Scalar mode quadrupole radiation from astronomical sources in $F(R)$ modified gravity

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We investigate the scalar mode quadrupole radiation of gravitational waves in $F(R)$ modified gravity. In $F(R)$ gravity a massive scalar mode appears in the gravitational waves. We find explicit expressions for the quadrupole radiation and the energy current of the scalar mode in general $F(R)$ gravity models. We consider a binary star and a bouncing star as astronomical sources of the gravitational waves and calculate the quadrupole radiation of the scalar and tensor modes. The scalar mode radiates under spherically symmetric conditions, but the tensor modes do not. The scalar mode mass is estimated for some typical energy scales. We show a possibility to detect the scalar mode in the future gravitational waves observation.

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I. INTRODUCTION

$F(R)$ gravity is a modified gravity theory in which the Einstein-Hilbert action, R, is replaced by a general function of R. It has been introduced as non-linear generalization of Einstein's theory [1]. One of the major applications of this idea has been made on the construction of cosmological models with an accelerating expansion [2]. Numerous models have been proposed to explain the early and late-time accelerating expansion of the universe, as a review, see, for example, [3–5].

There is potential to test the models of $F(R)$ gravity by looking at current observations attributed to the expansion of the universe, for example, type Ia supernovae [6,7], CMB fluctuations [8,9] and BAO [10,11]. Evidence of accelerating expansion alone is not sufficient, and other procedures to test the model of F(R) gravity are being explored. In fact, several studies have been done on the verification of F(R) gravity through the equation of state inside neutron stars [12,13] and its contribution to the solar system [14]. In this paper, we focus on the possibility of testing the models of $F(R)$ gravity in gravitational waves.

The first direct detection of gravitational waves (GWs) from a binary black hole was succeeded in 2015 by LIGO [15]. This is a new clue in examining the theory of gravity. In consequence, the observed gravitational waves were consistent with the predictions of general relativity (GR). It shows that GR can be adapted to strong gravity. However, GWs may directly reveal the existence of phenomena beyond GR. Exploring extra modes of GWs has already been done [16]. Expectations are growing for the development of future GWs detectors such as KAGRA [17], LIGO-India [18], LISA [19] and DECIGO [20].

One of the characteristics of F(R) gravity is that an extra degree of freedom appears in GWs [21–26]. The extra degree of freedom propagates as a scalar mode of GWs. The scalar mode of $F(R)$ gravity has a nonvanishing mass depending on $F(R)$ modification [27–33]. Thus, $F(R)$ gravity can be constrained through the scalar mode mass [34–39].

Here, we investigate the scalar mode propagation in $F(R)$ gravity in more detail. First of all, we try to solve the wave equation with a source. Applying the procedure in Ref. [40] to a general $F(R)$ gravity, the wave equation can be divided into tensor and scalar modes. Then we study the gravitational waves propagation from gravitational sources. We solve the wave equation for the scalar mode and evaluate the quadrupole radiation. The scalar mode radiation is considered from two typical sources, a binary star and a bouncing star that shrinks in size and bounces back. The amplitude of the scalar mode is suppressed by the mass correction. We calculate the suppression compared with the tensor modes. Then we estimate the possibility to detect the scalar mode in future gravitational wave observations. We also evaluate the delay of the massive scalar mode from the first signal according to the propagation speed.

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This paper is organized as follows. Section II describes the basic formulation for the tensor and scalar modes of the gravitational wave. We give expressions for the quadrupole radiation and energy current. In Sec. III we evaluate the scalar mode radiation from a binary star and a bouncing star and discuss the possibility to detect the scalar mode. Finally, we give some concluding remarks.

II. BASIC FORMULATION

A. Wave equation

$F(R)$ gravity is motivated by an exploration of cosmic accelerating expansion such as the inflation and dark energy by extension of the Ricci scalar, R to a general form, $F(R)$ in the action. It is expected that $F(R)$ gravity induces phenomena beyond GR. We focus on the possibility to test the model of $F(R)$ gravity through gravitational wave propagation.

We start from the $F(R)$ gravity action,

$$S = \int d^4x\sqrt{-g}\frac{1}{16\pi G}F(R) + S_{\mathrm{matter}}. \tag{1}$$

where G denotes the gravitational constant. The equation of motion is driven by varying the action (1) with respect to the metric tensor,

$$\mathcal{G}_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

where we introduce the modified Einstein tensor, $\mathcal{G}_{\mu\nu}$, defined by

$$\begin{aligned}\mathcal{G}_{\mu\nu} \equiv F'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F(R) \\ + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F',\end{aligned} \tag{3}$$

and $T_{\mu\nu}$ is the energy-momentum tensor derived from the matter action, S_{matter}. To find the gravitational wave equation, the metric perturbation is employed in Eq. (3). We consider the perturbation of the metric tensor around a flat Minkowski background, $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{4}$$

The perturbation of $F(R)$ and $F'(R)$ around the background curvature $\tilde{R}$ is given by

$$F(R) = F(\tilde{R}) + F'(\tilde{R})\delta R, \tag{5}$$

$$F'(R) = F'(\tilde{R}) + F''(\tilde{R})\delta R. \tag{6}$$

The scalar mode of GWs is identified with

$$\frac{F''(\tilde{R})}{F'(R)}\delta R = \Phi. \tag{7}$$

It should be noted that the curvature, $\tilde{R}$, vanishes in the flat Minkowski background.

