Cosmographic view on the H_0 and σ_8 tensions

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Measurements of the H_0 and σ_8 parameters within the standard cosmological model recently highlighted significant statistical tensions between the cosmic microwave background and low-redshift probes, such as local distance ladder, weak lensing, and galaxy clustering surveys. In this work, we frame geometrical distances in a model-independent way by means of cosmographic approximations in the range $z \in (0, 2.3)$ to take into account a robust dataset composed of baryon acoustic oscillations (BAOs), type Ia supernovae (SN), cosmic chronometer (CC) data, and measurements from redshift space distortions (RSDs). From the joint analysis BAO + SN + CC + RSD, we find an accuracy of ∼1.4% and ∼3.7% on H_0 and σ_8, respectively. Our result for H_0 is at 2σ tension with local measurements by the SH0ES team, while our σ_8 estimate is at 2.6σ tension with Planck-CMB analysis. This inference shows a tension statistically smaller when compared to those estimated via the Lambda cold dark matter (ΛCDM) model. We also find that the jerk parameter can deviate more than 3σ from the ΛCDM prediction. Under the same cosmographic setup, we also present results by considering a SH0ES Gaussian prior on H_0 that allows for improved accuracy of the parameter space of the models. The present work brings observational constraints on H_0 and σ_8 into a new model-independent perspective, which differs from the predictions obtained within the ΛCDM paradigm.

I. INTRODUCTION

Within the standard framework of general relativity, the acceleration of the Universe observed today could be attributed to dark energy under the form of the cosmological constant (Λ), which drives the late-time cosmic evolution and whose origins are traced back to early quantum fluctuations of vacuum [1–6]. The flat Lambda cold dark matter (ΛCDM) model is supported by robust observational evidence promoting such a paradigm to be the standard model of cosmology [7–9]. However, despite its success, theoretical shortcomings related to the nature of Λ on the one hand, and tensions among recent cosmological measurements, on the other hand, challenge the ΛCDM scenario as the ultimate model to describe the dynamics and evolution of the Universe [10,11]. The fine-tuning issues emerging from the huge discrepancy between the observed dark energy density and the predictions of quantum field theory plague the standard interpretation of Λ as the energy density of vacuum [10,12–15]. Moreover, the latest findings of the Planck Collaboration have confirmed, up to a high

accuracy level, that the most suitable scenario able to explain the primordial inflationary era is provided by the Starobinsky model [16], which contemplates corrections with respect to Einstein's gravity.

On the other hand, some tensions and anomalies became recently significant when analyzing different datasets, placing the ΛCDM cosmology at a crossroads. The most discussed and statistically significant tension in the literature concerns the estimate of the Hubble constant H_0 from the cosmic microwave background (CMB) and the direct local distance ladder measurements. Assuming the ΛCDM scenario, Planck-CMB data analysis provides $H_0 = (67.4 \pm 0.5)$ km/s/Mpc [9], which is in ∼5σ tension with the local measurement $H_0 = (73.30 \pm 1.04)$ km/s/Mpc found by the SH0ES team [17]. Additionally, many other late-time measurements are in agreement with a higher Hubble constant value [18,19]. Motivated by such discrepancies, unlikely to disappear completely by introducing multiple systematic errors, it has been widely discussed in the literature whether new physics beyond the standard cosmological model can solve the H_0 tension (see [11,20,21] and references therein). Still, in the context of the ΛCDM model, CMB measurements from Planck and ACT + WMAP indicate values of $S_8 = 0.834 \pm 0.016$ [9] and

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$S_8 = 0.840 \pm 0.030$ [22], respectively, where $S_8 \equiv \sigma_8 \sqrt{\Omega_{m0}/0.3}$, being σ_8 the amplitude of matter fluctuations averaged on spheres of radius 8 Mpc/h, and Ω_{m0} the matter density parameter today. These values of S_8 are typically higher than those obtained by weak lensing and galaxy clustering measurements, which range between 0.703 and 0.782, showing a tension of $\sim 3\sigma$ among these datasets.[1] Although this tension might be due to systematic errors [24], it is worthwhile to investigate the possibility of new physics beyond the standard model to explain the S_8 tension [25–28]. Moreover, a large tension between redshift space distortion (RSD) and CMB measurements has been identified by [29]. Disagreements between CMB and RSD measurements with other datasets, including the E_G statistic, are discussed in detail in [30], pointing out a tension up to 5σ.

The need to further investigate the nature of cosmic speedup appears therefore essential in order to cure the aforementioned issues. At the same time, the degeneracy among different paradigms proposed over the last years to describe the dark energy behavior has motivated the development of model-independent techniques that allow investigating the cosmic expansion without resorting to *a priori* assumed cosmological setups [31–35]. Among these, of remarkable interest is the cosmographic approach [36–41], which relies only upon the cosmological principle and involves series expansions of the luminosity distance around the present time. Cosmography represents a powerful method as it provides a set of observable quantities that can be directly compared to data, and assures the independence of any postulated equation of state for dark energy. Hence, the cosmographic method has been widely used with the aim of breaking the degeneracy among different theoretical scenarios that behave in the same manner when compared to observations [42–47]. Alternative robust methods to reconstruct the cosmological parameters in a model-independent way include, e.g., Gaussian process [48–55], principal component analysis [56–59], or machine learning algorithms [60–62].

Nevertheless, the cosmographic technique is affected by two main problems that may limit its use as an accurate descriptor of cosmic expansion. The first is due to the need for a wide number of data in order to be able to properly distinguish between Λ and an evolving dark energy component. This, indeed, is required for reducing the uncertainties over the cosmographic coefficients. A second issue is related to the use of high-redshift data, which is needed to explore possible departures from the ΛCDM model. However, this contrasts with the foundation of the standard cosmographic technique, which is based on the Taylor expansion series around the present time, namely,

$z = 0$. The resulting convergence problems often lead to significant error propagations that, consequently, lower the predictive power of the method itself [63].

