Bouncing and inflationary dynamics in quantum cosmology in the de Broglie–Bohm interpretation

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The quantum cosmology of the flat Friedmann-Lemaître-Robertson-Walker universe, filled with a scalar field, is considered in the de Broglie–Bohm (dBB) interpretation framework. A stiff-matter quantum bounce solution is obtained. The bouncing and subsequent pre-inflationary and inflationary dynamics are studied in details. We consider some representative primordial inflation models as examples, for which analytical expressions characterizing the dynamical quantities can be explicitly derived. The dependence of the inflationary dynamics on the quantum bounce parameters is then analyzed. The parameters emerging from our description are constrained by requiring the produced dynamics to be in accordance with some key cosmological quantities. The constraining conditions are also illustrated through regions of parameter space in terms of the bounce quantities.

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I. INTRODUCTION

The standard cosmological model predicts that the Universe began in a very hot and dense state, followed by a radiation-dominated expansion phase, which is in turn followed by a pressureless matter-dominated phase and then to a dark energy expansion phase more recently (in terms of cosmological time scales). This model responds to important observations of the current Universe [1], such as the cosmic microwave background (CMB) radiation, the large-scale structure, the cosmological redshift, among others. Despite its successes, the hot big bang (HBB) standard model suffers from the well-known flatness and the horizon problems. The standard solution for these problems is provided by inflation [2–6], which consists of an accelerated expansion phase, typically attributed to the vacuum energy of a scalar field, the inflaton. In addition of providing a simple explanation for the spatial flatness and homogeneity of the Universe, inflation models can also provide an origin for the primordial anisotropies (e.g., due to quantum fluctuations of the inflaton field, like in cold inflation [7], or due to classical thermal fluctuations, like in warm inflation [8]). These are the same anisotropies leading later to the origin of the large-scale structure of the Universe and the spectrum of temperature fluctuations measured in the CMB. Despite of all its success, inflation is not the only paradigm able to provide a solution for the aforementioned problems of the HBB standard model and also to give a way of producing scale-invariant perturbations (or very close to scale-invariant perturbations, as required by the observations [1]). In this context, bouncing models have been shown (see e.g., Refs. [9–20]) to be able to achieve those same features produced by inflation.[1] Furthermore, bouncing models can also avoid the issue faced by the theory of General Relativity (GR), the initial singularity, which is not addressed in the inflationary paradigm [22,23]. When a bounce is present, the Universe has a contraction phase and can go to an expansion phase without passing through a singularity.

Nevertheless, the existence of a bounce does not exclude the possibility of an inflationary period in the Universe and vice versa. In fact, a Universe filled with the inflaton field can have a bounce dominated by its kinetic energy and eventually to evolve to an inflationary phase. Such situation is common place in the context of loop quantum cosmology (LQC) for instance [17,24–30]. More precisely, after the bounce, in the presence of an inflaton field with an appropriate potential able to support inflation, an inflationary phase is almost inevitable [31–39]. In these descriptions, the bounce is then followed by inflation, after a short nonaccelerating regime, such that the postinflationary Universe can be sensible to the previous bounce

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[1]For a recent comparison of the inflationary and bouncing paradigms in terms of the explanatory depth provided by both, see Ref. [21].

dynamics (see e.g., Ref. [27]). One appealing feature of this combined dynamics is that the bounce dynamics plays the role of a preinflationary scenario, which can lead to observational signatures and connect the quantum bounce to observations. Furthermore, by combining the dynamics of bounce and inflation, it can offer an unique opportunity for exploring the positive points of both scenarios and, hence, for having a better hope towards our goal of understanding the primordial Universe.[2]

The main purpose of this work is to consider this same combined scenario of bounce and inflation dynamics, but using a different quantization strategy than that of LQC. Here, we discuss the preinflationary physics that follows from the contracting phase before the bounce to the end of inflation and how the existence of the bounce affects the inflationary dynamics. We constrain the bounce parameters in order to provide a minimum number of e-folds to solve the standard model puzzles and, additionally, guarantees that the bounce scales are consistent with the forthcoming processes after the bounce. For this proposal, we closely follow the strategy recently considered for the case of LQC and put forward in Ref. [42] and which will be extended to the problem we will study in this paper.

In order to establish the bounce model, we perform the canonical quantization of GR in the Arnowitt-Deser-Misner (ADM) formalism [43–45] and construct the Wheeler-DeWitt (WDW) equation [46–48]. Then, for the quantization of the resulting wavefunction, we consider the de Broglie-Bohm (dBB) interpretation [49–51], where no external agent and collapse postulate are necessary. In the dBB interpretation (also known as Bohmian quantum mechanics), particles have a deterministic trajectory and are guided by the wave function. It is important to mention that there are other suitable alternatives that can be applied to quantum cosmology, such as, e.g., the many-worlds interpretation and collapse models [52–54], but we will not consider them here. From the quantization of the model, a bounce dynamics emerges and in which the initial singularity is resolved. This is similar to the resolution of the singularity by the quantum bounce in LQC, but here performed in the context of the dBB quantum cosmology (see e.g., Refs. [20,55] for reviews). We write the WDW equation in the minisuperspace [56,57], which restricts all possible geometries of the full superspace to a homogeneous and isotropic scenario and preserves its main qualitative aspects. Quantum cosmology models in minisuperspace in the dBB interpretation have been extensively discussed in the literature, like, for example, in Refs. [58–85]. Among these papers, some authors consider bouncing models described by perfect fluids using the Schutz formalism [86,87] for different values for the equation of state parameter ω, whereas others consider bouncing models described by a scalar field. In the latter, the bounce is usually dominated by the kinetic part of the energy density of the scalar field, which behaves as a stiff-matter fluid. Particularly, in Ref. [70] it was introduced a background model for a bounce dominated by a free and massless scalar field, whose scalar spectrum, however, revealed to be incompatible with observations [72]. On the other hand, the authors in Ref. [80] introduced an observational consistent bouncing model involving a single scalar field with an exponential potential, which is responsible for a dust contracting phase, a stiff-matter bounce phase, radiation and dust phases, a dark-energy phase and a final dust phase. While the work in Ref. [72] indicated that a pure stiff-matter bounce is not compatible with observations, Ref. [80] showed that this same bounce is consistent only if a potential energy density dominates after the bounce phase. In other words, in the context of a stiff-matter bounce, a phase where the potential energy density dominates is required. This is sufficient to argue that an inflationary phase following the bounce is also a reasonable alternative, which will be the situation considered in this work.

The model presented here consists of a quantum Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology in minisuperspace in the dBB interpretation, whose material content is a scalar field. In the bounce phase, where the quantum effects are relevant, the kinetic part of the scalar field dominates, i.e., the bounce is dominated by stiff-matter. In the expanding phase far from the bounce, the quantum effects are negligible, the potential energy density eventually dominates and an inflationary phase occurs. In the bounce phase, following Ref. [75], we write an analytical solution for the scale factor, which provides an expression for the nonsingular Hubble parameter. On the other hand, in the inflationary phase, we consider some well-motivated inflationary potentials, which are the monomial chaotic potentials, namely, the quadratic, quartic and sextic power-law potentials, as well as the Starobinsky potential. The choice of the monomial power-law potentials is due more for the easy of treating our problem in terms of these potentials. Even though they are in strong tension with the observations, e.g., by predicting a too large tensor-to-scalar ratio, at least in the context of cold inflation, they are still viable potentials in the context of the warm inflation picture [8]. We do not address in the present work the cosmological perturbations aspects of the theory, which will be the subject of future research, and dedicate in this work, as an initial study, to first understand the background dynamics of the model.

The dynamics of our model will be traced back to the contracting phase, where the initial conditions are set, passing through the bounce point, where the amplitude of the inflaton field, $\phi(t_B) = \phi_B$, is uniquely determined [42] (t_B denotes here the cosmic time at the bounce). The bounce phase ends when the kinetic term stops being

[2]For other works combining the dynamics of both bounce and inflation, see also Refs. [40,41].

dominant, i.e., when the equation of state parameter ω is zero, and where a transition phase begins. This phase briefly ends when $\omega = -1/3$, where the inflationary phase begins. Each of these phases can be characterized analytically as we are going to see, allowing us, for example, to obtain the duration of each phase, including the determination of the number of inflationary e-folds, which is explicitly dependent on the bounce parameters. These results also help us when constraining the many parameters added in the quantization procedure, breaking the, in principle, arbitrariness in their choice.

