Alleviating the H_0 tension with new gravitational scalar tensor theories

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We investigate the cosmological applications of new gravitational scalar-tensor theories, and we analyze them in the light of H_0 tension. In these theories, the Lagrangian contains the Ricci scalar and its first and second derivatives in a specific combination that makes them free of ghosts, thus corresponding to healthy biscalar extensions of general relativity. We examine two specific models, and for particular choices of the model parameters, we find that the effect of the additional terms is negligible at high redshifts, obtaining a coincidence with ΛCDM cosmology; however, as time passes, the deviation increases, and thus, at low redshifts the Hubble parameter acquires increased values ($H_0 \approx 74$ km/s/Mpc) in a controlled way. The mechanism behind this behavior is the fact that the effective dark-energy equation-of-state parameter exhibits phantom behavior, which implies faster expansion, which is one of the sufficient conditions that are capable of alleviating the H_0 tension. Lastly, we confront the models with cosmic chronometer (CC) data, showing full agreement within 1σ confidence level.

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I. INTRODUCTION

Although the concordance ΛCDM paradigm is very successful in describing early- and late-time cosmological evolution at both background and perturbation levels, nevertheless during the last years, there have appeared some potential tensions with specific datasets, such as the H_0 and σ_8 ones. In particular, the estimation for the present Hubble parameter H_0 according to the Planck Collaboration and assuming ΛCDM scenario is $H_0 = (67.27 \pm 0.60)$ km/s/Mpc [1], which is in tension at about 4.4σ with the direct measurement of the 2019 SH0ES Collaboration (R19), namely $H_0 = (74.03 \pm 1.42)$ km/s/Mpc, obtained using long-period Cepheids [2]. On the other hand, the σ_8 tension arises from the fact that the parameter that quantifies the matter clustering within spheres of radius $8h^{-1}$ Mpc is found to be different from the cosmic microwave background (CMB) estimation [1] and from the SDSS/BOSS measurement [3–5]. These tensions, and especially the H_0 one, progressively seem not to be related to unknown systematics, opening the road to many modifications of the standard lore [6,7] (for a review, see Ref. [8]).

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One may follow two main ways to alleviate the H_0 tension. The first is to modify the Universe content and/or particle interactions while keeping general relativity as the underlying gravitational theory [9–41]. The second way is to construct gravitational modifications, which applied to cosmological framework would lead to altered expansion rate [42–71]. We mention here that modified gravity has additional advantages too, such as the improvement of the renormalizability behavior of general relativity as well as the description of inflationary and/or dark-energy phases, and thus, it might be more preferable. Finally, there is another way to alleviate H_0 tension, in the framework of the running vacuum models [72], based on quantum field theory in curved spacetime [73–75], without the need to acquire phantom behavior (for a review of both the theoretical and phenomenological situation, see Ref. [76] and references therein).

In the present work, we are interested in alleviating the H_0 tension in the framework of new gravitational scalar-tensor theories [77–79]. In such constructions, one uses Lagrangians with the Ricci scalar as well as its first and second derivatives, nevertheless in combinations that result to ghost-free theories. These theories are found to have $2+2$ propagating degrees of freedom, and thus, fall outside Horndeski/Galileon [80–82] and beyond-Horndeski theories [83]. However, although they are biscalar extensions of general relativity, they were named

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"new gravitational scalar-tensor theories" since they can still be expressed in pure geometrical terms [77].

The plan of the work is the following: In Sec. II, we briefly review the new gravitational scalar-tensor theories, and in Sec. III, we apply them to a cosmological framework, extracting the modified Friedmann equations. Then, in Sec. IV, we construct specific models that can alleviate the H_0 tension, and we compare the induced behavior to that of ΛCDM scenario as well as to cosmic chromometers (CC) data. Finally, in Sec. V, we provide the conclusions.

II. OVERVIEW

In this section, we give a brief overview of the gravitational scalar-tensor theories. The action of such constructions is given as [77,78]

$$S=\int d^4\sqrt{-g}f(R,(\nabla R)^2,\Box R), \tag{1}$$

with $(\nabla R)^2=g^{\mu\nu}\nabla_\mu R\nabla_\nu R$. In the following, we set the Planck mass $M_P=1/\kappa=1$, where κ is the gravitational constant, for simplicity. One can rewrite the above action by converting the Lagrangian using double Lagrange multipliers, resulting to actions of multiscalar fields coupled minimally to gravity. In order to achieve it, one fixes the dependence of f on $\Box R=\beta$.