The gravitational wave equation (2) contains a mixture of tensor and scalar modes. We extend the prescription separating these two modes in Ref. [40] to a general $F(R)$ gravity. To find a wave equation for the physical degrees of freedom we introduce, $\bar{h}_{\mu\nu}$,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} + \left(b\Phi - \frac{1}{2}h\right)\eta_{\mu\nu}, \tag{8}$$

and impose the following gauge conditions,

$$\nabla^\nu \bar{h}_{\mu\nu} = 0. \tag{9}$$

In these conditions the lowest order of Ricci tensor and scalar are expressed as

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\left[\Box\left(\bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu}\right) + b(\eta_{\mu\nu}\Box\Phi + 2\partial_\mu\partial_\nu\Phi)\right], \tag{10}$$

$$R^{(1)} = \frac{1}{2}\Box\bar{h} - 3b\Box\Phi. \tag{11}$$

We set $T_{\mu\nu} = 0$ and derive the perturbed equation of motion from Eq. (2),

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} + [\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu]\Phi = 0. \tag{12}$$

The perturbed equation is divided into the tensor and scalar parts,

$$-\frac{1}{2}\Box\bar{h}_{\mu\nu} + (b+1)[\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu]\Phi = 0. \tag{13}$$

To eliminate the scalar part from Eq. (13), we set $b = -1$. Then Eq. (13) reduces to

$$\Box\bar{h}_{\mu\nu} = 0. \tag{14}$$

The wave equation of the tensor mode is now successfully extracted and equivalent to the one in GR. On the other hand, the scalar mode equation can be obtained by tracing Eq. (12).

$$[\Box - m^2]\Phi = 0, \tag{15}$$

where the mass squared in Eq. (15) is expressed as,

$$m^2 = \frac{1}{3}\frac{F'(0)}{F''(0)}. \tag{16}$$

The existence of scalar mode is attributed to the $F(R)$ modification. In other words, physics beyond GR emerges. This is because the $F(R)$ modified gravity has an extra degree of freedom [22–24]. A gauge choice such that h vanishes is not possible due to the non-zero mass existence in Eq. (15). The degree of freedom of $h_{\mu\nu}$ is now reduced to six by the gauge condition in Eq. (9). There is still room for gauge choice in the tensor mode wave equation in Eq. (14). $\bar{h}_{\mu\nu}$ is further fixed by the gauge transformation generated by ξ^μ satisfying $\Box\xi^\mu = 0$. By imposing the transverse-traceless gauge conditions on $\bar{h}_{\mu\nu}$ as in GR,

$$\eta^{\mu\nu}\bar{h}_{\mu\nu} = 0, \qquad \bar{h}_{0i} = 0, \tag{17}$$

the remaining four gauge degrees of freedom of ξ^μ are fixed. The physical degree of freedom of the tensor modes, $\bar{h}_{\mu\nu}$ is two. Thus the GWs propagation of tensor mode in $F(R)$ gravity is nothing changes from GR. The two degrees of freedom corresponding to GR are the tensor modes, $\bar{h}_{\mu\nu}$ and the other is the scalar mode, Φ. The mass in Eq. (16) depends on the function $F(R)$ [27–29,31–33]. The scalar mode shows the verifiability of modified gravity theory through the GWs detections [36,37].

B. Tensor modes

To consider the phenomena of GWs, the energy-momentum tensor is induced in the wave equation for tensor mode (14) as a source of GWs,

$$\Box\bar{h}_{\mu\nu} = 8\pi\tilde{G}T_{\mu\nu}. \tag{18}$$

where we redefine the gravitational constant as $\tilde{G} = G/F'$. We find the radiation of gravitational waves from the solution of this equation.

It is more convenient to employ the Fourier representation of $\bar{h}$,

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}}\int dk^0 \bar{h}_{\mu\nu}(\mathbf{x}, k^0)e^{-ik^0 t}. \tag{19}$$

Green's function is defined by the solution of $\Box G(\mathbf{x}) = \delta(\mathbf{x})$. By using the Fourier representation of Green's function, the tensor mode solution of Eq. (18) is found to be

$$\bar{h}_{\mu\nu}(\mathbf{x}, k^0) = -16\pi\tilde{G}\int d^3\mathbf{x} G(\mathbf{x}-\mathbf{x}', k_0)T_{\mu\nu}(\mathbf{x}', k_0). \tag{20}$$

As is well-known, the Fourier representation of Green's function is given by

$$\begin{aligned} G(\mathbf{x}, k_0) &= \frac{1}{\sqrt{2\pi}^3}\int \frac{1}{\sqrt{2\pi}^3}\frac{1}{-\mathbf{k}^2 + k_0^2}e^{i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{k} \\ &= -\frac{1}{4\pi|\mathbf{x}|}e^{ik_0|\mathbf{x}|}. \end{aligned} \tag{21}$$

Substituting Eq. (21) into Eqs. (20) and (19), we obtain the retarded solution of the tensor mode,

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = 4\tilde{G}\int d^3\mathbf{x}' \frac{T_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}. \tag{22}$$

This solution shows that GWs emitted from the source travel at the speed of light. The only difference in tensor mode between GR and F(R) gravity is the gravitational constant. If $\tilde{G}$ is regarded as the observed constant, no difference appears.

C. Scalar mode

For the scalar mode propagation from gravitational sources, the trace of the energy-momentum tensor is introduced in the wave equation (15),

$$[\Box - m^2]\Phi = 8\pi\tilde{G}T. \tag{23}$$

Green's function for scalar mode is defined as the solution of $[\Box - m^2]\mathcal{G}(\mathbf{x}) = \delta(\mathbf{x})$. The difference from Green's function in the tensor mode is the nonvanishing mass. After the integral with respect to the wave vector, we obtain the Fourier representation of the Green's function,

$$\begin{aligned} \mathcal{G}(\mathbf{x}, k_0) &= \frac{1}{\sqrt{2\pi}^3}\int \frac{1}{\sqrt{2\pi}^3}\frac{1}{-\mathbf{k}^2 - m^2 + k_0^2}e^{i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{k} \\ &= -\frac{1}{4\pi|\mathbf{x}|}e^{i\sqrt{k_0^2-m^2}|\mathbf{x}|}. \end{aligned}$$

The Fourier representation of the scalar mode is given by

$$\Phi(\mathbf{x}, k^0) = -16\pi\tilde{G}\int d^3\mathbf{x}\mathcal{G}(\mathbf{x}-\mathbf{x}', k_0)T(\mathbf{x}', k_0). \tag{24}$$