Over the years, several alternatives to the standard cosmographic approach have been then investigated with the purpose of overcoming the aforementioned drawbacks. One possibility is to make use of auxiliary variables and provide expansion series of cosmological observables in terms of reparametrizations of the redshift that converge to a finite value for $z \to \infty$ [44,64–66]. Further methods involve the use of rational polynomials to stabilize the behavior of the cosmographic series at large z [67]. A relevant example of the latter class is offered by the Padé approximations, which have been recently employed to study the nature of the cosmic acceleration in different theoretical contexts, due to their ability to overcome typical convergence issues proper of the Taylor series and significantly reduce uncertainties on fitting coefficients [68–71].

The aim of the present work is to assess cosmological tensions in the measurements of H_0 and σ_8 from a model-independent perspective through cosmography. In particular, we adopt the y-redshift parametrization proposed by [64] and the (2,1) Padé approximation motivated by previous results obtained by [72]. We describe the main features of the cosmographic technique based on these parametrizations in Sec. II. In Sec. III, we present the methodology and the datasets employed to analyze the H_0 and σ_8 tensions. Then, in Sec. IV, we provide the main results of this work and discuss possible implications in view of the state of the art of the aforementioned tensions. Finally, Sec. V is dedicated to the summary of our findings and conclusive remarks. In this paper, we use natural units, $c = \hbar = 1$.

II. THE COSMOGRAPHIC APPROACH

The global evolution of the Universe can be studied by requiring the validity of the cosmological principle, according to which the Universe is assumed to be isotropic and homogeneous on the largest scales. Such a principle is supported by an overwhelming number of observations and formally leads to the Friedmann-Robertson-Walker metric:

$$ds^2 = dt^2 - a(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (1)$$

where a vanishing spatial curvature is assumed, as suggested by observations [9]. Here, $a(t)$ is the cosmic scale factor normalized such that $a(t_0) = 1$, being t_0 the present time.

Different from standard cosmological approaches based on the solutions of the Friedmann equations, cosmography allows for a kinematic study of the cosmic expansion that is totally independent of the underlying dynamics governing the Universe's evolution. Thus, by means of the cosmographic method, one can infer the history of $a(t)$ directly from observations, *de facto* avoiding the use of Einstein's field equations [36].

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[1]For a review and additional information on S_8 estimations, see [11,20,23] and references therein.

The cosmographic method is based on the Taylor expansion of the scale factor around the present time [73]

$$a(t) = 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \frac{1}{3!} j_0 H_0^3 (t - t_0)^3$$
$$+ \frac{1}{4!} s_0 H_0^4 (t - t_0)^4 + \frac{1}{5!} l_0 H_0^5 (t - t_0)^5 + \mathcal{O}((t - t_0)^6), \tag{2}$$

where H_0 is the Hubble constant and $\{q_0, j_0, s_0, l_0\}$ are the current values of the deceleration, jerk, snap, and lerk parameters, respectively, defining the so-called cosmographic series [74]

$$H(t) \equiv \frac{1}{a}\frac{da}{dt}, \qquad q(t) \equiv -\frac{1}{aH^2}\frac{d^2 a}{dt^2}, \qquad j(t) \equiv \frac{1}{aH^3}\frac{d^3 a}{dt^3},$$
$$s(t) \equiv \frac{1}{aH^4}\frac{d^4 a}{dt^4}, \qquad l(t) \equiv \frac{1}{aH^5}\frac{d^5 a}{dt^5}. \tag{3}$$

The ability to discriminate among different dark energy models should, in principle, increase when considering higher-order terms. However, the lack of large and accurate observational data at high redshifts somewhat restricts the use of cosmographic coefficients up to the lerk parameter.

Moving to the redshift variable through the relation $a = (1 + z)^{-1}$, one can write the luminosity distance for a flat universe as

$$d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')}. \tag{4}$$

In view of the above definitions, the latter provides

$$d_L(z) = H_0^{-1} \left[z + \frac{1}{2}(1 - q_0)z^2 - \frac{1}{6}(1 - q_0 - 3q_0^2 + j_0)z^3 + \frac{1}{24}(2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10q_0 j_0 + s_0)z^4 \right.$$
$$\left. + \left(-\frac{1}{20} - \frac{9j_0}{40} + \frac{j_0^2}{12} - \frac{l_0}{120} + \frac{q_0}{20} - \frac{11j_0 q_0}{12} + \frac{27q_0^2}{40} - \frac{7j_0 q_0^2}{8} + \frac{11q_0^3}{8} + \frac{7q_0^4}{8} - \frac{11s_0}{120} - \frac{q_0 s_0}{8} \right) z^5 \right] + \mathcal{O}(z^6), \tag{5}$$

which can be used to describe the late-time evolutionary stage of the Universe without any assumptions on the cosmological model. Furthermore, by inverting Eq. (4), it is possible to find the corresponding Hubble series.

Comparing Eq. (5) directly with observations provides numerical bounds over the cosmographic parameters and, thus, allows one to reconstruct the cosmic expansion history up to a desired z. However, it is worth to remark that truncating the cosmographic series at a given order may cause biases in the numerical outcomes. If, on one hand, taking into account only low-order expansions may compromise the accuracy of the method, then, on the other hand, considering higher orders induces decreasing convergence. The latter issue occurs when data at $z > 1$ are employed in the analysis, and is a consequence of the short convergence radius of the Taylor series. In what follows, we shall face this problem by considering two different strategies aiming to extend the suitability of the cosmographic technique up to high redshifts.

A. y redshift

The first possibility to overcome the convergence problem is to introduce auxiliary variables and reparametrize the redshift via a function that well behaves for $z > 1$. These new variables must possess some suitable properties, such as exhibiting smooth features throughout cosmic history thus avoiding any divergence within the domain, and being one-to-one invertible with the original z variable.

A relevant example in this respect is provided by the so-called y redshift introduced by [64]:

$$y = \frac{z}{1 + z}, \tag{6}$$

which allows encoding the cosmic evolution back to the big bang into the finite range $y \in (0, 1)$. Therefore, we expect that the luminosity distance expressed as a Taylor series in powers of y well behaves from the present time up to early epochs, and the convergence radius to be $|y| = 1$, implying the convergence of the series for $|y| < 1$.