The paper is organized as follows. In Sec. II, we introduce the classical background cosmological model, which is the FLRW universe filled with a canonical scalar field. We also present a canonical transformation, which results in a transformed Hamiltonian that is suitable for our calculations. A classical solution for the scale factor is obtained in a new time variable for a flat spatial section, $k = 0$, and when the kinetic term dominates over potential term, $V/(\dot{\phi}^2/2) \ll 1$, which describes a stiff-matter ($\omega = 1$) solution. In Sec. III, we quantize the FLRW cosmological model in the dBB interpretation using the WDW equation. The quantum solution for the scale factor is obtained in the dBB interpretation from the solution of the WDW equation. The nonsingular Hubble parameter and the critical density are also derived. In Sec. IV, we present the full background dynamics, starting from the prebounce contracting phase, passing through a quantum bounce, the subsequent transition phase and until the end of inflation. In Sec. V, we introduce the physical conditions that the quantum bounce and inflationary phases must attain for consistency. In Sec. VI, we present the results of our analysis. Regions of parameter space involving the quantum bounce are determined by requiring that consistent bounce and proper duration of the inflationary phase are produced. Finally, in Sec. VII, we present our conclusions along also with a discussion of perspectives for future research. Some of the most technical details of the calculations are presented in the Appendix. Throughout this work we use the natural units system, in which $c = \hbar = 1$. We will also be working in the context of a standard flat FLRW cosmology with the spacetime metric given by the line element $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$, where t is the physical time, $\mathbf{x}$ are the comoving coordinates and $a(t)$ is the scale factor.

II. CLASSICAL BACKGROUND MODEL

The geometry of spacetime is generated by a matter content. We consider here as the matter content a canonical scalar field ϕ with potential energy $V(\phi)$. The scalar field action in curved spacetime is

$$\mathcal{S} = \mathcal{S}_{\rm EH} + \mathcal{S}_{\rm mat} = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{2.1}$$

where $\kappa \equiv \sqrt{8\pi}/m_{\rm Pl}$, $m_{\rm Pl}$ is the Planck mass ($m_{\rm Pl} \simeq 1.22\times 10^{19}$ GeV), $g_{\mu\nu}$ is the metric tensor, g is the determinant of the metric tensor and R is the Ricci scalar. From the action $\mathcal{S}$, Eq. (2.1), the Lagrangian density for a homogeneous scalar field in the flat FLRW universe is given by

$$\mathcal{L} = -\frac{3}{\kappa^2}a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 - a^3V(\phi), \tag{2.2}$$

where the dot denotes the derivative with respect to cosmic time. The Hamiltonian density is given by

$$\mathcal{H}_\phi = -\kappa^2\frac{P_a^2}{12a} + \frac{P_\phi^2}{2a^3} + a^3V(\phi), \tag{2.3}$$

where P_a and P_ϕ are the canonical momenta conjugated to a and ϕ, respectively. The classical equations of motion with respect to cosmic time read,

$$\dot{a} = \{a, \mathcal{H}_\phi\} = -\kappa^2\frac{P_a}{6a}, \tag{2.4a}$$

$$\dot{P}_a = \{P_a, \mathcal{H}_\phi\} = -\kappa^2\frac{P_a^2}{12a^2} + \frac{3P_\phi^2}{2a^4} - 3a^2V(\phi), \tag{2.4b}$$

$$\dot{\phi} = \{\phi, \mathcal{H}_\phi\} = \frac{P_\phi}{a^3}, \tag{2.4c}$$

$$\dot{P}_\phi = \{P_\phi, \mathcal{H}_\phi\} = -a^3V_{,\phi}(\phi). \tag{2.4d}$$

By following for instance Ref. [88], it becomes convenient for the quantization procedure to be introduced shortly and, for making easy the technical analysis of the resulting equations, to introduce and work with a new set of variables (T, P_T) instead of (ϕ, P_ϕ). We perform the canonical transformation[3] $(\phi, P_\phi) \to (T, P_T)$,

$$T = \frac{\phi}{P_\phi}, \qquad P_T = \frac{P_\phi^2}{2}, \tag{2.5}$$

where P_T is a new canonical momentum conjugated to the new variable T. The transformed Hamiltonian density, from Eq. (2.3), now becomes

$$\mathcal{H}_T = -\kappa^2\frac{P_a^2}{12a} + \frac{P_T}{a^3} + a^3V(T, P_T). \tag{2.6}$$

The new system of equations in these variables becomes more involved for a nonzero potential. Since in this paper the objective is to explore the bounce for a kinetic

[3]This canonical transformation will be relevant only in the canonical quantization context, but it is important to consider the same variables in the classical solutions for comparison.

dominated (stiff matter) field, we set $V = 0$ from now on. In this case, in the variables (T, P_T) one then has that

$$\dot{T} = \{T, \mathcal{H}_T\} = \frac{1}{a^3}, \quad \text{and} \quad dt = a^3 dT, \tag{2.7}$$

from which T can be understood as a new time variable. From the relation between the cosmic time t and the new time variable T, the dynamical equations with respect to the time variable T are now given by

$$a' = -\frac{\kappa^2}{6} a^2 P_a, \tag{2.8a}$$

$$P'_a = -\frac{\kappa^2}{12} a P_a^2 + \frac{3P_T}{a}, \tag{2.8b}$$

$$P'_T = 0, \tag{2.8c}$$

where the prime denotes the derivative with respect to T. From Eq. (2.6), setting the super-Hamiltonian constraint $\mathcal{H}_T$ to zero, one obtains

$$-\kappa^2 \frac{P_a^2}{12a} + \frac{P_T}{a^3} = 0. \tag{2.9}$$

From Eqs. (2.8a) and (2.9), the Friedmann equation expressed in terms of the new variables (T, P_T) becomes

$$H^2 = \left(\frac{a'}{a^4}\right)^2 = \frac{\kappa^2}{3} \frac{P_T}{a^6}, \tag{2.10}$$

where H is the Hubble parameter.

This simple analytical solution for the scale factor can be seen as the case of a pure stiff-matter (i.e., a kination regime) case, which then leads to the solution

$$a(T) = a_0 e^{\pm\lambda T}, \tag{2.11}$$

where $\lambda = \kappa\sqrt{P_T/3}$ is a constant. In this particular case, from Eq. (2.8c), P_T is a constant of motion. The positive and negative signs in the exponential represent, respectively, expanding and contracting universe solutions. Therefore, these are separate classical solutions. Note that by using Eq. (2.5), the parameter λ can also be expressed like

$$\lambda = \frac{\kappa P_\phi}{\sqrt{6}}. \tag{2.12}$$

From the fact that P_T (and P_ϕ) is a constant of motion, the term P_T/a^6 in the Friedmann equation, Eq. (2.10), represents correctly a stiff-matter fluid. This is what is expected, since for a scalar field with zero potential energy the equation of state parameter is simply given by $\omega = 1$.

III. QUANTUM BACKGROUND MODEL: DE BROGLIE–BOHM INTERPRETATION

In order to obtain the quantum behavior of the model, we first make use of the WDW equation, $\hat{\mathcal{H}}\Psi(a, T) = 0$. In this equation, the super-Hamiltonian constraint $\mathcal{H}$, which is given by Eq. (2.6), is promoted to an operator $\hat{\mathcal{H}}$, and $\Psi(a, T)$ is the wave functional of the primordial universe and from which the Bohmian trajectories for the observables are assumed to follow. The canonical momenta (P_a, P_T) are replaced by operators $(-i\partial_a, -i\partial_T)$. From these requirements, the WDW equation is then given by

$$i\partial_T \Psi(a, T) = \frac{\kappa^2}{12} a^2 \partial_a^2 \Psi(a, T), \tag{3.1}$$

where we have considered $V = 0$, as in the classical case and discussed in the previous section. Note that when working with Eq. (3.1) and if for instance one tries to rescale the wave functional or change the time variable, a novel first-order derivative ∂_a term will appear. However, this first-order derivative term can always be eliminated. In a sense, we can see the possible appearance of such a term as an ambiguity in the ordering of the factors a and P_a in the first term in the right-hand-side of Eq. (3.1). Nevertheless, its presence is often useful in order to obtain an analytical result for the WDW equation. It then becomes more convenient to rewrite Eq. (3.1) as

$$i\partial_T \Psi(a, T) = \left[\frac{\kappa^2}{12} a^2 \partial_a^2 + \frac{s\kappa^2}{12} a \partial_a\right] \Psi(a, T), \tag{3.2}$$

where the parameter s represents the above mentioned ambiguity. As shown in the Appendix, a suitable choice for s is $s = (3\omega - 1)/2$, from which we can obtain an analytical solution for the WDW equation. In this case, for stiff-matter, Eq. (3.2) reduces to

$$i\partial_T \Psi(a, T) = \frac{\kappa^2}{12} (a^2 \partial_a^2 + a\partial_a) \Psi(a, T), \tag{3.3}$$

which reproduces the WDW equation for a single fluid when $\omega = 1$ (see e.g., Ref. [20]). An explicit analytical solution for the WDW equation in this case can be found (see the Appendix for details). To apply the Bohmian quantum mechanics, we first write the wavefunction solution for the WDW equation, Eq. (A12), which was obtained for $\omega = 1$, in the polar form as

$$\Psi(a, T) = \Omega(a, T) e^{iS(a,T)}, \tag{3.4}$$

where $\Omega(a, T)$ and $S(a, T)$ are real functions given by

$$\Omega(a, T) = \sqrt{|\Psi(a, T)|^2} = \left[\frac{8T_0}{\pi(T_0^2 + T^2)}\right]^{1/4} \exp\left[-\frac{3T_0 \ln^2(\epsilon a)}{\kappa^2(T_0^2 + T^2)}\right], \tag{3.5}$$

and

$$S(a,T) = -\frac{3T\ln^2(\epsilon a)}{\kappa^2(T_0^2+T^2)} - \frac{1}{2}\arctan\left(\frac{T_0}{T}\right) + \frac{\pi}{4}, \quad (3.6)$$

where T_0 and ϵ are constants coming from the solution of the WDW equation.