By the inverse Fourier transformation, the scalar mode is represented as

$$\begin{aligned} \Phi(\mathbf{x}, t) &= -16\pi\tilde{G}\int dk^0\int d^3\mathbf{x}'\mathcal{G}(\mathbf{x}-\mathbf{x}', k_0)T(\mathbf{x}', k_0)e^{-ik^0 t} \\ &= 4\tilde{G}\int d^4x' T(\mathbf{x}', t')\mathcal{G}(|\mathbf{x}-\mathbf{x}'|, t-t'). \end{aligned} \tag{25}$$

To simplify the expression we set $t - t' = \Delta t$. Then, Green's function, $\mathcal{G}(|\mathbf{x}-\mathbf{x}'|, \Delta t)$, is rewritten as

$$\mathcal{G}(|\mathbf{x}-\mathbf{x}'|, \Delta t) = \frac{1}{2\pi|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{\infty} e^{i(\sqrt{k^{02}-m^2}|\mathbf{x}-\mathbf{x}'| - k_0\Delta t)}dk^0. \tag{26}$$

The Green's function is represented by the Bessel function according to Refs. [41,42],

$$\mathcal{G}(|\mathbf{x}-\mathbf{x}'|,\Delta t)=\frac{\delta(\Delta t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}-\frac{m}{\sqrt{\Delta t^2-|\mathbf{x}-\mathbf{x}'|^2}}\times J_1(m\sqrt{\Delta t^2-|\mathbf{x}-\mathbf{x}'|^2})\theta(\Delta t-|\mathbf{x}-\mathbf{x}'|). \tag{27}$$

Substituting Eq. (27) to Eq. (25), the retarded solution of the scalar mode is found to be

$$\Phi(\mathbf{x},t)=4\tilde{G}\int d^3\mathbf{x}'\left[\frac{T(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}-\int_{-\infty}^{t-|\mathbf{x}-\mathbf{x}'|}dt'\frac{m}{\sqrt{\Delta t^2-|\mathbf{x}-\mathbf{x}'|^2}}\times J_1(m\sqrt{\Delta t^2-|\mathbf{x}-\mathbf{x}'|^2})T(\mathbf{x}',t')\right]. \tag{28}$$

We introduce the time-dependent parameters, $t_p=t-|\mathbf{x}-\mathbf{x}'|$, $t_f=t+|\mathbf{x}-\mathbf{x}'|$ and write $\tau=\sqrt{(t'-t_p)(t'-t_f)}$. Then the second term on the right-hand side in Eq. (28) is rewritten as

$$\int_V d^3\mathbf{x}'\int_0^\infty md\tau\frac{J_1(m\tau)}{m\sqrt{\tau^2+|\mathbf{x}-\mathbf{x}'|^2}}\times T\left(\mathbf{x}',t-\sqrt{\tau^2+|\mathbf{x}-\mathbf{x}'|^2}\right). \tag{29}$$

We transform the integral variable τ to ζ with $m\tau=m|\mathbf{x}-\mathbf{x}'|\sinh\zeta$. Equation (29) is simplified to

$$\int_V d^3\mathbf{x}'\int_0^\infty d\zeta J_1(m|\mathbf{x}-\mathbf{x}'|\sinh\zeta)\times T(\mathbf{x}',t-\cosh\zeta|\mathbf{x}-\mathbf{x}'|). \tag{30}$$

Therefore, the scalar mode is found to be

$$\Phi(\mathbf{x},t)=4\tilde{G}\int d^3\mathbf{x}'\int_0^\infty d\zeta\left[\frac{\delta(\zeta)}{|\mathbf{x}-\mathbf{x}'|}-mJ_1(m|\mathbf{x}-\mathbf{x}'|\sinh\zeta)\right]\times T(\mathbf{x}',t-\cosh\zeta|\mathbf{x}-\mathbf{x}'|). \tag{31}$$

It should be noted that $\cosh\zeta$ takes the value from 1 to ∞ for the interval of the integration, $\zeta:0\to\infty$. We regard $c_s\equiv 1/\cosh\zeta$ as the velocity of scalar mode propagation. For $\zeta=0$ the velocity is equal to the speed of light. At the limit $\zeta\to\infty$ the scalar mode does not propagate, i.e. $c_s=0$.

D. Quadrupole radiation

Here we focus on GW radiation whose source is sufficiently far away from the observer and the gravitational source is non-relativistic. In this case, the GW radiation is generated by the quadrupole and higher moments of the energy and momentum distributions.

First, we consider the tensor mode solution. For the observer, $|\mathbf{x}-\mathbf{x}'|\sim|\mathbf{x}|\equiv r$, Eq. (22) is written as

$$\bar{h}^{ij}(\mathbf{x},t)\sim\frac{4\tilde{G}}{r}\int T^{ij}(\mathbf{x}',t-r)d^3\mathbf{x}', \tag{32}$$

at the leading order. From the conservation law, $\partial_\mu T^{\mu\nu}=0$, we obtain

$$\partial_\mu\partial_\nu x^i x^j T^{\mu\nu}(x)=2T^{ij}(x).$$

The three-dimensional spatial integration of this equation gives

$$\int 2T^{ij}(x)d^3\mathbf{x}=\int d^3\mathbf{x}[\partial_m\partial_l x^i x^j T^{ml}+\partial_0\partial_0 x^i x^j T^{00}(x)+2\partial_k\partial_0 x^i x^j(T^{k0}(x)+T^{0k}(x))]=\partial_0\partial_0\int d^3\mathbf{x}x^i x^j T^{00}(x). \tag{33}$$

From the first line to the second line in this equation, we drop the surface terms. Substituting Eq. (33) into Eq. (32), the tensor mode solution is represented as the 2nd derivative of the quadrupole moment, I^{ij},

$$\bar{h}^{ij}(\mathbf{x},t)\sim\frac{2\tilde{G}}{r}\partial_0\partial_0\int T^{00}(\mathbf{x}',t-r)x^{i\prime}x^{j\prime}d^3\mathbf{x}'=\frac{2\tilde{G}}{r}\frac{d^2I^{ij}}{d^2t}. \tag{34}$$

It should be noted that the projection operators are necessary to describe the polarization of the tensor modes.

