Strong-field QED in the Furry-picture momentum-space formulation: Ward identities and Feynman diagrams

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The impact of a strong electromagnetic background field on otherwise perturbative QED processes is studied in the momentum-space formulation. The univariate background field is assumed to have finite support in time, thus being suitable to provide a model for a strong laser pulse in plane-wave approximation. The usually employed Furry picture in position space must be equipped with some regularizing procedure, also to ensure the Ward identity. The momentum space formulation allows generically for an easy and systematic account of such expressions, both globally and order-by-order in the weak-field expansion. In the limit of an infinitely long-acting (monochromatic) background field, these terms become gradually suppressed, and the standard perturbative QED Feynman diagrams are recovered in the leading-order weak-field limit. A few examples of three- and four-point amplitudes are considered to demonstrate the application of our Feynman rules which employ free Dirac spinors, the free photon propagator, and the free fermion propagator, while the external field impact is solely encoded in the fermion-fermion-photon vertex function. Properties of the latter vertex function are elaborated in some detail. The appearance of on/off shell contributions, singular structures, and Oleinik resonances is pointed out.

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I. INTRODUCTION

With the advent of permanently increasing laser intensities by advanced technologies [1] one gets access towards the strong-field regime of quantum electrodynamics (QED) [2] in a hitherto uncharted region of parameter space. Among the famous examples are the "vacuum break-down" by the Sauter-Schwinger effect (cf. [3–6] for recent activities and entries to extensive citations) and the unsettled implications [7–9] of the Ritus-Narozhny conjecture (cf. [10] for an introduction). While QED delivers unprecedentedly accurate results in certain regions of the parameter space, at high energies of the involved particles, and for processes in strong background fields there is still room for testing the theory for "physics beyond the Standard Model" or verifying long-standing predictions. For instance, the soft-photon theorems—at the heart of the infra-red (IR) structure of QED—seem to fail when many hadrons are involved in the final state of high-energy strong-interaction collisions. This issue triggered activities for new detector concepts and plans of experimental investigations at the Large Hadron Collider (LHC) [11]. In fact, Strominger's IR triangle [12] finds recently much interest culminating in "new symmetries of QED" [13], see e.g. [14].

The current standard approach to calculations of QED processes in strong background fields deploys the Furry picture in position space. Fermions (electrons and positrons) are dressed by accounting exactly for the (quasi-classical) interaction with the electromagnetic background field, while the interaction of fermions with photons is dealt with in order-α perturbative expansion ($\alpha \approx 1/137$; the fine-structure constant), see [15–17] for reviews. A convenient model of the laser field is accomplished in the plane-wave approximation. The external field, taken as given background (for backreaction dynamics, cf. [18,19]), depends then only on one variable, the invariant phase $\phi \coloneqq k \cdot x = \omega t^-$, where k^μ is the reference momentum of the laser field with central frequency $\omega \coloneqq k^0$ and $t^- = x_0 - x_\parallel$ is the light-front time in a coordinate system where $\vec{k} \| \vec{e}_z$. Due to the high symmetry, the Dirac equation can be solved exactly, delivering the Volkov solution, which depends trivially on three components of space-time and, generally, highly nontrivial on ϕ or t^- via the background. For a laser pulse, the background has finite support in the variables ϕ and t^-, and the details of the temporal structure shape the final phase-space distribution in a distinct manner. The limiting case of an infinitely long-acting external field is called a monochromatic field. It is to be contrasted with the sandwich field—where, like in the situation of a passing gravitational wave—an observer sees the vacuum, followed by the pulse, and is left afterward in the vacuum again, irrespective of memory effects imparted on test particles.

It is often taken for guaranteed that the weak-field limit of the background field (which is a classical field) and the monochromatic limit (i.e. infinite support at constant strength) yields the standard perturbative QED results obtained via Feynman diagrams in momentum space. The role of loops is less obvious in that correspondence. A particular situation is when considering a short weak (classical background) field pulse; a straightforward treatment by perturbative QED via Feynman diagrams does not catch the features caused by the higher/lower Fourier components of the weak field, see [20]. In particular, [21] points out that, to preserve the Ward identity, one has to supplement the Furry-picture position-space Feynman diagrams by some terms related to regularization and gauge invariance. It happens that a concise formulation in momentum space[1] provides, in a clear and systematic manner, such necessary terms, with their origin and relation to gauge invariance analogous to position space.

It is the goal of the present paper to dilate on the strong-field QED Furry picture in the momentum space. Our approach has been outlined in [23]. The key is to employ Ritus matrices [24] and to accumulate all dependencies on the external classical field in the dressed vertex [25,26], while keeping vacuum photon and fermion lines for propagators and in/out states. Our presentation enables easy access to the systematic expansion of amplitudes and probabilities/cross sections in powers of the classical laser intensity parameter a_0. In the lowest order of small a_0 we obtain "pulsed perturbative QED" which accounts for temporal pulse shape effects [20,23]; the very special case of a monochromatic external classical field recovers standard perturbative QED.

Our paper is organized as follows. In Sec. II we present the formal development of the momentum-space Feynman rules with emphasis on implementing gauge invariance and preserving the Ward identity, the definition of the fermion-fermion-photon vertex, and the graphical representation. Section III is dedicated to several limiting cases; the monochromatic limit is considered and some remarks on soft factors are supplied as well. Examples of applications are introduced in Sec. IV. The explication of the three-point amplitude, that is for the one-vertex processes nonlinear Compton/Breit-Wheeler/one-photon annihilation, is spelt out in Sec. V to demonstrate the path from our rules towards the basic formulation of already exhaustively analyzed phenomena. The two two-vertex processes and related four-point amplitudes are dealt with in Sec. VI with some details with respect to nonlinear two-photon Compton and nonlinear Møller scatterings and crossing channels. Special aspects of Oleinik resonances, singularity structures, and the monochromatic limit are uncovered here. We conclude in Sec. VII.

II. FEYNMAN RULES FOR FURRY PICTURE IN MOMENTUM SPACE

A. Background-field description

The following considerations apply to Lorenzian null fields, i.e. the classical background field has the structure $\vec{E} = -\partial_t \vec{A}$ (electric field) and $\vec{B} = \vec{\nabla} \times \vec{A}$ (magnetic field) with four-potential $A^\mu = (A^0, \vec{A})$ in Lorenz ($\partial_\mu A^\mu = 0$) and Weyl ($A^0 = 0$) gauge

$$A^\mu = \frac{m}{|e|} a_0 g(\phi, \Delta\phi)[\varepsilon_1^\mu \cos(\xi)\cos(\phi + \phi_{\mathrm{CEP}}) + \varepsilon_2^\mu \sin(\xi)\sin(\phi + \phi_{\mathrm{CEP}})], \tag{1}$$

where, in units with $c = \hbar = 1$, m and e denote the electron's rest mass and electric charge, $\varepsilon_{1,2}^\mu$ refer to the laser's polarization vector [e.g. $\varepsilon_1^\mu = (0,1,0,0)$ and $\varepsilon_2^\mu = (0,0,1,0)$ in a reference frame where $k^\mu = \omega(1,0,0,1)$], and the carrier envelope phase reads ϕ_{CEP}. Side conditions specify further this model of a laser pulse in plane-wave approximation:

$$k^2 \coloneqq k \cdot k = 0 \quad \text{(null field)}, \qquad k \cdot \varepsilon_{1,2} = 0 \quad \text{(transversality)}, \qquad \varepsilon_i \cdot \varepsilon_j = -\delta_{ij}, \tag{2}$$

$$\xi = \begin{cases} 0 \text{ or } \frac{\pi}{2} & \text{(lin. polarization)}, \\ \pm\frac{\pi}{4} & \text{(circ. polarization)}, \end{cases} \tag{3}$$

or elliptic polarization for other values of ξ. The quantity $g(\phi, \Delta\phi)$ denotes the pulse shape function (or envelope for a shortcut) with $\Delta\phi$ as the pulse width parameter. Scalar products of four-vectors are henceforth noted as dot products.

[1]The nontrivial relation of position-space Feynman rules and momentum-space Feynman rules is phrased in [22] as follows. "When deriving the momentum-space Feynman rules, we have formally integrated over the position of the photon emission vertex in spacetime, under the assumption that emission at long distances should be sufficiently suppressed. Unfortunately, it is not, a consequence of the fact that QED (like all unbroken gauge theories) has long-range interactions."

B. Dressed vertex decomposition

The dressed vertex is defined by [25,26]

$$\Delta^\mu \coloneqq \int \mathrm{d}^4 x \bar{E}_{p'}(x)(-ie\gamma^\mu)E_p(x)e^{ik'\cdot x} \tag{4}$$

$$= -\frac{ie}{2\pi}\int \mathrm{d}\ell \Gamma^\mu(\ell, p, p'|k) \times (2\pi)^4\delta^{(4)}(p+\ell k - p' - k')), \tag{5}$$

where p (p') is the in(out) going fermion four-momentum, and the outgoing photon four-vector is denoted by k'. By inserting the Ritus matrices, e.g. $E_p = (1 + e\frac{\not{k}\not{A}}{2k\cdot p})\exp\{iS_p(x)\}$ with the Hamilton-Jacobi action $S_p(x) = -p\cdot x - \frac{1}{2k\cdot p}\int_{\phi_0}^{\phi=k\cdot x}\mathrm{d}\phi'[2ep\cdot A(\phi') - e^2A^2(\phi')]$, the second line follows, defining the dressed vertex function:

$$\Gamma^\mu(\ell, p, p'|k) \coloneqq \int \mathrm{d}\phi \left(1 - e\frac{\not{k}\not{A}}{2k\cdot p'}\right)\gamma^\mu\left(1 + e\frac{\not{k}\not{A}}{2k\cdot p}\right) \times \exp\{S_{p'} - S_p + p'\cdot x - p\cdot x\} \tag{6}$$

$$= \gamma^\mu B_0(\ell) + \Gamma_1^{\mu\nu}B_{1\nu}(\ell) + \Gamma_2^\mu B_2(\ell). \tag{7}$$

Note (i) the crucial "photon number parameter" ℓ as Fourier conjugate of the phase ϕ and (ii) the decomposition into elementary vertices $\{\gamma^\mu, \Gamma_1^{\mu\nu}, \Gamma_2^\mu\}$, depending on $\{p, p', k\}$,

$$\gamma^\mu\text{: Dirac matrix obeying } [\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu}, \tag{8}$$

$$\Gamma_1^{\mu\nu} = e\left(\frac{\gamma^\mu\not{k}\gamma^\nu}{2k\cdot p} + \frac{\gamma^\nu\not{k}\gamma^\mu}{2k\cdot p'}\right), \tag{9}$$

$$\Gamma_2^\mu = -e^2\frac{\not{k}k^\mu}{2k\cdot pk\cdot p'}, \tag{10}$$

and phase integrals $\{B_0, B_1^\mu, B_2\}$, depending on ℓ as well

$$B_0 = \int_{-\infty}^{\infty}\mathrm{d}\phi \exp\{i\ell\phi + iG(\phi)\}, \tag{11}$$

$$B_1^\mu = \int_{-\infty}^{\infty}\mathrm{d}\phi \exp\{i\ell\phi + iG(\phi)\}A^\mu(\phi), \tag{12}$$

$$B_2 = \int_{-\infty}^{\infty}\mathrm{d}\phi \exp\{i\ell\phi + iG(\phi)\}A^2(\phi). \tag{13}$$

The function G reads

$$G(\phi, \phi_0) = \alpha_1^\mu \int_{\phi_0}^{\phi}\mathrm{d}\phi' A_\mu(\phi') + \alpha_2\int_{\phi_0}^{\phi}\mathrm{d}\phi' A^2(\phi'), \tag{14}$$

where $\phi_0 \to -\infty$ is a useful choice for pulses and

$$\alpha_1^\mu = e\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right), \quad \alpha_2 = -e^2\left(\frac{1}{k\cdot p'} - \frac{1}{k\cdot p}\right). \tag{15}$$

The phase integral (11) in Eq. (7) needs a regularization which takes care of the Ward identity $k'\cdot\Gamma = 0$. As explained in some detail in the next subsection, this is accomplished by

$$B_0(\ell) = \pi\mathcal{G}\delta(\ell) - \mathcal{P}\left(\frac{\alpha_1^\mu B_{1\mu}}{\ell} + \frac{\alpha_2 B_2}{\ell}\right), \tag{16}$$

where the instruction $\mathcal{P}$ means taking Cauchy's principle value and

$$\mathcal{G} = \exp\{iG^+\} + \exp\{iG^-\}, \quad G^\pm \coloneqq \lim_{\phi\to\infty} G(\pm\phi). \tag{17}$$