In terms of the variable y, up to the fifth order, we have

$$d_L(y) = H_0^{-1} \left[y + \frac{1}{2}(3 - q_0)y^2 + \frac{1}{6}(11 - j_0 - 5q_0 + 3q_0^2)y^3 + \frac{1}{24}(50 - 7j_0 - 26q_0 + 10j_0 q_0 + 21q_0^2 - 15q_0^3 + s_0)y^4 \right.$$
$$\left. + \frac{1}{120}(274 - 47j_0 + 10j_0^2 - l_0 - 154q_0 + 90j_0 q_0 + 141q_0^2 - 105j_0 q_0^2 - 135q_0^3 + 105q_0^4 + 9s_0 - 15q_0 s_0)y^5 \right], \tag{7}$$

while the Hubble parameter is given as

$$H(y) = H_0\left(1 + k_1 y + \frac{k_2 y^2}{2} + \frac{k_3 y^3}{6} + \frac{k_4 y^4}{24}\right), \quad (8)$$

where

$$k_1 = 1 + q_0, \quad (9a)$$

$$k_2 = 2 - q_0^2 + 2q_0 + j_0, \quad (9b)$$

$$k_3 = 6 + 3q_0^3 - 3q_0^2 + 6q_0 - 4q_0 j_0 + 3j_0 - s_0, \quad (9c)$$

$$k_4 = -15q_0^4 + 12q_0^3 + 25q_0^2 j_0 + 7q_0 s_0 - 4j_0^2 - 16q_0 j_0$$
$$- 12q_0^2 + l_0 - 4s_0 + 12j_0 + 24q_0 + 24. \quad (9d)$$

B. Padé parametrizations

The issue of stabilizing cosmographic expansions at high z can be also addressed by making use of rational approximations. These are constructed from the ratio between a generic nth degree polynomial and an mth degree polynomial, leading to (n, m)-order polynomials that can be used to approximate cosmic observables in terms of the cosmographic coefficients. The advantage of rational approximations relies on the fact that they can be calibrated by choosing the most suitable order for maximizing the convergence radius, thus allowing for a stable fitting procedure.

The good properties of rational polynomials are testified by the case of Padé approximations, which have been shown to overcome the high-redshift divergences plaguing the cosmographic analysis based on Talyor polynomials [75]. The (n, m) Padé approximation of a generic function $f(z)$ is given as [76]

$$P_{n,m}(z) = \frac{\sum_{i=0}^{n} a_i z^i}{1 + \sum_{j=1}^{m} b_j z^j}, \quad (10)$$

where the sets of coefficients a_i and b_i can be found by requiring the matching of the $n + m$ derivatives of $P_{n,m}$ evaluated at the origin and the corresponding derivatives obtained from the Taylor series of $f(z)$.

The matter of choosing the correct order of expansion is related to the best compromise between minimizing the number of free parameters and reducing error propagation in numerical analyses dealing with data beyond $z \simeq 1$. This issue has been recently addressed by [72], who performed a detailed study based on optimization procedures and mathematical considerations on the degeneracy among coefficients, showing that the most suitable Padé approximation for cosmographic purposes is the one of order (2,1). The latter is characterized by the following luminosity distance:

$$d_L(z) = \frac{1}{H_0}\left[\frac{6(-1+q_0)z + (-5 - 2j_0 + q_0(8+3q_0))z^2}{6(-1+q_0) + 2(-1 - j_0 + q_0 + 3q_0^2)z}\right]. \quad (11)$$

In this case, the Hubble parameter is given by

$$H(z) = \frac{2H_0(z+1)^2(j_0 z - 3q_0^2 z - q_0(z+3) + z + 3)^2}{p_0 + p_1 z + p_2 z^2}, \quad (12)$$

where

$$p_0 = 18(-1 + q_0)^2, \quad (13a)$$

$$p_1 = 6(-1 + q_0)(-5 - 2j_0 + 8q_0 + 3q_0^2), \quad (13b)$$

$$p_2 = 14 + 7j_0 + 2j_0^2 - 10(4 + j_0)q_0 + (17 - 9j_0)q_0^2 + 18q_0^3 + 9q_0^4. \quad (13c)$$

Motivated by the aforementioned arguments, in Sec. IV we shall take into account the y-redshift and (2,1) Padé approximations to investigate the expansion history of the Universe in a model-independent way so as to gain further insights into the cosmological tensions.

III. DATASETS AND METHODOLOGY

In the following, we define the datasets that will be used in our analysis.

(i) BAO: From the latest compilation of baryon acoustic oscillation (BAO) distance and expansion rate measurements from the SDSS Collaboration, we use 14 BAO measurements, viz., the isotropic BAO measurements of $D_V(z)/r_d$ [where $D_V(z)$ and r_d are the spherically averaged volume distance and sound horizon at baryon drag, respectively] and anisotropic BAO measurements of $D_M(z)/r_d$ and $D_H(z)/r_d$ [where $D_M(z)$ and $D_H(z) = c/H(z)$ are the comoving angular diameter distance and the Hubble distance, respectively], as compiled in Table 3 of [77]. This measurement sample covers the range $z \in [0.15, 2.33]$, where the line-of-sight comoving distance is calculated by assuming the expansion rate of the Universe given from the cosmographic expressions (8) and (12).

(ii) SN: We also consider the type Ia supernovae (SN) distance moduli measurements from the *Pantheon* sample, consisting of 1048 SNeIa in the range $0.01 < z < 2.3$ [78], used to constrain the normalized expansion rate $E(z) = H(z)/H_0$ [79].

(iii) CC: Our analysis involves the cosmic chronometer measurements of $H(z)$ from the differential age evolution of massive, early-time, passively evolving galaxies acting as standard clocks [80]. In particular,

TABLE I. 68% CL intervals on the cosmographic parameters within the y-redshift parametrization inferred from different combinations of datasets. The H_0 values are given in units of km/s/Mpc.

