Symplectic charges in the Yang-Mills theory of the normal conformal Cartan connection: Applications to gravity

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It is known that a source-free Yang-Mills theory with the normal conformal Cartan connection used as the gauge potential gives rise to equations of motion equivalent to the vanishing of the Bach tensor. We investigate the conformally invariant presymplectic potential current obtained from this theory and find that on the solutions to the Einstein field equations, it can be decomposed into a topological term derived from the Euler density and a part proportional to the potential of the standard Einstein-Hilbert Lagrangian. The pullback of our potential to the asymptotic boundary of an asymptotically de Sitter spacetime turns out to coincide with the current obtained from the holographically renormalized gravitational action. This provides an alternative derivation of a symplectic structure on scri without resorting to holographic techniques. We also calculate our current at the null infinity of an asymptotically flat spacetime and in particular show that it vanishes for variations induced by the Bondi–Metzner–Sachs group of asymptotic symmetries. In addition, we calculate the Noether currents and charges corresponding to gauge transformations and diffeomorphisms.

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I. INTRODUCTION

The interest in the conformal completions of Einstein spacetimes [1–8] stems from their applications in the analysis of the asymptotics of solutions, as well as the holographic theory. It leads to considerations that use both Einstein's equations and conformal geometry simultaneously. This is not straightforward since Einstein's equations are not conformally invariant. In fact, they become singular at the conformal boundary, which makes the analysis of the equations complicated. Moreover, natural structures such as the symplectic potential and Noether charges cannot be defined at the boundary in a straightforward way. For example, one needs to perform a renormalization procedure to obtain a symplectic potential at the conformal boundary [3,9]. The regularization is defined in a specific gauge, the so-called Fefferman-Starobinski coordinate system [3,5]. It is interesting to try to find another, more natural way of obtaining the pullback of the potential.

In four-dimensional spacetimes, a conformally invariant condition for satisfying vacuum Einstein's equations with a (possibly vanishing) cosmological constant Λ is the vanishing of the Bach tensor [10]. This way, the space of Einstein metrics is naturally immersed in the space of Bach flat metrics. Bach's equations were used to study the structure and stability of the conformal boundary in Einstein spacetimes [11–13]. They share many properties with Einstein's field equations. In particular, they form a well-posed evolution system [13,14] and are obtained from a Lagrangian [15]. The success of the application of these equations to general relativity leads to a natural question of whether the symplectic potential or Noether currents of this auxiliary theory can be of some use in the Einstein theory.

In this paper, we propose a possible way of exploiting the relation between the spaces of Einstein and Bach flat metrics by pulling back into the former the geometrical conformally invariant structure with which the latter is equipped. We construct a conformally invariant presymplectic potential current for the Bach theory and show that its pullback to the space of Einstein metric tensors with a nonvanishing cosmological constant differs from the presymplectic potential current for the Einstein theory by trivial terms. The advantage of this approach is that due to the conformal invariance, our current is automatically finite at the conformal boundary (scri) of asymptotically (anti–)de Sitter spacetimes. This way, one can obtain currents that are well defined at all spacetime points simultaneously, both in the bulk and on scri. We show, using the Fefferman-Graham coordinates, that the pullback of this current to scri of an asymptotically de Sitter spacetime coincides with the symplectic potential current obtained by the method of holographic renormalization [3,5]. For the sake of completeness, we also consider the case of an asymptotically flat spacetime and calculate the new symplectic potential on its null boundary. However, since in the case of a vanishing cosmological constant the potential only consists of a trivial, topological term, the

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result lacks a clear physical meaning and in fact vanishes for variations generated by the diffeomorphisms belonging to the Bondi–Metzner–Sachs (BMS) group of asymptotic symmetries of an asymptotically flat spacetime.

To achieve this, we will use two different Lagrangians for the Bach theory, both of which have interesting but different properties. They differ by a topological term (the Euler-Gauss-Bonnet term [16]), and thus their symplectic currents are related. The first Lagrangian is conformally invariant, which we ensure by using the normal conformal Cartan (NCC) connection [17–22] (closely related to the local twistor connection [21–23] as well as to the tractor calculus [20,24,25]) as the gauge potential in the (also conformally invariant) Yang-Mills Lagrangian. The corresponding symplectic potential current defined in terms of the Cartan connection is conformally invariant. The second Lagrangian is not conformally invariant; however, its symplectic current restricted to the space of solutions of Einstein field equations with a cosmological constant is proportional to the symplectic potential of general relativity. The pullback of the Yang-Mills symplectic potential current to the space of Einstein metric tensors with a nonvanishing cosmological constant becomes a linear combination of symplectic potential currents of the gravitational Lagrangian and the Euler Lagrangian, respectively.

A. Notation and conventions

We consider a four-dimensional manifold equipped with a (pseudo-)Riemannian metric. In Secs. II and III the metric can have any nondegenerate signature, while in Sec. IV we restrict ourselves to spacetimes with the Lorentzian signature $\mathrm{diag}(-,+,+,+)$.

To express fields on the manifold we will use both orthonormal tetrads, as well as holonomic tetrads corresponding to a coordinate system. In the first case, we shall use the indices $a, b, c, \ldots = 0, 1, 2, 3$ to enumerate the elements of the tangent and cotangent tetrads, components of tensors with respect to those tetrads, and tetrad connection one-forms. On the other hand, to express tensors and the metric connection in terms of a coordinate system we will use greek letters $\alpha, \beta, \gamma, \ldots = 0, 1, 2, 3$. The normal conformal Cartan connection and its curvature have values in a 6 by 6 matrix algebra—we will use the uppercase indices $I, J, K, \ldots = 0, \ldots, 5$ to identify their components. In Sec. IV we also use indices $i, j, k, \ldots = 1, 2, 3$ and $A, B, C, \ldots = 2, 3$ to identify subsets of the Fefferman-Graham and Bondi-Sachs coordinates, in a way which is explained in more details in the corresponding subsections.

II. NORMAL CONFORMAL CARTAN CONNECTION

A. Definition

We will now provide a working definition of the NCC connection. A more elegant geometric definition of this structure can be found in [18], and the relation between the two formulations is explained in [19]. It is also worth mentioning that Penrose's notion of the local twistor connection [21–23] is simply the spinorial version of the NCC connection.

We start by considering a four-dimensional manifold M endowed with a spacetime metric tensor

$$g = \eta_{ab}\theta^a \otimes \theta^b, \tag{1}$$

where η_{ab} is a constant, nondegenerate, symmetric matrix of signature (p, q) and $(\theta^0, \ldots, \theta^3)$ is a locally defined cotangent frame dual to a tangent frame $(e_0, \ldots, e_3)$. The matrix η_{ab} and its inverse η^{ab} will be used to raise and lower the tetrad indices $a, b, c, \ldots$. The choice of a coframe defines a volume form

$$\begin{aligned} \mathrm{Vol} &\coloneqq \frac{1}{4!}\sqrt{|\det\eta|}\varepsilon_{abcd}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d \\ &= \sqrt{|\det\eta|}\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3, \end{aligned} \tag{2}$$

whose components we will denote by $\epsilon_{abcd} = \sqrt{|\det\eta|}\varepsilon_{abcd}$.

The NCC connection corresponding to a given choice of tetrad θ^a is defined as a matrix of one-forms,

$$A = \begin{bmatrix} 0 & \theta_b & 0 \\ P^a & \Gamma^a{}_b & \theta^a \\ 0 & P_b & 0 \end{bmatrix}, \tag{3}$$

where $\Gamma^a{}_b$ are the tetrad connection one-forms defined by the properties

$$\mathrm{d}\theta^a + \Gamma^a{}_b \wedge \theta^b = 0, \qquad \Gamma_{ab} = -\Gamma_{ba}, \tag{4}$$

while the one-forms $P_a = P_{ab}\theta^b$ are defined by minus the Schouten tensor, that is,

$$P_a \coloneqq \left(\frac{1}{12}R\eta_{ab} - \frac{1}{2}R_{ab}\right)\theta^b, \tag{5}$$

where R_{ab} and R are the Ricci tensor and Ricci scalar, respectively. In this notation, the Riemann tensor of g can be represented as a two-form

$$\mathcal{R}^a{}_b \coloneqq \frac{1}{2}R^a{}_{bcd}\theta^c \wedge \theta^d = \mathrm{d}\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b. \tag{6}$$

B. Conformal rescalings and Lorenz transformations

The special property of the NCC connection is the way it transforms upon rescalings [19]

$$g' = f^2 g, \qquad \theta'^a = f\theta^a, \qquad f \in C^\infty(M), \tag{7}$$

namely

$$A' = h^{-1}Ah + h^{-1}\mathrm{d}h, \tag{8}$$

with

$$h = \begin{bmatrix} \frac{1}{f} & 0 & 0 \\ \frac{f_{,}{}^{a}}{f^2} & \delta^a{}_b & 0 \\ \frac{f_{,c}f_{,}{}^{c}}{2f^3} & \frac{f_{,b}}{f} & f \end{bmatrix}, \tag{9}$$

where

$$f_{,a}\theta^a \coloneqq \mathrm{d}f. \tag{10}$$

Of course, for the local (pseudo)rotations, that is

$$g' = g, \qquad \theta'^a = \Lambda^a{}_b\theta^b, \tag{11}$$

where $\eta_{ab}\Lambda^a{}_c\Lambda^b{}_d = \eta_{cd}$, the transformation law (8) still applies, with

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \Lambda^a{}_b & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

Both the matrix (9) corresponding to the conformal rescaling of the tetrad and (12) associated with a (pseudo) rotation represent elements of the group $SO(p+1, q+1)$ identified with the group $SO(Q)$ of 6 by 6 matrices h which preserve the form Q,

$$h^I{}_K h^J{}_L Q_{IJ} = Q_{KL}, \tag{13}$$

with

$$Q = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{14}$$

C. Curvature

The curvature of the NCC connection is represented by a matrix obtained from A by the standard formula

$$F = \mathrm{d}A + A \wedge A, \tag{15}$$

and a calculation yields

$$F = \begin{bmatrix} 0 & 0 & 0 \\ DP^a & C^a{}_b & 0 \\ 0 & DP_b & 0 \end{bmatrix}, \tag{16}$$

where

$$DP^a = \mathrm{d}P^a + \Gamma^a{}_b \wedge P^b, \qquad C^a{}_b = \frac{1}{2}C^a{}_{bcd}\theta^c \wedge \theta^d, \tag{17}$$

and $C^a{}_{bcd}$ stands for the Weyl tensor of g. The curvature satisfies the Bianchi identity

$$D_A F \coloneqq \mathrm{d}F + A \wedge F - F \wedge A = 0 \tag{18}$$

and the transformation law

$$F' = h^{-1}Fh \tag{19}$$

under a transformation of A given by (8) for any $h \in SO(Q)$. As a result, complex identities of Riemannian geometry satisfied by the Weyl tensor can be written in an elegant and graceful way.

D. Relation with the Bach tensor

Another conformally invariant operation in four-dimensional geometry is the Hodge dual of a differential two-form. We can apply it to the curvature F and find that [19,21]

$$D_A\star F \coloneqq \mathrm{d}\star F + A \wedge \star F - \star F \wedge A = \begin{bmatrix} 0 & 0 & 0 \\ B^{ac}\star\theta_c & 0 & 0 \\ 0 & B_b{}^c\star\theta_c & 0 \end{bmatrix}, \tag{20}$$

where B_{ab} is the Bach tensor,

$$B_{ab} = 2\nabla^c\nabla_{[b}P_{c]a} - 2P^{cd}C_{cadb}. \tag{21}$$

What is particularly important from the point of view of applications to Einstein's theory of gravity is that

$$R_{ab} = \Lambda\eta_{ab} \Rightarrow B_{ab} = 0 \tag{22}$$

and that the second equality above is conformally invariant. Hence, the Bach tensor is an obstacle to the metric tensor being conformally Einstein. Its vanishing is not a sufficient condition, though—there are known examples of spacetimes with $B_{ab} = 0$ which are not conformally Einstein [19,26,27], found among the homogeneous Fefferman metric tensors. As we can now see, in terms of the NCC connection, this obstacle becomes $D\star F$.