C. Ward identity and regularization of B_0

In the case of an external photon K attached to the vertex Γ, the condition $K\cdot\Gamma = 0$ ensures the independence on the arbitrary gauge function $q(K)$, when the polarization four vector is gauged according to $\varepsilon \to \varepsilon' + qK\cdot\Gamma$. Analogously, the photon propagator $\mathcal{D}_{\mu\nu} = \frac{i}{K^2+i\epsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{K_\mu K_\nu}{K^2-i\epsilon}\right)$, connecting two adjacent vertices, does not leave any dependence of the gauge parameter ξ on the structure $\Gamma\cdot\mathcal{D}\cdot\Gamma$ if the same Ward identity $K\cdot\Gamma = 0$ is fulfilled. Sandwiching the Ward identity by Dirac spinors yields, for $K = k'$,[2]

$$\begin{aligned} 0 &= \bar{u}_{p'}k'_\mu\Gamma^\mu u_p \\ &= [\bar{u}_{p'}k'_\mu\gamma^\mu u_p]B_0 + [\bar{u}_{p'}k'_\mu\Gamma_1^{\mu\nu}u_p]B_{1\nu} \\ &\quad + [\bar{u}_{p'}k'_\mu\Gamma_2^\mu u_p]B_2, \end{aligned} \tag{18}$$

where $\{B_0, B_{1\nu}, B_2\}$ denote the phase integrals (11), (12), (13) and $\{\gamma_0^\mu, \Gamma_1^{\mu\nu}, \Gamma_2^\mu\}$ are the elementary vertices (9), (10) (for both we suppress the momentum dependence for now). The free Dirac spinors in the side-condition (18) appear either if the vertex is attached to in/out fermion lines or by propagators, where the spin sum decomposition $\not{p} + m = \sum_\sigma u_{\sigma p}\bar{u}_{\sigma p}$ (analog for the other Dirac spinors) can be used. The energy-momentum balance $p + \ell k = p' + k'$ holds at each vertex, which is implied by the delta distribution in the fully dressed vertex Γ, and the Ward identity (18) of the dressed vertex function reads $0 = (\ell B_0(\ell) + \alpha_1^\mu B_{1\mu}(\ell) + \alpha_2 B_2(\ell))[\bar{u}_{p'}\not{k}u_p]$ upon using Dirac equations in momentum space, $(\not{p} - m)u_p = 0$ and $\bar{u}_{p'}(\not{p}' - m) = 0$, respectively. This implies a severe constraint for the phase integrals:

[2] Strictly speaking, this treatment refers essentially to on shell fermions. However, in gauge theories more general identities appear, known as Ward-Takahashi identities, where some of the external fermions are off shell. Such relations require further considerations with respect to dressed vertex.

$$0 = \ell B_0(\ell) + \alpha_1^\mu B_{1\mu}(\ell) + \alpha_2 B_2(\ell), \tag{19}$$

being equivalent to the Ward identity (18) and shows that the phase integrals $\{B_0, B_{1\nu}, B_2\}$ are not independent. The relation (19) is a well-known formula, which appears in several investigations of specific processes in strong-field QED, e.g. [27] in the case of Compton scattering or for the trident process [28]. However, the connection to the Ward identity and, therefore, to gauge invariance was not stressed there.

The Ward identity (18) must be solved for one of the phase integrals, e.g. B_0, in a distributional manner. This can be formulated as follows. Let be $b_0(\ell)$ a solution of Eq. (19), i.e. $0 = \ell b_0(\ell) + \alpha_1^\mu B_{1\mu}(\ell) + \alpha_2 B_2(\ell)$, then $\hat{b}_0(\ell) := b_0(\ell) + \mathcal{G}\delta(\ell)$ is a solution as well, where $\mathcal{G}$ is an arbitrary, but finite, function of the momenta. This seems trivial, because $\ell\delta(\ell) = 0$, but it turns out that this term leads to non-negligible contributions. However, the Ward identity (19) does not determine the delta distribution's prefactor $\mathcal{G}$. Instead, it can be derived by regulating the integral in the definition of B_0, Eq. (11). Adapting the procedure in [29] for Compton scattering in a generic case by inserting $e^{-\epsilon|\phi|}$ with $\epsilon > 0$ in the integral in Eq. (11) we get

$$B_0(\ell) = \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} \mathrm{d}\phi e^{-\varepsilon|\phi|} e^{i\ell\phi} e^{iG(\phi)} \tag{20}$$

$$\begin{aligned}
&= \lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{0} \mathrm{d}\phi e^{(i\ell+\varepsilon)\phi} e^{iG(\phi)} + \int_{0}^{\infty} \mathrm{d}\phi e^{(i\ell-\varepsilon)\phi} e^{iG(\phi)}\right] \\
&= \lim_{\varepsilon\to 0^+}\left[\frac{e^{(i\ell+\varepsilon)\phi} e^{iG(\phi)}}{i\ell+\varepsilon}\bigg|_{-\infty}^{0} - \frac{i}{i\ell+\varepsilon}\int_{-\infty}^{0} \mathrm{d}\phi e^{(i\ell+\varepsilon)\phi} G(\phi) e^{iG(\phi)} + \frac{e^{(i\ell-\varepsilon)\phi} e^{iG(\phi)}}{i\ell-\varepsilon}\bigg|_{0}^{\infty} \right.\\
&\quad \left. - \frac{i}{i\ell-\varepsilon}\int_{0}^{\infty} \mathrm{d}\phi e^{(i\ell-\varepsilon)\phi} G(\phi) e^{iG(\phi)}\right],
\end{aligned} \tag{21}$$

where we use partial integration and the shortcut $G' := \frac{\mathrm{d}}{\mathrm{d}\phi}G$. Considering the nonintegral terms of B_0, one gets

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\left[\frac{e^{(il\ell+\epsilon)\phi} e^{iG(\phi)}}{i\ell+\epsilon}\bigg|_{-\infty}^{0} + \frac{e^{(i\ell-\epsilon)\phi} e^{iG(\phi)}}{i\ell-\epsilon}\bigg|_{0}^{\infty}\right] \\
&\quad = 2\lim_{\epsilon\to 0^+}\left[\frac{\epsilon}{\epsilon^2+\ell^2}\right] e^{iG(0)} = 2\pi\delta(\ell) e^{iG(0)}.
\end{aligned} \tag{22}$$

In the very last step, we perform the limit in a distributional manner. To evaluate the other terms in Eq. (21), we consider the integral $\int_{-\infty}^{0} \mathrm{d}\phi e^{(i\ell+\epsilon)\phi} G'(\phi) e^{iG(\phi)}$ which is finite for every $\epsilon \geq 0$ due to the proportionality $G'(\phi) \sim A^\mu(\phi)$ for all ϕ, where A^μ is assumed to vanish at the lower limit of the integral, e.g. $\lim_{\phi\to -\infty} A^\mu(\phi) = 0$. The same holds for the other integral, so the limit in these integrals can be executed to get

$$\lim_{\epsilon\to 0^+}\left[\frac{i}{i\ell+\epsilon}\int_{-\infty}^{0} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG} + \frac{i}{i\ell-\epsilon}\int_{0}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG}\right] \tag{23}$$

$$\begin{aligned}
&= \lim_{\epsilon\to 0^+}\frac{1}{2}\left[\left(\frac{i}{i\ell+\epsilon} + \frac{i}{i\ell-\epsilon}\right)\int_{-\infty}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG} \right.\\
&\quad + \left(\frac{i}{i\ell+\epsilon} - \frac{i}{i\ell-\epsilon}\right)\\
&\quad \left.\times\left(\int_{-\infty}^{0} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG} - \int_{0}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG}\right)\right],
\end{aligned} \tag{24}$$

where we apply the identity $uw + vz = \frac{u+v}{2}(w+z) + \frac{u-v}{2}(w-z)$ with $\{u, v, w, z\} \in \mathbb{C}$. Starting with the first term in Eq. (24), we get

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\frac{1}{2}\left(\frac{i}{i\ell+\epsilon} + \frac{i}{i\ell-\epsilon}\right)\int_{-\infty}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG} \\
&\quad = \mathcal{P}\left[\frac{1}{\ell}\int_{-\infty}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG}\right]
\end{aligned} \tag{25}$$

by using

$$\lim_{\epsilon\to 0^+}\int_a^b \mathrm{d}x \frac{x^2}{x^2+\epsilon^2} H(x) = \mathcal{P}\int_a^b \mathrm{d}x H(x), \tag{26}$$

with an arbitrary function $H\colon (a, b) \to \mathbb{C}$. The second term in Eq. (24) contains again a delta-distribution,

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\frac{1}{2}\left(\frac{i}{i\ell+\epsilon} - \frac{i}{i\ell-\epsilon}\right) \\
&\quad\times\left(\int_{-\infty}^{0} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG} - \int_{0}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG}\right)
\end{aligned} \tag{27}$$

$$= i\pi\delta(\ell)\left(\int_{-\infty}^{0} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG} - \int_{0}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG}\right) \tag{28}$$

$$= i\pi\delta(\ell)\left(\int_{-\infty}^{0} \mathrm{d}\phi G' e^{iG} - \int_{0}^{\infty} \mathrm{d}\phi G' e^{iG}\right), \tag{29}$$

where we use $\delta(x)H(x) = \delta(x)H(0)$ in the last step. To evaluate these integrals, we use $G' e^{iG} = -i(e^{iG})'$,

$$\int_{-\infty}^{0} \mathrm{d}\phi G' e^{iG} - \int_{0}^{\infty} \mathrm{d}\phi G' e^{iG} = \frac{1}{i}\left(e^{iG}\Big|_{-\infty}^{0} - e^{iG}\Big|_{0}^{\infty}\right) = \frac{1}{i}(2e^{iG(0)} - e^{iG_+} - e^{iG_-}) \tag{30}$$

with the abbreviation $G_\pm \coloneqq \lim_{\phi\to\infty} G(\pm\phi)$. Finally, we insert Eqs. (22) and (24) in Eq. (21) and repeat the evaluation as above, to obtain

$$B_0(\ell) = (e^{iG_+} + e^{iG_-})\pi\delta(\ell) - \mathcal{P}\left[\frac{1}{\ell}\int_{-\infty}^{\infty} \mathrm{d}\phi e^{i\ell\phi} G' e^{iG}\right]. \tag{31}$$

Note the independence on both $G(0)$ and ϕ_0 as a consequence of Eqs. (21), (22), and (30). Considering $G'(\phi) = \alpha_1^\mu A_\mu(\phi) + \alpha_2 A^2(\phi)$ [cf. Eq. (14)] and the definitions of B_1^μ as well as B_2 in Eqs. (12) and (13), respectively, we can write B_0 in terms of the other phase integrals as

$$B_0(\ell) = \pi\delta(\ell)(e^{iG_+} + e^{iG_-}) - \mathcal{P}\left[\frac{\alpha_1^\mu}{\ell} B_{1\mu}(\ell) + \frac{\alpha_2}{\ell} B_2(\ell)\right] \tag{32}$$

$$= \pi\delta(\ell)\mathcal{G} + \hat{B}_0(\ell), \tag{33}$$

where we introduce the abbreviation

$$\mathcal{G}(p, p', k) \coloneqq \exp\left(iG^+(p, p', k)\right) + \exp\left(iG^-(p, p', k)\right) \tag{34}$$

with the asymptotic values $G^\pm$ (17) of the nonlinear phase (14) as well as the finite phase integral $\hat{B}_0(\ell) \coloneqq -\mathcal{P}[\frac{\alpha_1^\mu}{\ell} B_{1\mu}(\ell) + \frac{\alpha_2}{\ell} B_2(\ell)]$. Finally, it is easy to see that the regularized version (32) of the phase integral solves Eq. (18), which implies the solution of the Ward identity Eq. (19).

D. Final version of the vertex decompositon

Inserting the regularized version of the phase integral (32), the dressed vertex function (7) decomposes eventually as

$$\Gamma^\mu(\ell) = \Gamma^\mu_{\mathrm{div}}(\ell) + \Gamma^\mu_{\mathrm{reg}}(\ell), \tag{35}$$

$$\Gamma^\mu_{\mathrm{div}}(\ell) \coloneqq \gamma^\mu \pi \mathcal{G}\delta(\ell), \tag{36}$$

$$\Gamma^\mu_{\mathrm{reg}}(\ell) \coloneqq \left(\Gamma_1^{\mu\nu}(\ell) - \mathcal{P}\frac{\gamma^\mu \alpha_1^\nu}{\ell}\right) B_{1\nu}(\ell) + \left(\Gamma_2^\mu - \mathcal{P}\frac{\gamma^\mu \alpha_2}{\ell}\right) B_2(\ell). \tag{37}$$

We have suppressed the pertinent arguments p, p', k in all functions, but highlighted the ℓ dependence. The term labeled by "div" could be named "gauge invariance restoration term", since it emerges just from that requirement.