Datasets	H_0	q_0	j_0	s_0	l_0	σ_8
BAO + SN + CC	$69.21^{+0.97}_{-1.05}$	$-0.56^{+0.20}_{-0.17}$	$-0.4^{+2.0}_{-2.5}$	-12^{+11}_{-18}	> -30	...
BAO + SN + CC + SH0ES	$71.09^{+0.74}_{-0.77}$	$-0.69^{+0.16}_{-0.18}$	0.4 ± 2.0	-8^{+10}_{-15}	> -30	...
BAO + SN + CC + RSD	$69.21^{+0.91}_{-1.10}$	$-0.54^{+0.17}_{-0.20}$	$-0.2^{+1.8}_{-2.3}$	$-7.7^{+6.3}_{-19.7}$	> -28	$0.737^{+0.029}_{-0.027}$
BAO + SN + CC + RSD + SH0ES	$71.02^{+0.88}_{-0.67}$	$-0.70^{+0.12}_{-0.21}$	0.8 ± 1.8	$-1.9^{+3.1}_{-18.4}$	> -30	$0.725^{+0.026}_{-0.029}$

we make use of 31 CC measurements of $H(z)$ in the range $0.07 < z < 1.965$ compiled by [81–88].

(iv) SH0ES: A Gaussian prior on the Hubble constant as measured by the SH0ES Collaboration [17] is taken into account, i.e., $H_0 = (73.04 \pm 1.04)$ km/s/Mpc.

(v) RSD: The key dataset we use to constrain the parameter space σ_8-H_0 through cosmography is the redshift space distortion measurements. These represent a velocity-induced mapping from the real to the redshift space due to line-of-sight peculiar motions of objects, which introduce anisotropies in their clustering patterns [89]. This effect depends on the growth of structure, making RSD probes sensitive to the following combination [90,91]:

$$f\sigma_8(a) \equiv f(a)\sigma_8(a), \qquad (14)$$

where $f(a) \equiv d\delta_m/d\ln a$, with δ_m being the linear matter density contrast. Here, $\sigma_8(a) \equiv \sigma_8\delta_m(a)/\delta_m(1)$ is the linear amplitude of matter fluctuations averaged in spheres of radius $8h^{-1}$ Mpc, and σ_8 its present-day value. On subhorizon scales and in the linear regime, the evolution equation for $f(a)$ is given by

$$\frac{df(a)}{d\ln a} + f(a)^2 + \left(2 + \frac{1}{2}\frac{d\ln H(a)^2}{d\ln a}\right)f(a)$$
$$-\frac{3}{2}\Omega_m(a) = 0, \qquad (15)$$

where $\Omega_m(a) \equiv \Omega_{m0}a^{-3}H_0^2/H(a)^2$. Notice that $f(a)$ is dependent on $H(a)$ and, thus, on the cosmographic parameters. Since $\Omega_{m0} = 0.31$ is the mean value used in obtaining almost all RSD measurements, without loss of generality, we assume this value in our work, when RSD data are used. Because of the cosmographic approximations, all the other datasets have no dependence on Ω_{m0}, so it will not be possible to break the degeneracy in the matter density when they are combined with RSD data. We checked that by letting Ω_{m0} free, this parameter becomes unconstrained without affecting the best-fit values of the cosmographic baseline parameters.

Several measurements of $f\sigma_8(a)$ from a variety of different surveys, based on different assumptions (in particular, on the reference value of Ω_{m0}) and

subject to different systematics, exist in the literature. Before using any of them, it is imperative to assess their internal consistency. Such an analysis has been recently performed by [92] in the context of a Bayesian model comparison framework. There, it was possible to identify potential outliers as well as subsets of data affected by systematics or new physics. In this work, we shall make use of the RSD measurements of $f\sigma_8(z)$ provided in Table I by [92], consisting of 22 measurements of $f\sigma_8(z)$ in the redshift range $0.02 < z < 1.944$.

A. MCMC

We use the Markov chain Monte Carlo (MCMC) method to analyze the parameter set $\theta_i = \{H_0, q_0, j_0, l_0, s_0, \sigma_8\}$, building the posterior probability distribution function

$$\mathcal{P}(D|\theta) \propto e^{-\frac{\chi^2}{2}}, \qquad (16)$$

where χ^2 is the chi-squared function associated with each dataset. The goal of any MCMC approach is to draw N samples θ_i from the general posterior probability density

$$\mathcal{P}(\theta_i, \alpha|D) = \frac{1}{Z}\mathcal{P}(\theta, \alpha)\mathcal{P}(D|\theta, \alpha), \qquad (17)$$

where $\mathcal{P}(\theta, \alpha)$ and $\mathcal{P}(D|\theta, \alpha)$ are the prior distribution and the likelihood function, respectively. Here, D refers to the dataset, α accounts for possible nuisance parameters, and Z is a normalization term.

We perform our statistical analysis by means of the EMCEE algorithm [93], assuming the theoretical setups described in Sec. II and the following flat priors: $H_0 \in [10, 90]$, $q_0 \in [-2, 0]$, $j_0 \in [-10, 10]$, $s_0 \in [-100, 100]$, $l_0 \in [-100, 100]$, and $\sigma_8 \in [0.5, 1.5]$. We discard the first 20% steps of the chain as burn-in. We measure the convergence of the chains by checking that all parameters have $R - 1 < 0.01$, where R is the potential scale reduction factor of the Gelman-Rubin diagnostics [94]. The output from the chains is analyzed through the package ChainConsumer [95].