III. CARTAN-YANG-MILLS THEORY

A. Cartan-Yang-Mills Lagrangian

Inspired by (20) and (22) we define on the space of η-orthonormal coframes θ^a (1) a Lagrangian by inserting the NCC connection A into the standard Yang-Mills Lagrangian,

$$L_{\mathrm{CYM}}(\theta) = \frac{1}{2} F^I{}_J \wedge \star F^J{}_I, \tag{23}$$

and name it the Cartan-Yang-Mills Lagrangian. The coframes are defined locally on M; however, the Lagrangian is independent of the transformations (8)—hence, it is uniquely defined on the entire M. Moreover,

$$L_{\mathrm{CYM}}(\theta) = L_{\mathrm{CYM}}(f\theta), \qquad f \in C^3(M); \tag{24}$$

hence, the Lagrangian is manifestly conformally invariant. As a matter of fact, it follows from (16) that

$$\begin{aligned} L_{\mathrm{CYM}}(\theta) &= \frac{1}{2} F^I{}_J \wedge \star F^J{}_I = \frac{1}{2} C^a{}_b \wedge \star C^b{}_a \\ &= \frac{1}{4} C_{abcd} C^{abcd} \mathrm{Vol}, \end{aligned} \tag{25}$$

which makes it clear that despite L_{CYM} being defined as a function of θ^a, it depends only on $g = \eta_{ab}\theta^a\theta^b$. The right-hand side (RHS) of (25) is encountered in literature. However, we want to take advantage of the particular properties of the NCC connection and the Yang-Mills Lagrangian.

B. Variations and the field equations

Varying the Lagrangian with respect to θ^a and "integrating by parts," we obtain

$$\begin{aligned} \delta L_{\mathrm{CYM}}(\theta) &= \delta A^I{}_J \wedge D_A \star F^J{}_I + \frac{1}{2} F^I{}_J \wedge (\delta\star) F^J{}_I \\ &\quad + \mathrm{d}(\delta A^I{}_J \wedge \star F^J{}_I). \end{aligned} \tag{26}$$

The term involving $\delta\star$ vanishes due to the fact that the left and right duals of the Weyl tensor coincide [28]:

$$\star C^a{}_b = \frac{1}{2} \epsilon^a{}_{bc}{}^d C^c{}_d, \tag{27}$$

where $\epsilon_{abcd} = \sqrt{|\det\eta|}\varepsilon_{abcd}$; hence, it is constant, independent of θ^a. Breaking down the first term, we obtain [19]

$$\delta L_{\mathrm{CYM}}(\theta) = 2\delta\theta^a \wedge B_{ab} \star \theta^b + \mathrm{d}(\delta A^I{}_J \wedge \star F^J{}_I). \tag{28}$$

From the first term, we obtain the equations of the theory,

$$B_{ab} = 0, \tag{29}$$

equivalent to

$$D_A \star F = 0. \tag{30}$$

C. Symplectic current potential

From the second, exact term in the RHS of (28) we obtain the symplectic current potential

$$\Theta_{\mathrm{CYM}}(\theta; \delta\theta) = \delta A^I{}_J \wedge \star F^J{}_I. \tag{31}$$

It is conformally invariant due to (8) and (19),

$$\begin{aligned} &(\delta A^I{}_J \wedge \star F^J{}_I)(f\theta; f\delta\theta) \\ &= ((h^{-1}\delta A h)^I{}_J \wedge \star (h^{-1} F h)^J{}_I)(\theta; \delta\theta) \\ &= (\delta A^I{}_J \wedge \star F^J{}_I)(\theta; \delta\theta). \end{aligned} \tag{32}$$

A short calculation, using the explicit form of the NCC connection and its curvature associated with a given orthonormal tetrad (3), (16), gives the detailed form

$$\Theta_{\mathrm{CYM}}(\theta; \delta\theta) = 2\delta\theta^a \wedge \star D P_a + \delta\Gamma^a{}_b \wedge \star C^b{}_a. \tag{33}$$

D. Θ_{CYM} at Einstein metrics

When θ^a defines an Einstein metric tensor, that is when

$$R_{ab} = \Lambda \eta_{ab}, \tag{34}$$

our very symplectic current potential (33) takes a simpler form,

$$\Theta_{\mathrm{CYM}}(\theta; \delta\theta) = \delta\Gamma^a{}_b \wedge \star C^b{}_a, \tag{35}$$

where the first term in (33) vanishes due to the fact that (34) implies $P_a = -\frac{\Lambda}{6}\theta_a$ and $D\theta_a = 0$.

Since the Einstein metric tensors satisfy the equations of the Cartan-Yang-Mills theory (as their Bach tensor vanishes), all currents obtained from the three-form Θ_{CYM} also apply to them. Therefore, a natural question arises: what is the relation between Θ_{CYM} and the symplectic current potential Θ_{EH} of the Einstein-Hilbert action extended to the space of the η-orthonormal coframes?

To answer this question, we will decompose L_{CYM} into two parts,

$$L_{\mathrm{CYM}} = \frac{1}{4}\mathcal{E} + L_1, \tag{36}$$

where the Euler term [16]

$$\mathcal{E}(\theta) \coloneqq \epsilon^{abcd} \mathcal{R}_{ab} \wedge \mathcal{R}_{cd} \tag{37}$$

is a topological Lagrangian whose variation only yields the boundary term:

$$
\begin{aligned}
\delta\mathcal{E}(\theta) &= 2\epsilon^{abcd}\delta\mathcal{R}_{ab}\wedge\mathcal{R}_{cd} \\
&= 2\epsilon^{abcd}D(\delta\Gamma_{ab})\wedge\mathcal{R}_{cd} \\
&= 2\epsilon^{abcd}\mathrm{d}(\delta\Gamma_{ab}\wedge\mathcal{R}_{cd}) + 2\epsilon^{abcd}\delta\Gamma_{ab}\wedge D\mathcal{R}_{cd} \\
&= \mathrm{d}(2\epsilon^{abcd}\delta\Gamma_{ab}\wedge\mathcal{R}_{cd}),
\end{aligned} \tag{38}
$$

where we used the property $D\epsilon^{abcd} = 0$ which follows from metric compatibility $\Gamma_{ab} = -\Gamma_{ba}$ and the Bianchi identity $D\mathcal{R}^a{}_b = 0$. Let us also define

$$
\Theta_{\mathcal{E}}(\theta;\delta\theta) \coloneqq 2\epsilon^{abcd}\delta\Gamma_{ab}\wedge\mathcal{R}_{cd}, \tag{39}
$$

so that $\delta\mathcal{E} = \mathrm{d}\Theta_{\mathcal{E}}$. The remaining part of the Cartan-Yang-Mills Lagrangian is

$$
\begin{aligned}
L_1(\theta) &\coloneqq L_{\mathrm{CYM}} - \frac{1}{4}\mathcal{E} \\
&= \frac{1}{4}\epsilon^{abcd}(C_{ab}\wedge C_{cd} - \mathcal{R}_{ab}\wedge\mathcal{R}_{cd}) \\
&= \epsilon^{abcd}\theta_a\wedge P_b\wedge C_{cd} - \epsilon^{abcd}\theta_a\wedge P_b\wedge\theta_c\wedge P_d \\
&= -\epsilon^{abcd}P_a\wedge P_b\wedge\theta_c\wedge\theta_d \\
&= -4P^{[a}{}_a P^{b]}{}_b \mathrm{Vol}.
\end{aligned} \tag{40}
$$

That decomposition is not unique; however, this is the one that will work for us and provide a suitable decomposition of our symplectic current potential.

Because of (38), L_1 gives the same equations of motion as L_{CYM}, which are equivalent to the vanishing of the Bach tensor—see (28). The symplectic current potential Θ_{CYM} (31), on the other hand, splits into two parts:

$$
\Theta_{\mathrm{CYM}} = \frac{1}{4}\Theta_{\mathcal{E}} + \Theta_1, \tag{41}
$$

where

$$
\begin{aligned}
\Theta_1(\theta;\delta\theta) &\coloneqq \Theta_{\mathrm{CYM}}(\theta;\delta\theta) - \frac{1}{4}\Theta_{\mathcal{E}}(\theta;\delta\theta) \\
&= 2\delta\theta^a\wedge\star DP_a + \frac{1}{2}\epsilon^{abcd}\delta\Gamma_{ab}\wedge(C_{cd} - \mathcal{R}_{cd}) \\
&= 2\delta\theta^a\wedge\star DP_a + \epsilon^{abcd}\delta\Gamma_{ab}\wedge\theta_c\wedge P_d.
\end{aligned} \tag{42}
$$

From the decomposition (36) and variations (28), (38) follows

$$
\delta L_1(\theta) = 2\delta\theta^a\wedge B_{ab}\star\theta^b + \mathrm{d}\Theta_1. \tag{43}
$$

Hence, Θ_1 is a possible choice of the presymplectic potential current associated with the Lagrangian L_1.

Let us compare Θ_1 with a presymplectic potential current Θ_{EH} obtained from the Einstein-Hilbert Lagrangian $\star\frac{1}{16\pi G}(R - 2\Lambda)$, which can also be written as

$$
L_{\mathrm{EH}} = \frac{1}{16\pi G}\left(\frac{1}{2}\epsilon^{abcd}\theta_a\wedge\theta_b\wedge\mathcal{R}_{cd} - 2\Lambda\mathrm{Vol}\right). \tag{44}
$$

We have

$$
\begin{aligned}
16\pi G\delta L_{\mathrm{EH}}(\theta) &= \frac{1}{2}\epsilon^{abcd}(2\delta\theta_a\wedge\theta_b\wedge\mathcal{R}_{cd} + \theta_a\wedge\theta_b\wedge\delta\mathcal{R}_{cd}) - 2\Lambda\delta\theta_a\wedge\star\theta^a \\
&= \frac{1}{2}\epsilon^{abcd}(2\delta\theta_a\wedge\theta_b\wedge\mathcal{R}_{cd} + \theta_a\wedge\theta_b\wedge D\delta\Gamma_{cd}) - 2\Lambda\delta\theta_a\wedge\star\theta^a \\
&= \delta\theta_a\wedge(\epsilon^{abcd}\theta_b\wedge\mathcal{R}_{cd} - 2\Lambda\star\theta^a) + \mathrm{d}\left(\frac{1}{2}\epsilon^{abcd}\theta_a\wedge\theta_b\wedge\delta\Gamma_{cd}\right),
\end{aligned} \tag{45}
$$

where in the last equality we used the property $D\theta^a = 0$ which follows from $\Gamma^a{}_b$ being torsion-free. One can check that [29]

$$
\frac{1}{2}\epsilon^{abcd}\theta_b\wedge\mathcal{R}_{cd} = (R\eta^{ab} - 2R^{ab})\star\theta_b; \tag{46}
$$

therefore, the first term in (45) yields the vacuum Einstein equations, while the second gives the presymplectic potential current [30]

$$
\Theta_{\mathrm{EH}}(\theta;\delta\theta) = \frac{1}{32\pi G}\epsilon^{abcd}\theta_a\wedge\theta_b\wedge\delta\Gamma_{cd}. \tag{47}
$$

If $g = \eta_{ab}\theta^a\otimes\theta^b$ satisfies the vacuum Einstein equations, we have $P_a = P_{ab}\theta^b = -\frac{\Lambda}{6}\theta_a$. Therefore, $DP^a = 0$ and Θ_1 from (42) reduces to

$$
\Theta_1(\theta;\delta\theta)|_{\mathrm{EH}} = -\frac{\Lambda}{6}\epsilon^{abcd}\theta_a\wedge\theta_b\wedge\delta\Gamma_{cd}, \tag{48}
$$

and hence, for those tetrads, $\Theta_1 = -\frac{16\pi G\Lambda}{3}\Theta_{\mathrm{EH}}$. Therefore, on solutions of the vacuum Einstein's equations,

$$
\Theta_{\mathrm{CYM}} = \frac{1}{4}\Theta_{\mathcal{E}} - \frac{16\pi G\Lambda}{3}\Theta_{\mathrm{EH}}. \tag{49}
$$

In this way, on the Einstein spacetimes, $\Theta_{\mathcal{E}}$ can be thought of as a "correction" to the Einstein-Hilbert presymplectic potential which makes it conformally invariant.