One may interpret these parts of the dressed vertex as follows. The divergent part $\Gamma^\mu_{\mathrm{div}}$, Eq. (36), enforces $\ell = 0$. Therefore, it can be considered as part of the dressed vertex function with no momentum exchange with the background field. This has no contribution to one-vertex processes like nonlinear Compton scattering or nonlinear Breit-Wheeler pair production due to the vanishing physical phase space, i.e. there is neither single-photon absorption nor single-photon emission in perturbative QED. However, for processes with more than one vertex, e.g. the trident process, the vanishing momentum exchange from the background field to one vertex may eventually be compensated due to the momentum transfer at another vertex. Since the only dependence of this nontransfer term on the background field is condensed in the factor $\mathcal{G}$, the leading order in A^μ of the whole nonvanishing term is constant through $\mathcal{G} = 2 + \mathcal{O}(A^\mu)$. The finite part $\Gamma^\mu_{\mathrm{reg}}$, Eq. (37), may be interpreted as a part of the dressed vertex function with a genuine momentum transfer from the background field to the vertex, which is indicated by the occurring principal value in the finite part $\hat{B}_0(\ell)$ of the regularized phase integral (32), considering the other phase integrals are regular for $\ell \to 0$. Moreover, since the elementary vertices (9), (10) as well as the kinematic factors α_i (15) are independent of the background field, the leading order of the finite part $\Gamma^\mu_{\mathrm{reg}}$ of the vertex function Γ^μ is linear in A^μ, i.e. there is no A^μ-independent term in an expansion of Γ^μ with respect to the background field.

The decomposition of the vertex function implies the following modification of standard Feynman rules in momentum space: use $\int \frac{\mathrm{d}\ell}{2\pi}(-ie)\Gamma_\mu(\ell)$ for the fermion-fermion-photon vertex, instead of $-ie\gamma_\mu$, and integrate over internal momenta. While an N-vertex Furry-picture diagram in position space involves N space-time integrals (cf. [15] of how to process the expressions), in momentum space one meets N integrations over the respective vertex-attached "photon number parameter" ℓ.

E. Expansion in powers of a_0

One benefit of our momentum space formulation is the possibility of a straightforward series expansion of the amplitude in powers of a_0. The temporal pulse shape is imprinted transparently in the weak-field limit. To begin with, we introduce for the bookkeeping of powers of a_0 the tilde notation: every quantity with a tilde is free of any dependence on a_0, e.g. $A^\mu(a_0, \phi) = a_0 \tilde{A}^\mu(\phi)$, which implies

$$G = a_0 \alpha_1^\mu \int_{\phi_0}^{\phi} \mathrm{d}\phi' \tilde{A}_\mu(\phi') + a_0^2 \alpha_2 \int_{\phi_0}^{\phi} \mathrm{d}\phi' \tilde{A}^2(\phi') = a_0 \tilde{A}_1 + a_0^2 \tilde{A}_2, \tag{38}$$

$$\exp\{iG\} = \sum_{N=0}^{\infty} a_0^N \tilde{G}_N = \sum_{N=0}^{\infty} a_0^N \sum_{(m,n)} \tilde{G}_{mn}|_{m+2n=N},$$
$$\tilde{G}_{mn} := \frac{i^{m+2n}}{m!n!} \tilde{A}_1^m \tilde{A}_2^n, \tag{39}$$

when using the abbreviations $\tilde{A}_1 := \alpha_1^\mu \int_{\phi_0}^{\phi} \mathrm{d}\phi' \tilde{A}_\mu(\phi')$ and $\tilde{A}_2 := \alpha_2 \int_{\phi_0}^{\phi} \mathrm{d}\phi' \tilde{A}^2(\phi')$. The quantities $\tilde{G}_N = \sum_{m,n \in \mathcal{N}(m,n)} \tilde{G}_{mn}$ use the re-indexing with index set $\mathcal{N}(m,n) := \{(m,n) \in \mathbb{N}^2 | m + 2n = \mathbb{N}\}$, e.g. $\{(0,0)\}$ for $N = 0$, $\{(1,0)\}$ for $N = 1$, and $\{(2,0),(0,1)\}$ for $N = 2$.

Analogously, the phase integrals (12), (13) become sums of Fourier transforms:

$$B_1^\mu = \sum_{N=0}^{\infty} a_0^{1+N} \tilde{B}_{1,N}^\mu, \qquad \tilde{B}_{1,N}^\mu := \int_{-\infty}^{\infty} \mathrm{d}\phi e^{i\ell\phi} \tilde{G}_N \tilde{A}^\mu, \tag{40}$$

$$B_2 = \sum_{N=0}^{\infty} a_0^{2+N} \tilde{B}_{2,N}, \qquad \tilde{B}_{2,N} := \int_{-\infty}^{\infty} \mathrm{d}\phi e^{i\ell\phi} \tilde{G}_N \tilde{A}^2. \tag{41}$$

Proceeding in such a manner we represent the vertex functions (36), (37) as

$$\Gamma_{\mathrm{div}}^\mu = \sum_{N=0}^{\infty} a_0^N \tilde{\Gamma}_{\mathrm{div}N}^\mu, \qquad \tilde{\Gamma}_{\mathrm{div}N}^\mu = \pi\gamma^\mu \delta(\ell)(\tilde{G}_N^+ + \tilde{G}_N^-), \tag{42}$$

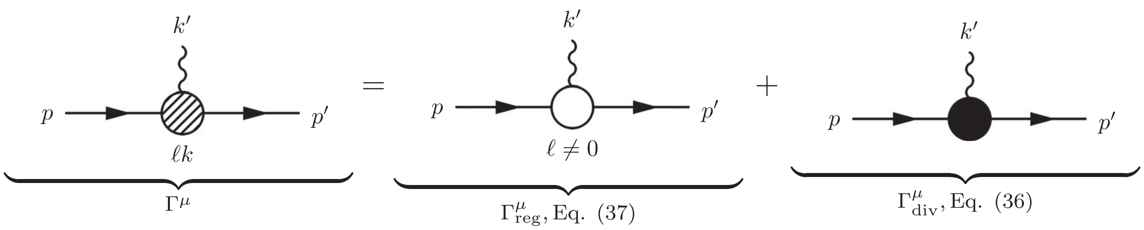
$$\Gamma_{\mathrm{reg}}^\mu = \sum_{N=1}^{\infty} a_0^N \tilde{\Gamma}_{\mathrm{reg}N}^\mu,$$
$$\tilde{\Gamma}_{\mathrm{reg}N}^\mu = \left(\Gamma_{1\nu}^\mu - \mathcal{P}\frac{\gamma^\mu \alpha_{1\nu}}{\ell}\right) \tilde{B}_{1,N-1}^\nu + \Theta(N-1)\left(\Gamma_2^\mu - \mathcal{P}\frac{\gamma^\mu \alpha_2}{\ell}\right) \tilde{B}_{2,N-2}, \tag{43}$$

where the Heaviside distribution ensures to take only $N > 1$ contributions in the last term. We note (i) $\tilde{\Gamma}_{\mathrm{div}N=0}^\mu = 2\pi\gamma^\mu \delta(\ell)$ due to $\tilde{G}_{N=0}^\pm = 1$; $\delta(\ell)$ enforces $\ell = 0$ meaning that this vertex part does not exchange energy-momentum with the external field, and (ii) the $N = 1$ contribution to $\Gamma_{\mathrm{reg}}^\mu$ is solely related to $\tilde{B}_1^\mu = \int_{-\infty}^{\infty} \mathrm{d}\phi e^{i\ell\phi} \tilde{A}^\mu(\phi)$, i.e. the Fourier transform of the background field; it is the only linear contribution (in a_0 and the background field), while contributions $\propto a_0^N$ with $N > 1$ are nonlinear in the background field.

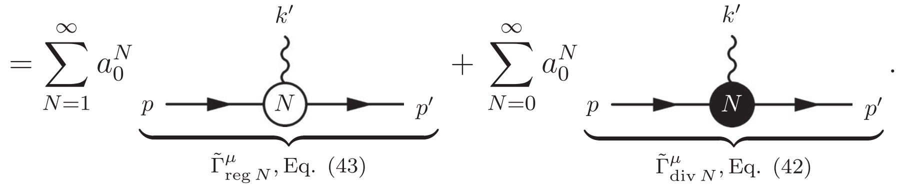
In a follow-up paper, we show in more detail that, in leading order of a series expansion in powers of a_0, the standard momentum Feynman rules are recovered and explicate the next-to-leading order (NLO) terms.

F. Graphical representations

The above vertex function decomposition facilitates the following graphical representation with the a_0 expansion in the bottom lines:



(44)



(45)

Each expression must be supplemented by a factor of $-ie\delta^{(4)}(p + \ell k - p' - k')$. The pertinent arguments ℓ, p, p', k are not displayed here. Note that the expanded vertex functions in (45) are $\propto a_0^N$, thus allowing for easy bookkeeping.

III. SOME LIMITING CASES: $\Delta\phi \to \infty$, $a_0 \to 0$, $k' \to 0$, $k \to 0$

Having prepared the momentum space formulation and related Feynman rules, we now turn to some limiting cases. These include the monochromatic limit $\Delta\phi \to \infty$, the weak-field limit $a_0 \to 0$ and recovery of standard perturbative Feynman rules, and the soft limits $k' \to 0$ and $k \to 0$ as well.

A. Monochromatic limit: $\Delta\phi \to \infty$

The above formalism is devised to include arbitrary shape functions $g(\phi, \Delta\phi)$. For specified (explicitly given) temporal pulse structures, $G(\phi)$ can be explicated and the phase integrals B_0, B_1^μ and B_2 can be executed to arrive at special functions (cf. [30] for such an example with respect to nonlinear Compton). A very special case is the monochromatic plane-wave background field, often named infinitely-extended plane-wave (IPW). Formally, $\Delta\phi \to \infty$ and the envelope function in Eq. (1) obeys $g(\phi, \Delta\phi) = 1$ for all $\phi \in \mathbb{R}$. Setting $\phi_0 = 0$ is a suitable choice for monochromatic plane-wave backgrounds, since then the initial condition is known due to $A^\mu(\phi_0) = a\varepsilon_1^\mu \cos\xi$ and the nonlinear phase defined in (14) is finite for finite values of ϕ. The distinction of $\Gamma^\mu_{\rm div}$ and $\Gamma^\mu_{\rm reg}$ is no longer suitable, since also B_0, B_1^μ and B_2 in $\Gamma^\mu_{\rm reg}$ contribute to the so-called δ-comb, which refers to individual harmonics. The carrier-envelope phase $\phi_{\rm CEP}$ is irrelevant and can be skipped.

Under these prepositions, Eq. (38) yields

$$G(\phi, \phi_0 = 0) = -|\alpha_-|[\sin(\phi + \Theta) - \sin(\Theta)] - \beta\left[\frac{1}{2}\cos(2\xi)\sin(2\phi) + \phi\right], \tag{46}$$

where $\alpha_- \coloneqq a\alpha_1 \cdot \varepsilon_-$ is represented as $\alpha_- = |\alpha_-|e^{i\Theta}$, and $a = a_0 m/|e|$, $\beta \coloneqq \frac{1}{2}a^2\alpha_2$. Insertion into Eqs. (11)–(13) results in

$$B_0(\ell) = 2\pi \exp\{i|\alpha_-|\sin\Theta\} \sum_{n=-\infty}^{\infty} \delta(\ell - \beta - n) C_n, \tag{47}$$

$$B_\pm(\ell) = 2\pi \exp\{i|\alpha_-|\sin\Theta\} \sum_{n=-\infty}^{\infty} \delta(\ell - \beta - n) C_{n\mp1}, \tag{48}$$

$$B_2(\ell) = 2\pi \exp\{i|\alpha_-|\sin\Theta\} \sum_{n=-\infty}^{\infty} \delta(\ell - \beta - n) \times \left[C_n + \frac{1}{2}\cos(2\xi)(C_{n+2} + C_{n-2})\right], \tag{49}$$

and allows the representation of the fully dressed vertex as

$$\Gamma^\mu(\ell) = \sum_{n=-\infty}^{\infty} \delta(\ell - \beta - n)\Gamma^\mu_{{\rm IPW}n}, \tag{50}$$

$$\Gamma^\mu_{{\rm IPW}n} \coloneqq 2\pi \exp\{i|\alpha_-|\sin\Theta\}\left\{\gamma^\mu C_n + \frac{1}{2}a[\Gamma^\mu_+ C_{n-1} + \Gamma^\mu_- C_{n+1}] + \Gamma^\mu_2\left[C_n + \frac{1}{2}\cos(2\xi)(C_{n+2} + C_{n-2})\right]\right\}, \tag{51}$$

where we use the abbreviation $\Gamma^\mu_\pm \coloneqq \varepsilon_{\pm,\nu}\Gamma_1^{\mu\nu}$; cf. Eqs. (66), (67) below for the definition of $\varepsilon^\mu_\pm$ and the decomposition of B_1^μ into $B_\pm$. To make some intermediate steps more obvious, we note

$$B_\pm(\ell) \coloneqq \int_{-\infty}^{\infty} d\phi \exp\{i(\ell \pm 1)\phi + iG(\phi)\} \tag{52}$$

$$= \exp\{i|\alpha_-|\sin\Theta\} \int_{-\infty}^{\infty} d\phi \exp\left\{i(\ell \pm 1 - \beta)\phi - i|\alpha_-|\sin(\phi + \Theta) - \frac{i}{2}\beta\cos(2\xi)\sin(2\phi)\right\}. \tag{53}$$

Using the Jacobi-Anger expansion of the non-Fourier part one gets

$$\exp\left\{-i|\alpha_-|\sin(\phi + \Theta) - \frac{1}{2}i\beta\cos(2\xi)\sin(2\phi)\right\} = \sum_{n=-\infty}^{\infty} C_n e^{-in\phi}, \tag{54}$$

$$C_n = \sum_{s=-\infty}^{\infty} J_{n-2s}(|\alpha_-|) J_s\left(\frac{1}{2}\beta\cos(2\xi)\right) e^{-i(n-2s)\Theta}, \tag{55}$$

where J_n stand for Bessel functions of the first kind. The phase integral (53) reads then

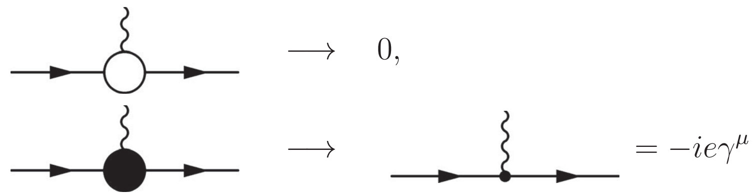
$$B_\pm(\ell) = \exp\{i|\alpha_-|\sin\Theta\} \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} J_{n-2s}(|\alpha_-|) J_s\left(\frac{1}{2}\beta\cos(2\xi)\right) e^{-i(n-2s)\Theta} \int_{-\infty}^{\infty} d\phi \exp\{i(\ell \pm 1 - \beta - n)\phi\}, \tag{56}$$

and the index shift $n \to n \pm 1$ yields Eq. (48).

