Algebraic approach to relativistic Landau levels in the symmetric gauge

Ulrich D. Jentschura

Department of Physics and LAMOR, Missouri University of Science and Technology, Rolla, Missouri 65409, USA

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We use an algebraic approach to the calculation of Landau levels for a uniform magnetic field in the symmetric gauge with a vector potential $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$, where $\vec{B}$ is assumed to be constant. The magnetron quantum number constitutes the degeneracy index. An overall complex phase of the wave function, given in terms of Laguerre polynomials, is a consequence of the algebraic structure. The relativistic generalization of the treatment leads to fully relativistic bispinor Landau levels in the symmetric gauge, in a representation which writes the relativistic states in terms of their nonrelativistic limit, and an algebraically accessible lower bispinor component. Negative-energy states and the massless limit are discussed. The relativistic states can be used for a number of applications, including the calculation of higher-order quantum electrodynamic binding corrections to the energies of quantum cyclotron levels.

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I. INTRODUCTION

This paper is about nonrelativistic and relativistic Landau levels for spin-$1/2$ particles, pertinent to a uniform magnetic field. We aim to develop algebraic relations, which describe the energetically degenerate Landau levels in terms of cyclotron and magnetron quantum numbers. Furthermore, we aim to generalize the treatment to the fully relativistic domain, where the Landau levels become eigenstates of the magnetically coupled Dirac equation. Landau levels, including the relativistic case, are important for many physical processes, first and foremost perhaps, in the context of the determination of fundamental physical constants, for quantum cyclotron processes in Penning traps [1–4]. Among the many other applications, we mention the description of the quantum Hall effect [5–9], 2D quantum dots [10], particle production in the magnetars [11], and processes related to synchrotron radiation [12,13].

The constant, uniform magnetic field is assumed to be directed along the z axis, and used in the form $\vec{B}_{\mathrm{T}} = B_{\mathrm{T}}\hat{\mathrm{e}}_z$, where we use the subscript T to denote the possible application to a Penning trap [1,2]. Landau levels are normally calculated [14] in the Landau gauge $\vec{A} = -By\hat{\mathrm{e}}_x$ [see Eq. (112.1) of Ref. [15]]. Upon solving the corresponding Schrödinger equation, one finds that in the Landau gauge, energy eigenstates are also eigenstates of the x component p_x of the momentum operator. The momentum component p_x becomes a parameter which shifts the center of oscillation for the effective harmonic oscillator potential, which acts onto the motion in the y direction [see Eq. (112.5) of Ref. [15]]. In the Landau gauge, the value of p_x does not affect the energy of the state and becomes a continuous degeneracy index.

Here, we use the so-called symmetric gauge [10,16], where the vector potential is taken as $\vec{A} = \frac{1}{2}(\vec{B}_{\mathrm{T}} \times \vec{r})$. In the symmetric gauge, for the nonrelativistic case, the eigenfunctions and energies are in principle known [10,16]. Here, we augment the theory by identifying the raising and lowering operators of the cyclotron and magnetron motions as the determining dynamic variables of the problem. This enables us to fully clarify the origin of the degeneracy of the (nonrelativistic) Landau levels in the symmetric gauge, and greatly simplify the evaluation of matrix elements of eigenstates.

We also find the fully relativistic generalization of the Landau levels, corresponding to the relativistic quantum cyclotron states in the limit of vanishing axial frequency. The relativistic states can be used as input for higher-order quantum electrodynamic calculations of quantized-field effects in Penning traps [4], and any physical systems where relativistic effects become important. Units with $\hbar = c = \epsilon_0 = 1$ are used throughout this paper.

II. NONRELATIVISTIC PROBLEM

We start our treatment of Landau levels in the symmetric gauge, inspired by an electron in a Penning trap [1,2,17], where an additional electric quadrupole potential leads to axial confinement. In the limit of a vanishing axial frequency, the cyclotron and magnetron levels of the Penning trap become equal to Landau levels. In general, quantum cyclotron states are described [1,2,17] by the quantum numbers $k\ell ns$, where $k = 0, 1, 2, \ldots$ is the axial excitation, $\ell = 0, 1, 2, \ldots$ is the magnetron quantum number, $n = 0, 1, 2, \ldots$ is the cyclotron quantum number, and $s = \pm 1$ is the spin projection quantum number.