Under the cosmographic approach, each analysis involves at least three free parameters. Thus, given the dimension of the parameter space, we consider BAO + SN + CC as our minimal baseline. We divided our analysis into two steps:
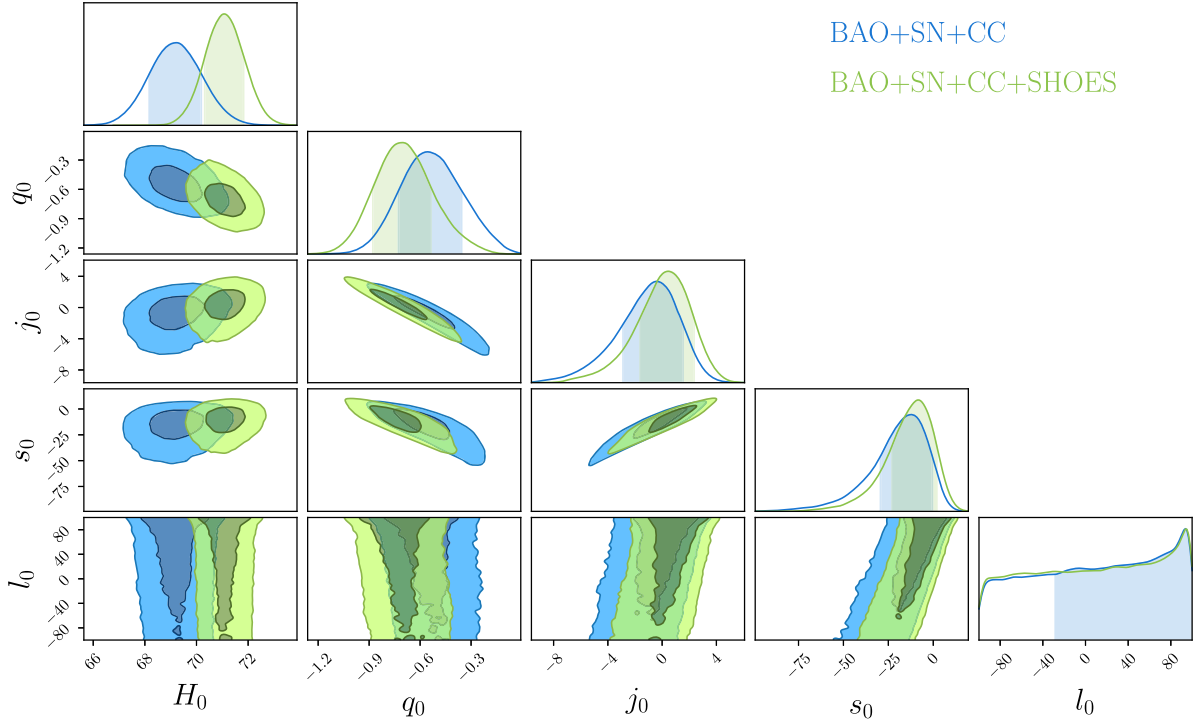
FIG. 1. Two-dimensional marginalized confidence regions (68% and 95% CL) and one-dimensional posterior distribution for the cosmographic coefficients obtained from the joint BAO + SN + CC and BAO + SN + CC + SH0ES analyses for the y-redshift parametrization. The H_0 values are expressed in units of km/s/Mpc.

(i) First, we analyze the BAO + SN + CC case. Then, we include SH0ES prior, i.e., BAO + SN + CC + SH0ES.

(ii) Second, we add the RSD measurements to the minimal baseline, i.e., BAO + SN + CC + RSD. Then, we also analyze the BAO + SN + CC + RSD + SH0ES combination.

The above joint analyses provide us with an overview of the observational constraints on the free parameters of both models under study in this work. In what follows, we present our main results.

IV. RESULTS AND DISCUSSION

Let us start by discussing the results emerging from the perspective of the y-redshift cosmography. We summarize in Table I our results at the 68% confidence level (CL).

We note that the first parameters of the cosmographic series, namely, H_0, q_0, and j_0, are well constrained. The model-independent estimate of the Hubble constant is $H_0 = 69.21^{+0.97}_{-1.05}$ km/s/Mpc, with 1.4% accuracy. This result is competitive with other current estimates performed by assuming the ΛCDM cosmology [96–99]. Also, the deceleration parameter is robustly constrained to $q_0 = -0.56^{+0.20}_{-0.17}$, suggesting the current accelerated expansion of the Universe at high statistical significance evidence. When the SH0ES prior is taken into account, the constraints over the Hubble constant are further improved: $H_0 = 71.09^{+0.74}_{-0.77}$ km/s/Mpc, with 1% accuracy.

In Fig. 1, we show the 68% CL and 95% CL contour regions for the cosmographic series obtained from the joint BAO + SN + CC and BAO + SN + CC + SH0ES analyses. It is important to emphasize that adding the SH0ES

TABLE II. 68% CL intervals on the cosmographic parameters within the (2,1) Padé approximation inferred from different combinations of datasets. The H_0 values are given in units of km/s/Mpc.