In the present work, we consider theories with the following f form:

$$f(R,(\nabla R)^2,\Box R)=\mathcal{K}((R,(\nabla R)^2)+\mathcal{G}(R,(\nabla R)^2)\Box R, \tag{2}$$

thus maintaining a linear form in $\Box R=\beta$. Generalizations to nonlinear forms are straightforward, although more complicated. In this case, (1) transforms to

$$\begin{aligned}S=\int d^4x\sqrt{-\hat{g}}\Bigg[&\frac{1}{2}\hat{R}-\frac{1}{2}\hat{g}^{\mu\nu}\nabla_\mu\chi\nabla_\nu\chi\\&-\frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\hat{g}^{\mu\nu}\mathcal{G}\nabla_\mu\chi\nabla_\nu\phi+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}\\&+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\widehat{\Box}\phi-\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}\phi\Bigg],\end{aligned} \tag{3}$$

where $\mathcal{K}=\mathcal{K}(\phi,B)$ and $\mathcal{G}=\mathcal{G}(\phi,B)$, with $B=2e^{\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The χ and ϕ fields are introduced through the conformal transformations $g_{\mu\nu}=\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\hat{g}_{\mu\nu}$, $\varphi\equiv f_\beta$, and they enter in a specific combination in a way that the final form of the action is equivalent to the original higher-derivative gravitational action.

Varying the action (3) with respect to the metric leads to the following field equations in the Einstein frame [77,78]:

$$\begin{aligned}\mathcal{E}_{\mu\nu}=&\frac{1}{2}G_{\mu\nu}+\frac{1}{4}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi-\frac{1}{2}\nabla_\mu\chi\nabla_\nu\chi+\frac{1}{4}g_{\mu\nu}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}g^{\alpha\beta}\mathcal{G}\nabla_\alpha\chi\nabla_\beta\phi-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\nabla_{(\mu}\chi\nabla_{\nu)}\phi\\&-\sqrt{\frac{2}{3}}g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\phi\mathcal{G}_B\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{4}g_{\mu\nu}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\Box\phi+\mathcal{G}_B(\Box\phi)\nabla_\mu\phi\nabla_\nu\phi+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\nabla_\mu\nabla_\nu\phi-\frac{1}{2}\nabla_\kappa\left(e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\delta^\lambda_{(\mu}\delta^\kappa_{\nu)}\nabla_\lambda\phi\right)\\&+\frac{1}{4}\nabla_\kappa\left(e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}g_{\mu\nu}\nabla^\kappa\phi\right)-\frac{1}{8}g_{\mu\nu}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{K}_B\nabla_\mu\phi\nabla_\nu\phi+\frac{1}{8}g_{\mu\nu}e^{-\sqrt{\frac{2}{3}}\chi}\phi=0.\end{aligned} \tag{4}$$

Additionally, varying (3) with respect to χ and ϕ gives rise to field equations as

$$\begin{aligned}\mathcal{E}_\chi=&\Box\chi+\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}\nabla_\mu\chi\nabla_\nu\phi-\frac{2}{3}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\phi\mathcal{G}_Bg^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi+\frac{1}{2}\sqrt{\frac{2}{3}}\nabla_\mu\left(e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}\nabla_\nu\phi\right)-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}\Box\phi\\&+\sqrt{\frac{2}{3}}\mathcal{G}_B\nabla_\mu\phi\nabla_\nu\phi g^{\mu\nu}\Box\phi-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{K}_B\sqrt{\frac{2}{3}}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi+\frac{1}{4}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\phi=0,\end{aligned} \tag{5}$$

and

$$\begin{aligned}\mathcal{E}_\phi=&-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}_\phi\nabla_\mu\chi\nabla_\nu\phi+2\sqrt{\frac{2}{3}}\nabla_\beta(g^{\mu\nu}\mathcal{G}_Bg^{\alpha\beta}\nabla_\alpha\phi\nabla_\mu\chi\nabla_\nu\phi)+\frac{1}{2}\sqrt{\frac{2}{3}}\nabla_\nu\left(e^{-\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\mathcal{G}\nabla_\mu\chi\right)\\&+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}_\phi\Box\phi-2\mathcal{G}_B(\Box\phi)^2-2\nabla_\nu\mathcal{G}_B\Box\phi\nabla^\nu\phi-\frac{1}{2}\sqrt{\frac{2}{3}}\nabla^\mu\left(e^{-\sqrt{\frac{2}{3}}\chi}\nabla_\mu\chi\mathcal{G}\right)+\frac{1}{2}\nabla^\mu\left(e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}_\phi\nabla_\mu\phi\right)\\&-\frac{1}{2}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\nabla^\mu\chi\mathcal{G}_B\nabla_\mu B+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\nabla^\mu\mathcal{G}_B\nabla_\mu B+\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}_B\nabla^\mu\left(e^{\sqrt{\frac{2}{3}}\chi}\nabla_\mu\chi\nabla^\nu\phi\nabla_\nu\phi\right)\\&+2e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{G}_B\nabla^\mu\left(e^{\sqrt{\frac{2}{3}}\chi}\nabla^\nu\phi\right)\nabla_\mu\nabla_\nu\phi+2\mathcal{G}_BR^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}_\phi-\nabla_\nu\left(e^{-\sqrt{\frac{2}{3}}\chi}\mathcal{K}_Bg^{\mu\nu}\nabla_\mu\phi\right)-\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}=0,\end{aligned} \tag{6}$$