The kinetic momentum $\vec{\pi}_{\rm T} = \vec{p} - e\vec{A}$, in the symmetric gauge, can be written as

$$\vec{\pi}_{\rm T} = \vec{\pi}_{\parallel} + \vec{p}_{\perp}, \tag{1a}$$

$$\vec{\pi}_{\parallel} = \vec{p}_{\parallel} - \frac{e}{2}(\vec{B}_{\rm T} \times \vec{r}), \qquad \vec{p}_{\parallel} = p_x \hat{\rm e}_x + p_y \hat{\rm e}_y, \tag{1b}$$

$$\vec{B}_{\rm T} = B_{\rm T} \hat{\rm e}_z. \qquad \vec{p}_{\perp} = p_z \hat{\rm e}_z. \tag{1c}$$

Here, e is the electron charge. As already mentioned, we use the symmetric gauge for the vector potential $\vec{A} = \frac{1}{2}(\vec{B}_{\rm T} \times \vec{r})$, rather than $\vec{A}' = -By\hat{\rm e}_x$, which yields the same magnetic field. The gauge transformation $\Lambda = -\frac{1}{2} B_{\rm T} xy$ leads from $\vec{A} = \frac{1}{2}(\vec{B}_{\rm T} \times \vec{r})$ to $\vec{A}'$. We note that the physical interpretation of a wave function can depend on the gauge (see p. 268 of Ref. [18]). The choice $\vec{A} = \frac{1}{2}(\vec{B}_{\rm T} \times \vec{r})$ leads to wave functions which are confined in the x and y directions, while, with the choice $\vec{A}'$, one obtains unconfined wave functions in the y direction.

Furthermore, the choice $\vec{A} = \frac{1}{2}(\vec{B}_{\rm T} \times \vec{r})$ is generalizable to the case of a nonvanishing additional electric quadrupole trap field [1,2]. The nonrelativistic Hamiltonian is given as follows:

$$H_0 = \frac{(\vec{\sigma} \cdot \vec{\pi}_{\rm T})^2}{2m} = H_{\parallel} + \frac{\omega_c}{2} \sigma_z + \frac{p_z^2}{2m}, \tag{2}$$

$$\omega_c = -\frac{e B_{\rm T}}{m} = \frac{|e| B_{\rm T}}{m}. \tag{3}$$

Here, $|e| = -e$ is the modulus of the electron charge. The spin-independent part of the Hamiltonian relevant for the xy plane is

$$H_{\parallel} = \frac{\vec{\pi}_{\parallel}^2}{2m} = \frac{\vec{p}_{\parallel}^2}{2m} + \frac{\omega_c}{2} L_z + \frac{m\omega_c^2}{8}\rho^2 = \omega_c \left(a_c^{\dagger} a_c + \frac{1}{2} \right), \tag{4}$$

where the $a_c^{\dagger}$ and a_c are the raising and lowering operators of the cyclotron motion. We have set the electric quadrupole potential of the trap equal to zero, wherefore, in the sense of Refs. [1,2], the magnetron frequency vanishes. The cyclotron (c) and magnetron (m) raising ($a_c^{\dagger}$ and ($a_{\rm m}^{\dagger}$) and lowering operators (a_c and $a_{\rm m}$) are given as follows:

$$a_c = \frac{1}{\sqrt{2}} \left(\tilde{a}_0 p_x - {\rm i}\tilde{a}_0 p_y - \frac{{\rm i}x}{2\tilde{a}_0} - \frac{y}{2\tilde{a}_0} \right), \tag{5a}$$

$$a_c^{\dagger} = \frac{1}{\sqrt{2}} \left(\tilde{a}_0 p_x + {\rm i}\tilde{a}_0 p_y + \frac{{\rm i}x}{2\tilde{a}_0} - \frac{y}{2\tilde{a}_0} \right), \tag{5b}$$

$$a_{\rm m} = \frac{1}{\sqrt{2}} \left(\tilde{a}_0 p_x + {\rm i}\tilde{a}_0 p_y - \frac{{\rm i}x}{2\tilde{a}_0} + \frac{y}{2\tilde{a}_0} \right), \tag{5c}$$

$$a_{\rm m}^{\dagger} = \frac{1}{\sqrt{2}} \left(\tilde{a}_0 p_x - {\rm i}\tilde{a}_0 p_y + \frac{{\rm i}x}{2\tilde{a}_0} + \frac{y}{2\tilde{a}_0} \right). \tag{5d}$$