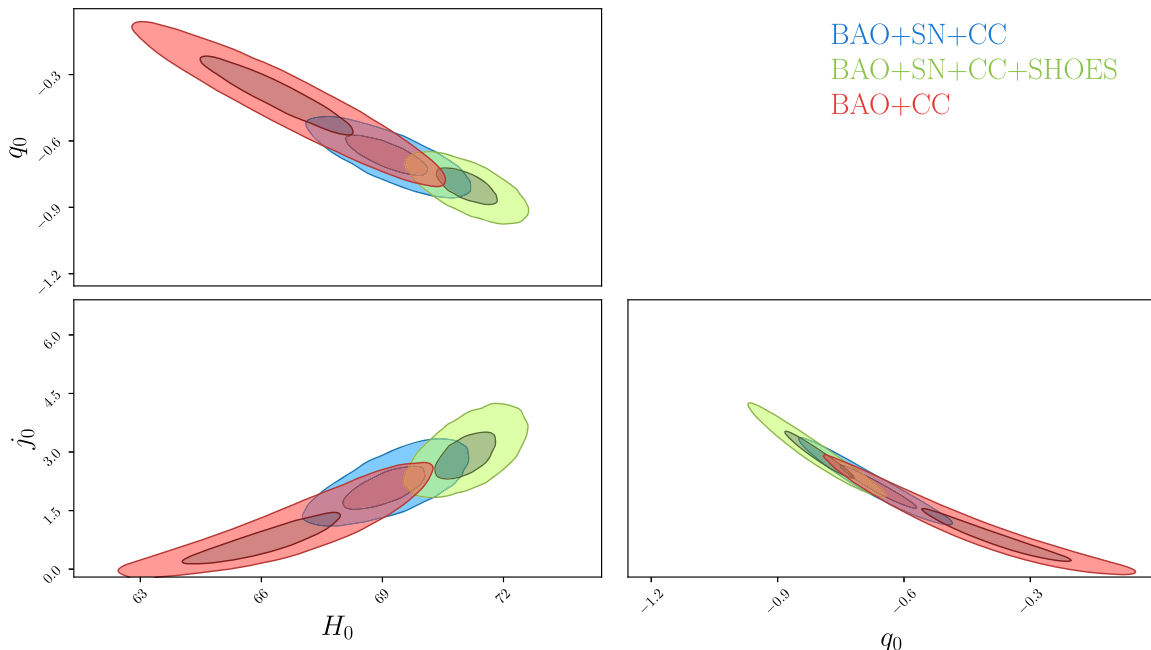| Datasets	H_0	q_0	j_0	σ_8
BAO + CC	$66.5^{+2.0}_{-1.9}$	$-0.38^{+0.16}_{-0.21}$	$0.59^{+0.93}_{-0.53}$	...
BAO + SN + CC	$69.11^{+1.06}_{-0.98}$	$-0.663^{+0.088}_{-0.095}$	$2.06^{+0.60}_{-0.51}$	...
BAO + SN + CC + SH0ES	$71.06^{+0.81}_{-0.71}$	$-0.803^{+0.079}_{-0.084}$	$2.88^{+0.63}_{-0.56}$	...
BAO + SN + CC + RSD	69.2 ± 1.0	$-0.663^{+0.086}_{-0.094}$	$2.05^{+0.59}_{-0.49}$	$0.740^{+0.027}_{-0.029}$
BAO + SN + CC + RSD + SH0ES	$71.09^{+0.73}_{-0.75}$	$-0.797^{+0.077}_{-0.084}$	$2.84^{+0.61}_{-0.54}$	$0.724^{+0.028}_{-0.026}$

FIG. 2. Two-dimensional marginalized confidence regions (68% and 95% CL) and one-dimensional posterior distribution for the cosmographic coefficients obtained from the BAO + SN + CC and BAO + SN + CC + SH0ES joint analyses for the y-redshift (2,1) Padé parametrization. The H_0 values are expressed in units of km/s/Mpc.

prior produces an increase of H_0, which, in turn, affects the other cosmographic parameters.

Now, we shall discuss the main results related to the (2,1) Padé parametrization summarized in Table II. With regard to accuracy and observational limits on the parameters H_0 and q_0, we do not notice any significant differences with respect to the y-redshift case. In short, we find 1.4% and 1% accuracy constraints on H_0 from the joint BAO + SN + CC and BAO + SN + CC + SH0ES analyses, respectively. These measurements show, respectively, 2σ and 1.4σ tensions with the estimates of H_0 provided by the SH0ES team. From the perspectives of both Padé and y-redshift parametrizations, the constraints on H_0 are in accordance with BAO, BBN, and CMB obtained in the standard cosmological context [96–99].

Figure 2 shows the 68% and 95% CL contour regions resulting from the three joint analyses, namely, BAO + CC, BAO + SN + CC, and BAO + SN + CC + SH0ES. In particular, the latter provides j_0 values that are more than 3σ away from the ΛCDM prediction, i.e., $j_0 = 1$. However, from the BAO + CC analysis, we find $j_0 = 0.59^{+0.93}_{-0.53}$, which is compatible with ΛCDM at 1σ. Thus, under the (2,1) Padé approximation, when considering the SN sample, the constraints on j_0 are not compatible with the standard model. Our result is consistent with the findings of [100] who, starting from another methodology and dataset, conclude that SN and $H(z)$ data are incompatible with the ΛCDM model at 2σ CL, and also with each other. Furthermore, using the Gaussian process, [101] finds that the jerk parameter is more than 3σ away from that of Λ

cosmology. Therefore, it seems that, from a model-independent point of view, ΛCDM may be incompatible with the jerk parameter in light of SN data. Similar conclusions are reported by [72,102].

As a guideline, we can compare our findings with the predictions of the flat ΛCDM model. Specifically, assuming the concordance value $\Omega_{m0} = 0.3$, one obtains the following values for the cosmographic parameters: $(q_0, j_0, s_0, l_0) = (-0.55, 1, -0.35, 3.115)$.[2] Thus, from the perspective of the y-redshift parametrization, our constraints on j_0 are compatible with the ΛCDM predictions. The high-order cosmographic parameters, namely, s_0 and l_0, although poorly constrained, are also fully compatible with the ΛCDM cosmology.

On the left panel of Fig. 4, we show the statistical reconstruction at the 1σ CL of the Hubble parameter for the (2,1) Padé model from the BAO + SN + CC + RSD analysis. The latter has been considered due to the fact that it is free of the SH0ES H_0 prior. Then, we can compare our results with the ΛCDM predictions inferred from some independent observations. In particular, we anchor the ΛCDM dynamics to the 1σ values obtained from the Planck-CMB data [9]. From a statistical point of view, the Padé reconstruction includes the ΛCDM model as a subset. The error bars for the Padé parametrization appear larger due to the presence of extra free degrees of freedom that are not robustly constrained when compared