where for simplicity, we have neglected the hats. Here, the subscripts in $\mathcal{G}$ and $\mathcal{K}$ denote the partial derivatives and the symmetrization is indicated by the parentheses in spacetime indices. The above equations reduce to GR for $\mathcal{K}=\phi/2$ and $\mathcal{G}=0$, with the conformal transformation in this case being $\chi=-\sqrt{\frac{3}{2}}\ln 2$. As we can see, the above equations do not contain any higher derivative terms, and therefore, the present theory is well-behaved. Lastly, note that since we have set the Planck mass to one, the field χ is dimensionless while ϕ has dimensions of $[M]^2$.

III. COSMOLOGICAL BEHAVIOR

We can now proceed to the study of the cosmological behavior of the present model. For this, we consider a flat Friedmann-Robertson-Walker (FRW) metric,

$$ds^2=-dt^2+a(t)^2\delta_{ij}dx^i dx^j, \tag{7}$$

with $a(t)$ the scale factor. We further assume that the two scalars are time dependent only.

Including the matter sector, considered to correspond to a perfect fluid, the metric field equations (4) become

$$\mathcal{E}_{\mu\nu}=\frac{1}{2}T_{\mu\nu}, \tag{8}$$

with $T_{\mu\nu}=\frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ representing the matter energy-momentum tensor.

With the above substitutions into Eqs. (4), we obtain the following Friedmann equations:

$$3H^2-\rho_m-\frac{1}{2}\dot\chi^2+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}+\frac{2}{3}\dot\phi^2[\dot\phi(\sqrt{6}\dot\chi-9H)-3\ddot\phi]\mathcal{G}_B-\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot B\,\dot\phi\,\mathcal{G}_B+\frac{\phi}{2}+\dot\phi^2(\mathcal{G}_\phi-2\mathcal{K}_B)\right]=0, \tag{9}$$

$$3H^2+2\dot H+p_m+\frac{1}{2}\dot\chi^2+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left(-\frac{\phi}{2}+\dot B\,\dot\phi\,\mathcal{G}_B+\dot\phi^2\mathcal{G}_\phi\right)=0, \tag{10}$$

with $B(t)=2e^{\sqrt{\frac{2}{3}}\chi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi=-2e^{\sqrt{\frac{2}{3}}\chi}\dot\phi^2$, and $H=\dot a/a$ the Hubble parameter, where dots denoting differentiation with respect to t. Similarly, the two scalar field equations (5) and (6) lead to

$$\mathcal{E}_\chi=\ddot\chi+3H\dot\chi-\frac{1}{3}\dot\phi^2[\dot\phi(3\sqrt{6}H-2\dot\chi)+\sqrt{6}\ddot\phi]\mathcal{G}_B+\frac{1}{2\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}[2\dot B\,\dot\phi\,\mathcal{G}_B-\phi+2\dot\phi^2(\mathcal{K}_B+\mathcal{G}_\phi)]+\frac{1}{\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}=0, \tag{11}$$

and

$$\begin{aligned}\mathcal{E}_\phi=&\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}[\dot\phi(-9H+\sqrt{6}\dot\chi)-3\ddot\phi]\mathcal{K}_B+\frac{1}{6}\dot B\{3e^{-\sqrt{\frac{2}{3}}\chi}\dot B+4\dot\phi[\dot\phi(9H-\sqrt{6}\dot\chi)+3\ddot\phi]\}\mathcal{G}_{BB}\\&+\frac{1}{3}e^{-\sqrt{\frac{2}{3}}\chi}[\dot\phi(9H-\sqrt{6}\dot\chi)+3\ddot\phi]\mathcal{G}_\phi+\left\{e^{-\sqrt{\frac{2}{3}}\chi}\dot B\,\dot\phi+\frac{2}{3}\dot\phi^2[\dot\phi(9H-\sqrt{6}\dot\chi)+3\ddot\phi]\right\}\mathcal{G}_{B\phi}\\&-e^{-\sqrt{\frac{2}{3}}\chi}\dot\phi^2\mathcal{K}_{B\phi}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\dot\phi^2\mathcal{G}_{\phi\phi}-e^{-\sqrt{\frac{2}{3}}\chi}\dot B\,\dot\phi\,\mathcal{K}_{BB}+\left[\frac{4}{3}\dot\phi(9H-2\sqrt{6}\dot\chi)\ddot\phi-\frac{1}{\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}\dot B\,\dot\chi\right.\\&\left.+\dot\phi^2\left(18H^2+6\dot H-3\sqrt{6}H\dot\chi-\frac{2}{3}\dot\chi^2-\sqrt{6}\ddot\chi\right)\right]\mathcal{G}_B-\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}_\phi+\frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}=0,\end{aligned} \tag{12}$$