One can interpret $\Omega^2 = |\Psi|^2$ as the probability density, whereas S is the phase, which guides the trajectory according to the dBB interpretation. In the present case, the trajectory is the evolution of the scale factor. Hence, following the dBB interpretation, the guidance equation can be defined as

$$P_a = \partial_a S. \quad (3.7)$$

Thus, from Eqs. (3.6) and (3.7), we have that Eq. (2.8a) becomes

$$a' = \frac{Ta\ln(\epsilon a)}{(T_0^2+T^2)}. \quad (3.8)$$

Working out the solution of Eq. (3.8), subject to the initial condition $a(0) = a_B$ and to the classical limit, Eq. (2.11), for $T \to \pm\infty$, we obtain that

$$a(T) = a_B e^{\lambda T_0[\sqrt{1+(T/T_0)^2}-1]}, \quad (3.9)$$

where λ has already been defined by Eq. (2.12), a_B is the scale factor at the bounce (i.e., at $T=0$) and we must impose $a_0 = a_B e^{-\lambda T_0}$ for consistency. This also allows us to express the constant ϵ appearing in the WDW wave function solution Eq. (3.4) in terms of λ, T_0 and a_B as

$$\epsilon = \frac{e^{\lambda T_0}}{a_B}. \quad (3.10)$$

The result given by Eq. (3.9), which is different from Eq. (2.11), represents a bouncing universe solution. Note that in the asymptotic limits $T \to \pm\infty$, we obtain the classical solutions,

$$a(T) = a_B e^{\pm\lambda(T \mp T_0)}. \quad (3.11)$$

One must notice that the quantum bounce solution, Eq. (3.9), is a natural match of the latter classical solutions, which happens at $T=0$. If there are other bounces phases related to our quantum solution, we do not address them in this work.

Finally, it is important to check the consistency of the quantum bounce solution and that it is indeed free of a classical big bang singularity at $T=0$. First of all, note that by considering the probability density, $|\Psi|^2 = \Omega^2$, from Eq. (3.5) and that $\epsilon > 0$, one obtains that the singularity $a=0$ is resolved for all values of T. Particularly, using Eq. (3.10), at $T=0$, we obtain that

$$|\Psi(a,0)|^2 = \left(\frac{8}{\pi T_0}\right)^{1/2}\exp\left[-\frac{6\ln^2\left(\frac{a}{a_B}e^{\lambda T_0}\right)}{\kappa^2 T_0}\right]. \quad (3.12)$$

One can notice that in the limit where $a \to 0$, the exponential in Eq. (3.12) tends to zero. Therefore, there is no probability of a quantum solution for a vanishing ($a=0$) scale factor.

From the analytical solution for the scale factor, Eq. (3.9), we can determine the Hubble parameter, which we find to be given by

$$H^2 = \frac{\lambda^2}{a^6}\left[1 - \frac{\lambda^2 T_0^2}{\ln^2\left(\frac{a}{a_B}e^{\lambda T_0}\right)}\right]. \quad (3.13)$$

The above result reproduces the expected bounce at $a = a_B$, where $H=0$. In the limit $a \to \infty$, we must reobtain the classical Friedmann equation, which, from Eq. (3.13), gives

$$H^2 = \frac{\lambda^2}{a^6} \equiv \frac{\kappa^2}{3}\rho. \quad (3.14)$$

Therefore, we notice that $\rho = 3\lambda^2/(\kappa^2 a^6)$. Additionally, at the bounce, where $a = a_B$, one obtains the so-called critical density (i.e., the energy density at the bounce), ρ_B, which is given by

$$\rho_B = \frac{3\lambda^2 m_{\rm Pl}^2}{8\pi a_B^6}. \quad (3.15)$$

Solving for λ, one obtains $\lambda = \sqrt{\frac{\kappa^2 a_B^6 \rho_B}{3}} = \sqrt{\frac{8\pi\rho_B}{3m_{\rm Pl}^2}}a_B^3$. Hence, Eq. (3.13) can be expressed as

$$H^2 = \frac{\kappa^2}{3}\rho\left\{1 - \frac{1}{\left[1 - \frac{1}{6\lambda T_0}\ln\left(\frac{\rho}{\rho_B}\right)\right]^2}\right\}. \quad (3.16)$$

Equation (3.16) is the quantum version of the Friedmann equation, Eq. (3.14) and our main result of this section. It is important to mention that the result for the quantum scale factor of this section, Eq. (3.9), has been already considered in Ref. [75], but our result is more general and obtained by a different method, which follows Ref. [69]. In Ref. [75], the solution for the scale factor is obtained from a solution of the WDW equation for an arbitrary separation of variables, whereas in our case the solution for the WDW equation is obtained by propagating an initial Gaussian wave packet as widely considered in the literature [55] for $\omega \neq 1$. Additionally, we obtain that $a(0) = a_B$, whereas in Ref. [75] this requirement is not addressed. In fact, our result is a novel extension of Ref. [69] for $\omega = 1$, where the analytical quantum scale factor was obtained for all values of ω except for $\omega = 1$.

Some relevant remarks must be made about the parameters in Eq. (3.16). The first one is that T_0 and ρ_B are novel parameters, which are integration constants that arise due to the WDW quantization in the dBB interpretation. The former comes from the wave function initial condition, Eq. (A10), whereas the latter arises from the nonsingular behavior of the bounce dynamics. When $\rho = \rho_B$, one obtains $H = 0$, which is the transition between contracting and expanding phases, connected by a bounce. On the other hand, when $T_0 \to 0$ one obtains the classical limit, Eq. (3.14). The second one is that λ, although explicitly present in the quantum Hubble parameter, it is a parameter already of the classical theory. From Eq. (3.15), λ is present in the definition of ρ_B. However, the definition of λ as a constant parameter, given by Eq. (2.12), holds only before the beginning of the inflationary phase. This can be noticed, e.g., from Eq. (2.4d), which shows that P_ϕ is constant only for negligible potential energy, i.e., around the bounce and according to what we will be considering throughout the next section. In all of our subsequent calculations in which will be involving λ, it is then well justified to assume its constancy, or more specifically, its value at the bounce instant, λ_B. Hence, only T_0 and ρ_B (or λ_B) will be constrained by cosmological considerations.[4] The bounce depth a_B, which also arises from the nonsingular behavior, will be set to unit without loss of generality.

IV. BACKGROUND DYNAMICS

In the previous section, we presented the quantum bounce solution in the case where, at the bounce, the matter content of the Universe is dominated by stiff matter, which is described by a massless scalar field. We now explore the implications of this result for the background dynamics of a scalar field that goes through the bounce phase. In bouncing inflationary cosmologies, the Hubble antifriction term in the contracting phase generally leads to dynamics where the scalar field is kinetic dominated, whereas in the expanding phase an inflationary potential is included when a classical description is possible. Once an appropriate potential is assigned to the scalar field, an inflationary phase far from the bounce becomes highly likely to happen. This prediction follows generically, much in the same way as it happens in the context of LQC and where similar conditions prevail at the bounce [27,31–35].

To study the background evolution for the inflaton field here, we will closely follow the same strategy implemented in Ref. [42]. The evolution is considered to start deep in the contracting phase, where the inflaton field is expected to be performing small oscillations around the minimum of its potential.[5] As the universe contracts towards the bounce and considering the evolution starting sufficiently in the past, the kinetic energy of the scalar field will tend to become dominant, realizing the conditions leading to the solutions obtained in the previous section for a quantum bounce dominated by stiff matter.

Here we present the different phases of the background evolution, starting from the prebounce (classical regime) contracting phase, the quantum bounce phase, and the subsequent expansion (again in the classical regime) of the pre-inflation transition and inflationary phases. For our analysis, we will focus on a scalar inflaton field described by the following primordial potentials: (a) the power-law monomial chaotic potential,

$$V = \frac{V_0}{2n}\left(\frac{\phi}{m_{\rm Pl}}\right)^{2n}, \tag{4.1}$$

and (b) the Starobinsky potential [90],

$$V = V_0\left(1 - e^{-\sqrt{\frac{16\pi}{3}}\frac{\phi}{m_{\rm Pl}}}\right)^2. \tag{4.2}$$

For the power-law monomial potential, for definiteness, we will consider the cases of the quadratic ($n = 1$), quartic ($n = 2$) and the sextic ($n = 3$) as working examples. In Eqs. (4.1) and (4.2), V_0 is fixed by the amplitude of the CMB scalar spectrum, which gives (see e.g., Ref. [42]), for the monomial potentials, for $n = 1$: $V_0/m_{\rm Pl}^4 = 1.355 \times 10^{-12}$; for $n = 2$: $V_0/m_{\rm Pl}^4 = 1.373 \times 10^{-13}$; and for $n = 3$: $V_0/m_{\rm Pl}^4 = 4.563 \times 10^{-15}$, whereas for the Starobinsky potential, $V_0/m_{\rm Pl}^4 = 1.497 \times 10^{-13}$.