Next, we move to the scalar mode solution (31). The change of variable, $w=mr\sinh\zeta$, makes the integral of the Bessel function easier to compute [43],

$$\int_0^\infty d\zeta mJ_1(mr\sinh\zeta)=\int_0^\infty dw\frac{mJ_1(w)}{\sqrt{w^2+(mr)^2}}=mI_{\frac{1}{2}}\left(\frac{mr}{2}\right)K_{\frac{1}{2}}\left(\frac{mr}{2}\right), \tag{35}$$

where $I_{\frac{1}{2}}$ and $K_{\frac{1}{2}}$ denote the modified Bessel functions and satisfy,

$$I_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sinh z,\qquad K_{\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}. \tag{36}$$

For the distant observer, it is assumed that the velocity of scalar mode propagation is almost constant and the energy-momentum tensor is independent of the value of $\cosh\zeta$.

In other words, assuming that the energy-momentum tensor does not depend on the velocity or the velocity changes a little. Then Eq. (31) becomes

$$\begin{aligned}\Phi &= 4\tilde{G}\int d^3\mathbf{x}'\int_0^\infty d\zeta\left[\frac{\delta(\zeta)}{r} - mJ_1(mr\sinh\zeta)\right]\\ &\quad\times T\left(\mathbf{x}', t-\frac{r}{c_s}\right)\\ &\sim 4\tilde{G}\int d^3\mathbf{x}'\left[\frac{1}{r} - \frac{1-e^{-mr}}{r}\right]T\left(\mathbf{x}', t-\frac{r}{c_s}\right)\\ &= \frac{4\tilde{G}e^{-mr}}{r}\int d^3\mathbf{x}' T\left(\mathbf{x}', t-\frac{r}{c_s}\right). \end{aligned} \tag{37}$$

The tracing of the energy-momentum tensor can be divided into 00 and spatial parts. T_{00} gives the mass density and T_{ij} is related to the 2nd derivative of the quadrupole moment as can be seen from the tensor mode analogy,

$$\begin{aligned}\Phi &= \frac{4\tilde{G}e^{-mr}}{r}\int T d^3\mathbf{x}'\\ &= \frac{4\tilde{G}e^{-mr}}{r}\int (T^0{}_0 + T^i{}_i) d^3\mathbf{x}' \end{aligned} \tag{38}$$

$$= \frac{4\tilde{G}e^{-mr}}{r}M + \frac{2\tilde{G}e^{-mr}}{r}\frac{d^2 I}{d^2 t}. \tag{39}$$

The scalar mode has a Yukawa-like potential depending on the total mass and the trace of the quadrupole moment, I. This property is due to the fact that the scalar mode is massive, which has been obtained in other studies [31,33]. The additional quadrupole radiation part enables us to understand dynamical phenomena.

E. Energy current

Following the procedure developed in [40], we calculate the effective energy-momentum tensor for a general form of Φ. The gravitational radiations carry energy and then act as a source of gravitational fields. In order to introduce the background metric arising from gravitational waves themselves we have to consider the perturbation around a curved background metric, $\gamma_{\mu\nu}$.

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}. \tag{40}$$

The perturbation of the modified Einstein tensor can be described as,

$$\mathcal{G}_{\mu\nu} = \mathcal{G}^{\mathrm{B}}_{\mu\nu} + \mathcal{G}^{(1)}_{\mu\nu} + \mathcal{G}^{(2)}_{\mu\nu},$$

where the number in the upper indices denotes the order of the expansion and $\mathcal{G}^{\mathrm{B}}_{\mu\nu}$ is the modified Einstein tensor for the background. For the GWs the 1st order term vanishes, $\mathcal{G}^{(1)}_{\mu\nu} = 0$, from the wave equation. Then the background satisfies

$$\mathcal{G}^{\mathrm{B}}_{\mu\nu} = -\mathcal{G}^{(2)}_{\mu\nu}. \tag{41}$$

Later, we will average over several wavelengths, $\langle\mathcal{G}^{(2)}_{\mu\nu}\rangle$, assuming that the background is on a large scale compared with the wavelengths of GWs. Equation (41) means that the background metric $\gamma_{\mu\nu}$ is $\mathcal{O}(h^2)$. So the background is decomposed

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + j_{\mu\nu}, \tag{42}$$

where $j_{\mu\nu}$ is the order $\mathcal{O}(h^2)$. The curvature tensor of the background is also $\mathcal{O}(h^2)$. Then the 2nd-order perturbation of the Ricci tensor is given by

$$\begin{aligned}R^{(2)}_{\mu\nu} &= \frac{1}{4}\nabla_\mu h^{\alpha\beta}\nabla_\nu h_{\alpha\beta} + \frac{1}{2}h^{\alpha\beta}(\nabla_\mu\nabla_\nu h_{\alpha\beta} + \nabla_\alpha\nabla_\beta h_{\mu\nu} - \nabla_\alpha\nabla_\nu h_{\mu\beta} - \nabla_\alpha\nabla_\mu h_{\beta\nu})\\ &\quad -\frac{1}{2}\left(\nabla_\beta h^{\alpha\beta} - \frac{1}{2}\nabla^\alpha h\right)(\nabla_\nu h_{\mu\alpha} + \nabla_\mu h_{\alpha\nu} - \nabla_\alpha h_{\mu\nu}) + \frac{1}{2}\nabla^\beta h^\alpha{}_\nu(\nabla_\beta h_{\mu\alpha} - \nabla_\alpha h_{\mu\beta}). \end{aligned} \tag{43}$$