In these formulas, the generalized (magnetic) Bohr radius is (we temporarily restore factors of $\hbar$ and c)

$$\tilde{a}_0 = \sqrt{\frac{mc^2}{\hbar\omega_c}} \frac{\hbar}{mc} = \frac{\hbar}{\alpha_c mc} = \sqrt{\frac{\hbar}{|e| B_{\rm T}}} = \ell_B, \tag{6}$$

where we have set $\alpha_c = \sqrt{\hbar\omega_c/(mc^2)}$ (α_c is a generalized fine-structure constant), and ℓ_B is the magnetic length [5]. The relations (5) can be inverted,

$$p_x = \frac{1}{2\sqrt{2}\tilde{a}_0} [a_c + a_c^{\dagger} + a_{\rm m} + a_{\rm m}^{\dagger}], \tag{7a}$$

$$p_y = \frac{\rm i}{2\sqrt{2}\tilde{a}_0} [a_c - a_c^{\dagger} - a_{\rm m} + a_{\rm m}^{\dagger}], \tag{7b}$$

$$x = \frac{{\rm i}\tilde{a}_0}{\sqrt{2}} [a_c - a_c^{\dagger} + a_{\rm m} - a_{\rm m}^{\dagger}], \tag{7c}$$

$$y = -\frac{\tilde{a}_0}{\sqrt{2}} [a_c + a_c^{\dagger} - a_{\rm m} - a_{\rm m}^{\dagger}]. \tag{7d}$$

The nonvanishing commutators are as follows:

$$[a_c, a_c^{\dagger}] = [a_{\rm m}, a_{\rm m}^{\dagger}] = 1. \tag{8}$$

All other commutators vanish, which means that, in particular, the cyclotron and magnetron excitation numbers, together with the z component of the momentum and the spin projection quantum number s, form a complete set of commuting observables for the nonrelativistic spin-independent problem. The eigenfunctions are given as follows:

$$H_{\rm NR} \psi_{kn\ell s}(\vec{r}) = E_{\rm NR} \psi_{kn\ell s}(\vec{r}), \tag{9a}$$

$$E_{\rm NR} = \omega_c \left(n + \frac{s+1}{2} \right) + \frac{k^2}{2m}. \tag{9b}$$

Here, $k \in \mathbb{R}$ is a continuous quantum number characterizing the momentum component in the z direction. The energy eigenvalue is independent of the magnetron quantum number ℓ. The momentum in the z component and the spin component of the eigenstate can be split off:

$$\psi_{k\ell ns}(\vec{r}) = \psi_{n\ell}(\vec{\rho})\frac{e^{ikz}}{\sqrt{2\pi}}\chi_s, \tag{10a}$$

$$\vec{\rho} = x\hat{e}_x + y\hat{e}_y, \tag{10b}$$

$$\chi_{+1} = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_{-1} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{10c}$$

According to Refs. [1,2,17], one uses the quantum numbers $k\ell ns$ (in this, alphabetically inspired, sequence) by convention. However, for the spin-independent, two-dimensional, nonrelativistic case, we use the designation $n\ell$, in correspondence with the hydrogen atom where the first index indicates the quantum number which determines the energy (for a comprehensive discussion, see Chap. 4 of Ref. [19]). One may search for the general formula for the nonrelativistic eigenfunction $\psi_{n\ell}(\vec{\rho})$, which fulfills the equation

$$H_{\parallel}\psi_{n\ell}(\vec{\rho}) = E_n\psi_{n\ell}(\vec{\rho}), \qquad E_n = \omega_c\left(n + \frac{1}{2}\right), \tag{11}$$

where $H_{\parallel}$ is given in Eq. (4). The solution reads as follows:

$$\psi_{n\ell}(\vec{\rho}) = \frac{2^{-\frac{1}{2}(|n-\ell|+1)}}{\sqrt{\pi}\tilde{a}_0}\sqrt{\frac{\min(n,\ell)!}{\max(n,\ell)!}}\left(\frac{\rho}{\tilde{a}_0}\right)^{|n-\ell|} \times i^{|n-\ell|}L^{|n-\ell|}_{\min(n,\ell)}\left(\frac{1}{2}\left(\frac{\rho}{\tilde{a}_0}\right)^2\right) \times e^{i(n-\ell)\varphi}\exp\left[-\frac{1}{4}\left(\frac{\rho}{\tilde{a}_0}\right)^2\right]. \tag{12}$$

Here, $\rho = |\vec{\rho}|$ and $\varphi = \arctan(y/x)$. The complex phase $i^{|n-\ell|}$ is a consequence of the application of the raising operators on the ground states [see also Eq. (13a)].