A. Setting the initial conditions

We set the initial conditions in the classical contracting phase. From the classical scale factor in the contracting phase, Eq. (3.11), the Hubble parameter is given by

$$H = \frac{\dot a}{a} = \frac{a'(T)}{a(T)^4} = -\frac{\lambda}{a_B^3 e^{-3\lambda(T+T_0)}}. \tag{4.3}$$

From Eqs. (2.7) and (3.11), in the contracting phase, hence the minus sign in Eq. (4.3), one obtains

$$t - t_B = -\frac{1}{3}\frac{a_B^3 e^{-3\lambda(T+T_0)}}{\lambda}, \tag{4.4}$$

[4] In other approaches, e.g., LQC, ρ_B is a free parameter of the theory, related to the Barbero-Immirzi parameter γ, which is in turn fixed by black hole entropy calculations in LQG [89]. The possibility of varying γ as a free parameter of the underlying quantum theory can also be considered [42]. However, in Bohmian quantum gravity, even if we can develop black hole entropy calculations, there remains a free parameter that must be constrained by observations.

[5] Here, we only consider inflaton potentials that have a minimum.

where t_B is the cosmic time at the bounce. On the other hand, from Ref. [42] one obtains that $t - t_B$ is also given by

$$t_\alpha - t_B \simeq -\frac{1+\bar{\alpha}}{3}\sqrt{\frac{3m_{\rm Pl}^2\bar{\alpha}}{8\pi(1+\bar{\alpha})V(\phi_\alpha)}}, \tag{4.5}$$

where $\alpha \equiv V/(\dot{\phi}^2/2)$ is the ratio between potential and kinetic energy densities for the scalar field. The quantum bounce, as assumed in the present study, is dominated by stiff-matter fluid. Thus, we can see the choice of α as indicating how far in the past we set the initial conditions for the inflaton field. The amplitude $\phi_\alpha \equiv \phi(t_\alpha)$ is the inflaton amplitude at the time where the fraction of potential and kinetic energy densities has some given value α. In the latter equation we also wrote $\bar{\alpha}$ instead of α, where $\bar{\alpha}$ is taken as the "average" value for α and we approximate it as a constant within the range (0,1) (see Ref. [42] for details). From this approximation, we estimate in the following T_α and $\phi_\alpha \equiv \phi(T_\alpha)$. In particular, from the Eqs. (4.4) and (4.5), we find that

$$T_\alpha \simeq \frac{1}{3\lambda}\ln\left[\frac{a_B^3}{(1+\bar{\alpha})\lambda}\sqrt{\frac{8\pi(1+\bar{\alpha})V(\phi_\alpha)}{3m_{\rm Pl}^2\bar{\alpha}}}\right] - T_0. \tag{4.6}$$

For the power-law potential, Eq. (4.1), and Starobinsky potential, Eq. (4.2), one can obtain ϕ_α as [42]

$$\phi_\alpha = \frac{n\sqrt{1+\alpha}}{2\sqrt{3\pi}}m_{\rm Pl}, \tag{4.7}$$

and

$$\phi_\alpha = \frac{1}{4}\sqrt{\frac{3}{\pi}}\ln\left(1 + 2\frac{\sqrt{1+\alpha}}{3}\right)m_{\rm Pl}, \tag{4.8}$$

respectively.

Let us now see how we can connect the solution ϕ_α with the one valid around the bounce phase. From Eqs. (2.4c), (2.4d), and (2.7) at the bounce phase (i.e., when $\dot{\phi}^2/2 \gg V$), we can integrate $\phi(T)$ with the initial condition $\phi(0) = \phi_B$ to obtain

$$\phi(T) = \phi_B \pm \frac{\sqrt{3}\lambda m_{\rm Pl}}{2\sqrt{\pi}}T, \tag{4.9}$$

where we have also used that $P_\phi = \sqrt{3/\pi}\lambda m_{\rm Pl}/2$ from Eq. (2.12). Finally, we can obtain ϕ_B evaluating Eq. (4.9) at $T = T_\alpha$ to obtain

$$\phi_B = \phi_\alpha \mp \frac{\sqrt{3}\lambda m_{\rm Pl}}{2\sqrt{\pi}}T_\alpha. \tag{4.10}$$

Explicit expressions for ϕ_B, Eq. (4.10), can be obtained analytically for specific potentials, in particular for the potentials we are considering in our study, Eqs. (4.1) and (4.2). Firstly, we substitute T_α given by Eq. (4.6), which depends on V. Secondly, we substitute ϕ_α for each potential. For the power-law potential, using Eq. (4.7), one obtains

$$\phi_B/m_{\rm Pl} = \frac{n\sqrt{1+\alpha}}{2\sqrt{3\pi}} \pm \frac{3\lambda T_0}{2\sqrt{3\pi}}\left\{1 - \frac{1}{3\lambda T_0}\ln\left[\frac{a_B^3}{\lambda}\sqrt{\frac{8\pi}{3m_{\rm Pl}^2\bar{\alpha}(1+\bar{\alpha})}\frac{V_0}{2n}\left(\frac{n\sqrt{1+\alpha}}{2\sqrt{3\pi}}\right)^n}\right]\right\}, \tag{4.11}$$

whereas for the Starobinsky potential, using Eq. (4.8), we obtain

$$\phi_B/m_{\rm Pl} = \frac{1}{4}\sqrt{\frac{3}{\pi}}\ln\left(1 + 2\frac{\sqrt{1+\alpha}}{\sqrt{3}}\right) \pm \frac{3\lambda T_0}{2\sqrt{3\pi}}\left\{1 - \frac{1}{3\lambda T_0}\ln\left[\frac{4\sqrt{2\pi}a_B^3}{(3+2\sqrt{1+\alpha})\lambda m_{\rm Pl}}\sqrt{\frac{V_0}{3\bar{\alpha}}}\right]\right\}, \tag{4.12}$$

where for both cases in $\pm$ the plus (minus) sign refers to $\dot{\phi}_c > 0$ ($\dot{\phi}_c < 0$).

The expressions of ϕ_B for both type of potentials depend on the parameters λ, T_0 and α. In the following, we will fix the value of α in order to establish the initial condition at the prebounce phase, whereas for λ and T_0, we will show that they can be constrained by appropriate cosmological scales. Therefore, additionally to the natural condition $(\dot{\phi}_B^2/2)/V(\phi_B) \gg 1$ at the bounce, ϕ_B will be constrained by λ and T_0.

B. Bounce phase

This phase is characterized by the dominance of the kinetic energy over the potential energy, $(\dot{\phi}^2/2)/V \gg 1$, i.e., $\alpha \ll 1$, where the scalar field behaves as stiff matter. From Eqs. (2.4c) and (2.4d) at the bounce phase and Eqs. (2.12) and (3.15), one obtains that

$$\dot{\phi}(t) = \pm\sqrt{2\rho_B}\left(\frac{a_B}{a(t)}\right)^3. \tag{4.13}$$

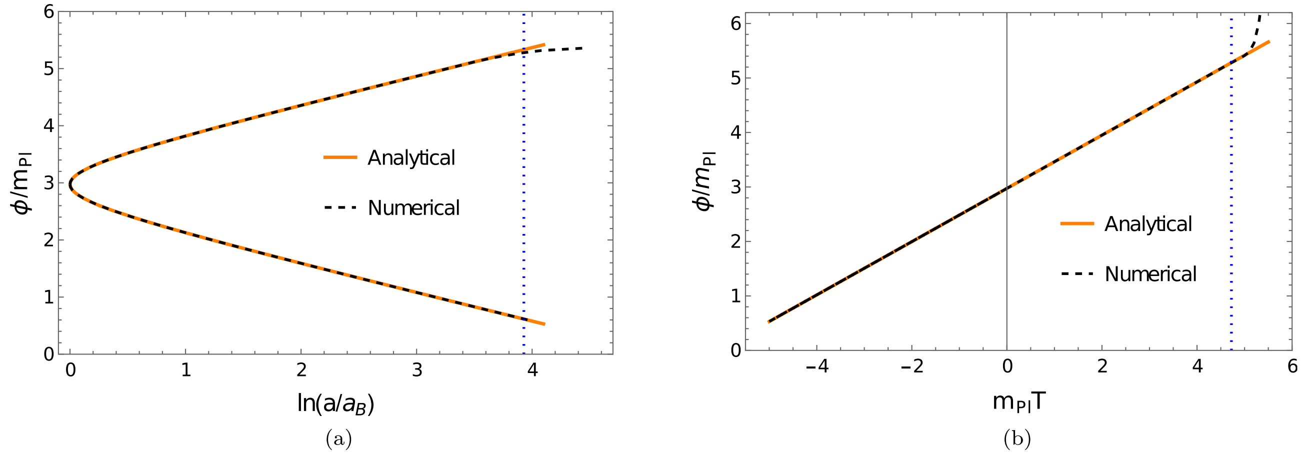

FIG. 1. Comparison between analytical and numerical results for the evolution of inflaton amplitude ϕ. The evolution coincide during the kinetic energy dominated regime ($\dot{\phi}^2/2 \gg V$) and breaks right after its end, indicated by the vertical blue dotted line, which marks the beginning of the transition phase ($\omega = 0$). For the transition phase, we have considered the quadratic power law potential. The evolution is given in terms of both the scale factor [panel (a)] and in terms of the time variable T [panel (b)].