E. Noether currents of L_{CYM} the Cartan-Yang-Mills Lagrangian

Let $L(\Phi)$ be a Lagrangian form describing a general theory of fields $(\Phi_i)_{i\in I}$. Its variation with respect to the fields gives rise to the equations of motion and the presymplectic potential current

$$\delta L(\Phi) = E(\Phi)_i \delta\phi^i + \mathrm{d}\Theta(\Phi;\delta\Phi). \tag{50}$$

A variation δ_{S} is a symmetry of the theory if the corresponding variation of the Lagrangian is an exact form

$$\delta_{\mathrm{S}} L(\Phi) = \mathrm{d}Z_{\mathrm{S}}(\Phi). \tag{51}$$

For each such symmetry, we can define an associated Noether current [31,32]

$$J_{\mathrm{S}}(\Phi) = \Theta(\Phi;\delta_{\mathrm{S}}\Phi) - Z_{\mathrm{S}}(\Phi). \tag{52}$$

If Φ_0 satisfies the equations of motion, $E(\Phi_0)=0$, from (50) and (51) follow that $\mathrm{d}J_{\mathrm{S}}(\Phi_0)=0$ for any symmetry δ_{S} of L. The Noether current is determined up to the addition of an exact three-form [33] and depends on the particular choice of the presymplectic potential and Z. If we consider a gauge transformation defined by a spacetime-dependent parameter λ, then by results of [33], as explained in [31],

$$J_\lambda(\theta) = \mathrm{d}Q_\lambda \tag{53}$$

for some two-form Q_λ when the background satisfies the variational equations. The form Q_λ, called the Noether charge, is not uniquely determined by J_λ, since one can freely add to it any closed two-form.

Let us now turn to our Lagrangian L_{CYM}. First, we will calculate the Noether current associated with the pseudorotations and conformal rescalings of the tetrad θ^a. Under such a transformation we have $\delta L = 0$; therefore, we only need to consider the presymplectic potential part of (52). As explained in Sec. II B, the effects of both pseudorotations and conformal rescalings on the Cartan connection $A^I{}_J$ can be encoded in a subgroup of the group $SO(p+1,q+1)$ of matrices preserving the quadratic form (14) according to the transformation law (8). Hence, the variation of the connection can be expressed by the element of the algebra $\mathfrak{so}(p+1,q+1)$ generating the particular transformation. Let us denote this generator by $\gamma^I{}_J$. Then

$$\begin{aligned}\delta_\gamma A^I{}_J &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\exp(-\gamma t)^I{}_K A^K{}_L \exp(\gamma t)^L{}_J \\ &\quad + \exp(-\gamma t)^I{}_K \mathrm{d}\exp(\gamma t)^K{}_J) \\ &= A^I{}_K \gamma^K{}_J - \gamma^I{}_K A^K{}_J + \mathrm{d}\gamma^I{}_J \\ &= D_A \gamma^I{}_J.\end{aligned} \tag{54}$$

Thus

$$\begin{aligned}J_\gamma(\theta) &= \Theta_{\mathrm{CYM}}(\theta;\delta_\gamma\theta) \\ &= D_A\gamma^I{}_J \wedge \star F^J{}_I \\ &= \mathrm{d}(\gamma^I{}_J \wedge \star F^J{}_I) + \gamma^I{}_J \wedge D_A \star F^J{}_I.\end{aligned} \tag{55}$$

In the case when γ generates a conformal rescaling $\theta^a \mapsto \exp(\alpha t)\theta^a$, which is when the matrix $\exp(\gamma t)$ is of the form (9), one can check that this current vanishes identically (including off-shell):

$$J_\alpha(\theta) = 0. \tag{56}$$

This follows from (16), (20), and the fact that such matrices γ have the form

$$\gamma = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & * & * \end{bmatrix}. \tag{57}$$

On the other hand, in the case when γ generates a matrix representing a pseudorotation of the form (12), we obtain

$$J_\omega(\theta) = \mathrm{d}(\omega^a{}_b \wedge \star C^b{}_a), \tag{58}$$

where $\omega \in \mathfrak{so}(p,q)$ is the generator of the pseudorotation. Here again we use (16), (20), as well as the form of the matrix γ:

$$\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \omega^a{}_b & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{59}$$

While not zero, this current is conserved for all tetrads, not only those satisfying the equations of motion $D_A \star F = 0$.

Next, we will calculate the Noether current associated with a diffeomorphism generated by a vector field ξ. Since our theory is diffeomorphically invariant, the variation of L_{CYM} induced by a variation $\delta_\xi\theta = \mathcal{L}_\xi\theta$ is simply the Lie derivative of L_{CYM}:

$$\begin{aligned}\delta_\xi L_{\mathrm{CYM}}(\theta) &= \mathcal{L}_\xi L_{\mathrm{CYM}}(\theta) = \xi \lrcorner L_{\mathrm{CYM}}(\theta) \\ &= \xi \lrcorner \left(\frac{1}{2} F^I{}_J \wedge \star F^J{}_I\right),\end{aligned} \tag{60}$$

where in the second equality we used the fact that the Lagrangian is a top form. Using the expression for the presymplectic potential current Θ_{CYM} in terms of the normal conformal Cartan connection and its curvature (23), (31), we obtain from (52)

$$J_\xi(\theta) = \mathcal{L}_\xi A^I{}_J \wedge \star F^J{}_I - \xi \lrcorner \left(\frac{1}{2} F^I{}_J \wedge \star F^J{}_I\right). \tag{61}$$

Note that this current is conformally invariant. Next, we use the fact that the left and right duals of the Weyl tensor coincide [28] [see also (27)] to show that

$$\begin{aligned}
\xi \lrcorner \left(\frac{1}{2} F^I{}_J \wedge \star F^J{}_I\right) &= \xi \lrcorner \left(\frac{1}{2} C^a{}_b \wedge \star C^b{}_a\right) \\
&= \xi \lrcorner \left(\frac{1}{4} \epsilon^b{}_{ac}{}^d C^a{}_b \wedge C^c{}_d\right) \\
&= \frac{1}{2} \epsilon^b{}_{ac}{}^d (\xi \lrcorner C^a{}_b) \wedge C^c{}_d \\
&= (\xi \lrcorner F^I{}_J) \wedge \star F^J{}_I
\end{aligned} \tag{62}$$

and hence,

$$\begin{aligned}
J_\xi(\theta) &= (\mathcal{L}_\xi A^I{}_J - \xi \lrcorner F^I{}_J) \wedge \star F^J{}_I \\
&= (\mathrm{d}(\xi \lrcorner A^I{}_J) + \xi \lrcorner \mathrm{d}A^I{}_J - \xi \lrcorner (\mathrm{d}A^I{}_J + A^I{}_K \wedge A^K{}_J)) \wedge \star F^J{}_I \\
&= (\mathrm{d}(\xi \lrcorner A^I{}_J) - A^K{}_J(\xi \lrcorner A^I{}_K) + A^I{}_K(\xi \lrcorner A^K{}_J)) \wedge \star F^J{}_I \\
&= D_A(\xi \lrcorner A^I{}_J) \wedge \star F^J{}_I.
\end{aligned} \tag{63}$$

Integrating by parts we obtain the decomposition

$$\begin{aligned}
J_\xi(\theta) &= \mathrm{d}((\xi \lrcorner A^I{}_J) \star F^J{}_I) - (\xi \lrcorner A^I{}_J) D_A \star F^J{}_I \\
&=: \mathrm{d}Q_\xi + \xi^a C_a,
\end{aligned} \tag{64}$$

where $C_a = -A^I{}_{Ja} D \star F^J{}_I$ are constraints that vanish whenever the equations of motion $D\star F = 0$ hold and

$$Q_\xi(\theta) = (\xi \lrcorner A^I{}_J) \star F^J{}_I \tag{65}$$

is a possible choice for the Noether charge associated with ξ. Using it, we can also calculate the associated Iyer-Wald charge Hamiltonian charge [7,31,32]

$$\begin{aligned}
\not{\delta} H_\xi(\theta, \delta\theta) &= \delta Q_\xi(\theta) - \xi \lrcorner \Theta_{\mathrm{CYM}}(\theta; \delta\theta) \\
&= (\xi \lrcorner A^I{}_J) \delta(\star F^J{}_I) + \delta A^I{}_J \wedge (\xi \lrcorner \star F^J{}_I).
\end{aligned} \tag{66}$$

Let us examine the behavior of this current under pseudorotations and conformal rescalings of the tetrad. Let $\theta^a \mapsto T^a{}_b \theta^b$ be such a transformation, that is, $T^a{}_b \theta^b = f\theta^a$ for $f \in C^\infty(M)$ or $T^a{}_b \in SO(p,q)$. Then we have

$$\begin{aligned}
J_\xi(T^a{}_b\theta^b) &= \Theta_{\mathrm{CYM}}(T^a{}_b\theta^b; \mathcal{L}_\xi(T^a{}_b\theta^b)) - \mathcal{L}_\xi L_{\mathrm{CYM}}(T^a{}_b\theta^b) \\
&= \Theta_{\mathrm{CYM}}(T^a{}_b\theta^b; T^a{}_b \mathcal{L}_\xi \theta^b) + \Theta_{\mathrm{CYM}}(T^a{}_b\theta^b; \xi(T^a{}_b)\theta^b) - \mathcal{L}_\xi L_{\mathrm{CYM}}(T^a{}_b\theta^b) \\
&= \Theta_{\mathrm{CYM}}(\theta^a; \mathcal{L}_\xi\theta^a) + \Theta_{\mathrm{CYM}}(\theta^a; (T^{-1})^a{}_c \xi(T^c{}_b)\theta^b) - \mathcal{L}_\xi L_{\mathrm{CYM}}(\theta^a) \\
&= J_\xi(\theta^a) + J_{T^{-1}\xi(T)}(\theta^a),
\end{aligned} \tag{67}$$

where $J_{T^{-1}\xi(T)}$ is the current J_γ from (55), where γ is the element of $\mathfrak{so}(p+1, q+1)$ corresponding to the generator of pseudorotations/rescalings $T^{-1}\xi(T)$. This, together with (56) and (58) means that $J_\xi(\theta)$ is invariant under conformal rescaling of the tetrad, while under a pseudorotation it changes by the exact three-form (58).