B. Recovery of Feynman rules of perturbative QED

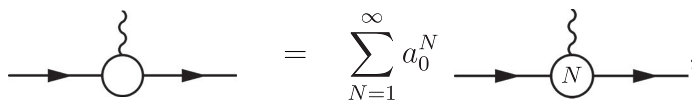
The relation to standard perturbative QED Feynman rules has two facets:

(i) Get the fermion-fermion-photon vertex from the dressed vertex by inspecting Eq. (45) and noting that, at $a_0 \to 0$ and for off shell legs, $\mathcal{G} \to 2$ and the leading-order term in the sum $\sum_{N=1}^{\infty} \cdots \to a_0 \to 0$, i.e.

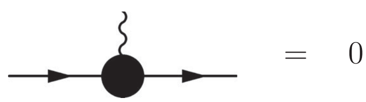


where the small dot represents the standard perturbative QED vertex.

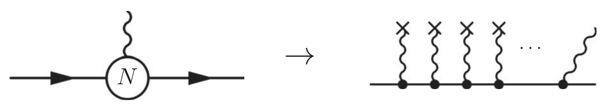
(ii) However, in the case of a monochromatic background and for on shell legs (e.g. for Compton), the leading-order terms in the a_0 expansion stem from the vertex



since (for on shell amplitude) we have

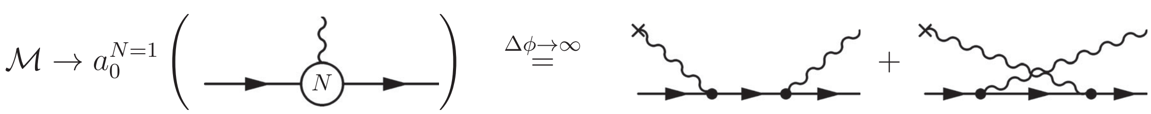


The powers of a_0 are expelled off the regularized vertex, which becomes



with N attached laser field lines in all permutations.

On the level of cross-section, $\mathrm{d}\sigma/\mathrm{d}\Omega' d\omega' = \mathcal{I}_\gamma^{-1} \sum |\mathcal{M}|^2 \mathrm{d}\Phi'$, with $\mathcal{I}_\gamma \propto a_0^2$. Therefore, for $a_0 \to 0$, only the term

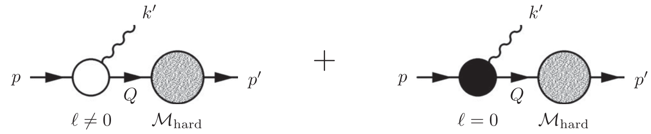


survives, making the cross section $\propto \sum |\mathcal{M}|^2 \delta\Big(\omega' - \frac{\omega}{1+\frac{\omega'}{m}(1-\cos\Theta')}\Big)$, where, upon executing the spin and polarization sums, $\sum$, the Klein-Nishina cross section is recovered; the energy-momentum balance via the delta distribution arises from phase space integration, $\mathrm{d}\Phi'$, and only for $\Delta\phi \to \infty$.

C. Soft photons: $k' \to 0$

Lowest-order soft theorems[3] allow for factorizing amplitudes as $\mathcal{M} = \mathcal{M}_{\mathrm{hard}} \times \mathcal{S}$, where $\mathcal{S}$ is the soft factor accounting for the emission of a soft photon and $\mathcal{M}_{\mathrm{hard}}$ is the amplitude of hard interaction. Focusing first on the soft-photon emission off an external leg, e.g. an incoming electron, the corresponding matrix element reads

[3]For the relation of soft-photon theorem and asymptotic symmetry (Ward identity) and memory effect within the infrared triangle, see [12] and citations therein.

$\mathcal{M} =$ 

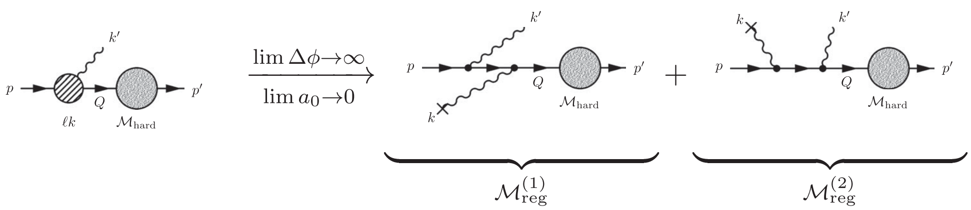
(57)

$$= -ie \int \frac{\mathrm{d}\ell}{2\pi} [\bar{u}(p')\mathcal{M}_{\mathrm{hard}}(p', p + \ell k - k')S_F(p + \ell k - k')\Gamma^\mu(\ell, p + \ell k - k'))\varepsilon^{*\prime}_\mu(k')u(p)], \tag{58}$$

where $\mathcal{M}_{\mathrm{hard}}$ is part of the matrix element which emerges from the interaction of the electron with other nonsoft particles, $\varepsilon'_\mu(k')$ denotes the polarization of the emitted soft photon and $S_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$ is Feynman's free fermion propagator. Since k' is small compared with the electron momenta, one has $\mathcal{M}_{\mathrm{hard}}(p', p+\ell k - k') \approx \mathcal{M}_{\mathrm{hard}}(p', p+\ell k)$. Furthermore, we observe $\Gamma^\mu(\ell, p, p + \ell k - k') \equiv \Gamma^\mu(\ell, p, p - k')$ since the outgoing momentum p' appears only in the product $p' \cdot k$ in $\Gamma^\mu(\ell, p, p')$. Inserting the decomposition (35) in the matrix element (58), the infrared behavior of k' must be examined for two parts:

(i) Gauge-restoration part $\Gamma^\mu_{\mathrm{div}}$: The corresponding matrix element becomes $\mathcal{M}_{\mathrm{div}} = -ie \int \frac{\mathrm{d}\ell}{2\pi} [\bar{u}(p') \mathcal{M}_{\mathrm{hard}}(p', p + \ell k)S_F(p + \ell k - k')\pi\mathcal{G}(p, p - k') \delta(\ell)\gamma^\mu\varepsilon^{*\prime}_\mu(k')u(p)]$, where the ℓ integration is executed by employing the δ-distribution leading to $\ell = 0$. Considering the nonlinear Volkov phase G defined in (14), we have $\lim_{k'\to 0} G(\phi, p, p - k') = G(\phi, p, p) \equiv 0$, which implies $\lim_{k'\to 0} \mathcal{G}(p, p - k') = 2$. That means this part of the matrix element has the same infrared behavior as ordinary one-photon bremsstrahlung (cf. Secs. 6.1 in [31] and 13.5 in [32]) $\mathcal{M}_{\mathrm{div}} \approx -[\bar{u}(p)\mathcal{M}_{\mathrm{hard}}(p', p) u(p)](e\frac{p\cdot\varepsilon'}{k'\cdot p})$, where e denotes the elementary electric charge and we use $S_F(p - k')\gamma^\mu\varepsilon^{*\prime}_\mu u(p) \to -i\frac{p\cdot\varepsilon'}{p\cdot k'}$ for $k' \to 0$.

(ii) Regularized finite part $\Gamma^\mu_{\mathrm{reg}}$: Considering the leading order in powers of a_0 and the monochromatic limit, $\Delta\phi \to \infty$, the diagram (58) recovers the standard perturbative case:



(59)

Therefore, we have a special case of a subleading Low theorem, where one part of the matrix element, $\mathcal{M}^{(1)}_{\mathrm{reg}}$, has an infrared pole and the other term, $\mathcal{M}^{(2)}_{\mathrm{reg}}$, does not. The elaboration of the general version of the expressions $\mathcal{M}^{(1,2)}_{\mathrm{reg}}$ in (59) is relegated to separate work.

D. Soft-background field: $k \to 0$

Similarly to Sec. III C, the soft interaction with the background field refers to the limit $k \to 0$, which is equivalent to $\omega \to 0$, where we set $k^\mu = \omega n^\mu$ with the normalized reference momentum n^μ. Consequently, this soft limit needs to be examined separately for the two parts of the dressed vertex:

(i) Regularized part $\Gamma^\mu_{\mathrm{reg}}$: Considering the elementary vertices defined in Eqs. (9) and (10), we find $\Gamma^{\mu\nu}_1 = e(\frac{\gamma^\mu\not{n}\gamma^\nu}{2n\cdot p} + \frac{\gamma^\nu\not{n}\gamma^\mu}{2n\cdot p'})$ and $\Gamma^\mu_2 = -e^2\frac{\not{n}n^\mu}{2n\cdot pn\cdot p'}$, which are both finite in the limit $\omega \to 0$. On the contrary, the kinematic factors α^μ_1 and α_2 defined in (15) have the typical form of Weinberg's soft factors [32], hence a divergence in the soft limit. Therefore, for $\omega \to 0$, we have $\Gamma^{\mu\nu}_1 \ll \frac{\gamma^\mu\alpha^\nu_1}{\ell}$ and $\Gamma^\mu_2 \ll \frac{\gamma^\mu\alpha_2}{\ell}$ for all finite ℓ. This leads to the soft limit of the finite part of the dressed vertex (37), $\lim_{\omega\to 0}\Gamma^\mu_{\mathrm{reg}}(\ell) = \gamma^\mu\mathcal{S}^{\mathrm{soft}}_{\mathrm{reg}}(\ell)$, where the regularized soft factor reads

$$\mathcal{S}^{\mathrm{soft}}_{\mathrm{reg}}(\ell) = -\mathcal{P}\left[\frac{\alpha^\nu_1}{\ell}B_{1\nu}(\ell) + \frac{\alpha_2}{\ell}B_2(\ell)\right]. \tag{60}$$

Linearizing the phase integrals in a_0, i.e. considering only $\tilde{B}^\mu_{1,1}(\ell)$ given in (40) and dropping

$B_2 \propto a_0^2$, this soft factor results in $\mathcal{S}_{\rm reg}^{\rm soft}(\ell) \to -\frac{\alpha_1^\mu}{\ell}\int_{-\infty}^{\infty} \mathrm{d}\phi A_\mu(\phi) e^{i\ell\phi}$, which recovers the soft-factor elaborated in [21] for the case of Bhabha scattering. Furthermore, considering linear polarization $A^\mu = a\varepsilon^\mu g(\phi, \Delta\phi)\cos\phi$ and performing the simultaneous limits $a_0 \to 0$ and $\Delta\phi \to \infty$, the soft factor reads $\mathcal{S}_{\rm reg}^{\rm soft}(\ell) \to -a\alpha_1^\mu \varepsilon_\mu = ae(\frac{p\cdot\varepsilon}{k\cdot p} - \frac{p'\cdot\varepsilon}{k\cdot p'})$, which indeed is, up to a constant normalization, Weinberg's well-known soft factor.