with $\mathcal{G}_{B\phi}=\mathcal{G}_{\phi B}\equiv\frac{\partial^2\mathcal{G}}{\partial B\partial\phi}$, etc.

The above Friedmann equations (9), (10) can be rewritten as

$$H^2=\frac{1}{3}(\rho_{\rm DE}+\rho_m) \tag{13}$$

$$2\dot H+3H^2=-(p_{\rm DE}+p_m), \tag{14}$$

with the effective dark energy and pressure defined as

$$\begin{aligned}\rho_{\rm DE}\equiv&\frac{1}{2}\dot\chi^2-\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}-\frac{2}{3}\dot\phi^2[\dot\phi(\sqrt{6}\dot\chi-9H)-3\ddot\phi]\mathcal{G}_B\\&+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left[\dot B\,\dot\phi\,\mathcal{G}_B+\frac{\phi}{2}+\dot\phi^2(\mathcal{G}_\phi-2\mathcal{K}_B)\right],\end{aligned} \tag{15}$$

$$p_{\rm DE}\equiv\frac{1}{2}\dot\chi^2+\frac{1}{4}e^{-2\sqrt{\frac{2}{3}}\chi}\mathcal{K}+\frac{1}{2}e^{-\sqrt{\frac{2}{3}}\chi}\left(\dot B\,\dot\phi\,\mathcal{G}_B+\dot\phi^2\mathcal{G}_\phi-\frac{\phi}{2}\right). \tag{16}$$

Hence, one can show that in the new gravitational scalar-tensor theories the effective dark-energy density satisfies

$$\dot{\rho}_{\mathrm{DE}} + 3H(\rho_{\mathrm{DE}} + p_{\mathrm{DE}}) = 0, \tag{17}$$

while one can define the corresponding dark-energy equation-of-state parameter as

$$w_{\mathrm{DE}} \equiv \frac{p_{\mathrm{DE}}}{\rho_{\mathrm{DE}}}. \tag{18}$$

IV. HUBBLE TENSION

In this section, we construct specific models of the theory in order to be able to alleviate the H_0 tension. We mention here that in modified gravity theories, one typically has arbitrary functions, and thus, she has a huge freedom to determine both their forms as well as their parameters. This freedom is similar to the freedom of choosing the arbitrary potentials in scalar-field cosmology. Hence, in the end of the day, the obtained models are phenomenological, aiming to be in agreement with observations. In the theories examined in the present manuscript, we consider specific ansatzes for the functions $\mathcal{K}(\phi, B)$ and $\mathcal{G}(\phi, B)$, and we select models that lead to higher Hubble function at low redshifts, while introducing negligible deviations in the Hubble parameter at high redshifts as compared to ΛCDM. The two phenomenological models with the best behavior related to the H_0 tension are presented in the following.

A. Model I

As a first example, we consider the following forms for $\mathcal{K}(\phi, B)$ and $\mathcal{G}(\phi, B)$:

$$\mathcal{K}(\phi, B) = \frac{1}{2}\phi - \frac{\zeta}{2}B \quad \text{and} \quad \mathcal{G}(\phi, B) = 0, \tag{19}$$

with ζ a coupling constant with dimensions $[M]^{-4}$. The corresponding Friedmann equations (9), (10) read as

$$3H^2 - \rho_m - \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}(\phi + \zeta\dot{\phi}^2) = 0, \tag{20}$$

$$3H^2 + 2\dot{H} + p_m + \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}(\phi - \zeta\dot{\phi}^2) = 0, \tag{21}$$

while the two scalar field equations (11) and (12) become

$$\ddot{\chi} + 3H\dot{\chi} + \frac{1}{2\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{2\sqrt{6}}e^{-\sqrt{\frac{2}{3}}\chi}(\phi - \zeta\dot{\phi}^2) = 0, \tag{22}$$

$$\zeta\ddot{\phi} + \frac{1}{3}\zeta\dot{\phi}(9H - \sqrt{6}\dot{\chi}) - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi} + \frac{1}{2} = 0. \tag{23}$$