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[2]It is worth noting that $j_0 = 1$ is a theoretical prediction, being independent of any cosmological parameter (see the Appendix).
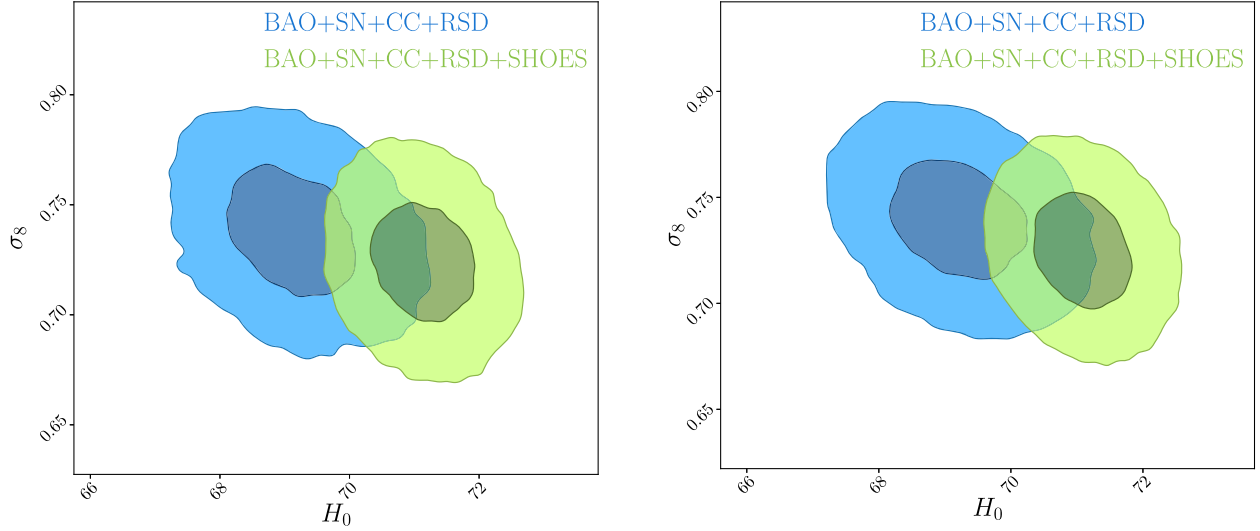
FIG. 3. 68% and 95% CL contours in the $\sigma_8 - H_0$ parameter space obtained from the dataset combinations $BAO + SN + CC + RSD$ and $BAO + SN + CC + RSD + SH0ES$, for the y-redshift (left panel) and (2,1) Padé parametrizations (right panel).
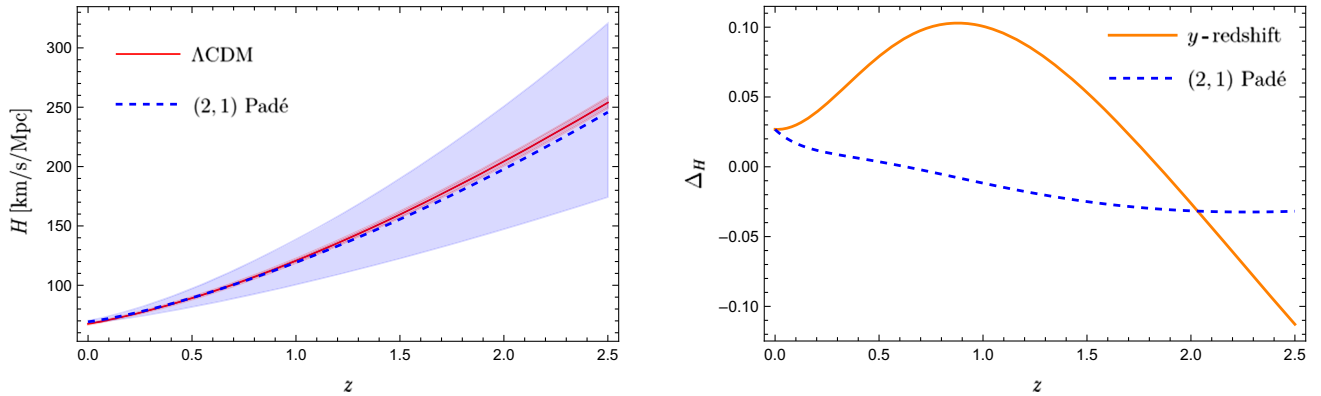

FIG. 4. Comparison between the results of our $BAO + SN + CC + RSD$ joint analysis of the cosmographic parametrizations and those of the ΛCDM model from Planck [9]. Left panel: 1σ reconstruction of the Hubble rate for the (2,1) Padé parametrization and the ΛCDM model. Right panel: relative difference of the Hubble rate from the best-fit values of the y-redshift and (2,1) Padé parametrizations, with respect to the best-fit values predicted by the ΛCDM model.

to a model-dependent analysis based on the ΛCDM paradigm. Anyway, the cosmic expansion rate given by the Padé parametrization is compatible with ΛCDM for the entire redshift range considered here. On the other hand, we do not show the 1σ reconstruction for the y-redshift model as this is very degenerate. Such a behavior is indeed expected as the high-order cosmographic terms are not properly constrained due to less accurate and few measurements at high redshifts [39,72]. Moreover, it is interesting to estimate the relative deviations with respect to the ΛCDM model. This can be done through the quantity

$$\Delta_H \equiv \frac{H_i}{H_{\Lambda CDM}} - 1, \quad (18)$$

where the index i labels the y-redshift and the (2,1) Padé parametrizations. In this regard, in the right panel of Fig. 4, we show the relative difference of the Hubble rate for both cosmographic approaches with respect to the Planck-ΛCDM baseline [9]. In this case, we assume the best-fit values from the $BAO + CC + SN + RSD$ analysis, while the lerk parameter is fixed to the reference value expected in the ΛCDM context, to overcome the unconstrained result of the y-redshift parametrization. For the (2,1) Padé parametrization, we find a discrepancy of $\lesssim 3\%$ with respect to ΛCDM over the whole redshift interval under consideration. This discrepancy increases up to $\sim 10\%$ in the case of the y-redshift parametrization.

It is possible to quantify the level of tension between two estimates i and j of H_0 by means of the simple 1D tension metric, which can be constructed as

$$T_{H_0} \equiv \frac{|H_{0,i} - H_{0,j}|}{\sqrt{\sigma_{H_{0,i}}^2 + \sigma_{H_{0,j}}^2}} \qquad (19)$$

measured in equivalent Gaussian standard deviations. In particular, we find that the results from the BAO + SN + CC and BAO + SN + CC + SH0ES combinations are in 1.9σ and 1.3σ tensions, respectively, with the local measurement by the SH0ES team.