(ii) Gauge restoration part $\Gamma_{\rm div}^\mu$: First we note that the integrals appearing in the prefactor of the gauge restoration part $\mathcal{G}$ defined in Eq. (17), $\int_{\phi_0}^{\pm\infty}\mathrm{d}\phi' A^\mu(\phi')$ and $\int_{\phi_0}^{\pm\infty}\mathrm{d}\phi' A^2(\phi')$, are finite in the soft limit $\omega \to 0$, whereas the exponents $G_\pm$ of $\mathcal{G}$ diverge due to the presence of the factors α_1^μ and α_2. This means, the factor $\mathcal{G}$ itself acts like a soft factor, which highly oscillates in the soft limit. However, considering the linearization of this soft factor in a_0, we find $\mathcal{G} \to i\alpha_1^\mu(\int_{\phi_0}^{+\infty}\mathrm{d}\phi' A_\mu(\phi') + \int_{\phi_0}^{-\infty}\mathrm{d}\phi' A_\mu(\phi'))$, which again recovers the form known from Weinberg's soft factors, but this time with the integrated fields acting as polarization vectors.

In summary, it can be stated that in both cases, soft photon emission and soft interaction with the background field, generalized versions of typical soft factors appear, which can be, in a suitable limit, connected to soft factors well-known from monochromatic QED. However, the cancellation of the soft factors shown here with higher-order vertex corrections, i.e. finding a generalized version of the Bloch-Nordsieck theorem, deserves separate work.

IV. EXAMPLES AND FUTURE APPLICATIONS

The above formalism is ready for direct numerical applications. Elements are Dirac spinors, Dirac matrices, the metric tensor, momentum, and polarization four-vectors; fermion and photon propagators are as in free-field and could be defined as standalone objects; most importantly, the nonlinear phase integrals encode solely the external field and require some care and numerical optimization. For a given exclusive reaction, these elements are to be connected by scalar and matrix products, thus delivering a few partial amplitudes (e.g. direct and exchange terms or the multitude of diagrams with the same out-state) to be summed up to one complex number—the amplitude $\mathcal{M}$. Its mod square, $|\mathcal{M}|^2$, is to be garnished to arrive eventually at probability or cross section which depend on spins, polarizations, and invariants referring to the initial state including the background field and final phase space. Partial or complete integration over the final phase-space variables need often specially adapted procedures, while spin/polarization summations, if required, are straightforward. Handling of the δ distributions is analogous to position space formulation. All ℓ dependence is integrated out before squaring the amplitude, and finally use $[(2\pi)^3\delta^{(3)}(p_i - p_f)]^2 \to (2\pi)^3 V \frac{p_i^0}{p_i^+}\delta^{(3)}(p_i - p_f)$, where $p^+ \coloneqq \frac{1}{2}(p^0 + p^3)$. We refrain here from such specific numerologies but instead stress the need for an in-depth understanding of the essential dependencies and singular structures of $\mathcal{M}$, as the core of $|\mathcal{M}|^2$, prior to numerical evaluations.

To sketch applications of the presented formalism we (re)consider one- and two-vertex processes related to three- and four-point amplitudes of nonlinear (one- and two-photon) Compton and Møller scattering processes. The detailed application to nonlinear trident is relegated to an accompanying paper. The following Sec. V recalls the one-vertex processes by demonstrating how the above rules lead to the known matrix elements, in particular for nonlinear Compton with a few supplementing remarks on nonlinear Breit-Wheeler. The next-to-one Sec. VI considers the two two-vertex processes with emphasis on nonlinear two-photon Compton and nonlinear Møller scattering.

V. ONE-VERTEX PROCESSES/THREE-POINT AMPLITUDE

The matrix element of one-vertex processes has, symbolically, the structure $\mathcal{M} \sim J^\mu \mathcal{A}_\mu(k)$ with current $J^\mu \sim [\bar{u}(p')\Gamma^\mu u(p)]$, where u is the free-field fermion wave function, $\bar{u}$ its adjoint, and $\mathcal{A}_\mu(k)$ stands for the photon (momentum k) wave function. Depending on the orientation of the four-momenta p, p', and k, it refers to the processes $e_L^- \to e_L^{-\prime} + \gamma$ (nonlinear Compton), $\gamma \to e_L^- + e_L^+$ (nonlinear Breit-Wheeler) and $e_L^- + e_L^+ \to \gamma$ (nonlinear one-photon annihilation), which are interrelated by crossing symmetry, where the label "L" is a reminder of the laser dressing of charged fermions. Due to CPT invariance, $e_L^\pm \to e_L^\mp$ applies.

A. Nonlinear Compton

The matrix element for nonlinear Compton (nlC) scattering reads

$$M_{\rm nlC} = \int \frac{\mathrm{d}\ell}{2\pi}\delta^{(4)}(p + \ell k - p' - k') \times [\bar{u}(p')(-ie)\Gamma^\mu(\ell, p, p', k)\varepsilon_\mu^{*\prime}(k')u(p)], \quad (61)$$

where $\varepsilon_\mu^{*\prime}(k')$ stands for the polarization four-vector of the outgoing photon, and we continue to mark tied fermion lines by $[\cdots]$. The vertex decomposition (35) facilitates two contributions, $M_{\rm nlC}^{\rm div}$ and $M_{\rm nlC}^{\rm reg}$. The one related to $\Gamma_{\rm div}^\mu$ (36) refers to the "gauge-restoration part",

$$M_{\mathrm{nlC}}^{\mathrm{div}} = \frac{1}{2}\mathcal{G}\int \frac{\mathrm{d}\ell}{2\pi}\delta^{(4)}(p+\ell k-p'-k')\delta(\ell)\times[\bar{u}(p')(-ie)\gamma^\mu \varepsilon_\mu^{*\prime}(k')u(p)] = 0, \tag{62}$$

which vanishes since the $\delta(\ell)$ term, upon ℓ integration, enforces the balance equation $p-p'-k'$ for on shell momenta, leaving no phase space. The nonzero term, related to $\Gamma_{\mathrm{reg}}^\mu$ (37), becomes

$$M_{\mathrm{nlC}}^{\mathrm{reg}} = \int \frac{\mathrm{d}\ell}{2\pi}\delta^{(4)}(p+\ell k-p'-k')\left[\bar{u}(p')(-ie)\left\{\left(\Gamma_1^{\mu\nu}-\mathcal{P}\frac{\gamma^\mu\alpha_1^\nu}{\ell}\right)B_{1\nu}(\ell)+\left(\Gamma_2^\mu-\mathcal{P}\frac{\gamma^\mu\alpha_2}{\ell}\right)B_2(\ell)\right\}\varepsilon_\mu^{*\prime}(k')u(p)\right], \tag{63}$$

$$= \frac{-ie}{2\pi k^+}\delta^{lf}(p-p'-k')\left[\bar{u}(p')\left\{\underbrace{\left(\Gamma_1^{\mu\nu}-\mathcal{P}\frac{\gamma^\mu\alpha_1^\nu}{\ell_0}\right)}_{\propto e}\underbrace{B_{1\nu}(\ell_0)}_{\propto\frac{a_0}{e}(\cdots)}+\underbrace{\left(\Gamma_2^\mu-\mathcal{P}\frac{\gamma^\mu\alpha_2}{\ell_0}\right)}_{\propto e^2}\underbrace{B_2(\ell_0)}_{\propto\frac{a_0^2}{e^2}(\cdots)}\right\}\varepsilon_\mu^{*\prime}(k')u(p)\right], \tag{64}$$

where in the last two lines the light cone coordinates are employed, $\delta^{lf}(q) \coloneqq \frac{1}{2}\delta(q^-)\delta^{(2)}(q^\perp)$. The photon number parameter is $\ell_0 = \frac{(p+k')^2-m^2}{2k\cdot p}$. The matrix element (64) is the starting point for many investigations of one-photon nonlinear Compton in a pulsed plane-wave background. To make this relation explicit we rewrite Eq. (14) by means of (1) and $a \coloneqq a_0 m/|e|$ as

$$G(\phi) = -\mathrm{Re}\alpha_- \int_{-\infty}^{\phi}\mathrm{d}\phi g(\phi')\exp\{i(\phi'+\phi_{\mathrm{CEP}})\} - \frac{1}{2}a^2\alpha_2\left(\cos 2\xi\int_{-\infty}^{\phi}d\phi' g(\phi')^2\cos(2[\phi'+\phi_{\mathrm{CEP}}]) + \int_{-\infty}^{\phi}d\phi' g(\phi')^2\right), \tag{65}$$

where $\alpha_\pm \coloneqq a\alpha_{1\mu}\varepsilon_\pm^\mu$ with $\varepsilon_\pm^\mu \coloneqq \varepsilon_1^\mu\cos\xi \pm i\varepsilon_2^\mu\sin\xi$. Using the abbreviation

$$B_{1\pm}(\ell) \coloneqq \int_{-\infty}^{\infty}\mathrm{d}\phi\exp\{\pm(i[\phi+\phi_{\mathrm{CEP}}])\} \times\exp\{-i[\ell\phi+G(\phi)]\} \tag{66}$$

the phase integrals [cf. (12), (13)] in (64) can be cast in the form

$$B_1^\mu(\ell) = \frac{1}{2}a\{\varepsilon_+^\mu B_{1-}(\ell)+\varepsilon_-^\mu B_{1+}(\ell)\}, \tag{67}$$

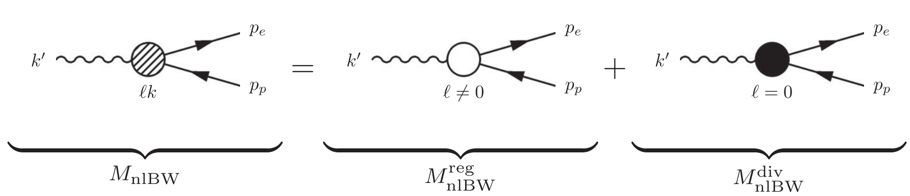
$$B_2(\ell) = a^2\int_{-\infty}^{\infty}\mathrm{d}\phi g(\phi)^2\{1+\cos 2\xi\cos(2[\phi+\phi_{\mathrm{CEP}}])\} \times\exp\{i(\phi+\phi_{\mathrm{CEP}})\} \tag{68}$$

yielding eventually Eq. (3.26) in [27] with many accompanying and subsequent works. The one-photon nonlinear Compton process based on the one-vertex diagram seems to be exhaustively analyzed (cf. [15] for a recent review). The special setup of multicolor laser background fields, e.g. the superposition of aligned optical and x-ray free-electron laser (XFEL) beams, i.e. x-ray scattering at an electron moving in the laser field [33,34], offer further interesting facets, up to polarization gating to produce a monoenergetic γ beam [35,36]). Furthermore, nonlinear Compton has nonperturbative contributions (analog to nonlinear Breit-Wheeler), and [37] shows how to isolate them.

With respect to power counting of e and a_0, the assignments are displayed in Eq. (64), where $(\cdots)$ stands for the series expansion of the $\exp\{iG\}$ term.

B. Nonlinear Breit-Wheeler

The matrix elements of nonlinear Breit-Wheeler (nlBW) refer to



(69)

and read, analogous to the nonlinear Compton as crossing channel in Eqs. (61)–(64),

$$M_{\mathrm{nlBW}} = \int \frac{\mathrm{d}\ell}{2\pi}[\bar{u}(p_e)(-ie)\Gamma^\mu(\ell, -p_p, p_e)\varepsilon'_\mu v(p_p)]\delta^{(4)}(k' + \ell k - p_p - p_e) \tag{70}$$

$$= \frac{-ie}{2\pi k^+}[\bar{u}(p_e)\Gamma^\mu(\ell_0, -p_p, p_e)\varepsilon'_\mu v(p_p)]\bigg|_{\ell_0 = \frac{(p_e+p_p)^2}{2k\cdot k'}}, \tag{71}$$

$$M_{\mathrm{nlBW}}^{\mathrm{div}} = \frac{-ie}{2\pi k^+}\pi\mathcal{G}(-p_p, p_e)\delta(\ell_0)\delta^{lf}(k' - p_p - p_e)[\bar{u}(p_e)\gamma^\mu\varepsilon'_\mu v(p_p)] \to 0, \tag{72}$$

$$M_{\mathrm{nlBW}}^{\mathrm{reg}} = \frac{-ie}{2\pi k^+}\delta^{lf}(k' - p_p - p_e)\left[\bar{u}(p_e)\left\{\left(\Gamma_1^{\mu\nu}(-p_p, p_e) - \mathcal{P}\frac{\gamma^\mu\alpha_1^\nu(-p_p, p_e)}{\ell}\right)B_{1\nu}(\ell, -p_p, p_e)\right.\right.$$
$$\left.\left. + \left(\Gamma_2^\mu(-p_p, p_e) - \mathcal{P}\frac{\gamma^\mu\alpha_2(-p_p, p_e)}{\ell}\right)B_2(\ell, -p_p, p_e)\right\}\varepsilon'_\mu v(p_p)\right], \tag{73}$$

where $M_{\mathrm{nlBW}}^{\mathrm{div}} \to 0$ is a result of combining the δ distributions $\delta(\ell_0)\delta^{lf}(k' - p_p - p_e) \to \delta^{(4)}(k' - p_e - p_e)$ which can not be satisfied by on shell momenta.