From Eq. (8) with $b = -1$ it is expressed by $\bar{h}$ and Φ as,

$$R^{(2)}_{\mu\nu} = \frac{1}{4}\nabla_\mu\bar{h}^{\alpha\beta}\nabla_\nu\bar{h}_{\alpha\beta} + \frac{1}{2}\bar{h}^{\alpha\beta}\nabla_\mu\nabla_\nu\bar{h}_{\alpha\beta} + \frac{3}{2}\nabla_\mu\Phi\nabla_\nu\Phi + \Phi\nabla_\mu\nabla_\nu\Phi + \frac{1}{2}\gamma_{\mu\nu}\Phi\Box\Phi. \tag{44}$$

Thus the 2nd-order modified Einstein tensor (3) is found to be

$$\begin{aligned}\mathcal{G}^{(2)}{}_{\mu\nu} &= F'\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R^{(2)} - \frac{1}{2}h_{\mu\nu}R^{(1)}\right] + F''\left[R^{(1)}R^{(1)}_{\mu\nu} - \frac{1}{4}\gamma_{\mu\nu}R^{(1)2}\right] + \gamma_{\mu\nu}\Box(F''R^{(2)}) - \gamma_{\mu\nu}h^{\alpha\beta}F'\partial_\alpha\partial_\beta\Phi\\ &\quad + h_{\mu\nu}\Box(F''R^{(1)}) - \gamma_{\mu\nu}\gamma^{\alpha\beta}\Gamma^{\rho(1)}_{\alpha\beta}F'\partial_\rho\Phi - \partial_\mu\partial_\nu(F''R^{(2)}) + \Gamma^{\rho(1)}_{\mu\nu}F'\partial_\rho\Phi, \end{aligned} \tag{45}$$

where the perturbation of the connection is

$$\Gamma^{\rho(1)}_{\mu\nu} = \frac{1}{2}\gamma^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}) \\ = \frac{1}{2}\gamma^{\rho\lambda}(\partial_\mu \bar{h}_{\lambda\nu} + \partial_\nu \bar{h}_{\mu\lambda} - \partial_\lambda \bar{h}_{\mu\nu}) \\ - \frac{1}{2}\gamma^{\rho\lambda}(\gamma_{\lambda\nu}\partial_\mu\Phi + \gamma_{\mu\lambda}\partial_\nu\Phi - \gamma_{\mu\nu}\partial_\lambda\Phi).$$

On a large-scale background curvature, the terms that remain after averaging over several wavelengths are

$$\langle R^{(2)}_{\mu\nu}\rangle = \left\langle -\frac{1}{4}\partial_\mu \bar{h}^{\alpha\beta}\partial_\nu \bar{h}_{\alpha\beta} + \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}\gamma_{\mu\nu}\Phi\Box\Phi \right\rangle, \quad (46)$$

and

$$\langle R^{(2)}\rangle = \langle \gamma^{\mu\nu}R^{(2)}_{\mu\nu} - h^{\mu\nu}R^{(1)}_{\mu\nu}\rangle = \left\langle \frac{9}{2}\Phi\Box\Phi \right\rangle. \quad (47)$$

The averages over several wavelengths for Eqs. (10) and (11) are

$$\langle R^{(1)}_{\mu\nu}\rangle = \frac{1}{2}\langle \gamma_{\mu\nu}\Box\Phi + 2\partial_\mu\partial_\nu\Phi\rangle, \\ \langle R^{(1)}\rangle = \langle 3\Box\Phi\rangle,$$

where we use the wave equation Eq. (14) so we take $\Box\bar{h}_{\mu\nu} = 0$. Then we get

$$\langle R^{(1)}R^{(1)}_{\mu\nu}\rangle = \left\langle 3\Box\Phi\partial_\mu\partial_\nu\Phi + \frac{3}{2}\gamma_{\mu\nu}(\Box\Phi)^2 \right\rangle.$$

Thus, we obtain the average of the 2nd order perturbation of modified Einstein tensor (45),

$$\langle \mathcal{G}^{(2)}_{\mu\nu}\rangle = F'\left\langle -\frac{1}{4}\partial_\mu \bar{h}^{\alpha\beta}\partial_\nu \bar{h}_{\alpha\beta} - \frac{3}{2}\partial_\mu\Phi\partial_\nu\Phi \right\rangle. \quad (48)$$

The effective energy-momentum tensor is defined by

$$F'T^G_{\mu\nu} \equiv -\frac{1}{8\pi\tilde{G}}\langle \mathcal{G}^{(2)}_{\mu\nu}\rangle. \quad (49)$$

Substituting Eq. (48) into Eq. (49), we successfully derived the effective energy-momentum tensor including the scalar mode in the general case, Φ.

$$T^G_{\mu\nu} = \frac{1}{8\pi\tilde{G}}\left\langle \frac{1}{4}\partial_\mu \bar{h}^{\alpha\beta}\partial_\nu \bar{h}_{\alpha\beta} + \frac{3}{2}\partial_\mu\Phi\partial_\nu\Phi \right\rangle. \quad (50)$$

By the replacement of t and r, the energy current is given by

$$\frac{dE_{\mathrm{GW}}}{dt} = -\int \langle T^G_{0r}(t-r)\rangle r^2 d\Omega \\ = \int \langle T^G_{00}(t-r)\rangle r^2 d\Omega, \quad (51)$$

where we take the propagation speed of the scalar mode to almost light speed, $c_s \sim 1$. We will see the validity of this assumption in the later section. The tensor and scalar modes, $\bar{h}$ and Φ, are described by the 2nd derivative of the quadrupole moment. For a distant observer the total mass, M, is conserved, and the time derivative of the first term in Eq. (39) drops. Then the energy current is written in the quadrupole representation,

$$\frac{dE_{\mathrm{GW}}}{dt} = \left\langle \frac{\tilde{G}}{5}\dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij} + 12\tilde{G}e^{-2mr}\dddot{I}^2 \right\rangle, \quad (52)$$

Eq. (52) shows that the scalar mode emerges in gravitational radiation in addition to the tensor modes.