From Eq. (2.7), with respect to variable T, Eq. (4.13) will read as

$$\phi'(T) = \pm\sqrt{2\rho_B}a_B^3, \tag{4.14}$$

whose solution, for the initial condition $\phi(0) = \phi_B$, is given by Eq. (4.9).

It is useful to verify how good is the solution Eq. (4.9) for $\phi(T)$, which was obtained by neglecting the inflaton potential in its derivation, with the full numerical solution for $\phi(T)$, obtained by solving the inflaton equation of motion and keeping explicitly the potential. In Fig. 1 we show that the analytical approximation is indeed very good when compared with the numerical result. This agreement extends from the contracting bounce phase up until right after the beginning of the transition phase (when the potential starts dominating the kinetic energy), indicated by the vertical blue dotted line, and already in the expanding phase. Actually, we can confidently extend this approximation until the beginning of inflationary phase, which occurs immediately after the end of the transition phase and, for this reason, was omitted from the graph.

C. Postbounce transition phase

The post-bounce transition phase begins at the end of the bounce phase, when the potential and kinetic energies become of equal magnitude, i.e., when $\dot{\phi}^2/2 \simeq V$ or $\omega \simeq 0$. Therefore, the transition time occurs at the instant where

$$\dot{\phi}(T_c) = \pm\sqrt{2V(\phi(T_c))}, \tag{4.15}$$

where $T = T_c$ is the transition time. In order to obtain T_c, we need to specify the potential energy. For the power-law potential given by Eq. (4.1), after some algebra, we find the result

$$T_c = \frac{nW_0\left(\frac{3\lambda m_{\rm Pl} e^{\frac{3\lambda}{n}\left(\frac{\phi_B}{\sqrt{2\rho_{\rm B}}a_B^3}+T_0\right)}}{\sqrt{n}a_B^3 V_0^{\frac{1}{2n}}(2n\rho_{\rm B})^{\frac{1}{2}(1-\frac{1}{n})}}\right)}{3\lambda} - \frac{\phi_B}{\sqrt{2\rho_{\rm B}}a_B^3}, \qquad \dot{\phi}_c > 0, \tag{4.16a}$$

$$T_c = \frac{nW_{-1}\left(-\frac{3\lambda m_{\rm Pl} e^{\frac{3\lambda}{n}\left(-\frac{\phi_B}{\sqrt{2\rho_{\rm B}}a_B^3}+T_0\right)}}{\sqrt{n}a_B^3 V_0^{\frac{1}{2n}}(2n\rho_{\rm B})^{\frac{1}{2}(1-\frac{1}{n})}}\right)}{3\lambda} + \frac{\phi_B}{\sqrt{2\rho_{\rm B}}a_B^3}, \qquad \dot{\phi}_c < 0, \tag{4.16b}$$

where $W_0(x)$ and $W_{-1}(x)$ are Lambert functions. On the other hand, for the Starobinsky potential, given by Eq. (4.2), T_c is obtained by solving the equation

$$\sqrt{2V_0}\left(1 - e^{-\sqrt{\frac{16\pi}{3}}\frac{\phi_B \pm (\sqrt{2\rho_B}a_B^3)T_c}{m_{\rm Pl}}}\right) = \frac{\sqrt{2\rho_B}}{e^{3\lambda(T_c - T_0)}}, \tag{4.17}$$

where the plus (minus) signal refers $\dot{\phi}_c > 0$ ($\dot{\phi}_c < 0$). The analytical solution for T_c, when $\dot{\phi}_c > 0$, is found to be given by

$$T_c = \frac{\ln\left\{\frac{2\sqrt[6]{\frac{\pi}{2}}e^{-4\sqrt{\frac{\pi}{3}}(\frac{\phi_B}{m_{\rm Pl}})}\left[1+\frac{1}{4}\sqrt[3]{\frac{2}{\pi}}e^{4\sqrt{\frac{\pi}{3}}(\frac{\phi_B}{m_{\rm Pl}})}(f(\phi_B))^{2/3}\right]}{\sqrt{3}(f(\phi_B))^{1/3}}\right\}}{\lambda}, \tag{4.18}$$

whereas the solution for T_c when $\dot{\phi}_c < 0$ can only be obtained numerically. In Eq. (4.18), $f(\phi_B)$ is defined as

$$f(\phi_B) = \sqrt{27\left(\frac{8\pi\rho_B}{3V_0}\right)e^{6\lambda T_0} - 32\pi e^{-4\sqrt{\frac{\pi}{3}}\left(\frac{\phi_B}{m_{\rm Pl}}\right)}} + 9e^{3\lambda T_0}\sqrt{\frac{8\pi\rho_B}{3V_0}}. \tag{4.19}$$

As was shown in Fig. 1, our analytical approximated expressions are good even up to the transition time, indicated by the vertical blue dotted line in that figure. Therefore, we can use the bounce phase expressions to calculate $\phi(T_c) = \phi_c$, $\dot{\phi}(T_c) = \dot{\phi}_c$, $a(T_c) = a_c$ and also the number of e-folds lasting from the bounce instant up to the transition point, $N_e(T_c) = N_c$.

In the following we perform the next stage in our derivation, which is connecting the solution obtained at the transition point to the one at the start of the inflationary phase.

D. Beginning of the slow-roll inflationary phase

The beginning of the inflationary phase happens when $\ddot{a} > 0$, i.e., at the start of the accelerating phase and where $\omega = -1/3$, which means that $\dot{\phi}^2 = V(\phi)$. We define as $T = T_i$ this instant where the inflationary phase begins. Due to the fact that the transition phase is very short (see e.g., Ref. [27]), we can expand the relevant variables around T_c. For instance, we obtain that

$$\phi(T) \simeq \phi_c + T_c\phi'_c \ln\frac{T}{T_c}, \tag{4.20}$$

$$a(T) \simeq a_c\left(1 + T_c a_c^3 \mathcal{H}_c \ln\frac{T}{T_c}\right), \tag{4.21}$$

$$V(\phi) \simeq V(\phi_c) + T_c\phi'_c V_\phi(\phi_c) \ln\frac{T}{T_c}, \tag{4.22}$$

$$\mathcal{H}_c = \frac{a'_c}{a_c^4} \simeq \sqrt{\frac{\kappa^2}{3}\left(\frac{\phi_c'^2}{a_c^6} + V(\phi_c)\right)}. \tag{4.23}$$

From Eq. (4.20) and using Eq. (2.7), one obtains

$$\dot{\phi}_i = \frac{T_c}{T_i}\frac{\phi'_c}{a_i^3}. \tag{4.24}$$

Using Eq. (4.24) together with $\dot{\phi}_i^2 = V(\phi_i)$, one then obtains for T_i the result

$$T_i = \frac{1}{a_c^3\left(1 + \frac{6a_c^3\mathcal{H}_c V(\phi_c)}{\phi'_c V_\phi(\phi_c)}\right)} \frac{2\sqrt{V(\phi_c)}}{V_\phi(\phi_c) W_0\left(\frac{2\sqrt{V(\phi_c)}}{a_c^3 T_c V_\phi(\phi_c)\left(1+\frac{6a_c^3\mathcal{H}_c V(\phi_c)}{\phi'_c V_\phi(\phi_c)}\right)} e^{\frac{2V(\phi_c)}{T_c\phi'_c V_\phi(\phi_c)\left(1+\frac{6a_c^3\mathcal{H}_c V(\phi_c)}{\phi'_c V_\phi(\phi_c)}\right)}}\right)}, \qquad \dot{\phi}_c > 0, \tag{4.25a}$$

$$T_i = -\frac{1}{a_c^3\left(1 + \frac{6a_c^3\mathcal{H}_c V(\phi_c)}{\phi'_c V_\phi(\phi_c)}\right)} \frac{2\sqrt{V(\phi_c)}}{V_\phi(\phi_c) W_{-1}\left(\frac{-2\sqrt{V(\phi_c)}}{a_c^3 T_c V_\phi(\phi_c)\left(1+\frac{6a_c^3\mathcal{H}_c V(\phi_c)}{\phi'_c V_\phi(\phi_c)}\right)} e^{\frac{2V(\phi_c)}{T_c\phi'_c V_\phi(\phi_c)\left(1+\frac{6a_c^3\mathcal{H}_c V(\phi_c)}{\phi'_c V_\phi(\phi_c)}\right)}}\right)}, \qquad \dot{\phi}_c < 0, \tag{4.25b}$$

which is valid for an arbitrary potential.