While, in nonlinear Compton, the initial electron momentum may be zero, $\vec{p} = 0$, due to the action of the external classical field, the electron can emit a real photon(s), whatever the external-field central-frequency $\omega > 0$ is. One can imagine this as shaking off photons due to the quiver motion in the external field. The crossing channel, i.e. nonlinear Breit-Wheeler as one-photon decay, $\gamma' \to e_L^+ e_L^-$ with matrix element $\propto k'_\mu[\bar{u}_p\Gamma^\mu v_{p'}]$), faces a severe threshold, making nonlinear Compton and nonlinear Breit-Wheeler quite distinctive, even the amplitudes are related by crossing symmetry. The balance equations in the monochromatic case read

$$\ell k \pm k' = \pm q + q' \tag{74}$$

with quasimomenta $q = p + a_0^2 m^2/(2p\cdot k)$ and $q' = p' + a_0^2 m^2/(2p'\cdot k)$ which facilitate $q^2 = q'^2 = m_*^2$, lead to

$$\ell k \cdot k' = q\cdot q + m_*^2 \quad (\mathrm{nlBW,\ upper\ sign}), \tag{75}$$

$$\ell k \cdot p = k' \cdot q \quad (\mathrm{nlC,\ lower\ sign}). \tag{76}$$

Explication for nonlinear Compton (head-on laser-electron collisions) reads

$$k = (\omega, \omega, 0, 0), \tag{77}$$

$$p = (m\cosh y, -m\sinh y, 0, 0), \tag{78}$$

$$k' = (\omega', \omega'\cos\Theta', \omega'\sin\Theta', 0), \tag{79}$$

$$p' = (E', \sqrt{E'^2 - m^2 - \omega'^2\sin^2\Theta'}, -\omega'\sin\Theta', 0), \tag{80}$$

where $E = m\cosh y$ and $|\vec{p}| = m\sinh y$ relates energy E and momentum $\vec{p}$, $E^2 + \vec{p}^2 = m^2$, with rapidity y, and

$$\omega'(\ell, \cos\Theta') = \frac{\ell\omega}{1 + e^{-y}\kappa(1 - \cos\Theta')},$$
$$\kappa \coloneqq \ell\frac{\omega}{m} - \sinh y + \frac{1}{2}a_0^2 e^{-y}, \tag{81}$$

$$E' = \frac{m^2 - \omega'(m - \omega')(1 - \cos\Theta')}{m - \omega'(1 - \cos\Theta')}, \tag{82}$$

express ω' and E' as a function of $\cos\Theta'$. Forward (backward) scattering is defined by $\cos\Theta' = 1$ (-1). The out-electron angle is determined by $\sin\theta' = -\omega'\sin\Theta'/|p'|$.

In nonlinear Breit-Wheeler, the quasimomenta q and q' symmetrically enter the corresponding kinematic equations. The threshold energy, for the monochromatic case, is determined by $k\cdot k' = 2m_*^2$. However, the subthreshold pair production is enabled in short pulses, as emphasized in [38–42]. Temporal double pulses or bichromatic pulses enhance further the pair rate, as suggested in [43–45].

While Compton has a classical analog (shaking off the e.m. field accompanying an accelerated charge in the form of asymptotically outgoing waves), Breit-Wheeler is said to be a quantum process, i.e. “converting light into matter”. A particularly interesting aspect is the relation to vacuum birefringence, see [46], which is experimentally searched for in dedicated and highly specialized and optimized setups, e.g. pursued by Helmholtz International Beamline for Extreme Fields (HIBEF) [47–50].

VI. TWO-VERTEX PROCESSES/FOUR-POINT AMPLITUDE

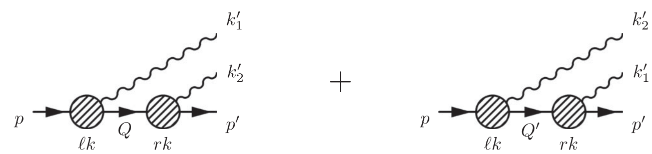
The two two-vertex diagrams have the symbolic matrix elements (i) $\mathcal{M} \sim [\bar{u}(p')\Gamma^\mu S_F(Q)\Gamma^\nu u(p)]\mathcal{A}_\mu(k_1)\mathcal{A}_\nu(k_2)$ and (ii) $\mathcal{M} \sim [\bar{u}(p')\Gamma^\mu u(p)]\mathcal{D}_{\mu\nu}[\bar{u}(P')\Gamma^\nu u(P)]$ with S_F and $\mathcal{D}_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu}}{k^2+i\epsilon}$ as fermion and photon propagators, and the Minkowski metric $\eta_{\mu\nu} \coloneqq \mathrm{diag}(1,-1,-1,-1)$. They have one (i) and two (ii) tied fermion lines. Again, depending on the orientation of the four-momenta, several processes related by crossing symmetry are conceivable: (i) $e^-_L \to e^{-\prime}_L + \gamma_1 + \gamma_1$ (nonlinear two-photon Compton, cf. [51–54]), $e^-_L + \gamma \to e^{-\prime}_L + \gamma'$ (nonlinear Compton scattering, i.e. x-ray Compton scattering at an electron moving in a nonaligned optical laser), $\gamma_1 + \gamma_2 \to e^-_L + e^+_L$ (nonlinear two-photon Breit-Wheeler) and time-reversed processes as well, in particular $e^-_L + e^+_L \to \gamma_1 + \gamma_2$; (ii) $e^-_{1L} + e^-_{2L} \to e^{-\prime}_{1L} + e^{-\prime}_{2L}$ (nonlinear Møller scattering), $e^-_L \to e^{-\prime}_L + e^{-\prime\prime}_L + e^+_L$ (nonlinear trident) and several crossing channels as well (e.g. nonlinear Bhabha scattering with s and t channel diagrams). Also, the involvement of two different lepton species is conceivable, e.g. electrons and muons.

We explicate now our momentum-space Furry-picture Feynman-rules for nonlinear two-photon Compton (Sec. VI A) and nonlinear Møller (Sec. VI B).

A. Two-photon nonlinear Compton

1. Diagrams and matrix element

The two-photon nonlinear Compton (2nlC) $e^-(p) +$ laser $\to \gamma(k_1') + \gamma(k_2') + e^-(p') +$ laser, in short $e^-_L \to e'^-_L + \gamma_1 + \gamma_2$, as a two-vertex tree-level diagram



(83)

requires a somewhat more intricate treatment, see [27,51–54], despite the simple matrix element (direct term, i.e. left diagram; the exchange term, i.e. right diagram, is to be processed analogously)

$$S_{2\mathrm{nlC}} = \int \frac{\mathrm{d}\ell}{2\pi}\frac{\mathrm{d}r}{2\pi}\frac{\mathrm{d}^4Q}{(2\pi)^4}\delta^{(4)}(p+\ell k - k_1' - Q)\delta^{(4)}(Q + rk - k_2' - p') \times [\bar{u}(p')(-ie)\Gamma^\mu(r,Q,p'|k)\varepsilon^{*\prime}_\mu(k_2')S_F(Q)(-ie)\Gamma^\nu(\ell,p,Q|k)\varepsilon^{*\prime}_\nu(k_1')u(p)]. \tag{84}$$

Collinear divergence and infra-red behavior as well as the on/off shell behavior of the fermion propagator provide some challenges. In addition, the soft-photon theorem (cf. [55] for contemporary reasoning) might be explicated here.

We consider now only the structure of the direct matrix element (84), where p and p' denote the momenta of the in and outgoing electrons, and $k_{1,2}'$ are momenta of the outgoing photons with polarization four-vectors $\varepsilon^{*\prime}_{\mu,\nu}(k_{1,2}')$; Q refers to the intermediate electron. Using one of the δ-distributions, the integral over the intermediate-electron momentum Q can be solved analytically:

$$S_{2\mathrm{nlC}} = \frac{-e^2}{(2\pi)^6}\int \mathrm{d}\ell \int \mathrm{d}r\delta^{(4)}(p + (r+\ell)k - k_1' - k_2' - p')\varepsilon'^*_{2\mu}(k_2')\varepsilon'^*_{1\nu}(k_1')[\bar{u}(p')\Gamma^\mu(r,Q,p'|k)S_F(Q)\Gamma^\nu(\ell,p,Q|k)u(p)], \tag{85}$$

where Q is now related to the external momenta and the photon number parameters via

$$p + \ell k - k_1' = Q = k_2' + p' - rk. \tag{86}$$

The remaining δ-distribution in (85) can be used to solve one of the photon-number parameter integrals by applying light cone coordinates to the involved momenta,

$$\delta^{(4)}(p + (r+\ell)k - k_1' - k_2' - p') = \delta^{\mathrm{lf}}(p - k_1' - k_2' - p')\delta(p^+ + (r+\ell)k^+ - k_1'^+ - k_2'^+ - p'^+), \tag{87}$$

where $\delta^{\mathrm{lf}(q)} = \frac{1}{2}\delta^{(2)}(q^\perp)\delta(q^-)$. The second δ distribution in (87) can be used to solve one of the integrals over the photon number parameter, e.g. the r integral, which leads to

$$S_{2\mathrm{nlC}} = \frac{-e^2}{(2\pi)^6 k^+}\delta^{\mathrm{lf}}(p - k_1' - k_2' - p')\varepsilon'^*_{2\mu}(k_2')\varepsilon'^*_{1\nu}(k_1')\int \mathrm{d}\ell[\bar{u}(p')\Gamma^\mu(r_\ell,Q,p'|k)S_F(Q)\Gamma^\nu(\ell,p,Q|k)u(p)], \tag{88}$$

where the two-photon number parameters are no longer independent, but related by $r_\ell \coloneqq \ell_0 - \ell$ with

$$\ell_0 \coloneqq \frac{k_1'^+ + k_2'^+ + p'^+ - p^+}{k^+} = \frac{(k_1' + k_2' + p')^2 - m^2}{2p\cdot k}. \quad (89)$$

2. Singularity structures

One obvious source of singularities is the vanishing denominator of the field-free electron propagator $S_F(Q)$. Therefore, the general resonance condition is given if the intermediate electron goes on shell, i.e. $Q^2 = m^2$. One convenient way to keep control of the propagator singularity is to define the virtuality of the intermediate electron, $v \coloneqq Q^2 - m^2$. Deploying Eq. (86), the virtuality is a function of either one of the two photon-number parameters,[4]

$$\nu_\ell \coloneqq Q^2(\ell) - m^2 = (p - k_1')^2 + 2\ell k\cdot(p - k_1'), \quad (90)$$

$$\nu_r \coloneqq Q^2(r) - m^2 = (k_2' + p')^2 - 2rk\cdot(k_2' + p'), \quad (91)$$

where we have $\nu_\ell \equiv \nu_{r_\ell = \ell_0 - \ell}$. The resonance condition $\nu \coloneqq Q^2 - m^2$ is then equivalent to $\nu_\ell = \nu_r = 0$. Written with the virtuality ν_ℓ, the denominator of the field-free fermion propagator reads

$$\frac{1}{Q^2 - m^2 + i\epsilon} = \frac{1}{\nu_\ell + i\epsilon} = \frac{1}{2k\cdot(p - k_1)}\left(\frac{1}{\ell_{\rm on} - \ell + i\epsilon}\right), \quad (92)$$

where we employ the replacement $\frac{\epsilon}{2k\cdot(p-k_1')} \to \epsilon$ and we use the abbreviation

$$\ell_{\rm on} = \frac{(p - k_1')^2 - m^2}{2k\cdot(p - k_1')} \neq 0. \quad (93)$$

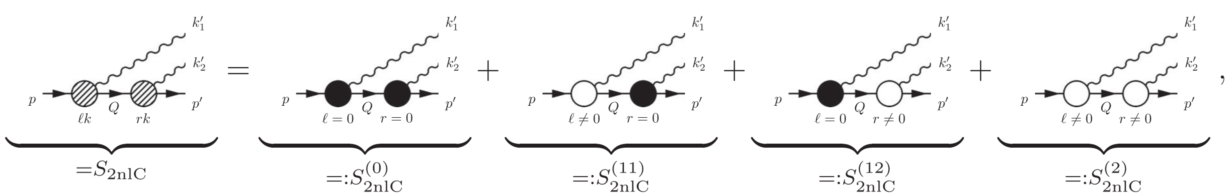
Analogously, written with the virtuality ν_r, one gets

$$\frac{1}{Q^2 - m^2 + i\epsilon} = \frac{1}{\nu_r + i\epsilon} = \frac{1}{2k\cdot(k_2' + p')}\left(\frac{1}{r_{\rm on} - r + i\epsilon}\right) \quad (94)$$

with $r_{\rm on} = \frac{(k_2'+p')^2 - m^2}{2k\cdot(k_2'+p')} \neq 0$. Consequently, the singularity structure of the electron propagator in the matrix element (88) is directly related to the values of the photon-number parameters at the respective vertex, i.e. $\nu_\ell = 0 \Leftrightarrow \ell = \ell_{\rm on}$ or equivalently $\nu_r = 0 \Leftrightarrow r = r_{\rm on}$. As we will show in the sequel, these types of singularities are the only ones, which may appear in the matrix element (88).