One could also try to obtain an invariant current by using a covariant Lie derivative [30]

$$\mathcal{L}^\Gamma_\xi \theta^a := \mathcal{L}_\xi \theta^a + (\xi \lrcorner \Gamma^a{}_b)\theta^b. \tag{68}$$

The geometric meaning of this operation is that we lift the vector field ξ to the frame bundle of M horizontally with respect to $\Gamma^a{}_b$, consider the pullback of the differential forms θ^a to this frame, and calculate the Lie derivative on the bundle instead of on the base space M. The variation of the tetrad connection associated with $\delta\theta^a = \mathcal{L}^\Gamma_\xi \theta^a$ is

$$\begin{aligned}
\mathcal{L}^\Gamma_\xi \Gamma^a{}_b &= \mathcal{L}_\xi \Gamma^a{}_b + (\xi \lrcorner \Gamma^a{}_c)\Gamma^c{}_b - (\xi \lrcorner \Gamma^c{}_b)\Gamma^a{}_c \\
&= \xi \lrcorner \left(\frac{1}{2} R^a{}_{bcd} \theta^c \wedge \theta^d\right),
\end{aligned} \tag{69}$$

which one can verify either by considering the tetrad connection as a one-form on the frame bundle or by directly checking that

$$\mathrm{d}(\mathcal{L}^{\Gamma}_{\xi}\theta^a) + \Gamma^a{}_b \wedge (\mathcal{L}^{\Gamma}_{\xi}\theta^b) + (\mathcal{L}^{\Gamma}_{\xi}\Gamma^a{}_b) \wedge \theta^b = 0,$$
$$\eta_{ac}\mathcal{L}^{\Gamma}_{\xi}\Gamma^c{}_b = -\eta_{bc}\mathcal{L}^{\Gamma}_{\xi}\Gamma^c{}_a. \tag{70}$$

Consider now a pair of orthonormal tetrads θ^a and $\theta'^a = T^a{}_b\theta^b$. The corresponding tetrad connections are related by the transformation

$$\Gamma'^a{}_b = T^a{}_c\Gamma^c{}_d(T^{-1})^d{}_b - (T^{-1})^c{}_b\mathrm{d}T^a{}_c. \tag{71}$$

Using (68), we obtain

$$\mathcal{L}^{\Gamma'}_{\xi}\theta'^a = T^a{}_b\mathcal{L}^{\Gamma}_{\xi}\theta^b. \tag{72}$$

Hence, the presymplectic potential current Θ_{CYM} associated with the variation $\delta\theta^a = \mathcal{L}^{\Gamma}_{\xi}\theta^a$ is invariant with respect to arbitrary (pseudo)rotations and conformal rescalings. Moreover, since the modified Lie derivative $\mathcal{L}^{\Gamma}_{\xi}$ agrees with $\mathcal{L}_{\xi}$ on objects without free tetrad indices, we have

$$\mathcal{L}^{\Gamma}_{\xi}L_{\mathrm{CYM}}(\theta) = \mathcal{L}_{\xi}L_{\mathrm{CYM}}(\theta) = \mathrm{d}\left(\xi \lrcorner \frac{1}{2}F^I{}_J \wedge \star F^J{}_I\right)$$
$$= \mathrm{d}((\xi \lrcorner F^I{}_J) \wedge \star F^J{}_I) \tag{73}$$

(see (62)). Hence, the associated Noether current

$$\tilde{J}_{\xi}(\theta) \coloneqq \Theta_{\mathrm{CYM}}(\theta; \mathcal{L}^{\Gamma}_{\xi}\theta) - \mathcal{L}^{\Gamma}_{\xi}L_{\mathrm{CYM}}(\theta)$$
$$= 2\mathcal{L}^{\Gamma}_{\xi}\theta^a \wedge \star DP_a + (\mathcal{L}^{\Gamma}_{\xi}\Gamma^a{}_b - \xi \lrcorner C^a{}_b) \wedge \star C^b{}_a \tag{74}$$

depends only on $g = \eta_{ab}\theta^a\theta^b$ and is conformally invariant. Since

$$\mathcal{L}^{\Gamma}_{\xi}\Gamma_{ab} - \xi \lrcorner C_{ab} = \xi \lrcorner (\mathcal{R}_{ab} - C_{ab}) = \xi \lrcorner 2\theta_a \wedge P_b$$
$$= 2\theta_{[a}P_{b]c}\xi^c - 2\xi_{[a}P_{b]} \tag{75}$$

and

$$\mathcal{L}^{\Gamma}{}_{\xi}\theta^a = \mathcal{L}_{\xi}\theta^a + (\xi \lrcorner \Gamma^a{}_b)\theta^b$$
$$= \mathrm{d}\xi^a + \xi \lrcorner \mathrm{d}\theta^a + (\xi \lrcorner \Gamma^a{}_b)\theta^b = D\xi^a, \tag{76}$$

we can rewrite the current (74) as

$$\tilde{J}_{\xi}(\theta) = 2D\xi^a \wedge \star P_a - 2\xi^a P_b \wedge \star C^b{}_a$$
$$= \mathrm{d}(2\xi^a \star DP_a) - 2\xi^a B_{ab}\star\theta^b$$
$$\eqqcolon \mathrm{d}\tilde{Q}_{\xi} - \xi^a\tilde{C}_a, \tag{77}$$

where

$$\tilde{Q}_{\xi} \coloneqq 2\xi^a \star DP_a \tag{78}$$

is the Noether charge and $\tilde{C}_a \coloneqq B_{ab}\star\theta^b$ are constraints that vanish when the Bach tensor vanishes. Note that on the solutions of the vacuum Einstein equations $DP_a = 0$ and thus the current vanishes.

IV. CARTAN-YANG-MILLS PRESYMPLECTIC POTENTIAL CURRENT ON THE CONFORMAL BOUNDARY

A. Asymptotically de Sitter spacetimes

In this section, we restrict our considerations to metric tensors of the signature $(-,+,+,+)$ on the manifold M. Moreover, we assume that the corresponding spacetime is asymptotically de Sitter, and therefore, the cosmological constant is positive throughout this section,

$$\Lambda > 0. \tag{79}$$

A metric tensor g that is asymptotically de Sitter can be written (or defined) in the Fefferman-Graham gauge, that is [5],

$$g = \frac{\ell^2}{\rho^2}\left(-\mathrm{d}\rho^2 + \sum_{n=0}^{\infty}\rho^n g^{(n)}_{ij}(\rho, x^1, x^2, x^3)\mathrm{d}x^i\mathrm{d}x^j\right), \tag{80}$$

where the asymptotic expansion amounts to expanding in $\rho > 0$ around the boundary $\mathcal{I}$ defined by the equation

$$\rho = 0, \tag{81}$$

contained in the conformal completion of (M, g). The goal of this section is to calculate the symplectic current potential Θ_{CYM} at $\mathcal{I}$. By comparison, the symplectic current potential Θ_{EH} is known to be ill-defined in that limit. However, it follows from the conformal invariance that Θ_{CYM} is well defined at $\mathcal{I}$.

It will be convenient to use a conformally rescaled metric tensor that is finite at $\rho = 0$:

$$\hat{g} \coloneqq \frac{\rho^2}{\ell^2}g = -\mathrm{d}\rho^2 + \sum_{n=0}^{\infty}\rho^n g^{(n)}_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{82}$$

and its inverse

$$\hat{g}^{\alpha\beta}\partial_\alpha \otimes \partial_\beta = -\partial_\rho \otimes \partial_\rho + \sum_{n=0}^{\infty}\rho^n g^{ij}_{(n)}\partial_i \otimes \partial_j, \tag{83}$$

where the matrix $g^{ij}_{(0)}$ is the inverse of $g^{(0)}_{ij}$, while

$$g^{ij}_{(n)} = -g^{ik}_{(0)}\sum_{m=1}^{n}g^{(m)}_{kl}g^{lj}_{(n-m)}. \tag{84}$$

We will use the metric $\hat{g}$ and its inverse to raise and lower indices $\alpha, \beta, \gamma, \ldots$ on tensors with a hat and g on the unhatted ones.

We will denote the pullback of the rescaled metric $\hat{g}_{\alpha\beta}$ to $\mathcal{I}$ by $\mathring{g}_{ij}$. If $\hat{g}_{\alpha\beta}$ has the Fefferman-Graham form (82), we have

$$\mathring{g}_{ij} = g^{(0)}_{ij}. \tag{85}$$

Einstein's equations (34) imposed on the metric tensor (82) imply [5,7]

$$g^{(1)}_{ij} = 0, \qquad g^{(2)}_{ij} = \mathring{R}_{ij} - \frac{1}{4}\mathring{g}_{ij}\mathring{R} =: \mathring{S}_{ij},$$
$$g^{(0)ij} g^{(3)}_{ij} = \mathring{D}^i g^{(3)}_{ij} = 0, \tag{86}$$

where $\mathring{S}_{ij}$, $\mathring{R}_{ij}$, $\mathring{R}$ are, respectively, the Schouten tensor, Ricci tensor, and Ricci scalar of $\mathring{g}_{ij}$, and $\mathring{D}_i$ is the metric-compatible, torsion-free connection defined by the metric tensor $\mathring{g}_{ij}$. We will also use the notation

$$\mathring{T}_{ij} \coloneqq g^{(3)}_{ij}. \tag{87}$$

Equation (84) implies

$$g^{ij}_{(1)} = 0, \qquad g^{ij}_{(2)} = -\mathring{S}^{ij}, \qquad g^{ij}_{(3)} = -\mathring{T}^{ij}. \tag{88}$$

(Indices $i, j, \ldots$ on $\mathring{S}_{ij}$ and $\mathring{T}_{ij}$ are raised with $\mathring{g}^{ij}$ and lowered with $\mathring{g}_{ij}$).

Let us consider a coframe θ^a such that

$$g = \eta_{ab}\theta^a\theta^b, \tag{89}$$

where $\eta_{ab} = \mathrm{diag}(-,+,+,+)$. We will denote its dual frame by e_a. From (80) it follows that for any such tetrad $\theta^a = \frac{\ell}{\rho}\hat{\theta}^a$ and $e_a = \frac{\rho}{\ell}\hat{e}_a$, where $\hat{\theta}^a$ and $\hat{e}_a$ are the orthonormal coframe and the dual frame associated with the rescaled metric $\hat{g}$ from (82). Let us calculate the pullback of Θ_{CYM} to $\mathcal{I}$ for this particular choice of θ^a. Since we are considering an Einstein spacetime (34), we can use the formula (35)

$$\Theta_{\mathrm{CYM}}(\theta^a, \delta\theta^a) = \delta\Gamma^b{}_c \wedge *C^c{}_b. \tag{90}$$

The tetrad connection of θ^a is [34]

$$\Gamma^a{}_b = \eta^{ac} e^\alpha_c e^\beta_b (c_{\alpha\beta\gamma} + c_{\beta\gamma\alpha} - c_{\gamma\alpha\beta})\mathrm{d}x^\gamma, \tag{91}$$

where

$$c_{\alpha\beta\gamma} = \eta_{ab}\theta^a_\alpha \partial_{[\beta}\theta^b_{\gamma]}. \tag{92}$$

Let us expand the one-forms $\Gamma^a{}_b$ in terms of ρ. First of all, since $\theta^a = \frac{\ell}{\rho}\hat{\theta}^a$ and $e_a = \frac{\rho}{\ell}\hat{e}_a$, we have

$$\begin{aligned} c_{\alpha\beta\gamma} &= -\frac{\ell^2}{\rho^3}\eta_{ab}\hat{\theta}^a_\alpha \delta^\rho{}_{[\beta}\hat{\theta}^b_{\gamma]} + \mathcal{O}(\rho^{-2}) \\ &= -\frac{\ell^2}{\rho^3}\delta^\rho{}_{[\beta}\hat{g}_{\gamma]\alpha} + \mathcal{O}(\rho^{-2}) \end{aligned} \tag{93}$$

and from (91) follows

$$\begin{aligned} \Gamma^a{}_b &= -\rho^{-1}\eta^{ac}\hat{e}^\alpha_c\hat{e}^\beta_b(\delta^\rho{}_{[\beta}\hat{g}_{\gamma]\alpha} + \delta^\rho{}_{[\gamma}\hat{g}_{\alpha]\beta} - \delta^\rho{}_{[\alpha}\hat{g}_{\beta]\gamma})\mathrm{d}x^\gamma + \mathcal{O}(1) \\ &= 2\rho^{-1}\eta^{ac}\hat{e}^\alpha_c\hat{e}^\beta_b\delta^\rho{}_{[\alpha}\hat{g}_{\beta]\gamma}\mathrm{d}x^\gamma + \mathcal{O}(1) \\ &= 2\rho^{-1}\eta^{ac}\hat{e}^\alpha_{[c}\hat{e}^\beta_{b]}\delta^\rho{}_\alpha\hat{g}_{\beta\gamma}\mathrm{d}x^\gamma + \mathcal{O}(1) \\ &= 2\rho^{-1}\eta^{ac}\hat{e}^\rho_{[c}\hat{e}^\beta_{b]}\hat{g}_{\beta\gamma}\mathrm{d}x^\gamma + \mathcal{O}(1). \end{aligned} \tag{94}$$