Below the gravitational constant $\tilde{G}$ is written as G.

III. SCALAR MODE QUADRUPOLE RADIATION

A. Binary star

At present, GWs from compact binary stars are the most promising source for observations. We focus on the scalar mode GWs from binary stars. It is assumed that the binary star rotates on the xy plane and these masses have m_1, m_2 and the stellar distance is L. The distances from the center of gravity to each star are given by $(r_1, r_2) = (m_2 L/M, m_1 L/M)$.

The quadrupole moment is defined by

$$I_{ij} = \int d^3x' \rho(x') x'_i x'_j. \quad (53)$$

The density and position of the binary star are represented as,

$$\rho(x) = m_1\delta(\mathbf{x} - \mathbf{x_1}) + m_2\delta(\mathbf{x} - \mathbf{x_2}), \\ \mathbf{x_1} = (r_1\cos\omega t, r_1\sin\omega t, 0), \\ \mathbf{x_2} = (-r_2\cos\omega t, -r_2\sin\omega t, 0), \quad \left(\omega = \sqrt{GM/L^3}\right).$$

After the spatial integration, the quadrupole moment of the binary star is derived

$$I_{ij} = \begin{pmatrix} L^2\mu\cos^2\omega t & L^2\mu\cos\omega t\sin\omega t & 0 \\ L^2\mu\cos\omega t\sin\omega t & L^2\mu\sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (54)$$

where μ denotes the reduced mass, $\mu \equiv m_1 m_2/(m_1 + m_2)$. When we take the typical velocity of the stars v and the distance between Earth and the binary star r, the amplitude of tensor mode from a binary star is evaluated by Eq. (34),

$$|h_{ij}| = \frac{4G}{rc^4}\frac{\mu L^2(2\pi f)^2}{r} \sim \frac{4G}{rc^4}\frac{\mu v^2}{r} \sim 5\times 10^{-23}\left(\frac{100\ \mathrm{Mpc}}{r}\right)\left(\frac{\mu}{10M_\odot}\right)\left(\frac{v}{0.1c}\right)^2. \quad (55)$$

On the other hand, the trace of the quadrupole moment (54) becomes $I = L^2\mu$. If the trace of the quadrupole moment does not have time dependence, the scalar mode does not radiate from a binary star. However, we have not taken into account the energy carried out by GWs. For consistency with current GW observations, we assume tensor mode GW radiation to be dominant. Since the tensor modes GW carries away the energy of a binary star, the interstellar distance L decreases monotonically with time as $L = L_0(1 - t/t_{\mathrm{coal}})^{1/4}$ where t_{coal} is the time of coalescence [44,45],

$$t_{\mathrm{coal}} = \frac{5}{256}\frac{c^5}{G^3}\frac{L_0{}^4}{\mu M^2}.$$

The scalar mode may have a chirp signal that does not oscillate from Eq. (39). Thus the amplitude is calculated to be

$$\Phi \sim \frac{4G\mu L_0{}^2}{c^4 t_{\mathrm{coal}}^2 r} \sim \frac{2^{18}}{5^2}\frac{G\mu^3}{c^2 M r}\left(\frac{v}{c}\right) \sim 5\times 10^{-31}\left(\frac{100\ \mathrm{Mpc}}{r}\right)\left(\frac{10M_\odot}{M}\right)^2 \times \left(\frac{\mu}{10M_\odot}\right)^2\left(\frac{v}{0.1c}\right)^{14}. \quad (56)$$

The strain of the amplitude is extremely small and it is consistent with the assumption. It increases over time but is not quite sufficient for observation. There is little hope to observe the scalar mode GWs from a binary star. However, we considered only the inspiral phase. The compact binary coalescence has the phases such as merger and ringdown phases [46]. It is an interesting topic, although it requires more precise analysis [47,48].

B. Bouncing star

Let us now study a toy model that we call a bouncing star. It is far from a real phenomenon such as a supernova explosion. However, it does provide some clues about the scalar mode propagation in spherically symmetric gravitational sources.

We consider a star with the radius $R(t)$ and the density $\rho = M/(\frac{4\pi}{3}R(t)^3)$, where M is the total mass of the star and does not depend on time. The trace of the quadrupole moment becomes

$$I = \int d^3x'\rho(x')x'_i x'^i = 4\pi\int_0^{R(t)} dr'\rho r'^4 = \frac{3}{5}MR(t)^2. \quad (57)$$

We assume that the star shrinks and bounces once and write the time evolution of radius as,

$$R(t) = R_0\left(1 - be^{\frac{-(t-t_0)^2}{\tau^2}}\right).$$

The star shrinks to $R_0(1-b)$ and bounces at $t = t_0$. The bouncing time interval is characterized by τ. In this situation, the scalar mode from Eq. (39) is given by

$$\Phi = \frac{48\tilde{G}MR_0{}^2 b}{5\tau^2}\frac{e^{-mr}}{r}\left[\left(1 - \frac{2(t-t_0)^2}{\tau^2}\right) - b\left(1 - \frac{4(t-t_0)^2}{\tau^2}\right)e^{\frac{-(t-t_0)^2}{\tau}}\right]e^{\frac{-(t-t_0)^2}{\tau^2}}, \quad (58)$$

where the static potential is neglected. The scalar mode is emitted from the star with spherical symmetry. This result is interesting because the tensor modes do not radiate from spherically symmetric objects.