A comparison of Eq. (12) with the literature is indicated. In Eq. (2) of Ref. [20], one obtains a corresponding result. The spin-independent wave function obtained in Ref. [20] is equal to our wave function if one replaces k (in the notation of Ref. [20] by $\min(n,\ell)$, and m (in the notation of Ref. [20]) by $n-\ell$ (in our notation). Furthermore, one may point out that the wave functions given in Eq. (2) of Ref. [20] are specialized to the case $\omega_c = 2$. In Ref. [8], this restriction is lifted. The parameter l_0 defined in Eq. (28) of Ref. [8] is equal to our parameter $\tilde{a}_0$.

In Eq. (A31) of Ref. [10], the authors obtain a corresponding result. Their parameter ℓ (which, in the conventions of Ref. [10] can become negative) is equal to our $n-\ell$. The inclusion of additional electric fields into the formalism is discussed in Refs. [1,2,17] and in Ref. [21].

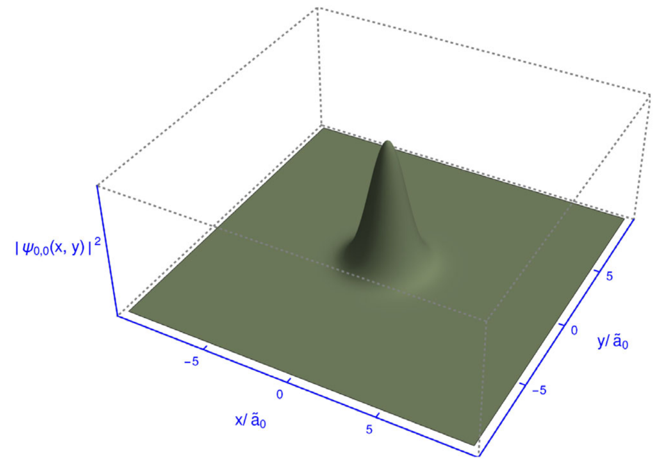

FIG. 1. We display the probability density $|\psi_{(n=0),(\ell=0)}(\vec{\rho})|^2$ of the cyclotron ground state, as given in Eq. (12). It is radially symmetric and nonvanishing at the origin.

Our formalism clearly identifies the cyclotron excitation, which is linked to the quantum number n that determines the energy, and the magnetron quantum number ℓ, which does not shift the energy but shifts the orbit further outward with increasing ℓ for given n (see Figs. 1 and 2). When one increases ℓ by 1, the wave function acquires a phase factor $\exp(-i\varphi)$. The cyclotron and magnetron excitations can be associated with the action of mutually commuting raising operators, acting on the ground state, and the quantum numbers n and ℓ belong to mutually commuting observables.

Indeed, the nonrelativistic eigenfunctions can be obtained from the ground state by the operation of the raising operators,

$$\psi_{n\ell}(\vec{\rho}) = \frac{1}{\sqrt{n!\ell!}}(a_c^\dagger)^n(a_m^\dagger)^\ell\psi_{00}(\vec{\rho}), \tag{13a}$$

$$\psi_{00}(\vec{\rho}) = \frac{1}{\tilde{a}_0\sqrt{2\pi}}\exp\left(-\frac{\rho^2}{4\tilde{a}_0^2}\right). \tag{13b}$$

The number operators count the number of cyclotron and magnetron excitations,

$$a_c^\dagger a_c|n,\ell\rangle = n|n,\ell\rangle, \qquad a_m^\dagger a_m|n,\ell\rangle = \ell|n,\ell\rangle. \tag{14}$$

The wave functions $\psi_{n\ell}(\vec{\rho})$ are eigenfunctions of the Hamiltonian $H_{\parallel}$, while their energy eigenvalue only depends on the cyclotron quantum number n, and is independent of the magnetron quantum number ℓ. The magnetron quantum number $\ell = 0, 1, 2, \ldots$ acts as a degeneracy index for the Landau level. Here, the magnetron degeneracy index is countable and leads to a more intuitive form of the wave function, which is normalizable to unity.