Moreover, the Weyl tensor is conformally invariant,

$$C^\alpha{}_{\beta\gamma\delta} = \hat{C}^\alpha{}_{\beta\gamma\delta}, \tag{95}$$

and the relation between the volume forms of $\hat{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ is as follows:

$$\hat{\epsilon}_{\alpha\beta\gamma\delta} = \sqrt{|\det\hat{g}|}\varepsilon_{\alpha\beta\gamma\delta} = \frac{\rho^4}{\ell^4}\sqrt{|\det g|}\varepsilon_{\alpha\beta\gamma\delta} = \frac{\rho^4}{\ell^4}\epsilon_{\alpha\beta\gamma\delta}. \tag{96}$$

Therefore,

$$\hat{\epsilon}^{\alpha\beta}{}_{\gamma\delta} = \epsilon^{\alpha\beta}{}_{\gamma\delta} \tag{97}$$

(since we use $\hat{g}^{\alpha\beta}$ to raise indices on the hatted quantities and $g^{\alpha\beta}$—on the unhatted). Consequently

$$\begin{aligned} \star C^\alpha{}_\beta &= \frac{1}{4}C^\alpha{}_{\beta\gamma\delta}\epsilon^{\gamma\delta}{}_{\epsilon\zeta}\mathrm{d}x^\epsilon \wedge \mathrm{d}x^\zeta \\ &= \frac{1}{4}\hat{C}^\alpha{}_{\beta\gamma\delta}\hat{\epsilon}^{\gamma\delta}{}_{\epsilon\zeta}\mathrm{d}x^\epsilon \wedge \mathrm{d}x^\zeta = \hat{\star}\hat{C}^\alpha{}_\beta. \end{aligned} \tag{98}$$

Since

$$\hat{g} = -\mathrm{d}\rho^2 + \mathring{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathcal{O}(\rho^2), \tag{99}$$

we have

$$\det\hat{g} = -\det\mathring{g} + \mathcal{O}(\rho^2), \tag{100}$$

and thus

$$\sqrt{|\det\hat{g}|} = \sqrt{\det\mathring{g}} + \mathcal{O}(\rho^2). \tag{101}$$

Therefore,

$$\hat{\epsilon}_{\rho ijk} = \sqrt{|\det \hat{g}|}\epsilon_{\rho ijk} = \sqrt{\det \mathring{g}}\epsilon_{ijk} + \mathcal{O}(\rho^2) = \mathring{\epsilon}_{ijk} + \mathcal{O}(\rho^2), \tag{102}$$

where $\mathring{\epsilon}_{ijk} \coloneqq \sqrt{\mathring{g}}\epsilon_{ijk}$ is a three-form defined on a neighborhood of $\mathcal{I}$, whose pullback to $\mathcal{I}$ is the volume form on $\mathcal{I}$ induced by the conformal metric $\hat{g}$. As a consequence of (102), we have

$$\frac{1}{2}\hat{C}^{\beta}{}_{\alpha}{}^{\gamma\delta}\hat{\epsilon}_{\gamma\delta jk} = \hat{C}^{\beta}{}_{\alpha}{}^{\rho l}\mathring{\epsilon}_{ljk} + \mathcal{O}(\rho^2) = -\hat{C}^{\beta}{}_{\alpha\rho}{}^{l}\mathring{\epsilon}_{ljk} + \mathcal{O}(\rho^2). \tag{103}$$

The pullback of (90) to $\mathcal{I}$ is

$$\tilde{\Theta}_{\rm CYM} = \lim_{\rho\to 0}\ \delta\Gamma^{a}{}_{bi}\hat{e}^{\alpha}_{a}\hat{\theta}^{b}_{\beta}\mathrm{d}x^i \wedge \left(\frac{1}{2}\hat{C}^{\beta}{}_{\alpha}{}^{\gamma\delta}\hat{\epsilon}_{\gamma\delta jk}\mathrm{d}x^j \wedge \mathrm{d}x^k\right). \tag{104}$$

Using the formula for the variation of the tetrad connection (94), we get

$$\tilde{\Theta}_{\rm CYM} = \lim_{\rho\to 0}\ \delta(2\rho^{-1}\eta^{ac}\hat{e}^{\rho}_{[c}\hat{e}^{\zeta}_{b]}\hat{g}_{\zeta i})\hat{e}^{\alpha}_{a}\hat{\theta}^{b}_{\beta}\mathrm{d}x^i \wedge \left(\frac{1}{2}\hat{C}^{\beta}{}_{\alpha}{}^{\gamma\delta}\hat{\epsilon}_{\gamma\delta jk}\mathrm{d}x^j \wedge \mathrm{d}x^k\right). \tag{105}$$

Next, we use (103) to obtain

$$\begin{aligned}
\tilde{\Theta}_{\rm CYM} &= -\lim_{\rho\to 0}\ \delta(2\rho^{-1}\eta^{ac}\hat{e}^{\rho}_{[c}\hat{e}^{\zeta}_{b]}\hat{g}_{\zeta i})\hat{e}^{\alpha}_{a}\hat{\theta}^{b}_{\beta}\mathrm{d}x^i \wedge (\hat{C}^{\beta}{}_{\alpha\rho}{}^{l}\mathring{\epsilon}_{ljk}\mathrm{d}x^j \wedge \mathrm{d}x^k)\\
&= -\lim_{\rho\to 0}\ \delta(2\rho^{-1}\eta^{ac}\hat{e}^{\rho}_{[c}\hat{e}^{\zeta}_{b]}\hat{g}_{\zeta i})\hat{e}^{\alpha}_{a}\hat{\theta}^{b}_{\beta}\hat{C}^{\beta}{}_{\alpha\rho}{}^{i}\mathring{\rm Vol}\\
&= -\lim_{\rho\to 0}\ \delta(2\rho^{-1}\hat{e}^{\rho}_{[a}\hat{e}^{\zeta}_{b]}\hat{g}_{\zeta i})\hat{\theta}^{a}_{\alpha}\hat{\theta}^{b}_{\beta}\hat{C}^{\beta\alpha}{}_{\rho}{}^{i}\mathring{\rm Vol}\\
&= \lim_{\rho\to 0}\ \delta(2\rho^{-1}\hat{e}^{\rho}_{a}\hat{e}^{\zeta}_{b}\hat{g}_{\zeta i})\hat{\theta}^{a}_{\alpha}\hat{\theta}^{b}_{\beta}\hat{C}^{\beta\alpha\rho i}\mathring{\rm Vol}\\
&= \lim_{\rho\to 0}\ 2\rho^{-1}(\delta\hat{e}^{\rho}_{a}\hat{e}^{\zeta}_{b}\hat{g}_{\zeta i} + \hat{e}^{\rho}_{a}\delta\hat{\theta}^{c}_{i}\eta_{cb})\hat{\theta}^{a}_{\alpha}\hat{\theta}^{b}_{\beta}\hat{C}^{\beta\alpha\rho i}\mathring{\rm Vol}\\
&= \lim_{\rho\to 0}\ 2\rho^{-1}(\delta\hat{e}^{\rho}_{a}\hat{g}_{\beta i}\hat{\theta}^{a}_{\alpha}\hat{C}^{\beta\alpha\rho i} + \delta\hat{\theta}^{c}_{i}\eta_{cb}\hat{\theta}^{b}_{\beta}\hat{C}^{\beta\rho\rho i})\mathring{\rm Vol},
\end{aligned} \tag{106}$$

where $\mathring{\rm Vol} \coloneqq \frac{1}{3!}\mathring{\epsilon}_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$. Using the fact that the Weyl tensor is traceless in every pair of indices, we get

$$\hat{g}_{\beta i}\hat{C}^{\beta\alpha\rho i} = \hat{g}_{\beta\gamma}\hat{C}^{\beta\alpha\rho\gamma} - \hat{g}_{\beta\rho}\hat{C}^{\beta\alpha\rho\rho} = -\hat{g}_{\beta\rho}\hat{C}^{\beta\alpha\rho\rho} = 0. \tag{107}$$

Thus the first term in the last line of (106) vanishes, and we are left with

$$\begin{aligned}
\tilde{\Theta}_{\rm CYM}(\theta;\delta\theta) &= \lim_{\rho\to 0}\ 2\rho^{-1}\delta\hat{\theta}^{c}_{i}\eta_{cb}\hat{\theta}^{b}_{\beta}\hat{C}^{\beta\rho\rho i}\mathring{\rm Vol}\\
&= \lim_{\rho\to 0}\ 2\rho^{-1}\delta\hat{\theta}^{c}_{i}\eta_{cb}\hat{\theta}^{b}_{j}\hat{C}^{j\rho\rho i}\mathring{\rm Vol}\\
&= \lim_{\rho\to 0}\ \rho^{-1}\delta\hat{g}_{ij}\hat{C}^{i\rho\rho j}\mathring{\rm Vol}\\
&= \frac{3}{2}\delta\mathring{g}_{ij}\mathring{T}^{ij}\mathring{\rm Vol},
\end{aligned} \tag{108}$$

where in the last step we used Eq. (A22) from Appendix A. Notice that although in general, the presymplectic potential current depends on the tetrad used to define the normal conformal Cartan connection and its variation, it turns out that at the conformal boundary of de Sitter spacetime, this dependence only manifests through the metric, $g = \eta_{ab}\theta^a\theta^b$, and the variation of the metric induced on $\mathcal{I}$ by the conformally rescaled metric $\hat{g}$.

Moreover, note that here $\delta\mathring{g}_{ij}$ means simply the variation of the intrinsic metric on $\mathcal{I}$ induced by a general variation $\delta g = \frac{\ell^2}{\rho^2}\delta\hat{g}$. It does not mean that δg has to preserve the Fefferman-Graham gauge (in which case we could write $\delta\mathring{g}_{ij} = \delta g^{(0)}_{ij}$).