Next, we perform the final stage of our calculations, where, by using the above expressions, we can calculate $\phi(T_i) = \phi_i$, $\dot{\phi}(T_i) = \dot{\phi}_i$, $a(T_i) = a_i$, and $N_e(T_i) = N_i$. From these results, we can finally derive the total duration of the phase lasting from the instant of the bounce up to the end of the inflationary phase.

E. Inflationary phase

In the inflationary phase, the evolution can be parametrized by the slow-roll parameter ϵ_V,

$$\epsilon_V = \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{V_{,\phi}}{V}\right)^2, \tag{4.26}$$

where $\epsilon_V \ll 1$ holds during inflation and $\epsilon_V \approx 1$ when inflation ends. By setting $\epsilon_V = 1$, we can obtain $\phi_{\rm end}$, the inflaton amplitude at the end of inflation. The number of e-folds of inflation is defined as

$$N_{\rm inf} \equiv \ln\left(\frac{a_{\rm end}}{a_i}\right) \approx \frac{8\pi}{m_{\rm Pl}^2}\int_{\phi_{\rm end}}^{\phi_i} \frac{V}{V'}\,d\phi, \tag{4.27}$$

TABLE I. Comparison between numerical and analytical results for inflaton field amplitudes and number of e-folds values for the power-law monomial and Starobinsky potentials for the case of $\dot{\phi}_c > 0$. The bounce parameters were fixed at the values $\lambda/m_{\rm Pl} = 1$ and $m_{\rm Pl}T_0 = 1$ and parameter α fixed at the value $\alpha = 1/3$.

Model	$\phi_B/m_{\rm Pl}$	$\phi_i^{\rm num}/m_{\rm Pl}$	$\phi_i^{\rm theory}/m_{\rm Pl}$	$\phi_{\rm end}^{\rm num}/m_{\rm Pl}$	$\phi_{\rm end}^{\rm theory}/m_{\rm Pl}$	$N_{\rm pre}^{\rm num}$	$N_{\rm pre}^{\rm theory}$	$N_{\rm inf}^{\rm num}$	$N_{\rm inf}^{\rm theory}$
Quadratic	2.99	5.27	5.32	0.20	0.28	3.87	3.87	175.39	177.46
Quartic	3.47	5.69	5.74	0.47	0.56	3.75	3.76	101.89	102.65
Sextic	3.93	6.14	6.19	0.64	0.85	3.73	3.74	78.93	78.80
Starobinsky	2.91	5.59	5.62	0.19	0.19	4.68	4.64	6.56×10^9	7.49×10^9

where ϕ_i is computed by using Eqs. (4.20) and (4.25a) (for $\dot{\phi} > 0$) or (4.25b) (for $\dot{\phi} < 0$). For the power-law potential, Eq. (4.1),

$$N_{\rm inf} \approx \frac{2\pi}{n m_{\rm Pl}^2}(\phi_i^2 - \phi_{\rm end}^2), \tag{4.28}$$

where

$$\phi_{\rm end} \approx \frac{n}{2\sqrt{\pi}} m_{\rm Pl}. \tag{4.29}$$

On the other hand, for the Starobinsky potential, we have that

$$N_{\rm inf} \approx \frac{3}{4}\left(e^{\sqrt{\frac{16\pi}{3}}\frac{\phi_i}{m_{\rm Pl}}} - e^{\sqrt{\frac{16\pi}{3}}\frac{\phi_{\rm end}}{m_{\rm Pl}}}\right) + \frac{\sqrt{3\pi}}{m_{\rm Pl}}(\phi_i - \phi_{\rm end}), \tag{4.30}$$

where

$$\phi_{\rm end} \approx \sqrt{\frac{3}{16\pi}} \ln\left(1 + \frac{2}{\sqrt{3}}\right) m_{\rm Pl}. \tag{4.31}$$

From the expression for $N_{\rm inf}$ obtained for each potential, we can determine under which conditions the minimum number of e-folds required for a successful inflationary model is achieved (e.g., $N_{\rm infl} \sim 60$). The number of e-folds depends on ϕ_i, which in turn depends on ϕ_c, which in turn depends on ϕ_B, i.e., it depends on the bounce dynamics. In the next section, we will explore the consequences of all of this.

Before closing this section, let us compare the validity of the above derived analytical equations with the numerical ones. This way we can certify the accuracy of these equations when we apply them when constraining the quantum bounce parameters. In Table I, we show the comparison between numerical and theory values for the inflaton field at the beginning (ϕ_i) and end of inflation ($\phi_{\rm end}$), for the number of e-folds between the bounce and the start of inflation ($N_{\rm pre}$) and for the duration of inflation ($N_{\rm inf}$) obtained for the power-law monomial and Starobinsky potentials. For illustration, we have considered the case of $\dot{\phi}_c > 0$ and the bounce parameters were fixed by assuming the values $\lambda/m_{\rm Pl} = 1$ and $m_{\rm Pl}T_0 = 1$. We have started the numerical evolution at the bounce considering the value of ϕ_B fixed according to Eq. (4.10) with parameter α fixed in the value $\alpha = 1/3$. This value for α has been shown in Ref. [42] to provide the best match between the analytical and numerical results and it is the same value we have adopted in this study. The results for the total number of inflationary e-folds, which is the relevant quantity in our subsequent analysis, are nevertheless weakly dependent on other choices made for α around the chosen value. Similar results shown in the Table I can also be obtained for $\dot{\phi}_c < 0$ when assuming the derived equations for this case.[6] As shown in Table I, the differences between the analytical and numerical quantities are small, specially for the number of inflationary e-folds, which differ by around one percent or less for the monomial power-law potentials, except for the Starobinsky potential, which displays a relatively larger variation, but still acceptable, given the much larger number of e-folds produced for the given initial conditions. The larger e-folding number in the Starobinsky model makes it also to be more sensitive to the value of the inflaton field at the beginning of inflation as a consequence of the much longer evolution. Typically, we find that the smaller the number of e-folds is for any of the models, the more accurate are the analytical expressions. We also note from the results shown in Table I that the preinflationary phase between the bounce and the start of inflation lasts around four e-folds, which is much similar to the case also seen in LQC [42].

V. COSMOLOGICAL CONSTRAINTS ON THE MODEL PARAMETERS

Let us now consider how the bounce parameters ρ_B and T_0, which were defined and introduced in Sec. III, can be constrained. For this objective, we can use the fact that the bounce length and energy scales, as well as the inflationary dynamics (e.g., its duration) are not arbitrary, but need to be set in such a way to guarantee the consistency of our model and also to satisfy the constraints coming from the HBB cosmology.

[6] Note that while the monomial power-law potentials analyzed here are symmetrical, the Starobinsky potential is asymmetric. However, in the Starobinsky potential inflation is assumed to happen along the plateau region, which resides in the right-hand side of the potential, while the left-hand side of the potential is too steep to favor inflation in general.

Firstly, we note that the bounce energy scale must be at least larger than that of big bang nucleosynthesis (BBN) (likewise, inflation must also occur at least at the energy scale of the BBN). Weaker conditions can be imposed by also demanding that the bouncing and inflationary energy scales be at least of those corresponding to the quark-gluon plasma phase transition of quantum chromodynamics (QCD) and electroweak (EW) phase-transition energy scales. We can associate each one of these relevant cosmological eras as corresponding to temperatures $T_{\rm BBN} \simeq 10$ MeV, $T_{\rm QCD} \simeq 100$ MeV, and $T_{\rm EW} \simeq 100$ GeV, respectively.

Note that for these temperature regimes, we are in the radiation dominated regime (considering that at the end of inflation the universe quickly reheats and transits into the HBB radiation dominated phase). Then, we can associate the energy density with that of radiation, i.e.,

$$\rho(T) = \frac{g_* \pi^2}{30} T^4, \tag{5.1}$$

where g_* is the number of relativistic degrees of freedom at the temperature T. From Ref. [91], we find for instance that $g_*^{(\rm BBN)} \simeq 10.76$, $g_*^{(\rm QCD)} \simeq 17.75$ and $g_*^{(\rm EW)} \simeq 102.85$, respectively. By imposing the bounce energy density to satisfy $\rho_B \gg \rho_i$, where $i = (\rm BBN, QCD, EW)$, then

$$\rho_B \gg \frac{g_*^{(\rm i)} \pi^2}{30} T_i^4. \tag{5.2}$$

On the other hand, it is reasonable to impose that the bounce energy density scale must not exceed that of the Planck scale, i.e., $\rho_B \ll m_{\rm Pl}^4$, assuming the Planck scale to be the largest energy scale which our (effective) quantum dynamics can likewise be applicable. Note that here we are conservatively assuming, following some points of view in the literature [92,93], the Wheeler-DeWitt equation to be a valid approximation for any fundamental quantum gravity theory at scales not so close to the Planck length, while for energy scales close or above the Planck scale, a more involved theory of quantum gravity is not excluded. Therefore, here we assume the physical ρ_B to lie within the range

$$\frac{g_*^{(\rm i)} \pi^2}{30} T_i^4 \ll \rho_B \ll m_{\rm Pl}^4. \tag{5.3}$$

The second condition [93,94] is imposed on the length scale of the bounce. From the Ricci scalar for our scale factor, Eq. (3.9), at the bounce instant (where the time variable T is valued $T = 0$),

$$R = \frac{6\lambda}{a_B^6 T_0}, \tag{5.4}$$

we can define the curvature scale at the bounce as

$$L_B = \frac{1}{\sqrt{R}}\bigg|_{T=0} = a_B^3 \sqrt{\frac{T_0}{6\lambda}}. \tag{5.5}$$