We estimate the amplitude of the bouncing star. Applying the bouncing of the core in a supernova explosion, we find

$$|\Phi|_{\mathrm{typical}} = \frac{48\tilde{G}MR_0{}^2 b}{5\tau^2}\frac{e^{-mr}}{r} = 7.93\times 10^{-44}\frac{MR_0{}^2 b}{\tau^2}\frac{e^{-mr}}{r} = 2\times 10^{-20}\left(\frac{10\,\mathrm{kpc}}{r}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{R_0}{6000\,\mathrm{km}}\right)^2\left(\frac{1\ \mathrm{s}}{\tau}\right)^2,$$

where the exponential term is dropped by assuming that the mass is sufficiently small and b approximated to 1. It shows that a core collapse of a supernova explosion in our galaxy may emit the detectable scalar mode GWs. Also, the energy current in this event is estimated from Eq. (52),

$$\frac{dE_{\mathrm{GW}}}{dt} \sim \frac{12G}{c^5}\frac{M^2R_0{}^4 b}{\tau^8} \sim 2\times 10^{43}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_0}{6000\,\mathrm{km}}\right)^4\left(\frac{1\ \mathrm{s}}{\tau}\right)^8 \mathrm{erg/s}. \quad (59)$$

The gravitational potential energy released in the supernova collapse is estimated in the order of 10^{53} erg [49–51]. The scalar mode GWs cost only $10^{-8}\%$ of total energy emission. The existence of a scalar mode does not have a significant contribution to supernova explosions and subsequent growth.

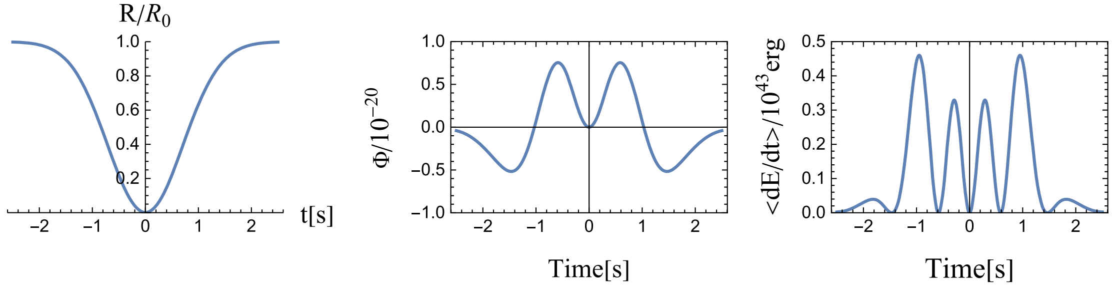

FIG. 1. Time dependence of radius(left), amplitude(middle), energy current(right) at $b = 1$.

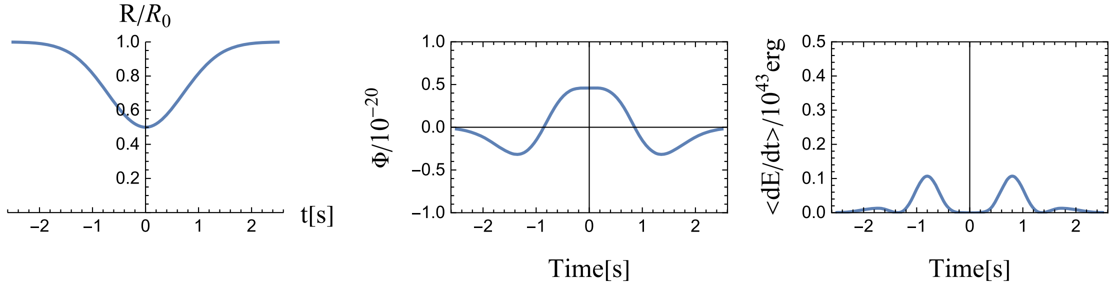

FIG. 2. Time dependence of radius(left), amplitude(middle), energy current(right) at $b = 0.5$.

Figure 1 shows the time dependence of the radius of the bouncing star, amplitude, and energy current of the scalar mode GWs for $b = 1$. We also show the figure in the case of $b = 0.5$ in Fig. 2 for comparison. The parameter b produces a large difference in energy release. Therefore, a dramatic event, such as the collapse of a star, is necessary to generate detectable scalar mode GWs.

C. Scalar-tensor ratio

We compare the amplitudes for the tensor and scalar modes. The scalar-tensor ratio of GWs is defined by

$$\mathcal{R} = \frac{|\Phi|}{|h_{ij}|}. \tag{60}$$

We assume that the quadrupole radiation intensities of both modes are equivalent, $\ddot{I} = \ddot{I}_{ij}$. The scalar-tensor ratio only depends on the exponential term in Eq. (39),

$$\mathcal{R} = e^{-mr}. \tag{61}$$

Below we estimate the ratio in some mass scale of modified gravity. The scale of mass depends on the modification scale of gravity theory. For instance, the R^2 model [2], $F(R) = R + R^2/M^2$ has the scalar mode mass, $m = M/\sqrt{6}$ from Eq. (16).

For the dark energy scale, $m = 10^{-33}\,\mathrm{eV} \sim (4200\,\mathrm{Mpc})^{-1}$ we obtain

$$\mathcal{R} \sim (0.999998)^{\frac{r}{10\,\mathrm{kpc}}},$$
$$(0.976)^{\frac{r}{100\,\mathrm{Mpc}}},$$
$$(0.368)^{\frac{r}{4200\,\mathrm{Mpc}}},$$

where the distances are assumed as 10 kpc for the scale of the Galaxy, 100 Mpc for the scale of galaxy clusters, and 4200 Mpc for the scale of primordial gravitational waves. Since the attenuation is a few to 60 percent, they are not a major obstacle to observation. On the other hand, we obtain

$$\mathcal{R} \sim (10^{-7\times 10^{50}})^{\frac{r}{10\,\mathrm{kpc}}}, \tag{62}$$

for the inflation scale, $m = 10^{15}$ GeV $\sim (2 \times 10^{-31}\ \mathrm{m})^{-1}$. In this case, the scalar mode rapidly suppresses. It seems very difficult to observe the scalar mode with the inflation scale. Therefore, the scalar mode GWs are interesting observable physical quantities when the typical scale of the modified gravity is at the dark energy scale.