The standard definition of the holographic energy-momentum tensor on the boundary is [6]

$$T_{ij} = \frac{3\ell}{16\pi G}\mathring{T}_{ij}, \tag{109}$$

which implies

$$\tilde{\Theta}_{\rm CYM} = \frac{8\pi G}{\ell}\delta\mathring{g}_{ij}T^{ij}\mathring{\rm Vol}. \tag{110}$$

On the other hand, the variation of the holographically renormalized Einstein-Hilbert action,

$$S_{\rm GR} = \frac{1}{16\pi G}\int_M (R-2\Lambda){\rm Vol} + \frac{1}{16\pi G}\int_{\mathcal{I}}\left(2K+\frac{4}{\ell}-\mathring{R}\right)\mathring{\rm Vol}, \tag{111}$$

yields the following presymplectic potential current on $\mathcal{I}$ [6]:

$$\tilde{\Theta}_{\rm GR} = -\frac{\ell}{2}\delta\mathring{g}_{ij}T^{ij}\mathring{\rm Vol}. \tag{112}$$

Hence,

$$\tilde{\Theta}_{\rm CYM} = -\frac{16\pi G\Lambda}{3}\tilde{\Theta}_{\rm GR}. \tag{113}$$

B. Asymptotically flat spacetimes

We describe an asymptotically flat spacetime using the Bondi-Sachs gauge (in coordinates Ω, u, x^A, where $\frac{1}{\Omega}$ is the luminosity distance) [35–37]:

$$\begin{aligned} g_{\Omega\Omega} &= g_{\Omega A} = 0, \qquad g_{uu} = -1 + 2M\Omega + \mathcal{O}(\Omega^2), \qquad g_{AB} = \Omega^{-2}\gamma_{AB} + \Omega^{-1}C_{AB} + \mathcal{O}(1), \\ g_{u\Omega} &= \Omega^{-2} - \frac{1}{16}C^{AB}C_{AB} + \mathcal{O}(\Omega), \qquad g_{uA} = \frac{1}{2}D^B C_{BA} + \mathcal{O}(\Omega), \end{aligned} \tag{114}$$

$$\begin{aligned} g^{uu} &= g^{uA} = 0, \qquad g^{\Omega\Omega} = \Omega^4 - 2M\Omega^5 + \mathcal{O}(\Omega^6), \qquad g^{AB} = \Omega^2\gamma^{AB} - \Omega^3 C^{AB} + \mathcal{O}(\Omega^4), \\ g^{u\Omega} &= \Omega^2 + \frac{1}{16}C^{AB}C_{AB}\Omega^4 + \mathcal{O}(\Omega^5), \qquad g^{\Omega A} = -\frac{1}{2}D_B C^{BA}\Omega^4 + \mathcal{O}(\Omega^5). \end{aligned} \tag{115}$$

Here M is the Trautman-Bondi mass aspect, $N_{AB} \coloneqq \partial_u C_{AB}$ is the Bondi news tensor, γ_{AB} is a unit sphere metric, D is its covariant derivative, and indices $A, B, \ldots$ on those objects are raised with γ^{AB}. Similar to the asymptotically de Sitter case, let us define $\hat{g} \coloneqq \Omega^2 g$. Let us also use indices $i, j, \ldots$ to go over the three coordinates u, x^A and define $\mathring{g}$ to be the pullback of $\hat{g}$ to $\mathcal{I}$ (i.e., the surface $\Omega = 0$).

From (114) and $\hat{g} \coloneqq \Omega^2 g$, we obtain

$$\hat{g} = 2{\rm d}u{\rm d}\Omega + \gamma_{AB}{\rm d}x^A{\rm d}x^B + \mathcal{O}(\Omega), \tag{116}$$

so the pullback of $\hat{g}$ to $\mathcal{I}$, which we denote by $\mathring{g}$, is

$$\mathring{g} = \gamma_{AB}{\rm d}x^A{\rm d}x^b. \tag{117}$$

Therefore,

$$\det\hat{g} = -\det\mathring{g} + \mathcal{O}(\Omega), \tag{118}$$

$$\sqrt{|\det\hat{g}|} = \sqrt{\det\gamma} + \mathcal{O}(\Omega), \tag{119}$$

$$\hat{\epsilon}_{\Omega ijk} = \sqrt{|\det\hat{g}|}\varepsilon_{\Omega ijk} = \sqrt{\det\gamma}\varepsilon_{ijk} + \mathcal{O}(\Omega) = \mathring{\epsilon}_{ijk} + \mathcal{O}(\Omega), \tag{120}$$

where, as in the previous section,

$$\begin{aligned} \mathring{\rm Vol} &\coloneqq \frac{1}{3!}\mathring{\epsilon}_{ijk}{\rm d}x^i \wedge {\rm d}x^j \wedge {\rm d}x^k \\ &\coloneqq \frac{1}{3!}\sqrt{\det\gamma}\varepsilon_{ijk}{\rm d}x^i \wedge {\rm d}x^j \wedge {\rm d}x^k \end{aligned} \tag{121}$$

is a three-form defined in the neighborhood of $\mathcal{I}$, whose pullback to $\mathcal{I}$ is the volume form induced on $\mathcal{I}$ by $\hat{g}$.

Since our spacetime satisfies $R_{ab} = 0$, we can calculate $\Theta_{\rm CYM}$ using

$$\Theta_{\rm CYM} = \delta\Gamma^a{}_b \wedge \star C^b{}_a. \tag{122}$$

The peeling theorem implies that $C^\alpha{}_{\beta\gamma\delta} = \hat{C}^\alpha{}_{\beta\gamma\delta} = \mathcal{O}(\Omega)$. Also,

$$\begin{aligned} \Gamma^a{}_b &= -\Omega^{-1}\eta^{ac}\hat{e}^\alpha_c\hat{e}^\beta_b(\delta^\Omega{}_{[\beta}\hat{g}_{\gamma]\alpha} + \delta^\Omega{}_{[\gamma}\hat{g}_{\alpha]\beta} - \delta^\Omega{}_{[\alpha}\hat{g}_{\beta]\gamma}){\rm d}x^\gamma \\ &\quad + \mathcal{O}(1) \\ &= 2\Omega^{-1}\eta^{ac}\hat{e}^\Omega_{[c}\hat{e}^\beta_{b]}\hat{g}_{\beta\gamma}{\rm d}x^\gamma + \mathcal{O}(1). \end{aligned} \tag{123}$$

Hence,

$$\begin{aligned}\tilde{\Theta}_{\mathrm{CYM}} &= \lim_{\Omega\to 0}\delta(2\Omega^{-1}\hat{e}^{\Omega}_{a}\hat{e}^{\zeta}_{b}\hat{g}_{\zeta i}\mathrm{d}x^i)\hat{\theta}^{a}_{\alpha}\hat{\theta}^{b}_{\beta}\wedge\left(\frac{1}{2}\hat{C}^{\beta\alpha\gamma\delta}\hat{\epsilon}_{\gamma\delta jk}\mathrm{d}x^j\wedge\mathrm{d}x^k\right)\\ &= \lim_{\Omega\to 0}2\Omega^{-1}\left(\hat{\theta}^{a}_{\alpha}\delta\hat{e}^{\Omega}_{a}\hat{g}_{\beta i}+\delta^{\Omega}{}_{\alpha}\eta_{cb}\delta\hat{\theta}^{c}_{i}\hat{\theta}^{b}_{\beta}\right)\mathrm{d}x^i\wedge(\hat{C}^{\beta\alpha\Omega l}\mathring{\epsilon}_{ljk}\mathrm{d}x^j\wedge\mathrm{d}x^k)\\ &= \lim_{\Omega\to 0}2\Omega^{-1}\left(\hat{\theta}^{a}_{\alpha}\delta\hat{e}^{\Omega}_{a}\hat{g}_{\beta i}+\delta^{\Omega}{}_{\alpha}\eta_{cb}\delta\hat{\theta}^{c}_{i}\hat{\theta}^{b}_{\beta}\right)\hat{C}^{\beta\alpha\Omega i}\mathring{\mathrm{Vol}}\\ &= \lim_{\Omega\to 0}\Omega^{-1}\delta\hat{g}_{ji}\hat{C}^{j\Omega\Omega i}\mathring{\mathrm{Vol}}\\ &= \lim_{\Omega\to 0}\Omega^{-1}\delta\mathring{g}_{ji}\hat{C}^{j}{}_{uu}{}^{i}\mathring{\mathrm{Vol}}.\end{aligned} \tag{124}$$

Inserting the components of the Weyl tensor calculated in Appendix B, we obtain

$$\tilde{\Theta}_{\mathrm{CYM}} = \left[\delta\mathring{g}_{uu}\left(\frac{1}{4}C^{AB}N_{AB}+2M\right)+\frac{1}{2}\delta\mathring{g}_{AB}\partial_u N^{AB}\right]\mathring{\mathrm{Vol}}. \tag{125}$$

In this case there is no contribution from the Einstein-Hilbert potential, since that part of the decomposition (49) has a proportionality constant of Λ, and thus vanishes for asymptotically flat spacetimes.

In the special case when $\delta\mathring{g} = \mathcal{L}_\xi\mathring{g}$ for some BMS vector ξ, we have

$$\mathcal{L}_\xi\mathring{g} = 2\alpha\mathring{g}, \tag{126}$$

so

$$\tilde{\Theta}_{\mathrm{CYM}}(\mathcal{L}_\xi\mathring{g}) = \alpha\gamma_{AB}\partial_u N^{AB}\mathring{\mathrm{Vol}} = 0, \tag{127}$$

since the Bondi news tensor is traceless.

V. SUMMARY

Using the normal conformal Cartan connection as a gauge potential of a Yang-Mills theory allowed us to obtain a conformally invariant presymplectic potential current (33). Because of the fact that the Yang-Mills current of our theory encodes the Bach tensor of the underlying spacetime (20), the theory's equations of motion are satisfied by any tetrad that generates a metric conformally equivalent to a solution to the vacuum Einstein field equations. This made it viable to use our potential current to derive Noether currents and charges (52), (55), (64) conserved on such tetrads. In particular, we showed that the current associated with diffeomorphisms is conformally invariant. Additionally, we described the way in which, on the solutions to the vacuum Einstein's equations, the potential decomposes into a part proportional to the Einstein-Hilbert presymplectic potential and the topological (Euler) term (49). As an example of an application of this formalism, we calculated the presymplectic potential current induced by our theory on the conformal boundary of asymptotically de Sitter spacetime and found out that the result is proportional to the potential one obtains using a holographically renormalized gravity action (113). This provides a mathematically elegant way of obtaining a symplectic structure on that boundary without resorting to renormalization techniques. On the other hand, the potential obtained at the null infinity of an asymptotically flat spacetime does not seem to have a clear physical relevance and in fact vanishes for variations induced by the BMS diffeomorphism group (127), which is to be expected since in the case of $\Lambda = 0$ only the topological part of the decomposition (49) remains.

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APPENDIX A: ASYMPTOTICALLY DE SITTER SPACETIMES IN THE FEFFERMAN-GRAHAM GAUGE

We consider the metric

$$g = \frac{\ell^2}{\rho^2}\left(-\mathrm{d}\rho^2+\sum_{n=0}^{\infty}\rho^n g^{(n)}_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) \tag{A1}$$

with an inverse

$$g^{\alpha\beta}\partial_\alpha\otimes\partial_\beta = \frac{\rho^2}{\ell^2}\left(-\partial_\rho\otimes\partial_\rho+\sum_{n=0}^{\infty}\rho^n g^{ij}_{(n)}\partial_i\otimes\partial_j\right), \tag{A2}$$

where the matrix $g^{ij}_{(0)}$ is the inverse of $g^{(0)}_{ij}$ and for $n \geq 1$ we have

$$g^{ij}_{(n)} = -g^{ik}_{(0)} \sum_{m=1}^{n} g^{(m)}_{kl} g^{lj}_{(n-m)}. \tag{A3}$$

Imposing vacuum Einstein's equations $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$ on (A1) with $\Lambda = \frac{3}{\ell^2}$ leads to the following constraints [5,7]:

$$g^{(1)}_{ij} = 0, \qquad g^{(2)}_{ij} = \mathring{R}_{ij} - \frac{1}{4}\mathring{R} g^{(0)}_{ij}, \qquad g^{(0)ij} g^{(3)}_{ij} = \mathring{D}^i g^{(3)}_{ij} = 0, \tag{A4}$$

where $\mathring{R}_{ij}$, $\mathring{R}$, and $\mathring{D}_i$ are, respectively, the Ricci tensor, Ricci scalar, and the metric-compatible, torsion-free covariant derivative of $g^{(0)}_{ij}$.