The corresponding effective dark-energy energy density and pressure (15), (16) become

$$\rho_{\mathrm{DE}} = \frac{1}{2}\dot{\chi}^2 - \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi + \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}(\phi + \zeta\dot{\phi}^2), \tag{24}$$

$$p_{\mathrm{DE}} = \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}\phi - \frac{1}{4}e^{-\sqrt{\frac{2}{3}}\chi}(\phi - \zeta\dot{\phi}^2). \tag{25}$$

In order to obtain the behavior of the Hubble parameter, we first set $z = -1 + a_0/a$, with the current value of the scale factor being set to $a_0 = 1$. It is well known that the behavior of the Hubble parameter in ΛCDM cosmology is given by

$$H_{\Lambda\mathrm{CDM}}(z) = H_0\sqrt{\Omega_{m_0}(1+z)^3 + 1 - \Omega_{m_0}}, \tag{26}$$

where H_0 is the present value of the Hubble parameter and Ω_{m_0} is the present value of matter density parameter defined as $\Omega_{m_0} = \frac{\rho_m}{3H^2}$ in Planck units. We set $\Omega_{m_0} = 0.31$ and $H_0 = 67.3$ km/s/Mpc. We then solve Eqs. (20)–(23) numerically to obtain the solutions for the scale factor and hence, for the Hubble parameter. In order to achieve this, we set the initial conditions such that the evolution of $H(z)$ that we obtain for $z = z_{\mathrm{CMB}} \approx 1100$ coincides with $H_{\Lambda\mathrm{CDM}}$, namely $H(z \to z_{\mathrm{CMB}}) \approx H_{\Lambda\mathrm{CDM}}$ while $H(z \to 0) > H_{\Lambda\mathrm{CDM}}(z \to 0)$. For our present analysis, we have one model parameter, i.e., ζ, which determines the late-time deviation of the model from ΛCDM scenario.

In Fig. 1, we plot the evolution of the dark-energy equation-of-state parameter in terms of the redshift. As we can see from the figure, $w_{\mathrm{DE}} < -1$ most of the time, thereby depicting phantom evolution which implies faster expansion. The phantom behavior is one of the mechanisms that can lead to the Hubble tension alleviation [84,85] (see also the discussion in [8]), and as we will see in the following, this is exactly what happens.

In Fig. 2, we present the normalised combination $H(z)/(1+z)^{3/2}$ as a function of the redshift for ΛCDM

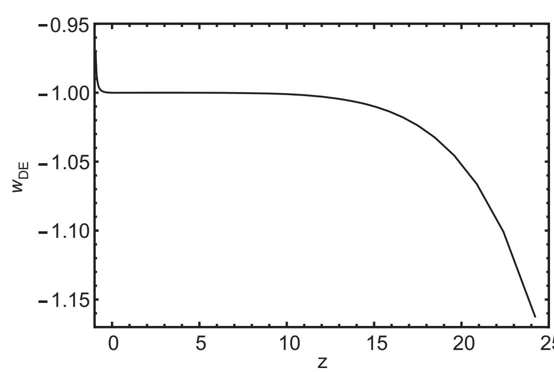

FIG. 1. The effective dark-energy equation-of-state parameter w_{DE} as a function of the redshift, for model I for $\zeta = -10$ in Planck units.

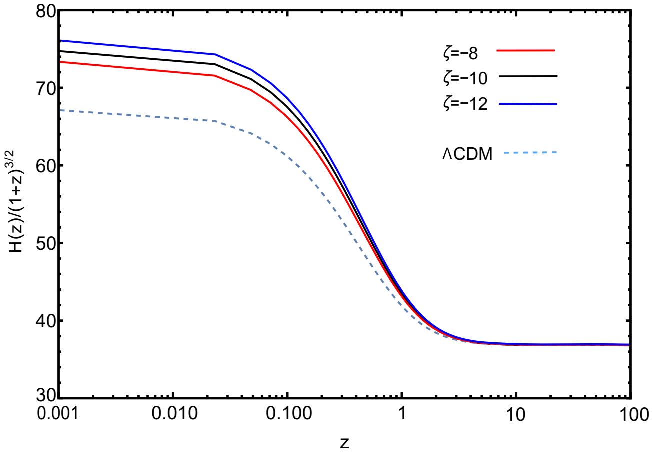

FIG. 2. The normalized combination $H(z)/(1+z)^{3/2}$ as a function of the redshift, for ΛCDM cosmology (blue dotted line), and for model I for $\zeta = -12$ (solid blue line), for $\zeta = -10$ (solid black line), and for $\zeta = -8$ (solid red line), in Planck units.