D. Constraints from propagation speed

We obtain the speed of the scalar mode propagation in Eq. (31). Constraints from the propagation speed also help in the verification of the scalar mode GWs as well as the scalar-tensor ratio. The mass constraints are found from the propagation speed in some observation periods. The propagation speed for a wave packet is derived as the group velocity,

$$c_s = \frac{\partial\omega}{\partial k}. \tag{63}$$

The dispersion of the scalar mode is $\omega = \sqrt{k^2 + m^2}$. The propagation speed of the scalar mode becomes

$$c_s = \sqrt{\frac{k^2}{k^2 + m^2}} = \sqrt{\frac{\omega^2 - m^2}{\omega^2}}. \tag{64}$$

The tensor modes propagate at light speed and the scalar mode does at c_s. From Eq. (64) the scalar mode mass is estimated as

$$m = \omega\sqrt{\left(1 - \left(\frac{c_s}{c}\right)^2\right)}$$
$$= 4.14 \times 10^{-15}\left(\frac{f}{1\ \mathrm{Hz}}\right)\sqrt{1 - \left(\frac{c_s}{c}\right)^2}\ [\mathrm{eV}/c^2], \tag{65}$$

where we denote $\omega = 2\pi f$ and the light speed c is not omitted.

We write Δt as the difference between the arrival time of the tensor and scalar modes at a distance r. It is described as,

$$\Delta t = \frac{r}{c_s} - \frac{r}{c}, \tag{66}$$

Then the ratio of the propagation speeds is estimated as,

$$\frac{c_s}{c} = \frac{1}{1 + \frac{c\Delta t}{r}}. \tag{67}$$

If we detected GWs from inside the Galaxy, $r \sim 10$ kpc, and the maximum delay is a century, the lower bound of c_s/c is determined from Eq. (67),

$$\frac{c_s}{c} \geq 0.99695. \tag{68}$$

In a century-long observation, the upper bound of the scalar mass is $3.2 \times 10^{-16}\ \mathrm{eV}/c^2 \geq m$ from Eq. (65).

In Table. I, the mass constraints are summarized for several cases of distance and observation period. In especially, the scalar mode GW has $\Delta t \sim 3 \times 10^{-26}$ s delay from the tensor modes when the mass is $10^{-33}\ \mathrm{eV}/c^2$. The scalar mode mass in the dark energy scale is difficult to observe because of the tiny delay from the tensor modes.

TABLE I. The upper bound of scalar mode mass $m[\mathrm{eV}/c^2]$ at 1 Hz.

Period	10 kpc	100 Mpc	4200 Mpc
A second	5.8×10^{-21}	6×10^{-23}	9×10^{-24}
A day	1.7×10^{-18}	1.7×10^{-20}	2.6×10^{-21}
A year	3.2×10^{-17}	3.2×10^{-19}	5.0×10^{-20}
A century	3.2×10^{-16}	3.2×10^{-18}	5.0×10^{-19}

IV. CONCLUSION

We have investigated the quadrupole radiation of GWs in $F(R)$ gravity. $F(R)$ gravity has an extra degree of freedom in the wave equations beyond GR. Thus the scalar mode also radiates in addition to the tensor modes. The scalar mode has a mass that depends on the $F(R)$ modification. We have derived the retarded solution in Eq. (31). The quadrupole radiation in the scalar mode is represented as a function of the trace of the quadrupole momentum. It has been shown that the amplitude of the scalar mode is suppressed exponentially. Also, we have derived the GW energy current including the scalar mode for a general $F(R)$ form in Eq. (52).

We have considered the scalar mode radiation from several astronomical sources. The radiation from binary stars is currently the most successful gravity source for tensor modes but the amplitude is too weak to detect the scalar mode GWs. However, there is not enough research on the moment of star coalescence and there is room for the observation of the scalar mode radiation.

We have evaluated a simple model of the bouncing star. The model is not appropriate to adapt to real stars, but it provides some clues to understand the scalar mode radiation. Spherically symmetric sources emit the scalar mode GWs, not the tensor modes. Applying the supernova explosions to the bouncing star, we show that the scalar mode radiation from the events inside the Galaxy is possible to detect in future GWs observations. This phenomenon is expected to be a promising candidate for the detection of scalar mode GWs. We have calculated the ratio of the amplitude for the scalar and tensor modes and found it proportional to e^{-mr}. If the scalar mode mass is at the dark energy scale, it does not suppress even for a cosmological distance. On the other hand, it is promptly suppressed at the inflation scale.

It is known as the chameleon mechanism that matter-energy density increases the mass of a scalar mode, making them unobservable. The mechanism is applied to the modified gravity. It has been shown that the signal is screened in ground-based detectors such as LIGO, while space-based ones such as LISA have a chance to observe [31]. In this work, we have focused on the GWs radiation from gravitational sources with a small matter energy density. The contribution from the background matter field is not taken into account. We are interested in evaluating the screening mechanism for the scalar mode in the solar system and on Earth.

The upper bounds on the scalar modes mass have been estimated from the propagation speed constraints in the observation period. It is much smaller than the inflation scale and larger than the dark energy scale. In addition to that, in phenomena where tensor modes are hardly radiated, we are able to obtain similar constraints from the photon instead of the massless tensor modes.

We conclude that the verification of $F(R)$ modified gravity using GWs is hopeful for the mode of the current accelerating expansion. It is difficult to obtain evidence of modification on a high-energy scale such as inflation in the current GW detectors. The scalar mode can be radiated from a spherically symmetric gravitational source, which is not predicted by GR. The observation of the scalar mode directly proves the necessity of an extension of GR.

There are other sources of GWs. We are interested in GWs from the early universe such as bubble collisions [52–55]. GWs from high-energy events in the early universe may directly or indirectly influence observations of cosmological phenomena [56]. These phenomena will become important with the next generation of GW observations [57–64]. We will continue the work and compare the results in $F(R)$ gravity with other modified gravity theories such as $F(T)$ [65,66], $F(\mathcal{G})$ [67–69], other formalisms, Palatini $F(R)$ [70] and Cartan $F(R)$ [71]. By examining these in detail, we hope to find the potential of the modified gravity.

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