To calculate the potential Θ_{CYM}, we need the Weyl tensor of the metric $g_{\alpha\beta}$. Since it is conformally invariant, we will calculate the Weyl tensor of a simpler metric,

$$\hat{g} = \frac{\rho^2}{\ell^2} g = -\mathrm{d}\rho^2 + \sum_{n=0}^{\infty} \rho^n g^{(n)}_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{A5}$$

First, let us introduce the following notation to simplify expressions:

$$\mathring{g}_{ij} \coloneqq g^{(0)}_{ij}, \qquad \mathring{S}_{ij} \coloneqq g^{(2)}_{ij} = \mathring{R}_{ij} - \frac{1}{4}\mathring{R}\mathring{g}_{ij}, \qquad \mathring{T}_{ij} \coloneqq g^{(3)}_{ij}. \tag{A6}$$

From (A5) we derive the following Christoffel symbols (below $\mathring{D}_i$ is the covariant derivative of $\mathring{g}_{ij}$ and $\mathring{\Gamma}^i{}_{jk}$ are its Christoffel symbols):

$$\hat{\Gamma}^\rho{}_{\rho\rho} = \hat{\Gamma}^\rho{}_{\rho i} = \hat{\Gamma}^i{}_{\rho\rho} = 0, \tag{A7}$$

$$\hat{\Gamma}^\rho{}_{ij} = \rho \mathring{S}_{ij} + \frac{3}{2}\rho^2 \mathring{T}_{ij} + \mathcal{O}(\rho^3), \tag{A8}$$

$$\hat{\Gamma}^i{}_{j\rho} = \rho \mathring{S}^i{}_j + \frac{3}{2}\rho^2 \mathring{T}^i{}_j + \mathcal{O}(\rho^3), \tag{A9}$$

$$\begin{aligned}\hat{\Gamma}^i{}_{jk} &= \mathring{\Gamma}^i{}_{jk} + \rho^2\left(\mathring{S}^{il}\mathring{\Gamma}_{ljk} + \frac{1}{2}\mathring{g}^{il}(2\partial_{(j}\mathring{S}_{k)l} - \partial_l \mathring{S}_{jk})\right) + \mathcal{O}(\rho^3) \\ &= \mathring{\Gamma}^i{}_{jk} + \rho^2\left(\mathring{D}_{(j}\mathring{S}_{k)}{}^i - \frac{1}{2}\mathring{D}^i\mathring{S}_{jk}\right) + \mathcal{O}(\rho^3). \end{aligned}\tag{A10}$$

Next, we calculate the Riemann tensor:

$$\hat{R}^\rho{}_{i\rho j} = \partial_\rho \hat{\Gamma}^\rho{}_{ij} - \hat{\Gamma}^\rho{}_{kj}\hat{\Gamma}^k{}_{\rho i} = \mathring{S}_{ij} + 3\rho\mathring{T}_{ij} + \mathcal{O}(\rho^2), \tag{A11}$$

$$\begin{aligned}\hat{R}^\rho{}_{ijk} &= 2\partial_{[j}\hat{\Gamma}^\rho{}_{k]i} + 2\hat{\Gamma}^\rho{}_{l[j}\hat{\Gamma}^l{}_{k]i} \\ &= \rho(\partial_j \mathring{S}_{ki} - \partial_k \mathring{S}_{ji} + \mathring{S}_{lj}\mathring{\Gamma}^l{}_{ki} - \mathring{S}_{lk}\mathring{\Gamma}^l{}_{ji}) + \mathcal{O}(\rho^2) \\ &= \rho(\mathring{D}_j \mathring{S}_{ki} - \mathring{D}_k \mathring{S}_{ji}) + \mathcal{O}(\rho^2) \\ &= 2\rho \mathring{D}_{[j}\mathring{S}_{k]i} + \mathcal{O}(\rho^2), \end{aligned}\tag{A12}$$

$$\hat{R}^i{}_{jkl} = 2\partial_{[k}\hat{\Gamma}^i{}_{l]j} + 2\hat{\Gamma}^i{}_{m[k}\hat{\Gamma}^m{}_{l]j} = \mathring{R}^i{}_{jkl} + \mathcal{O}(\rho^2). \tag{A13}$$

Lowering the first index, we get the only (up to symmetries) nonzero components of the Riemann tensor:

$$\hat{R}_{\rho i\rho j} = -\mathring{S}_{ij} - 3\rho\mathring{T}_{ij} + \mathcal{O}(\rho^2), \tag{A14}$$

$$\hat{R}_{\rho ijk} = -2\rho\mathring{D}_{[j}\mathring{S}_{k]i} + \mathcal{O}(\rho^2), \tag{A15}$$

$$\hat{R}_{ijkl} = \mathring{R}_{ijkl} + \mathcal{O}(\rho^2). \tag{A16}$$

By virtue of (A14), the fact that $\mathring{g}^{ij}\mathring{T}_{ij} = 0$ [7] and $\mathring{S} = \mathring{g}^{ij}\mathring{S}_{ij} = \frac{1}{4}\mathring{R}$, we have

$$\hat{R}_{\rho\rho} = \hat{g}^{ij}\hat{R}_{i\rho j\rho} = -\frac{1}{4}\mathring{R} + \mathcal{O}(\rho^2). \tag{A17}$$

Furthermore, from (A15) and the Bianchi identity $\mathring{D}_{[i}\mathring{R}_{jk]lm} = 0$ follows that

$$\hat{R}_{\rho i} = \hat{g}^{jk}\hat{R}_{\rho jik} = -\rho(\mathring{D}_i\mathring{S} - \mathring{D}^j\mathring{S}_{ji}) + \mathcal{O}(\rho^2) = \mathcal{O}(\rho^2). \tag{A18}$$

Finally,

$$\begin{aligned}\hat{R}_{ij} &= \hat{g}^{\rho\rho}\hat{R}_{\rho i\rho j} + \hat{g}^{kl}\hat{R}_{kilj} = \mathring{S}_{ij} + 3\rho\mathring{T}_{ij} + \mathring{R}_{ij} + \mathcal{O}(\rho^2) \\ &= 2\mathring{R}_{ij} - \frac{1}{4}\mathring{g}_{ij}\mathring{R} + 3\rho\mathring{T}_{ij} + \mathcal{O}(\rho^2) \end{aligned}\tag{A19}$$

and

$$\hat{R} = \hat{g}^{\rho\rho}\hat{R}_{\rho\rho} + \hat{g}^{ij}\hat{R}_{ij} = \frac{3}{2}\mathring{R} + \mathcal{O}(\rho^2). \tag{A20}$$

Next, we calculate the Weyl tensor using

$$\hat{C}_{\alpha\beta\gamma\delta} = \hat{R}_{\alpha\beta\gamma\delta} - (\hat{g}_{\alpha[\gamma}\hat{R}_{\delta]\beta} - \hat{g}_{\beta[\gamma}\hat{R}_{\delta]\alpha}) + \frac{1}{3}\hat{R}\hat{g}_{\alpha[\gamma}\hat{g}_{\delta]\beta}, \tag{A21}$$

$$\begin{aligned}\hat{C}_{\rho i\rho j} &= \hat{R}_{\rho i\rho j} + \frac{1}{2}\hat{R}_{ij} - \frac{1}{2}\hat{g}_{ij}\hat{R}_{\rho\rho} - \frac{1}{6}\hat{R}\hat{g}_{ij} \\ &= -\mathring{S}_{ij} - 3\rho\mathring{T}_{ij} + \mathring{R}_{ij} - \frac{1}{8}\mathring{R}\mathring{g}_{ij} + \frac{3}{2}\rho\mathring{T}_{ij} \\ &\quad + \frac{1}{8}\mathring{R}\mathring{g}_{ij} - \frac{1}{4}\mathring{R}\mathring{g}_{ij} + \mathcal{O}(\rho^2) \\ &= -\frac{3}{2}\rho\mathring{T}_{ij} + \mathcal{O}(\rho^2), \end{aligned} \tag{A22}$$

$$\hat{C}_{\rho ijk} = \hat{R}_{\rho ijk} + \hat{g}_{i[j}\hat{R}_{k]\rho} = -2\rho\mathring{D}_{[j}\mathring{S}_{k]i} + \mathcal{O}(\rho^2), \tag{A23}$$

$$\begin{aligned}\hat{C}_{ijkl} &= \hat{R}_{ijkl} - (\hat{g}_{i[k}\hat{R}_{l]j} - \hat{g}_{j[k}\hat{R}_{l]i}) + \frac{1}{3}\hat{R}\hat{g}_{i[k}\hat{g}_{l]j} \\ &= \mathring{R}_{ijkl} - 2(\mathring{g}_{i[k}\mathring{R}_{l]j} - \mathring{g}_{j[k}\mathring{R}_{l]i}) + \mathring{R}\mathring{g}_{i[k}\mathring{g}_{l]j} \\ &\quad - 3\rho(\mathring{g}_{i[k}\mathring{T}_{l]j} - \mathring{g}_{j[k}\mathring{T}_{l]i}) + \mathcal{O}(\rho^2) \\ &= \mathring{C}_{ijkl} - 3\rho(\mathring{g}_{i[k}\mathring{T}_{l]j} - \mathring{g}_{j[k}\mathring{T}_{l]i}) + \mathcal{O}(\rho^2) \\ &= -3\rho(\mathring{g}_{i[k}\mathring{T}_{l]j} - \mathring{g}_{j[k}\mathring{T}_{l]i}) + \mathcal{O}(\rho^2), \end{aligned} \tag{A24}$$

where we used the fact that the Weyl tensor vanishes in three dimensions.

APPENDIX B: ASYMPTOTICALLY FLAT SPACETIMES IN BONDI-SACHS GAUGE

One of the ways of describing an asymptotically flat spacetime in the neighborhood of $\mathcal{I}$ (where $\mathcal{I}$ is either the future or past null infinity) is using the Bondi-Sachs coordinates, where the physical metric satisfies [35–37]

$$\begin{aligned} &g_{rr} = g_{rA} = 0, \qquad g_{uu} = -1 + \frac{2M}{r} + \mathcal{O}(r^{-2}), \\ &g_{AB} = r^2\gamma_{AB} + rC_{AB} + \mathcal{O}(1), \\ &g_{ur} = -1 + \frac{1}{16r^2}C^{AB}C_{AB} + \mathcal{O}(r^{-3}), \\ &g_{uA} = \frac{1}{2}D^B C_{BA} + \mathcal{O}(r^{-1}), \end{aligned} \tag{B1}$$

where M is the Bondi mass aspect and C_{AB} contains the information about radiation at $\mathcal{I}$ (it is equivalent to the asymptotic shear σ° in the notation of Ashtekar). The radial coordinate r satisfies the condition

$$\det(g_{AB}) = r^4\det(\gamma_{AB}), \tag{B2}$$

from which follows

$$\gamma^{AB}C_{AB} = 0. \tag{B3}$$

Moreover, the Bondi news tensor is

$$N_{AB} \coloneqq \partial_u C_{AB}. \tag{B4}$$

This formalism is equivalent to the method of conformal completion [1,2,38,39]. After attaching the null boundary, one can introduce a coordinate system (u, Ω, x^A) on a neighborhood of $\mathcal{I}$ which is related to the Bondi-Sachs by keeping the functions u and x^A the same and taking $\Omega = \frac{1}{r}$. In those coordinates, we have

$$\begin{aligned} &g_{\Omega\Omega} = g_{\Omega A} = 0, \qquad g_{uu} = -1 + 2M\Omega + \mathcal{O}(\Omega^2), \\ &g_{AB} = \Omega^{-2}\gamma_{AB} + \Omega^{-1}C_{AB} + \mathcal{O}(1), \\ &g_{u\Omega} = \Omega^{-2} - \frac{1}{16}C^{AB}C_{AB} + \mathcal{O}(\Omega), \\ &g_{uA} = \frac{1}{2}D^B C_{BA} + \mathcal{O}(\Omega), \end{aligned} \tag{B5}$$

so the conformally rescaled metric $\hat{g} = \Omega^2 g$ extends to $\mathcal{I}$. The inverse metric components satisfy

$$\begin{aligned} &g^{uu} = g^{uA} = 0, \qquad g^{\Omega\Omega} = \Omega^4 - 2M\Omega^5 + \mathcal{O}(\Omega^6), \\ &g^{AB} = \Omega^2\gamma^{AB} - \Omega^3 C^{AB} + \mathcal{O}(\Omega^4), \\ &g^{u\Omega} = \Omega^2 + \frac{1}{16}C^{AB}C_{AB}\Omega^4 + \mathcal{O}(\Omega^5), \\ &g^{\Omega A} = -\frac{1}{2}D_B C^{BA}\Omega^4 + \mathcal{O}(\Omega^5). \end{aligned} \tag{B6}$$