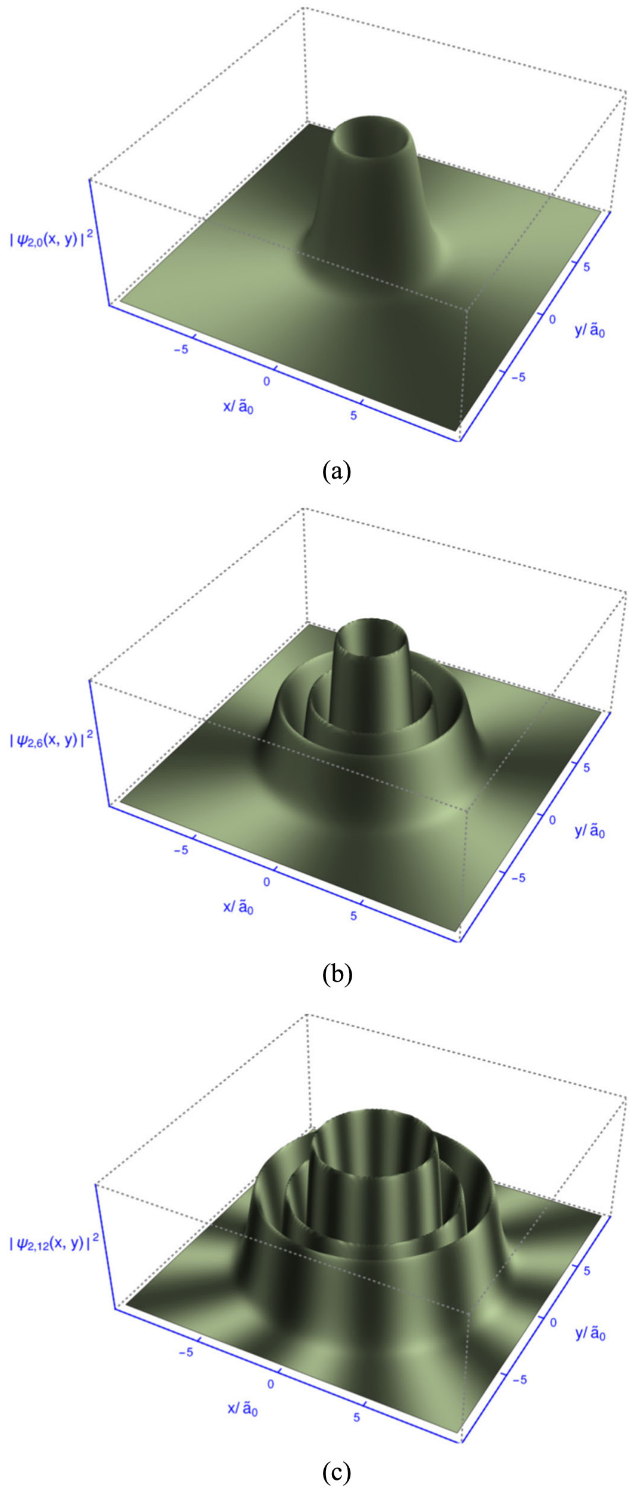

FIG. 2. The cyclotron states with $n = 2$ all have the same nonrelativistic energy $\frac{5}{2}\omega_c$. The probability density $|\psi_{(n=2)\ell}(\vec{\rho})|^2$ vanishes at the origin. With increasing magnetron excitation $\ell = 0$, 6, 12, the states spread away from the origin. The azimuthal dependence of the wave function is given by the factor $\exp[\mathrm{i}(n-\ell)\varphi]$. The plot surfaces are shaded according to the imaginary part of the wave function, with lighter regions indicating a positive imaginary part, and darker regions indicating a negative imaginary part of the wave function.

Wave functions are orthonormal,

$$\int \mathrm{d}^2\rho\,\psi^*_{n\ell}(\vec{\rho})\psi_{n'\ell'}(\vec{\rho}) = \delta_{nn'}\delta_{\ell\ell'}. \tag{15}$$

A total of four example wave functions are plotted in Figs. 1 and 2. Furthermore, one has the relation

$$\int \mathrm{d}^3r\,\psi^\dagger_{k\ell ns}(\vec{\rho})\psi_{k'\ell'n's'}(\vec{\rho}) = \delta(k-k')\delta_{nn'}\delta_{\ell\ell'}\delta_{ss'}. \tag{16}$$

(For the spinor case, we need to use $\psi^\dagger$ because it is a two-component wave function.) With increasing magnetron quantum number, the states spread further away from the origin, as is evident from the expectation value

$$\int \mathrm{d}^2\rho\,\rho^2|\psi_{n\ell}(\vec{\rho})|^2 = (2+2n+2\ell)\tilde{a}_0^2. \tag{17}$$

This trend is depicted in Fig. 2.