Let us now turn our attention to the implications of the cosmographic frameworks under consideration on the parameter σ_8. In particular, the inclusion of the RSD sample in our dataset combination allows us to obtain direct constraints on σ_8. In the context of the y-redshift parametrization, from the joint BAO + SN + CC + RSD and BAO + SN + CC + RSD + SH0ES analyses, we find $\sigma_8 = 0.737^{+0.029}_{-0.027}$ and $\sigma_8 = 0.725^{+0.026}_{-0.029}$, respectively (cf. Table I). Applying the 1D tension metric (19) to the σ_8 parameter, we note that these estimates are, respectively, at 2.6σ and 3σ tensions with Planck-CMB data inferred from the ΛCDM cosmology [9]. Our constraints are, however, in agreement with those from weak lensing surveys DES [103], KiDS [104,105], and galaxy clustering measurements [106,107].

Very similar results are found when using the (2,1) Padé parametrization. We can see this in Fig. 3, which shows the σ_8-H_0 parameter space, together with the corresponding 68% CL and 95% CL contours obtained from the combinations BAO + SN + CC + RSD and BAO + SN + CC + RSD + SH0ES for both cosmographic approaches. Therefore, we can conclude that the amplitude of matter fluctuations analyzed through cosmographic expansion at late times shows a significant tension with CMB measurements. Finally, we note that the effects of RSD data provide an improvement in the accuracy of the j_0 parameter, whose constraints are, in any case, much more stringent for the (2,1) Padé approximation compared to the y-redshift parametrization.

V. FINAL REMARKS

In this work, we showed how model-independent approaches may reveal a promising tool for investigating tensions among cosmological parameters when inferred from different datasets. For our purposes, we specifically considered the cosmographic technique based on a series expansion of a cosmological observable around $z = 0$, and describing the late-time evolution of the Universe through a set of kinematic parameters to be constrained directly by data.

In particular, motivated by suitable properties able to heal the convergence problems typical of standard cosmography, we considered the improved y-redshift and Padé parametrizations of the Hubble expansion rate to accurately describe the dynamics of the Universe independent of cosmological model choices. Under these theoretical

frameworks, we chose a robust data sample consisting of recent BAO, SN, CC, and RSD measurements in the redshift range $z \in (0, 2.3)$ to explore possible inconsistencies with the ΛCDM model predictions.

We thus performed an MCMC numerical analysis on the combination BAO + SN + CC + RSD, obtaining H_0 and σ_8 estimates with an accuracy of $\sim 1.4\%$ and $\sim 3.7\%$, respectively. It is worth stressing that our model-independent constraints are competitive with those inferred by assuming specific cosmological backgrounds. Without including the SH0ES prior on H_0, our measurements are at 2σ tension with the local measurements, which represents a significant reduction compared to the Planck-CMB estimate assuming the ΛCDM model. On the other hand, our measurements on σ_8 are at 2.6σ tension with the Planck-ΛCDM cosmology.

Furthermore, we found that the jerk parameter can deviate $>3\sigma$ from the prediction of the ΛCDM model. Since different dark energy models predict different values and behaviors for j_0, a confirmation of our results by future data, possibly through further improved cosmographic modeling, may pose a new challenge and internal tension within the standard cosmological model. Also, since heterogeneous measurements have been combined to constrain H_0 and σ_8, it might be worth analyzing the impact of different systematic errors and statistical weights on the final results. We leave this subject for future work.

APPENDIX: ΛCDM PREDICTION FOR THE COSMOGRAPHIC PARAMETERS

In this appendix, we show the theoretical expressions of the cosmographic parameters as expected for the flat ΛCDM model, whose Hubble expansion rate at late times is given by

$$H(z) = H_0 \sqrt{\Omega_{m0}(1 + z)^3 + 1 - \Omega_{m0}}. \qquad (A1)$$

To do that, we start from the definitions (3) and convert the time derivatives into derivatives with respect to the redshift by means of the relation

$$dt = -\frac{dz}{(1 + z)H(z)}. \qquad (A2)$$

In so doing, one finds the following expressions:

$$q(z) = -1 + (1+z)\frac{H'}{H}, \tag{A3}$$

$$j(z) = 1 - 2(1+z)\frac{H'}{H} + (1+z)^2\left(\frac{H'}{H}\right)^2 + (1+z)^2\frac{H''}{H}, \tag{A4}$$

$$s(z) = 1 - 3(1+z)\frac{H'}{H} + 3(1+z)^2\left(\frac{H'}{H}\right)^2 - (1+z)^3\left(\frac{H'}{H}\right)^3$$
$$+ (1+z)^2\frac{H''}{H} - (1+z)^3\frac{H^{(3)}}{H} - 4(1+z)^3\frac{H'H''}{H^2}, \tag{A5}$$

$$l(z) = 1 - 4(1+z)\frac{H'}{H} + 6(1+z)^2\left(\frac{H'}{H}\right)^2 - 4(1+z)^3\left(\frac{H'}{H}\right)^3$$
$$+ (1+z)^4\left(\frac{H'}{H}\right)^4 + 2(1+z)^2\frac{H''}{H} + (1+z)^3\frac{H^{(3)}}{H}$$
$$+ (1+z)^4\frac{H^{(4)}}{H} - (1+z)^3\frac{H'H''}{H^2} + 7(1+z)^4\frac{H'H^{(3)}}{H^2}$$
$$+ 11(1+z)^4\frac{H'^2H''}{H^3} + 4(1+z)^4\left(\frac{H''}{H}\right)^2. \tag{A6}$$

Thus, inserting Eq. (A1) into Eqs. (A3)–(A6), and evaluating the results at $z = 0$, we obtain

$$q_0 = -1 + \frac{3}{2}\Omega_{m0}, \tag{A7}$$

$$j_0 = 1, \tag{A8}$$

$$s_0 = 1 - \frac{9}{2}\Omega_{m0}, \tag{A9}$$

$$l_0 = 1 + 3\Omega_{m0} + \frac{27}{2}\Omega_{m0}^2. \tag{A10}$$

Notice that all the cosmographic parameters are independent of the value of the Hubble constant. Moreover, j_0 is fixed to the unity in the ΛCDM scenario, regardless of the Ω_{m0} value.

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