3. Asymptotically vanishing-field case

As illustrated in Sec. II B in the case of plane-wave pulses, i.e. asymptotically vanishing fields, the manifestly gauge-invariant dressed vertex function Γ^μ decomposes into a finite $\Gamma^\mu_{\rm reg}$ and a gauge-restoration $\Gamma^\mu_{\rm div}$ part. Consequently, inserting the decomposition by Eqs. (35)–(37) into Eq. (88), the matrix element of strong-field two-photon-Compton scattering becomes



, (95)

where the vertex structure of the particular matrix element introduces constraints on the respective photon-number parameter. We consider term-by-term:

(i) No energy-momentum transfer: The first part of (95) corresponds to the case, where both vertices are given by the divergent part of the dressed vertex, which implies that for this diagram, there is no energy-momentum transfer with respect to the background field on neither of the vertices. This follows also directly from the corresponding part of the 2nlC matrix element, which reads

$$S^{(0)}_{\rm 2nlC} = \frac{-e^2\pi^2}{(2\pi)^6 k^+}\delta^{\rm lf}(p - k_1' - k_2' - p')\epsilon'^*_{2\mu}(k_2')\epsilon'^*_{1\nu}(k_1')\int {\rm d}\ell[\bar{u}(p')\delta(r_\ell)\mathcal{G}(Q,p')\gamma^\mu S_F(Q)\gamma^\nu\delta(\ell)\mathcal{G}(p,Q)u(p)] \quad (96)$$

[4]There is only one *virtuality* for the intermediate electron, $\nu \coloneqq Q^2 - m^2$. However, here we phrase the dependence of the virtuality on the photon-number parameters as if they are independent, because the choice, which of the photon-number parameter integral in equation (85) one wants to solve, is arbitrary. Consequently, both of the *definitions* (90) and (91) are equivalent, if and only if $r \equiv r_\ell = \ell_0 - \ell$.

$$= \frac{-e^2\pi^2}{(2\pi)^6 k^+} \delta^{\rm lf}(p - k_1' - k_2' - p')\delta(r_0)\mathcal{G}(Q, p')\mathcal{G}(p, Q)\varepsilon'^*_{2\mu}(k_2')\varepsilon'^*_{1\nu}(k_1')[\bar{u}(p')\gamma^\mu S_F(Q)\gamma^\nu u(p)], \tag{97}$$

where the δ-distributions from the gauge-restoration parts are solved by $\ell = 0 = r_0 \equiv \ell_0$. The remaining δ-distributions only depend on the external particles. Therefore, using (87) and (89) leads to

$$\frac{1}{k^+}\delta^{\rm lf}(p - k_1' - k_2' - p')\delta(\ell_0) = \delta^{(4)}(p - k_1' - k_2' - p'). \tag{98}$$

However, there is no physical phase space solving $p - k_1' - k_2' - p' = 0$ with all on shell momenta. Furthermore, for the first part of (95), the virtualities read

$$\nu_{\ell=0} = (p - k_1')^2 - m^2 \neq 0, \tag{99}$$

$$\nu_{r=0} = (k_2' + p')^2 - m^2 \neq 0. \tag{100}$$

Therefore, there is no singularity to *cancel* the vanishing phase space, thus there is no contribution of $S^{(0)}_{\rm 2nlC}$ to the matrix element.

(ii) Contribution from the left vertex: The second term in the decomposition (95) represents the case, where at the left vertex, the photon-number parameter does not vanish, i.e. $\ell \neq 0$, whereas, on the right vertex, the photon-number parameter is identically zero: $r_\ell = \ell_0 - \ell = 0$. The corresponding part of the strong-field 2nlC matrix element is given by

$$S^{(11)}_{\rm 2nlC} = \frac{-e^2\pi}{(2\pi)^6 k^+} \delta^{\rm lf}(p - k_1' - k_2' - p')\varepsilon'^*_{2\mu}(k_2')\varepsilon'^*_{1\nu}(k_1') \int d\ell [\bar{u}(p')\delta(r_\ell)\mathcal{G}(Q, p')\gamma^\mu S_F(Q)\Gamma^\nu_{\rm reg}(\ell, p, Q|k)u(p)] \tag{101}$$

$$= \frac{-e^2\pi}{(2\pi)^6 k^+} \delta^{\rm lf}(p - k_1' - k_2' - p')\varepsilon'^*_{2\mu}(k_2')\varepsilon'^*_{1\nu}(k_1')\mathcal{G}(Q, p')[\bar{u}(p')\gamma^\mu S_F(Q)\Gamma^\nu_{\rm reg}(\ell_0, p, Q|k)u(p)], \tag{102}$$

where ℓ_0 is given by Eq. (89) and $\Gamma^\mu_{\rm reg}$ is the finite part of the dressed vertex defined in Eq. (37). The Cauchy principal-value operator in $\Gamma^\mu_{\rm reg}$ ensures $l_0 \neq 0$. For this part of the matrix element, the virtualities (90) and (91) read

$$\nu_{r_\ell=0} = (k_2' + p')^2 - m^2 \neq 0, \tag{103}$$

$$\nu_{\ell=\ell_0} \equiv \nu_{r=0} \neq 0, \tag{104}$$

thus, there is no singularity in $S^{(11)}_{\rm 2nlC}$.

(iii) Contribution from the right vertex: The third term in the decomposition (95) represents the case, where at the right vertex, the photon-number parameter does not vanish, i.e. $r_l \neq 0$, whereas, at the left vertex, the photon-number parameter is identically zero, $\ell = 0$. The corresponding part of the strong-field 2nlC matrix element is given as

$$S^{(12)}_{\rm 2nlC} = \frac{-e^2\pi}{(2\pi)^6 k^+} \delta^{\rm lf}(p - k_1' - k_2' - p')\varepsilon'^*_{2\mu}(k_2')\varepsilon'^*_{1\nu}(k_1') \times \int d\ell [\bar{u}(p')\Gamma^\mu_{\rm reg}(r_\ell, Q, p'|k)S_F(Q)\gamma_\mu\delta(\ell)\mathcal{G}(p, Q)u(p)] \tag{105}$$

$$= \frac{-e^2\pi}{(2\pi)^6 k^+} \delta^{\rm lf}(p - k_1' - k_2' - p')\varepsilon'^*_{2\mu}(k_2')\varepsilon'^*_{1\nu}(k_1')\mathcal{G}(p, Q)[\bar{u}(p')\Gamma^\mu_{\rm reg}(r_0, Q, p'|k)S_F(Q)\gamma_\mu u(p)], \tag{106}$$

where $r_0 \equiv \ell_0$ is given by Eq. (89) and $\Gamma^\mu_{\rm reg}$ is again the regularized finite part of the dressed vertex defined in Eq. (37) and the Cauchy principal-value operator in $\Gamma^\mu_{\rm reg}$ ensures $r_0 \neq 0$. For this part of the matrix element, the virtualities (90) and (91) read

$$\nu_{\ell=0} = (p - k_1')^2 - m^2 \neq 0, \tag{107}$$

$$\nu_{r=r_0=\ell_0} \equiv \nu_{\ell=0} \neq 0, \tag{108}$$

thus, there is no singularity in $S^{(12)}_{\rm 2nlC}$.

(iv) On shell and off shell contributions: The fourth term in the decomposition (95) represents the case, where on both vertices the photon-number parameters are nonzero, i.e. $\ell \neq 0$ and $r_\ell \neq 0$. The corresponding part of the strong-field 2nlC matrix element reads

$$S^{(2)}_{2\mathrm{nlC}} = \frac{-e^2}{(2\pi)^6 k^+}\delta^{\mathrm{lf}}(p-k'_1-k'_2-p')\varepsilon'^*_{2\mu}(k'_2)\varepsilon'^*_{1\nu}(k'_1)\int \mathrm{d}\ell[\bar{u}(p')\Gamma^{\mu}_{\mathrm{reg}}(r_\ell, Q, p'|k)S_F(Q)\Gamma^{\nu}_{\mathrm{reg}}(\ell, p, Q|k)u(p)], \qquad (109)$$

where $\Gamma^{\mu}_{\mathrm{reg}}$ is the finite part of the dressed vertex defined in Eq. (37). The Cauchy principal value operator in $\Gamma^{\mu}_{\mathrm{reg}}$ ensures $\ell \neq 0 \neq r_\ell$. However, according to Eq. (92), the condition $\ell = \ell_{\mathrm{on}}$, with ℓ_{on} defined in (93), is equivalent to $Q^2 - m^2 = 0$ inducing the propagator $S_F(Q) = \frac{i(\not{Q}+m)}{Q^2-m^2+i\epsilon}$ to diverge in the limit $\epsilon \to 0$. This divergence can be handled by applying the Sokhotski-Plemelj theorem,

$$\lim_{\epsilon\to 0^+}\frac{1}{Q^2-m^2+i\epsilon} = -i\pi\delta(Q^2-m^2) + \mathcal{P}\frac{1}{Q^2-m^2}, \qquad (110)$$

where, on the rhs, in the first term the denominator of the propagator is removed and Q is set on shell,[5] i.e. $Q^2 \stackrel{!}{=} m^2$, or equivalently $\ell \stackrel{!}{=} \ell_{\mathrm{on}}$, where ℓ_{on} is given in Eq. (93). In the second term of the rhs of Eq. (110), the Cauchy principal-value operator ensures that the denominator of the propagator never vanishes, which removes the singularity caused by $Q^2 = m^2$. In the language of virtualities introduced in Sec. VI A 2, this means for the first term of the rhs of (110), the virtualities read $\nu_\ell = \nu_{r_\ell} = 0$, i.e. the intermediate electron goes on-shell. However, since for this term, the denominator of the propagator is removed, the on-shell *intermediate* electron does not induce a pole. The second term of the rhs of Eq. (110) does not induce a pole neither, because the Cauchy principal-value operator protects the denominator of the electron propagator from vanishing, i.e. $\mathcal{P}\frac{1}{Q^2-m^2} = \mathcal{P}\frac{1}{\nu}$, which excludes the value $\nu = 0$ from the integration region. Therefore, there is no singularity in $S^{(2)}_{2\mathrm{nlC}}$.

In summary, it can be said, therefore, that in the case of asymptotically vanishing background fields, $\Delta\phi < \infty$, the matrix element (88) of two-photon Compton scattering has no singularities except for a single light-front δ-distribution, which ensures the conservation of the transverse and the minus-components of the external momenta.

4. Infinitely extended plane-wave/Oleinik resonances

The special case of a monochromatic plane-wave background field that is infinitely extended [see Eq. (1) with $\Delta\phi \to \infty$ or $g = 1$] should be considered separately. Now, the two-photon Compton matrix element (88) reads

$$S^{\mathrm{IPW}}_{2\mathrm{nlC}} = \frac{-e^2}{(2\pi)^6 k^+}\delta^{\mathrm{lf}}(p-k'_1-k'_2-p')\varepsilon'^*_{2\mu}(k'_2)\varepsilon'^*_{1\nu}(k'_1)\int \mathrm{d}\ell \sum_{n,n'=-\infty}^{\infty}\delta(\ell-\beta(p,Q)-n)\delta(r_l-\beta(p',Q)-n')$$
$$\times[\bar{u}(p')\Gamma^{\mu}_{\mathrm{IPW}n'}(r_\ell, Q, p'|k)S_F(Q)\Gamma^{\nu}_{\mathrm{IPW}n}(\ell, p, Q|k)u(p)], \qquad (111)$$

where the mode-wise dressed vertex function $\Gamma^{\mu}_{\mathrm{IPW}n}$ is given in Eq. (51). We mention that the functions $\beta(p,Q)$ and $\beta(Q,p')$ do not depend on the photon-number parameter ℓ, since $\beta(p,Q) = \beta(p, p-k'_1)$ and $\beta(Q,p') = \beta(p'+k'_2, p')$, respectively. Therefore, exchanging the integration and the two summations, one can solve the integral by using one of the δ-distributions, e.g. the first one, which leads to $\ell = \ell^{\mathrm{reso}}_n = \beta(p, p-k'_1) + n$. Then, the respective virtuality (90) results in

$$\nu_{\ell=\ell^{\mathrm{reso}}_n} = -k'_1\cdot(p+\{\beta_p+n\}k) + nk\cdot p, \qquad (112)$$

where we use the abbreviation $\beta_p = \frac{a^2e^2}{2}\frac{1}{k\cdot p}$. Finally, if one employs the resonance condition $\nu_{\ell=\ell^{\mathrm{reso}}_n} = 0$, for the emitted photon with four-momentum $k'^{\mu}_1 = \omega'_1 n'^{\mu}_1$, one finds for the resonance energy[6]

$$\omega'^{\mathrm{reso}}_{1,n} = \frac{nk\cdot p}{(p+\{\beta_p+n\}k)\cdot n'_1}. \qquad (113)$$

Singularities of this type are called Oleinik resonances [56,57], which were already identified for the nonlinear two-photon Compton process in [33,53,54]. For further investigations of the diagrams (114) with respect to Oleinik resonances we refer the interested reader to [58–61] and further citations therein, where one of the photon lines is attributed to a "field photon" of a nucleus.