III. RELATIVISTIC LANDAU LEVELS

Before we indulge in the calculations, let us remember that the problem of a relativistic electron coupled to a uniform electric or magnetic field, has been discussed in the literature before, notably, in a comprehensive treatise given in Ref. [22]. In unnumbered equations between Eqs. (10) and (11) of Sec. 3 of Chap. 3 of Ref. [22], the relativistic eigenstate of the magnetically coupled Dirac equation is given in terms of parabolic cylinder functions (Weber functions, denoted as D_n). The vector potential is taken in the Landau gauge. Alternative discussions of the relativistic states are given in Sec. 4 of Chap. 26 of Ref. [12] and in Sec. 6 of Chap. 2 of Ref. [23]. Our goal here is to find a representation of the relativistic eigenstates which allows us to clearly identify the connections of the nonrelativistic limit with the fully relativistic state.

We here aim to find the relativistic Landau levels in the symmetric gauge. Let us recall the nonrelativistic (NR) Hamiltonian and its eigenstates,

$$H_{\mathrm{NR}}\psi_{\mathrm{NR}} = E_{\mathrm{NR}}\psi_{\mathrm{NR}}, \qquad H_{\mathrm{NR}} = \frac{(\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}})^2}{2m}, \tag{18}$$

$$E_{\mathrm{NR}} = \omega_c\left(n + \frac{s+1}{2}\right) + \frac{k^2}{2m}. \tag{19}$$

The kinetic momentum in the trap is $\vec{\pi}_{\mathrm{T}} = \vec{p} - e\vec{A}_{\mathrm{T}}$, while the vector potential $\vec{A}_{\mathrm{T}}$ for the uniform magnetic field of the trap can be written as

$$\vec{\pi}_{\mathrm{T}} = \vec{p} - e\vec{A}_{\mathrm{T}}, \qquad \vec{A}_{\mathrm{T}} = \frac{1}{2}(\vec{B}_{\mathrm{T}}\times\vec{r}). \tag{20}$$

The Dirac Hamiltonian for a uniform magnetic field is

$$H_{\mathrm{D}} = \vec{\alpha}\cdot\vec{\pi}_{\mathrm{T}} + \beta m, \tag{21}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}, \tag{22}$$

where we use the Dirac matrix in the Dirac representation. We search for bispinor solutions of the form

$$H_{\mathrm{D}}\Psi = E_{\mathrm{D}}\Psi, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \Psi_{k\ell ns}. \tag{23a}$$

We note that Ψ denotes the relativistic bispinor wave function, while ψ is the nonrelativistic counterpart. We use the following *ansatz* for the relativistic eigenstates:

$$\psi_1 = \mathcal{N}\psi_{\mathrm{NR}}, \qquad \psi_2 = \mathcal{N}\frac{\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}}{E_D + m}\psi_{\mathrm{NR}}. \tag{23b}$$

Acting with H_{D} on Ψ, one obtains

$$\begin{aligned} H_{\mathrm{D}}\Psi &= \mathcal{N}\begin{pmatrix} m\mathbb{1}_{2\times 2} & \vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}} \\ \vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}} & -m\mathbb{1}_{2\times 2} \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{NR}} \\ \frac{\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}}{E_{\mathrm{D}}+m}\psi_{\mathrm{NR}} \end{pmatrix} \\ &= \mathcal{N}\begin{pmatrix} m\left(1+\frac{2E_{\mathrm{NR}}}{E_{\mathrm{D}}+m}\right)\psi_{\mathrm{NR}} \\ \vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}\left(1-\frac{m}{E_{\mathrm{D}}+m}\right)\psi_{\mathrm{NR}} \end{pmatrix} \\ &\overset{!}{=} \mathcal{N}E_{\mathrm{D}}\begin{pmatrix} \psi_{\mathrm{NR}} \\ \frac{\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}}{E_{\mathrm{D}}+m}\psi_{\mathrm{NR}} \end{pmatrix} = E_D\Psi. \end{aligned} \tag{24}$$

From the secular equation for the upper component, one obtains

$$E_{\mathrm{D}} = m\left(1+\frac{2E_{\mathrm{NR}}}{E_{\mathrm{D}}+m}\right) = m + \frac{2mE_{\mathrm{NR}}}{E_{\mathrm{D}}+m}. \tag{25}$$