cosmology, and for model I, for different values of ζ. Here we used $\zeta = -8, -10, -12$ in Planck units. We find that the present value of H_0 depends on the model parameter ζ as expected. For $\zeta = -10$, the present value of the Hubble parameter is around $H_0 \approx 74$ km/s/Mpc, which is consistent with the direct measurement of the present Hubble parameter. Values of ζ higher or lower than this lead to higher or lower values for H_0, respectively, and positive ζ corresponds to H_0 values lower than the value of H_0 in ΛCDM scenario; thus, they are not relevant for our present analysis. Note that in natural units $\zeta \sim -10$ corresponds to a typical value $\zeta^{1/4} \sim -10^{-19}$ GeV^{-1}. Hence, such values are the ones that needed in order to bring H_0 from its ΛCDM value to the local-measurement value; in other words, the magnitude and the sign of the modified gravity modification is phenomenologically determined by the distance of $H_0 = 67.3$ km/s/Mpc and $H_0 \approx 74$ km/s/Mpc.

For completeness, in Fig. 3, we depict the evolution of the deceleration parameter $q \equiv -1 - \dot{H}/H^2$ as a function of the redhsift. As we see, the redshift at which the

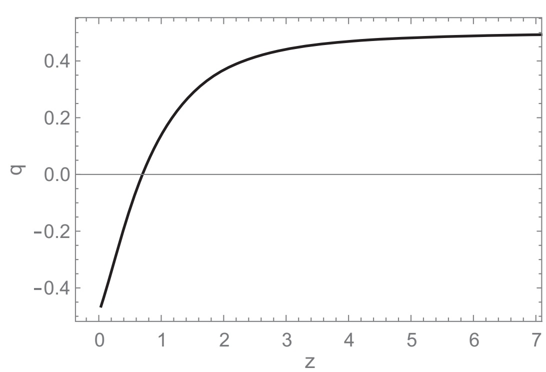

FIG. 3. The deceleration parameter q as a function of redshift z, for model I with $\zeta = -10$ in Planck units.

transition from deceleration to acceleration occurs is around $z_{\rm tr} = 0.68$, in agreement with current observations.

In summary, as we observe, there exist a range of the free model parameter ζ that is able to reproduce a Hubble function evolution that coincides with ΛCDM cosmology at high redshifts, but at late times, it alleviates the H_0 tension. The reason that this happens is the fact that the effective dark-energy equation-of-state parameter exhibits a phantom behavior (following the general requirements of [8,85]).

B. Model II

As a next, we consider the case where

$$\mathcal{K}(\phi, B) = \frac{1}{2}\phi \quad \text{and} \quad \mathcal{G}(\phi, B) = \xi B, \tag{27}$$

with ξ the corresponding coupling constant with dimensions $[M]^{-8}$. Thus, the Friedmann equations (9), (10) become

$$3H^2 - \rho_m - \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}(1 - 2e^{\sqrt{\frac{2}{3}}\chi})\phi + \xi\dot{\phi}^3(\sqrt{6}\dot{\chi} - 6H) = 0, \tag{28}$$

$$3H^2 + 2\dot{H} + p_m + \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}(1 - 2e^{\sqrt{\frac{2}{3}}\chi})\phi - \frac{1}{3}\xi\dot{\phi}^2(\sqrt{6}\dot{\phi}\dot{\chi} + 6\ddot{\phi}) = 0, \tag{29}$$

while the two scalar field equations (11) and (12) read as

$$\ddot{\chi} + 3H\dot{\chi} + \frac{1}{2\sqrt{6}}e^{-2\sqrt{\frac{2}{3}}\chi}(1 - e^{\sqrt{\frac{2}{3}}\chi})\phi - \sqrt{6}\xi\dot{\phi}^2(H\dot{\phi} + \ddot{\phi}) = 0, \tag{30}$$

$$\xi\dot{\phi}\{2(-6H + \sqrt{6}\dot{\chi})\ddot{\phi} + \dot{\phi}[-6\dot{H} + 3H(-6H + \sqrt{6}\dot{\chi}) + \sqrt{6}\ddot{\chi}]\} + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}(1 - 2e^{\sqrt{\frac{2}{3}}\chi}) = 0. \tag{31}$$

Therefore, in this case, the effective dark-energy energy density and pressure (15), (16) write as

$$\rho_{\rm DE} = \frac{1}{2}\dot{\chi}^2 - \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}(1 - 2e^{\sqrt{\frac{2}{3}}\chi})\phi - \xi\dot{\phi}^3(\sqrt{6}\dot{\chi} - 6H), \tag{32}$$

$$p_{\rm DE} = \frac{1}{2}\dot{\chi}^2 + \frac{1}{8}e^{-2\sqrt{\frac{2}{3}}\chi}(1 - 2e^{\sqrt{\frac{2}{3}}\chi})\phi - \frac{1}{3}\xi\dot{\phi}^2(\sqrt{6}\dot{\phi}\dot{\chi} + 6\ddot{\phi}). \tag{33}$$