We impose that the bounce curvature scale must be larger than the Planck length, $L_B \gg m_{\rm Pl}^{-1}$. This condition is also required such that the Wheeler-DeWitt equation is considered a valid approximation (this can also be seen as equivalent to the previous imposition for the validity of the WDW equation below the Planck energy density). At the same time, the length scale must be smaller that H^{-1} (causal condition). We consider the Hubble scale H in the BBN, QCD, and EW phase transition scales. Using that $H = \sqrt{8\pi\rho(T_i)/(3m_{\rm Pl}^2)}$, it results on the condition on L_B,

$$L_B \ll \sqrt{\frac{45}{4\pi^3 g_*^{(\rm i)}}} m_{\rm Pl} T_i^{-2}. \tag{5.6}$$

Therefore, the physical L_B lies within the range given by

$$m_{\rm Pl}^{-1} \ll L_B \ll \sqrt{\frac{45}{4\pi^3 g_*^{(\rm i)}}} m_{\rm Pl} T_i^{-2}. \tag{5.7}$$

The third condition that we can impose is related to the amplitude of the inflaton field at the bounce, ϕ_B. In order to have a stiff-matter bounce, we require that $\dot\phi_B^2/2 \gg V(\phi_B)$. From $\rho_B = \dot\phi_B^2/2 + V(\phi_B)$, one can write that

$$\rho_B \gg 2V(\phi_B). \tag{5.8}$$

Finally, we consider the condition related to the number of e-folds of inflation, $N_{\rm inf}$, which must satisfy

$$N_{\rm inf} \gtrsim 60, \tag{5.9}$$

in order for inflation to provide a solution for the HBB model flatness and horizon problems. Note that even though bouncing models can by themselves provide a solution for these problems, we still need some minimal number of inflationary e-foldings as been around $N_{\rm inf} \sim$ 50–60 if inflation is to provide consistent observables and as far as the primordial cosmological perturbations generated during inflation are concerned [1].

In the next section, we explore how each one and the combination of the above constraining conditions help in restricting the quantum bounce parameters generated in the present study.

VI. RESULTS

In this section we present our results for the physical region of the bounce parameters that give a successful realization of inflation and that are derived from the analytical expressions derived ealier in Sec. IV. From the Hubble parameter endowed by quantum corrections, Eq. (3.16), one notices that the free parameters are ρ_B and T_0. Note that parameter λ directly controls the energy density scale of the bounce, ρ_B, while the product of both λ and T_0 controls the magnitude of the quantum correction in the Hubble parameter. Additionally, the bounce curvature

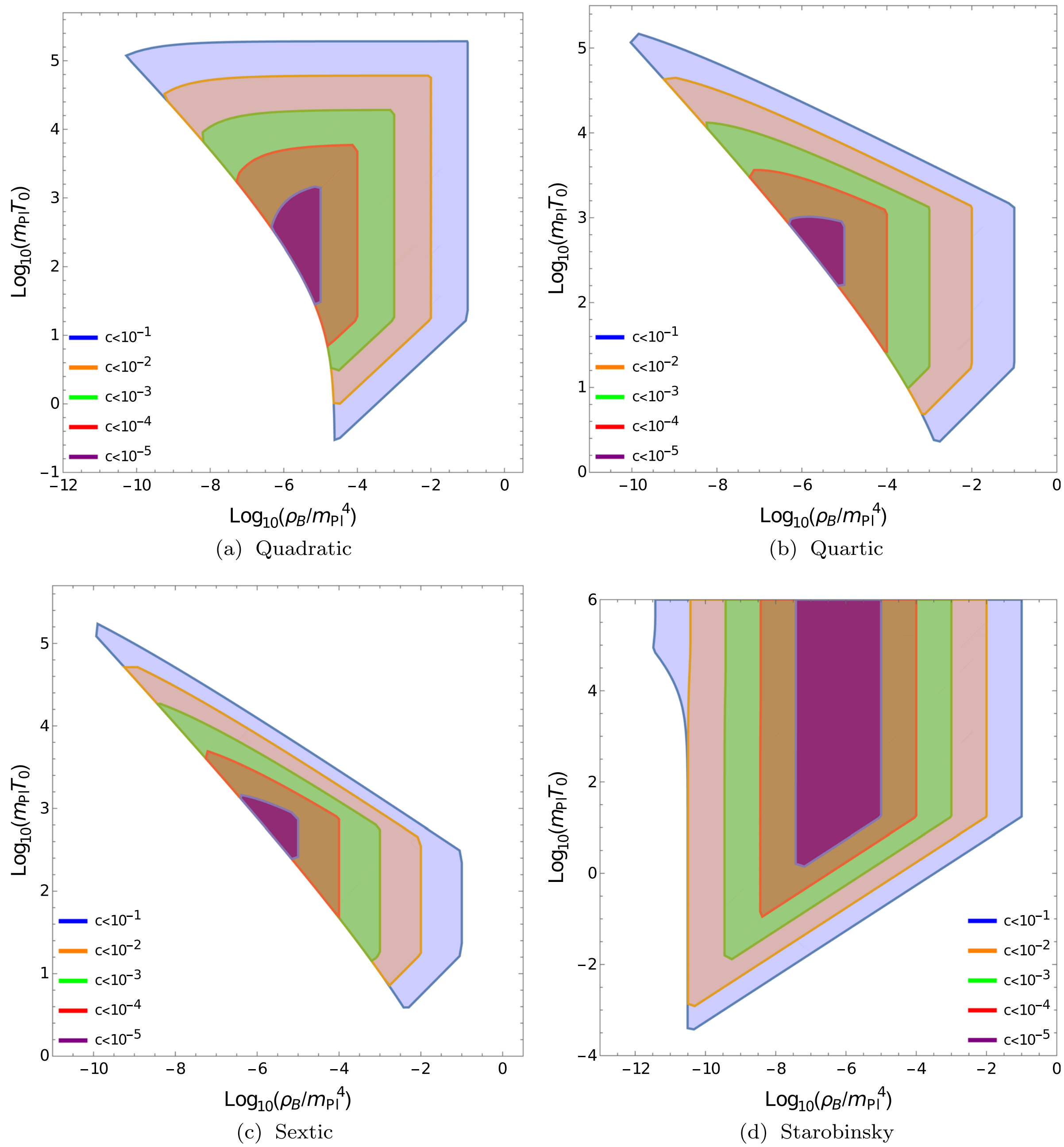

FIG. 2. Regions of allowed bounce parameters ρ_B and T_0 in the logarithm scale for quadratic ($n = 1$) [panel (a)], quartic ($n = 2$) [panel (b)], sextic ($n = 3$) [panel (c)] power-law potentials, and for the Starobinsky potential [panel d]. All cases are for $\dot{\phi}_c > 0$. The results for different regions are constrained by $(\rho_B/m_{\rm Pl}^4) \ll c$ and $(L_B m_{\rm Pl}) \gg c^{-1/4}$ in terms of the parameter c for different values of c.

scale, L_B, as well as the number of inflationary e-folds, $N_{\rm inf}$, also depend on these parameters. Hence, ρ_B and T_0 are subjected to the conditions set in the previous section and can be constrained by them. Note also that ρ_B and L_B are also dependent on the scale factor at the bounce, a_B, which we can set to unit without loss of generality.[7] We restrict our analysis to the case $\dot{\phi}_c > 0$ due to the fact that the results for $\dot{\phi}_c < 0$ are qualitatively similar.

For all the potentials considered, we display in Fig. 2 the region of parameters constrained by conditions (5.3) and (5.8), which limit the bounce energy density by (5.7), which limits the bounce length scale; and by (5.9), which imposes the minimum number of e-folds of inflation. The regions are restricted to avoid the Planck scale by the conditions $(\rho_B/m_{\rm Pl}^4) \ll c$ and $(L_B m_{\rm Pl}) \gg c^{-1/4}$, for the

[7]For $a_B \neq 1$, we can rescale the term λ/a_B^6 present in ρ_B and L_B and obtain a new equivalent condition.

representative values of $c = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$, and 10^{-5}. The value of c indicates how far from the Planck scale we consider our results.

Additionally to the Planck scale restrictions, the regions are limited also by each specific condition. The minimum number of e-folds, given by condition (5.9), restricts the left side region of the plots. The higher the minimum number of e-folds required, the more restricted the region. The conditions for ρ_B and L_B, given respectively by (5.2) and (5.6), from BBN, QCD, and EW phase transitions are also relevant. The former restricts a small tip on the right hand side of each region, whereas the latter avoids that the bottom of each region continues for arbitrarily smaller values of T_0 for the same range of ρ_B. Finally, the condition on ϕ_B, given by (5.8), restricts the upper tip of each region to continue to grow to the upper left.