This can be rewritten as follows:

$$E_{\mathrm{D}} - m = \frac{2mE_{\mathrm{NR}}}{E_{\mathrm{D}}+m}, \qquad E_{\mathrm{D}}^2 - m^2 = 2mE_{\mathrm{NR}}. \tag{26}$$

The Dirac energy E_{D} is thus obtained as follows,

$$\begin{aligned} E_{\mathrm{D}} &= \sqrt{m^2 + 2mE_{\mathrm{NR}}} = m\sqrt{1+\frac{2E_{\mathrm{NR}}}{m}} \\ &= m\sqrt{1+\frac{\omega_c}{m}(2n+s+1)+\frac{k^2}{m^2}}. \end{aligned} \tag{27}$$

Somewhat surprisingly, the state with quantum numbers $n=0$ and $s=-1$, and $k=0$, has exact rest mass energy even in the fully relativistic formalism. A check of the eigenvector property for the lower component is successful. We start from the expression in round brackets (lower component) in the second line of Eq. (24) and obtain:

$$\left(1-\frac{m}{E_{\mathrm{D}}+m}\right) = \frac{E_{\mathrm{D}}}{E_{\mathrm{D}}+m} = E_{\mathrm{D}}\frac{1}{E_{\mathrm{D}}+m}. \tag{28}$$

In view of the Taylor expansion $\sqrt{1+\epsilon} = 1+\frac{\epsilon}{2}-\frac{\epsilon^2}{8}+\frac{\epsilon^3}{16}+\mathcal{O}(\epsilon^4)$, we can write for positive energy in the limit $k\to 0$

$$\begin{aligned} E_{\mathrm{D}}(k\to 0) = m + \frac{\omega_c}{2m}(2n+s+1) - \frac{1}{8}\left(\frac{\omega_c}{m}(2n+s+1)\right)^2 \\ + \frac{1}{16}\left(\frac{\omega_c}{m}(2n+s+1)\right)^3. \end{aligned} \tag{29}$$

This is consistent with Eqs. (90), (96) and (105) of Ref. [17]. We set $\omega_c = \alpha_c^2 m$, where α_c is the cyclotron coupling constant. The normalization of the relativistic states according to $\int \mathrm{d}^3 r|\psi_{\mathrm{NR}}|^2 = \int \mathrm{d}^3 r|\Psi|^2 = 1$ leads to the relation

$$\int \mathrm{d}^3 r\Psi^\dagger(\vec{r})\Psi(\vec{r}) = \mathcal{N}^2\left[1+\frac{2mE_{\mathrm{NR}}}{(E_{\mathrm{D}}+m)^2}\right] \overset{!}{=} 1, \tag{30}$$

$$\mathcal{N} = \left[1+\frac{2mE_{\mathrm{NR}}}{(E_{\mathrm{D}}+m)^2}\right]^{-1/2}. \tag{31}$$

In the above formulas for the relativistic states, the dependence on the quantum numbers $k\ell ns$ has been suppressed for notational simplicity. Restoring the quantum numbers, one finds that

$$\int \mathrm{d}^3 r\Psi^\dagger_{k\ell ns}(\vec{\rho})\Psi_{k'\ell'n's'}(\vec{\rho}) = \delta(k-k')\delta_{nn'}\delta_{\ell\ell'}\delta_{ss'}. \tag{32}$$

For completeness, it is perhaps useful to remark that, if desired, one can easily write the entire operator $\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}$ in terms of cyclotron, magnetron and spin ladder operators, and express the lower component ψ_2 of the fully relativistic state, in terms of nonrelativistic eigenstates of raised and lowered quantum numbers.

IV. NEGATIVE-ENERGY STATES

In Dirac theory, there is an energy gap between positive-energy states with $E\geq m$ and negative-energy states with $E\leq -m$. It is relatively straightforward to check that the states [cf. Eqs. (23a) and (23b)]

$$\Phi = \mathcal{N}\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}}{E_{\mathrm{D}}+m}\psi_{\mathrm{NR}} \\ \psi_{\mathrm{NR}} \end{pmatrix} \tag{33}$$

with the same normalization factor $\mathcal{N}$ as for positive-energy states [see Eq. (31)], have the property

$$H_{\mathrm{D}}\Phi = -E_{\mathrm{D}}\Phi \tag{34}$$

and thus constitute the negative-energy (antiparticle) state of Dirac theory. As compared with the positive-energy state, the upper and lower components of the Dirac bispinor are interchanged, and the upper spinor receives a minus sign.