Let us now proceed to the numerical investigation of the above equations. Similarly to the previous model I, we choose the initial conditions such that our scenario matches

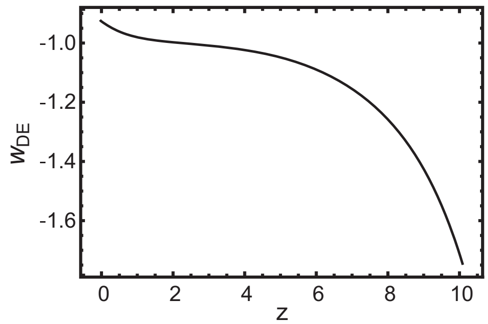

FIG. 4. The effective dark-energy equation-of-state parameter w_{DE} as a function of the redshift, for model II for $\xi = -10$ in Planck units.

ΛCDM cosmology for $z \approx 1100$. In Fig. 4, we depict the evolution of the dark-energy equation-of-state parameter with the redshift. As in the case of the previous subsection, here we also see that $w_{\mathrm{DE}} < -1$ for most redshifts, thereby depicting phantom evolution, thus serving as a mechanism for Hubble tension alleviation.

In Fig. 5, we present the normalized $H(z)/(1+z)^{3/2}$ as a function of the redshift for ΛCDM cosmology, and for model II, for different values of ξ, namely $\xi = -8, -10, -12$. As expected, we find that the present Hubble value H_0 depends on the model parameter ξ. Specifically, for $\xi = -10$, it is around $H_0 \approx 74$ km/s/Mpc, which is consistent with the directly measured value of the Hubble parameter. Values of ξ higher or lower than this give higher or lower values for H_0, respectively. Note that in natural units $\xi \sim -10$ corresponds to a typical value $\xi^{1/8} \sim -10^{-19}$ GeV^{-1}. Hence, similarly to model I above, such values are the ones that are phenomenologically needed in order to bring H_0 from $H_0 = 67.3$ km/s/Mpc to $H_0 \approx 74$ km/s/Mpc.

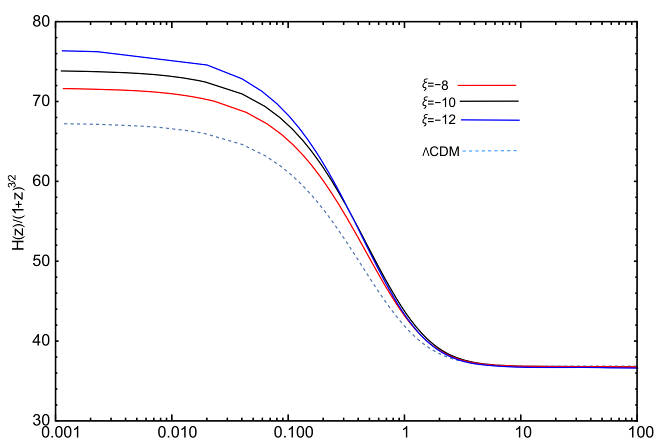

FIG. 5. The normalized combination $H(z)/(1+z)^{3/2}$ as a function of the redshift, for ΛCDM cosmology (blue dotted line), and for model II for $\xi = -12$ (solid blue line), for $\xi = -10$ (solid black line), and for $\xi = -8$ (solid red line), in Planck units.

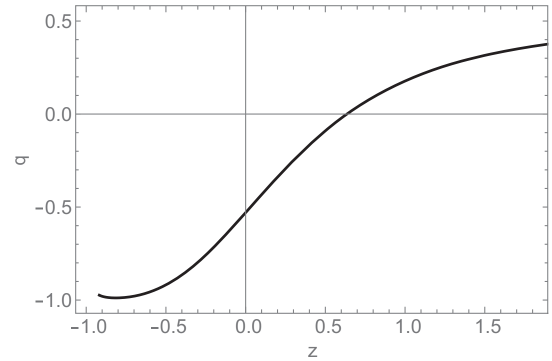

FIG. 6. The deceleration parameter q as a function of redshift z, for model II with $\xi = -10$ in Planck units.

In Fig. 6, we depict the evolution of the deceleration parameter q in terms of z. The transition redshift between deceleration and acceleration for this case is around $z_{\mathrm{tr}} = 0.65$, in agreement with current observations, too.

We close our analysis by confronting the two examined models with cosmic chronometer (CC) cosmological data. This dataset is based on the $H(z)$ measurements through the relative ages of passively evolving galaxies and the corresponding estimation of dz/dt [86]. In Fig. 7, we confront the predicted $H(z)$ evolution of our models, alongside the one of ΛCDM scenario, with the $H(z)$ cosmic chronometer data [87] at 1σ confidence level. As we deduce, the agreement is very good, and the theoretical $H(z)$ evolution lies within the direct measurements of the $H(z)$ from the CC data.