On the other hand, the condition $L_B \gg m_{\rm Pl}^{-1}$, set on (5.7), from Eq. (5.5) for $a_B = 1$ reads $T_0/\lambda \gg 6\sqrt{c}m_{\rm Pl}^{-2}$, imposing that for a fixed T_0, there is an upper limit for λ (and vice versa), which in turn constrains ρ_B. Likewise, the condition $\rho_B \ll m_{\rm Pl}^4$ that was set in Eq. (5.3) and considering Eq. (3.15) for $a_B = 1$, we have that $\lambda \ll \sqrt{\frac{8\pi}{3c}}m_{\rm Pl}$, which is also an upper limit for λ, but in this case independent of T_0.

The constrained region of parameters for the quadratic, quartic, and sextic potentials are shown, respectively, in Figs. 2(a)–2(c). The results show that the range of the parameters are similar for all of those potentials. However, we can also notice that the larger is the power n, the allowed regions tends to shrink. In fact, as seen in both panels shown in Fig. 2, as we move away from the Planck scale, the region of allowed parameters decreases, until it eventually vanishes. For the monomial power-law potentials, in particular, the strongest restraining condition is on the minimal number of e-folds that is required and which produces the saturation of the regions at lower values of ρ_B and T_0, as seen in Figs. 2(a)–2(c). For bounce energy density scales that are smaller than approximately $\rho_B \sim 10^{-6} m_{\rm Pl}^4$, inflation no longer becomes viable for these type of potentials. In Fig. 2(d), we notice that the Starobinsky potential has no significant differences along the horizontal axis when compared to the results obtained with the power-law potentials. The Starobinsky potential, as it generically predicts a much larger number of e-folds, it can as a consequence allow much smaller energy scales. In addition, the restrictions that we have imposed only constrain T_0 at very high values, $T_0 \sim 10^{67}$.

VII. CONCLUSIONS

In this work, we have considered the nonsingular quantum cosmology of the flat FLRW universe filled with a scalar field in the dBB interpretation. The bounce solution is considered to be due to the stiff-matter behavior of a scalar field (the inflaton). The bounce is then followed by an inflationary phase when the potential energy density dominates over the kinetic one. The intermediate preinflationary dynamics is established and provides a link between the bouncing and inflationary phases. The entire evolution from the contracting phase to the end of inflation is analytically determined when the initial conditions are set in the deep contracting phase and where the inflaton field oscillates around the minimum of its potential.

We have investigated the dependence of the inflationary dynamics on the quantum bounce parameters. Furthermore, we have shown how the quantum bounce parameters can be restricted by looking at the specific cosmological epochs where the bounce and inflation regimes can possibly happen, e.g., at the BBN, QCD, and EW phase transition epochs. Likewise, the bounce energy density and length scales have an upper constraint set by the Planck scale in order to the WDW equation to be able to provide a valid description. All relevant conditions were discussed and we have presented the restraining conditions through different values for $\rho_B/m_{\rm Pl}^4$, as illustrated in Fig. 2. The results shown in Fig. 2 indicate that the broader regions of valid parameters are the ones closer to the Planck scale. In this work, we have focused and performed our analysis for the quadratic, quartic and sextic monomial power-law potentials, as well as for the Starobinsky potential, but our study can as well be generalized for other appropriate primordial inflaton potentials. The results for the potentials that we have considered here are qualitatively similar when we look for the allowed regions in terms of the magnitude of the quantum bounce parameters ρ_B and T_0, with the exception for the Starobinsky potential, which allows for a much larger region of parameter values in general.

It would also be interesting to consider the bounce effects in the postinflationary dynamics, as far as its effects on the perturbations and on the observables derived from the power spectrum. The study of the perturbations in the context of bouncing models is more delicate, as earlier studies have shown (as examples, see e.g., Refs. [17,27,29,69,72,80,95,96]). The study of the perturbations in the context of the models here studied will be presented in a future work.

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APPENDIX: SINGLE FLUID IN DE BROGLIE–BOHM THEORY FOR $\omega = 1$ AND $\omega \neq 1$

In this appendix we present the general wave function solutions for an ideal fluid with arbitrary equation of state parameter ω. From these solutions, a deterministic dynamics for the scale factor can be obtained in the dBB interpretation.

Using the Schutz formalism [86,87], after some canonical transformations [97], the Hamiltonian for a single fluid with equation of state $\omega = p/\rho$ in the minisuperspace for a flat FLRW background reads as

$$\mathcal{H} = -\kappa^2 \frac{P_a^2}{12a} + \frac{P_T}{a^{3\omega}}. \tag{A1}$$

In this case, the classical equations of motion, for a given value of ω, reads

$$\dot{a} = -\kappa^2 \frac{P_a}{6a}, \tag{A2a}$$

$$\dot{P}_a = -\kappa^2 \frac{P_a^2}{12a^2} + \frac{3\omega P_T}{a^{3\omega+1}}, \tag{A2b}$$

$$\dot{T} = \frac{1}{a^{3\omega}}, \tag{A2c}$$

$$\dot{P}_T = 0. \tag{A2d}$$

From Eq. (A1), $\mathcal{H} = 0$ together with Eq. (A2) leads to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3} \frac{P_T}{a^{3\omega+3}}. \tag{A3}$$

From the quantum point of view, $\hat{\mathcal{H}}\Psi(a,T) = 0$, replacing (P_a, P_T) by the operators $(-i\partial_a, -i\partial_T)$, we obtain that

$$i\partial_T \Psi(a,T) = \frac{\kappa^2}{12} a^{3\omega-1} \partial_a^2 \Psi(a,T). \tag{A4}$$

Due to the ambiguity in the ordering of factors a and P_a in the right-hand side of the latter equation, we can rewrite it as

$$i\partial_T \Psi(a,T) = \frac{\kappa^2}{12} a^{3\omega-1} \left(\partial_a^2 + \frac{s}{a}\partial_a\right) \Psi(a,T). \tag{A5}$$

For $\omega \neq 1$, choosing $s = (3\omega - 1)/2$ and performing the change of variable

$$\chi = \frac{2}{\sqrt{3}\kappa} \frac{a^{\frac{3(1-\omega)}{2}}}{(1-\omega)}, \tag{A6}$$

one obtains that

$$i\partial_T \Psi(\chi,T) = \frac{1}{4} \partial_\chi^2 \Psi(\chi,T). \tag{A7}$$

On the other hand, due to the fact that Eq. (A6) is singular for $\omega = 1$, in this particular case we choose $s = 1$ and consider the following logarithmic change of variable,

$$\chi = \frac{\sqrt{3}}{\kappa} \ln(\epsilon a), \tag{A8}$$

where $\epsilon > 0$ is an arbitrary constant. Under these circumstances, we obtain Eq. (A7).

Equation (A7) can be interpreted as a Schrödinger-type equation for a free particle in one dimension, with mass $m = 2$ and negative kinetic energy. Also, due to the fact that the $a > 0$ (hence, $\chi > 0$), this is restricted to the half axis [98]. Unitary solutions can be obtained by performing a self-adjoint extension, which are given by the boundary condition (for more details, see Ref. [83]),

$$(\Psi^* \partial_\chi \Psi - \Psi \partial_\chi \Psi^*)|_{\chi=0} = 0. \tag{A9}$$

Considering a Gaussian initial condition,

$$\Psi(\chi, 0) = \left(\frac{8}{\pi T_0}\right)^{1/4} e^{-\frac{\chi^2}{T_0}}, \tag{A10}$$

which satisfies the boundary condition given by Eq. (A9), one obtains the following general solution:

$$\Psi(\chi,T) = \left[\frac{8T_0}{\pi(T_0^2+T^2)}\right]^{1/4} \exp\left[-\frac{T_0\chi^2}{T_0^2+T^2}\right] \times \exp\left\{-i\left[\frac{T\chi^2}{T_0^2+T^2} + \frac{1}{2}\arctan\left(\frac{T_0}{T}\right) - \frac{\pi}{4}\right]\right\}, \tag{A11}$$

where T_0 is an arbitrary constant.

The wave function $\Psi(a,T)$, Eq. (A11), can be obtained for $\omega \neq 1$ and $\omega = 1$, using Eqs. (A6) and (A8), respectively. Particularly, for $\omega = 1$ it reads

$$\Psi(a,T) = \left[\frac{8T_0}{\pi(T_0^2+T^2)}\right]^{1/4} \exp\left[-\frac{3T_0 \ln^2(\epsilon a)}{\kappa^2(T_0^2+T^2)}\right] \times \exp\left\{-i\left[\frac{3T\ln^2(\epsilon a)}{\kappa^2(T_0^2+T^2)} + \frac{1}{2}\arctan\left(\frac{T_0}{T}\right) - \frac{\pi}{4}\right]\right\}. \tag{A12}$$

The arbitrary parameter ϵ, in addition to the parameter T_0, must be chosen to ensure that χ in Eq. (A8) is always positive.

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