V. MASSLESS LIMIT

In the limit $m \to 0$, the gap between positive-energy and negative-energy states vanishes. We recall that

$$\omega_c m = |e| B_{\mathrm{T}} = \ell_B^{-2}. \tag{35}$$

The Dirac energy (27) attains the massless limit

$$E_{\mathrm{D}} \overset{m\to 0}{=} \sqrt{\ell_B^{-2}(2n+s+1)+k^2}\,. \tag{36}$$

The square-root dependence on the quantum numbers confirms the corresponding dependence, obtained on the basis of the Weyl equation, given in Eq. (2.23) of Ref. [5]. The appropriate zero-mass limit of the normalization factor is

$$\mathcal{N} \overset{m\to 0}{=} \frac{1}{\sqrt{2}}, \tag{37}$$

and the positive-energy solution is

$$\Psi \overset{m\to 0}{=} \begin{pmatrix} \frac{1}{\sqrt{2}}\psi_{\mathrm{NR}} \\ \frac{1}{\sqrt{2}}\frac{\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}}{E_D}\psi_{\mathrm{NR}} \end{pmatrix}. \tag{38}$$

The negative-energy solution is

$$\Phi \overset{m\to 0}{=} \begin{pmatrix} -\frac{1}{\sqrt{2}}\frac{\vec{\sigma}\cdot\vec{\pi}_{\mathrm{T}}}{E_D}\psi_{\mathrm{NR}} \\ \frac{1}{\sqrt{2}}\psi_{\mathrm{NR}} \end{pmatrix}. \tag{39}$$

VI. CONCLUSIONS

We have considered the calculation of both nonrelativistic as well as fully relativistic Landau levels in the symmetric gauge where $\vec{A} = \frac{1}{2}(\vec{B}_{\mathrm{T}} \times \vec{r})$, based on an algebraic approach. The spin-independent part of the nonrelativistic states finds a natural form in terms of the cyclotron quantum number n, and of the magnetron quantum number ℓ. We find a universal representation of the spinless nonrelativistic quantum state wave function with quantum numbers n (cyclotron) and ℓ (magnetron) in Eq. (12). After including the spin and the axial motion, we find the nonrelativistic spinor state with quantum numbers n and ℓ, and k (axial) and s (spin), as given in Eq. (10). For the nonrelativistic wave function $\psi = \psi_{k\ell ns}$, the spin-independent form is given in Eq. (12), and the nonrelativistic spinor form is given in Eq. (10). The orthonormality relations are given in Eqs. (15) (for the spin-independent part) and in Eq. (16) (for the spinor wave function). While the magnetron quanta are raised from the ground state by the magnetron raising operator $a_{\mathrm{m}}^{\dagger}$ [see Eq. (14)], the magnetron quantum number ℓ does not enter the formula for the energy of the Landau level [see Eqs. (11) and (9)]. Inverting the relations given in Eq. (5), one obtains a representation of the x, y, p_x and p_y operators in terms of the raising and lowering operators of the cyclotron and magnetron motions [see Eq. (7)].

In the context of the quantum Hall effect, Landau levels have been considered in both the Landau as well as the symmetric gauges [5]. They are also relevant to quantum cyclotron states in Penning traps [1,2]. Indeed, in the limit of vanishing axial confinement, the quantum cyclotron states approach the Landau levels in the symmetric gauge, and our formulas are essential elements in the discussion of higher-order quantum electrodynamic corrections to quantum cyclotron energy levels [4].

Based on an essential generalization of the nonrelativistic problem (see Sec. II and Ref. [5]), we calculate the fully relativistic Landau levels in the symmetric gauge, in Sec. III. These relativistic states constitute solutions of the magnetically coupled relativistic Dirac equation. The same quantum numbers k, ℓ, n and s, that we found for the nonrelativistic problem, characterize the relativistic state. The relativistic wave function is given in Eqs. (23a) and (23b), with the normalization given in Eq. (30). The fully relativistic form is needed as an essential ingredient in the calculation of quantum electrodynamic corrections to quantum cyclotron energy levels [4]. We mention relativistic quantum dynamics in synchrotrons as a further potential area of application. Negative-energy states are discussed in Sec. IV, and the massless limit of the Dirac bispinor solutions is discussed in Sec. V.

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