorization to Eq. (8), which now reads

$$\{(\phi')^2 - \phi''\}\{re^{2\phi}[2\beta_4 + 3\beta_5 e^{\phi}] - 4\phi'[\alpha_4(2 + r\phi') + \alpha_5 e^{\phi}(4 + 3r\phi')]\} = 0. \tag{D1}$$

By opting for the second branch, one finds that there is a one-parameter family of theories, parametrized by a real number ϵ, such that a flat spacetime solution exists. The coupling constants read

$$\lambda_4 = \frac{3\epsilon(4-\epsilon)\beta_4^2}{16\alpha_4}, \qquad \lambda_5 = \frac{3\beta_5^2}{4\alpha_5}, \qquad \beta_5 = \frac{\epsilon\beta_4\alpha_5}{\alpha_4}, \tag{D2}$$

while the scalar field is given by

$$\phi = \ln\left(\frac{16\alpha_4\alpha_5(2-\epsilon)}{r^2\alpha_4\beta_4(2-\epsilon)^2 - 16\epsilon\alpha_5^2}\right). \tag{D3}$$

Comparing the latter expression of λ_5 with its expression required for the black holes solution, either Eq. (28) or Eq. (32), one can see that both solutions cannot coexist. For the first branch, one gets the same negative conclusion in which the stealth solution is given by

$$\phi = \ln\left(\frac{\eta}{r}\right), \qquad \eta = \sqrt{\frac{4\alpha_5}{3\beta_5}}, \tag{D4}$$

with the couplings satisfying

$$\beta_4 = 0, \qquad \lambda_5 = \frac{9\beta_5^2}{4\alpha_5}, \qquad \lambda_4 = \frac{9\alpha_4\beta_5^2}{16\alpha_5^2}. \tag{D5}$$

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