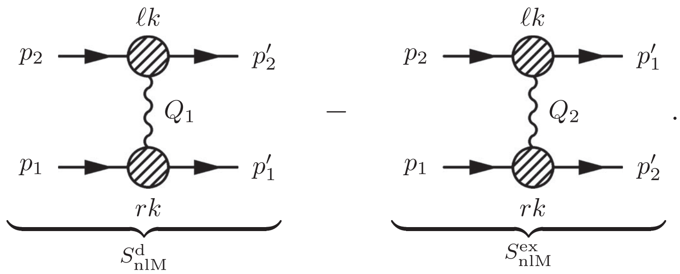
The multiphoton nonlinear Compton with more than two vertices, e.g. [62,63], perpetuates this line of arguments and offers a test bed of gluing techniques, such as developed in [64,65].

[5] The symbolic expression $\stackrel{!}{=}$ indicates the *ad hoc* requirement for "must be equal".

[6] This formula for the resonance energy is already known, see, e.g., [33,54].

B. Nonlinear Møller

As a further application of the momentum-space Furry-picture Feynman rules to two-vertex processes we consider nonlinear Møller scattering (nlM), i.e. $e_L^-(p_1) + e_L^-(p_2) \to e_L^-(p_1') + e_L^-(p_2')$ in the laser background field (1). Here, the Oleinik resonances are attributed to the on shell contributions of the photon propagator, cf. [66]. The leading-order tree-level two-vertex diagrams are



(114)

The direct term (first diagram) corresponds to the matrix element

$$S_{\mathrm{nlM}} = \int \frac{\mathrm{d}\ell}{2\pi}\frac{\mathrm{d}r}{2\pi}\frac{\mathrm{d}^4Q}{(2\pi)^4}[\bar{u}(p_1')(-ie)\Gamma^\mu(r, p_1, p_1'|k)u(p_1)]\delta^{(4)}(p_1 + rk - p_1' - Q)$$
$$\times \mathcal{D}_{\mu\nu}(Q)[\bar{u}(p_2')(-ie)\Gamma^\nu(\ell, p_2, p_2'|k)u(p_2)]\delta^{(4)}(p_2 + \ell k - p_2' - Q), \tag{115}$$

where $\mathcal{D}_{\mu\nu}$ is the free-photon propagator. Executing the Q integration with one of the $\delta^{(4)}$ distributions yield

$$S_{\mathrm{nlM}} = \frac{-e^2}{(2\pi)^6}\int \frac{\mathrm{d}\ell}{2\pi}\frac{\mathrm{d}r}{2\pi}\delta^{(4)}(p_1 + p_2 + [r+\ell]k - p_1' - p_2')$$
$$\times [\bar{u}(p_1')\Gamma^\mu(r, p_1, p_1'|k)u(p_1)]\mathcal{D}_{\mu\nu}(Q)[\bar{u}(p_2')\Gamma^\nu(\ell, p_2, p_2'|k)u(p_2)], \tag{116}$$

where $Q = p_1 + rk - p_1' = p_2' - \ell k - p_2$. The intermediate photon's virtuality is defined by $\nu \coloneqq Q^2$, which is related to the photon number parameters

$$r(\nu) = \frac{\nu - \delta p_1^2}{2\delta p_1 \cdot k}, \qquad \ell(\nu) = \frac{\nu - \delta p_2^2}{2\delta p_2 \cdot k}, \tag{117}$$

with $\delta p_n \coloneqq p_n - p_n'$, $n = 1, 2$. Thus, the photon number parameters ℓ and r are intervened. With aid of light cone variables, the four-momentum balance can be rewritten as

$$\delta^{(4)}(p_1 + p_2 + [r+\ell]k - p_1' - p_2') = \delta^{lf}(p_1 + p_2 - p_1' - p_2')\delta(p_1^+ + p_2^+ + [r+\ell]k^+ - p_1'^+ - p_2'^+) \tag{118}$$

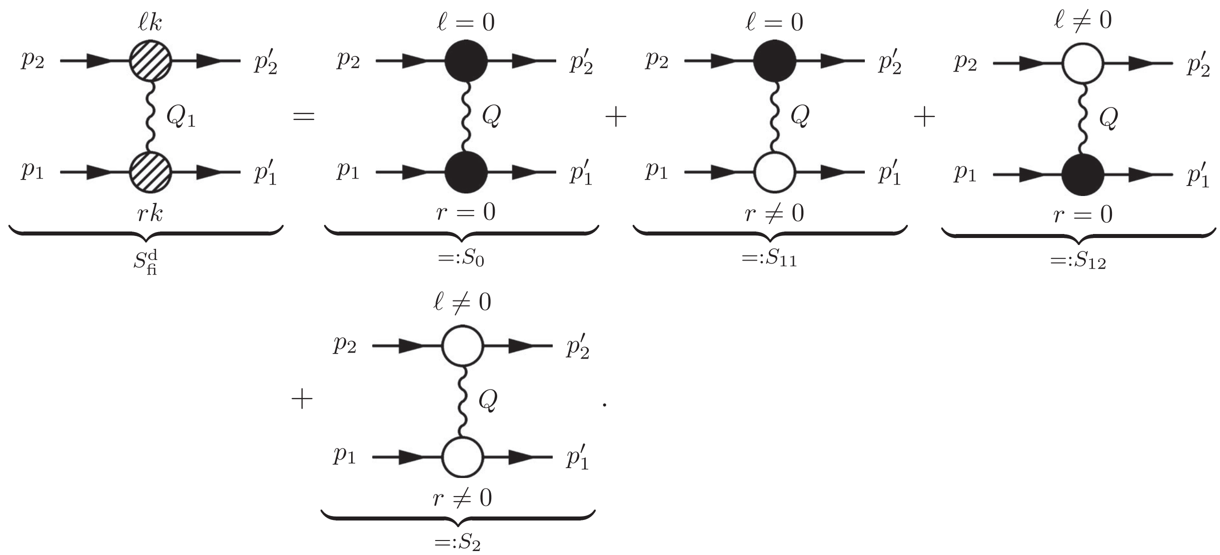
to execute the ℓ integral in Eq. (116) with the result

$$S_{\mathrm{nlM}} = \frac{-e^2}{(2\pi)^6 k^+}\delta^{lf}(p_1 + p_2 - p_1' - p_2')\int \mathrm{d}r[\bar{u}(p_1')\Gamma^\mu(r, p_1, p_1'|k)u(p_1)]\mathcal{D}_{\mu\nu}(Q)[\bar{u}(p_2')\Gamma^\nu(\ell_r, p_2, p_2'|k)u(p_2)], \tag{119}$$

where $\ell_r \coloneqq r_0 - r$ with $r_0 = (p_1^+ + p_2^+ - p_1'^+ - p_2'^+)/k^+$ or

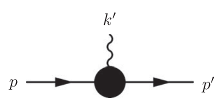
$$\ell_r = \frac{(p_1' + p_2' - p_2)^2 - m^2}{2k \cdot p_1} - r. \tag{120}$$

The decomposition (35) facilitates four contributions to the direct term:



(121)

Clearly, with respect to Eq. (36), the black-bullet vertices do dependent on the background field since



$\propto \mathcal{G} = e^{iG_+} + e^{iG_-}$

and $G_\pm = \alpha_1^\mu \int_{\phi_0}^{\pm\infty} \mathrm{d}\phi A_\mu(\phi) + \alpha_2 \int_{\phi_0}^{\pm\infty} \mathrm{d}\phi A(\phi)^2$, meaning that the first diagram, S_0, links to the Møller scattering in vacuum with momentum balance $p_1 + p_2 = p_1' + p_2'$, thus reproducing standard perturbative QED. The background field, in particular, its temporal shape encoded in $g(\phi, \Delta\phi)$, enters the other three diagrams. A suggestive interpretation is proposed in [21]-Fig. 2 by attributing a temporal ordering to the diagrams. For instance, the second diagram, S_{11}, would refer to virtual Compton under the influence of the background field (corresponding to the hatched vertex $\Gamma^\mu_{\rm reg}$), while the subsequent virtual-photon absorption in the black vertex $\Gamma^\mu_{\rm div}$ would proceed after the impact of the external field. Such an interpretation would ascribe the first diagram to proceeding before or after the action of the external field, while the last diagram would refer to both subprocesses within the action of the field; the third diagram, S_{12}, would be accordingly interpreted as virtual Compton prior to the external impact. Independent of such an interpretation, the fourth diagram, S_2, facilitates on and off shell contributions.

Analog to the sequence of steps in elaborating the matrix elements in Sec. V A one can easily explicate the above diagrams to obtain the decomposition of the four-point amplitude corresponding to Eq. (2.23) in [21]. As pointed out in Sec. III D, in the case of soft interactions with the background field, the finite $\Gamma^\mu_{\rm reg}$ and gauge restoration $\Gamma^\mu_{\rm div}$ parts of the dressed vertices factorize into a hard-scattering part and a generalized soft factor. Consequently, this soft/hard factorization also sets in for each diagram in the decomposition (121). Therefore, in the simultaneous limits $a_0 \to 0$ and $\Delta\phi \to \infty$, the corresponding soft versions of the five-point functions in perturbative monochromatic QED appear, where soft photons couple to each fermion line.

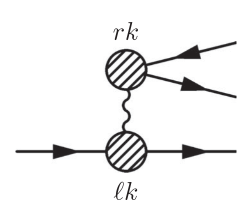
Considering again monochromatic plane-wave background fields and inserting (51) in the Møller matrix element (119), we arrive at

$$\mathcal{S}^{\rm IPW}_{\rm nlM} = \frac{-e^2}{(2\pi)^6 k^+} \sum_{n,n'} \int \mathrm{d}r \delta(r - \beta(p_1, p_1', k) - n) \times \delta(\ell_r - \beta(p_2, p_2', k) - n') \tag{122}$$

$$\times [\bar{u}(p_1')\Gamma^\mu_{{\rm IPW}n}(r, p_1, p_1')u(p_1)]\mathcal{D}_{\mu\nu}(Q) \times [\bar{u}(p_2')\Gamma^\mu_{{\rm IPW}n'}(\ell_r, p_2, p_2')u(p_2)]. \tag{123}$$

Solving the r integral, we find $r = \beta(p_1, p_1', k) + n$ and, therefore, the virtuality reads $\nu(r = \beta + n) = (\beta(p_1, p_1', k) + n)2(p_1 - p_1') \cdot k + (p_1 - p_1')^2$ meeting the resonance condition $\nu = 0$ reveals similar Oleinik resonances as shown in Sec. VI A 4.

Via crossing symmetry, the trident amplitude



has a similar decomposition as given above by the four direct-term diagrams, to be supplemented by the exchange terms. The changed in- and out-phase space, however, modifies the treatment/interpretation of individual contributions, to be dealt with in a follow-up paper. The nonlinear Møller scattering is laser-assisted, while the nonlinear trident is laser-enabled. Oleinik resonances show up in both channels [67].

VII. SUMMARY

Following [20,23] we present the comprehensive momentum-space Furry-picture Feynman rules for QED in an external classical background field. Our emphasis is on formal aspects of gauge invariance and Ward identity, thus providing a general framework well-suited for n-point amplitudes. The special case of four-point amplitudes, dealt with in [21] within a somewhat different formulation, emerges naturally. Three-point amplitudes are considered exhaustively in the past and are uncovered as well. The benefit of our formalism is a systematic approach to the weak-field approximation, i.e. the series expansion of the scattering matrix element in powers of the laser intensity parameter a_0. The leading-order term represents "pulsed perturbative QED" which accounts, in contrast to standard perturbative QED, for the temporal structure of the external classical field, where the Fourier transform of that field enters decisively. The limiting case of a monochromatic external classical field recovers the standard perturbative QED in terms of Feynman graphs and their rules of translation into amplitudes for processes with one incoming photon impinging on a target, e.g. an electron or a scattering electron-electron/positron system. The next-to-leading order terms in a_0 are determined by Fourier transforms of the external classical field in various (nonlinear) combinations. The monochromatic limiting case describes processes with two incoming photons impinging on the target, thus referring to the two-photon channel, e.g. $k_1 + k_2 + e^- \to X$, with $X = e^{-\prime} e^+ e^-$ for trident. When considering one monochromatic laser field, $k_1 = k_2$. One may also deal with $k_1 \neq k_2$, e.g. for the superposition of optical laser and XFEL beams. We leave the explication of such processes with respect to trident for separate work.

Elements of our QED momentum-space Furry-picture Feynman rules are free Dirac spinors, free fermion propagator, and free photon propagator, and the external field impact is solely encoded in the fermion-fermion-photon vertex function. By providing a suitable framework for the evaluation of the latter vertex function, standard platforms for the calculation of Feynman diagrams can be used for strong-field QED processes.

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