From (B5) and (B6) we calculate the Christoffel symbols of g:

$$\Gamma^u{}_{u\Omega} = \Gamma^u{}_{\Omega\Omega} = \Gamma^u{}_{\Omega A} = 0, \tag{B7}$$

$$\begin{aligned}\Gamma^u{}_{uu} &= \frac{1}{2}g^{u\Omega}(2\partial_u g_{u\Omega} - \partial_\Omega g_{uu}) \\ &= \frac{1}{2}\Omega^2\left(-\frac{1}{4}C^{AB}N_{AB} - 2M\right) + \mathcal{O}(\Omega^3), \end{aligned} \tag{B8}$$

$$\Gamma^u{}_{uA} = \frac{1}{2}g^{u\Omega}(\partial_A g_{u\Omega} - \partial_\Omega g_{uA}) = \mathcal{O}(\Omega^2), \tag{B9}$$

$$\Gamma^u{}_{AB} = -\frac{1}{2}g^{u\Omega}\partial_\Omega g_{AB} = \mathcal{O}(\Omega^{-1}), \tag{B10}$$

$$\begin{aligned}\Gamma^\Omega{}_{uu} &= \frac{1}{2}g^{\Omega u}\partial_u g_{uu} + \frac{1}{2}g^{\Omega\Omega}(2\partial_u g_{u\Omega} - \partial_\Omega g_{uu}) \\ &\quad + \frac{1}{2}g^{\Omega B}(2\partial_u g_{uB} - \partial_B g_{uu}) = \mathcal{O}(\Omega^3), \end{aligned} \tag{B11}$$

$$\begin{aligned}\Gamma^\Omega{}_{uA} &= \frac{1}{2}g^{\Omega u}\partial_A g_{uu} + \frac{1}{2}g^{\Omega\Omega}(\partial_A g_{u\Omega} - \partial_\Omega g_{uA}) \\ &\quad + \frac{1}{2}g^{\Omega B}(\partial_u g_{AB} + 2\partial_{[A}g_{B]u}) = \mathcal{O}(\Omega^3), \end{aligned} \tag{B12}$$

$$\Gamma^A{}_{uu} = \frac{1}{2} g^{A\Omega}(2\partial_u g_{u\Omega} - \partial_\Omega g_{uu}) = \mathcal{O}(\Omega^4), \tag{B13}$$

$$\Gamma^A{}_{u\Omega} = \frac{1}{2} g^{AB}(\partial_\Omega g_{uB} - \partial_B g_{u\Omega}) = \mathcal{O}(\Omega^2), \tag{B14}$$

$$\Gamma^A{}_{uB} = \frac{1}{2} g^{AC}(\partial_u g_{BC} + 2\partial_{[B} g_{C]u}) = \frac{1}{2} N^A{}_B \Omega + \mathcal{O}(\Omega^2), \tag{B15}$$

$$\Gamma^A{}_{\Omega B} = \frac{1}{2} g^{AC}\partial_\Omega g_{BC} = \mathcal{O}(\Omega^{-1}), \tag{B16}$$

$$\Gamma^A{}_{BC} = \frac{1}{2} g^{AD}(2\partial_{(B} g_{C)D} - \partial_D g_{BC}) - \frac{1}{2} g^{A\Omega}\partial_\Omega g_{BC} = \mathcal{O}(1). \tag{B17}$$

Next, we derive some components of the Riemann tensor of g_{ab}:

$$\begin{aligned} R^u{}_{uu\Omega} &= 2\partial_{[u}\Gamma^u{}_{\Omega]u} + 2\Gamma^u{}_{\alpha[u}\Gamma^\alpha{}_{\Omega]u} \\ &= \Omega\left(\frac{1}{4} C^{AB} N_{AB} + 2M\right) + \mathcal{O}(\Omega^2), \end{aligned} \tag{B18}$$

$$R^u{}_{uuA} = 2\partial_{[u}\Gamma^u{}_{A]u} + 2\Gamma^u{}_{\alpha[u}\Gamma^\alpha{}_{A]u} = \mathcal{O}(\Omega^2), \tag{B19}$$

$$R^A{}_{uuB} = 2\partial_{[u}\Gamma^A{}_{B]u} + 2\Gamma^A{}_{\alpha[u}\Gamma^\alpha{}_{B]u} = \mathcal{O}(\Omega^2) = \frac{1}{2}\Omega\partial_u N^A{}_B. \tag{B20}$$

Since $R_{\alpha\beta} = 0$, we have $C^\alpha_{\beta\gamma\delta} = R^\alpha_{\beta\gamma\delta}$. Therefore,

$$C^u{}_{uu}{}^u = \hat{C}^u{}_{uu\Omega} + \mathcal{O}(\Omega^2) = \Omega\left(\frac{1}{4} C^{AB} N_{AB} + 2M\right) + \mathcal{O}(\Omega^2), \tag{B21}$$

$$\begin{aligned} C^u{}_{uu}{}^A &= \hat{g}^{AB}\hat{C}^u{}_{uuB} = (\gamma^{AB} + \mathcal{O}(\Omega))\hat{C}^u{}_{uuB} + \mathcal{O}(\Omega^2) \\ &= \mathcal{O}(\Omega^2), \end{aligned} \tag{B22}$$

$$\begin{aligned} C^A{}_{uu}{}^B &= \hat{g}^{BC}\hat{C}^A{}_{uuC} = (\gamma^{BC} + \mathcal{O}(\Omega))\hat{C}^A{}_{uuC} + \mathcal{O}(\Omega^2) \\ &= \frac{1}{2}\Omega\partial_u N^{AB} + \mathcal{O}(\Omega^2). \end{aligned} \tag{B23}$$

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[1] R. Penrose, Asymptotic Properties of Fields and Space-Times, Phys. Rev. Lett. **10**, 66 (1963).
[2] R. Geroch, in *Asymptotic Structure of Space-Time*, edited by F. P. Esposito and L. Witten (Springer US, Boston, MA, 1977), pp. 1–105.
[3] G. Compère and D. Marolf, Setting the boundary free in AdS/CFT, Classical Quantum Gravity **25**, 195014 (2008).
[4] A. Ashtekar, B. Bonga, and A. Kesavan, Asymptotics with a positive cosmological constant: I. Basic framework, Classical Quantum Gravity **32**, 025004 (2014).
[5] G. Compère, A. Fiorucci, and R. Ruzziconi, The Λ-BMS$_4$ group of dS$_4$ and new boundary conditions for AdS$_4$, Classical Quantum Gravity **36**, 195017 (2019).
[6] G. Compère, A. Fiorucci, and R. Ruzziconi, The Λ-BMS$_4$ charge algebra, J. High Energy Phys. 10 (2020) 205.
[7] M. Kolanowski and J. Lewandowski, Hamiltonian charges in the asymptotically de Sitter spacetimes, J. High Energy Phys. 05 (2021) 063.
[8] J. M. M. Senovilla, Gravitational radiation at infinity with non-negative cosmological constant, Universe **8**, 478 (2022).
[9] S. de Haro, K. Skenderis, and S. N. Solodukhin, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, Commun. Math. Phys. **217**, 595 (2001).
[10] C. N. Kozameh, E. T. Newman, and K. P. Tod, Conformal Einstein spaces, Gen. Relativ. Gravit. **17**, 343 (1985).
[11] M. T. Anderson, Existence and stability of even-dimensional asymptotically de Sitter spaces, Ann. Henri Poincaré **6**, 801 (2005).
[12] M. T. Anderson and P. T. Chruściel, Asymptotically simple solutions of the vacuum Einstein equations in even dimensions, Commun. Math. Phys. **260**, 557 (2005).
[13] W. Kamiński, Well-posedness of the ambient metric equations and stability of even dimensional asymptotically de Sitter spacetimes, arXiv:2108.08085.
[14] P. Günther, über das Cauchysche Problem für die Bachschen Feldgleichungen, Mathematische Nachrichten **69**, 39 (1975).
[15] R. Bach, Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs, Math. Z. **9**, 110 (1921).
[16] C. Cherubini, D. Bini, S. Capozziello, and R. Ruffini, Second order scalar invariants of the Riemann tensor: Applications to black hole spacetimes, Int. J. Mod. Phys. D **11**, 827 (2002).
[17] É. Cartan, Les espaces à connexion conforme, Ann. Soc. Pol. Math. **2**, 171 (1923).
[18] S. Kobayashi, *Transformation Groups in Differential Geometry* (Springer, Berlin, Heidelberg, 1972).
[19] M. Korzyński and J. Lewandowski, The normal conformal Cartan connection and the Bach tensor, Classical Quantum Gravity **20**, 3745 (2003).
[20] S. N. Curry and A. R. Gover, An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity, in *Asymptotic Analysis in General*

Relativity, edited by D. Häfner, J.-P. Nicolas, and T. Daudé (Cambridge University Press, Cambridge, England, 2018), pp. 86–170.
[21] S. A. Merkulov, A conformally invariant theory of gravitation and electromagnetism, Classical Quantum Gravity **1**, 349 (1984).
[22] H. Friedrich, Twistor connection and normal conformal Cartan connection, Gen. Relativ. Gravit. **8**, 303 (1977).
[23] R. Penrose and W. Rindler, *Spinors and Space-Time: Volume 2: Spinor and Twistor Methods in Space-Time Geometry* (Cambridge University Press, Cambridge, England, 1986), Vol. 2.
[24] Y. Herfray and C. Scarinci, Einstein gravity as a gauge theory for the conformal group, Classical Quantum Gravity **39**, 025011 (2022).
[25] Y. Herfray, Asymptotic shear and the intrinsic conformal geometry of null-infinity, J. Math. Phys. (N.Y.) **61**, 072502 (2020).
[26] J. Lewandowski, On the Fefferman class of metrics associated with a three-dimensional CR space, Lett. Math. Phys. **15**, 129 (1988).
[27] P. Nurowski and J. F. Plebański, Non-vacuum twisting type-N metrics, Classical Quantum Gravity **18**, 341 (2001).
[28] C. Batista, Weyl tensor classification in four-dimensional manifolds of all signatures, Gen. Relativ. Gravit. **45**, 785 (2013).
[29] N. Straumann, *General Relativity with Applications to Astrophysics* (Springer, Berlin, 2004).
[30] E. De Paoli and S. Speziale, A gauge-invariant symplectic potential for tetrad general relativity, J. High Energy Phys. 07 (2018) 040.
[31] V. Iyer and R. M. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D **50**, 846 (1994).
[32] G. Compère, *Advanced Lectures on General Relativity* (Springer, Cham, 2019).
[33] R. M. Wald, On identically closed forms locally constructed from a field, J. Math. Phys. (N.Y.) **31**, 2378 (1990).
[34] K. Krasnov, *Formulations of General Relativity: Gravity, Spinors and Differential Forms* (Cambridge University Press, Cambridge, England, 2020).
[35] R. K. Sachs, Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time, Proc. R. Soc. A **270**, 103 (1962).
[36] T. Mädler and J. Winicour, Bondi-Sachs formalism, Scholarpedia **11**, 33528 (2016).
[37] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory* (Princeton University Press, Princeton, NJ, 2018).
[38] L. A. Tamburino and J. H. Winicour, Gravitational fields in finite and conformal Bondi frames, Phys. Rev. **150**, 1039 (1966).
[39] J. Tafel and S. Pukas, Comparison of the Bondi-Sachs and Penrose approaches to asymptotic flatness, Classical Quantum Gravity **17**, 1559 (2000).