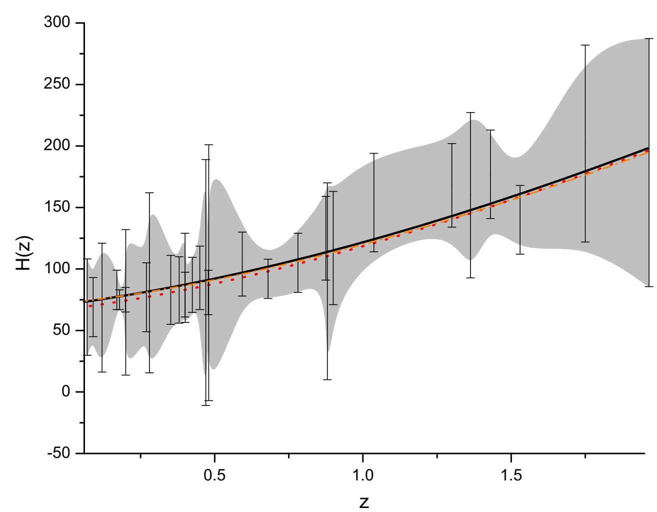

FIG. 7. The $H(z)$ in units of km/s/Mpc as a function of the redshift, for ΛCDM scenario (red dotted line), for model I with $\zeta = -10$ (orange dashed-dotted), and for model II with $\xi = -10$ (black solid line) in H_0 units, on top of the cosmic chronometers data points from [87] at 1σ confidence level. We have imposed $\Omega_{m_0} = 0.31$.

V. CONCLUSIONS

New gravitational scalar-tensor theories are novel modifications of gravity, consisting of a Lagrangian with the Ricci scalar and its first and second derivatives in a specific combination that makes the theory free of ghosts. Such constructions propagate $2+2$ degrees of freedom, thus forming a subclass of biscalar extensions of general relativity.

In the present work, we investigated the possibility of resolving the Hubble tension using these new gravitational scalar tensor theories. Considering a homogenoeus and isotropic background, we extracted the Friedmann equations, as well as the evolution equations of the new extra scalar degrees of freedom. We obtained an effective dark energy sector that consists of both extra scalar degrees of freedom.

We then studied the cosmological behavior of two specific models, imposing as initial conditions at high redshifts the coincidence of the behavior of the Hubble function with that predicted by ΛCDM cosmology. However, we showed that as time passes, the effect of biscalar modifications become important and thus, at low redshifts, the Hubble function acquires increased values in a controlled way. In particular, the present value of the Hubble parameter is sensitive to the choice of the model parameters.

In both models, we showed that at high and intermediate redshifts the Hubble function behaves identically to that of ΛCDM scenario; however, at low redshifts, it acquires increased values, resulting to $H_0 \approx 74$ km/s/Mpc for particular parameter choices. Hence, these new gravitational scalar tensor theories can alleviate the H_0 tension. The mechanism behind this behavior is the fact that the effective dark-energy equation-of-state parameter exhibits phantom behavior, which implies faster expansion, and it is one of the sufficient theoretical requirements that are capable of alleviating the H_0 tension [8,85] (although it is not a necessary requirement as we mention in the Introduction). Finally, we further confronted our models with cosmic chromometer data, and we found they are viable and in agreement with observations.

It would be interesting to investigate what is the situation of the other famous tension, namely the σ_8 one (there seems to be a disagreement between the amount of matter clustering, quantified by σ_8, predicted by ΛCDM cosmology and the local measurements of the matter distribution [8]) in the scenario at hand. In particular, a suggested solution for the H_0 tension does not guarantee an alleviation for the σ_8 one. There are models in which H_0 alleviation does impinge positively on the σ_8 tension, such as the running vacuum ones [73,88,89] or $f(T)$ gravity ones [84,90]; however, there are others in which it leads to a worsening of the latter. That is why it is necessary to perform a σ_8 analysis, too. Since such an analysis requires the investigation of perturbations and the evolution of matter overdensity δ, it is left for a separate project; however, the obtained phantom behavior is expected to lead to an increase in the friction term in the Jeans equation for δ, which is qualitatively expected to lead to a smaller σ_8.

In conclusion, in this first work on the subject, we deduced that the H_0 tension can be alleviated in the framework of new geometric gravitational theories. Definitely, the full verification of the above result requires a complete observational analysis, using data from supernovae type Ia (SNIa), baryonic acoustic oscillations (BAO), redshift space distortion (RSD), and cosmic microwave background (CMB) observations. Such a full and detailed observational confrontation is left for a future project.

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