Perturbative *S*-matrix unitarity and higher-order Lorentz violation

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We investigate the preservation of unitarity in a Lorentz and *CPT*-violating QED model containing higher-order operators. In particular, we consider modifications in the fermion sector with dimension-five operators. The higher-order operators lead to an indefinite metric and a pseudo-unitarity relation for the *S*-matrix. However, we show that the pseudo-unitarity condition can be promoted to a genuine unitarity relation by (i) restricting the energies to the effective region far below the Planck mass and (ii) considering stable particles to have a positive metric. In the context of the optical theorem, we focus on the one-loop Bhabha and Compton scattering processes. We show that no ghost states get propagated through the cuts, thus satisfying the unitarity condition. Further, we show that discontinuities of propagators are equivalent to replacing physical Dirac functionals in the cutting equation. The physical Dirac functionals are defined to select only mode solutions of stable particles. The provided extension of the Cutkosky rule may be helpful for analyzing perturbative unitarity in higher-order diagrams.

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I. INTRODUCTION

Over the past two decades, significant progress has been made in studying the possible breakdown of *CPT* and Lorentz invariance in extension to quantum field theories (QFT) and gravity. The combined efforts of theory, phenomenology, and ultrahigh precision experiments have allowed one to shape a robust effective framework known as the Standard-Model Extension (SME) [1,2]. The SME is an effective framework that accommodates the most general parametrizations of *CPT*, local Lorentz, and diffeomorphism symmetry violations, extending both the standard model of particles and gravity. The SME has established stringent limits on Lorentz violations and has identified the most promising sectors for detecting low-energy signatures of quantum gravity [3].

Extensions in QFT are typically achieved by introducing a privileged tensor that couples to both derivatives and fields. The effective terms are kept small by a high degree of suppression of the Planck scale. On the other hand, in gravity, the breaking of local Lorentz symmetry and diffeomorphism has been searched more with the mechanism of spontaneous symmetry breaking. In the case of spontaneous symmetry breaking, the background fields acquire dynamics and introduce extra ingredients, such as massless excitations or Nambu-Goldstone modes [4,5]. The two mechanisms have been called explicit and spontaneous symmetry breaking, respectively. In both cases, a background field with or without dynamics may arise by a nontrivial vacuum in a more fundamental theory such as strings [6,7].

The effective field theories of the SME can be classified according to the mass dimensions of the operators introduced to describe Lorentz symmetry breaking. Specifically, they can be divided into a minimal sector with operators of mass dimensions up to four, and a nonminimal sector with higher-order operators. Higher-order dimension operators have been a natural extension to include effects at higher energies in the effective framework. The exploration with nonrenormalizable operators are given in several sectors of the nonminimal SME: photons [8], fermions [9], and neutrinos [10], and also in linearized gravity [11]. Several works study radiative corrections [12–14], vacuum Cherenkov radiation [15,16], and explicit diffeomorphism breaking in gravity [17–19] to mention some.

A potential drawback of higher-order operators is that they may lead to the nonconservation of probability and to the loss of unitarity of the *S*-matrix [20]. However, it has been shown that there is no inherent contradiction in

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having unitarity conserved, as certain prescriptions introduced by Lee and Wick can be followed [21,22]. The basic idea of the Lee-Wick prescription is to restrict the asymptotic indefinite complex vector space, so as to consider only positive metric particles to be stable. In this work, we discuss the main ingredients and the principal assumption under which an indefinite metric arises and how it leads to the modification of the unitarity equation for the S-matrix, sometimes called the pseudo-unitarity relation. In recent years, many approaches have been developed to deal with the issue of unitarity conservation in Lorentz and CPT violating theories [23–25]. Using similar cutting techniques it has been possible to factorize amplitudes for hadronic processes; see [26]. The connection between the preservation of unitarity and Lorentz violation has been around for many years. A few years after the works of Lee and Wick, Nakanishi pointed out that a modification in the contour of integration in Feynman diagrams may lead to the loss of covariance [27,28]. This may result since momentum remains real, while in some regimes, the energy can become complex. Some further discussions on this context can be found in [29].

Recently the C-even part of the Myers and Pospelov model [30] has been studied for testing perturbative unitarity at tree level [31]. Considering the Compton scattering process at tree level, it was shown that unitarity is preserved. Here we extend the analysis to include the next natural step, which is to study perturbative unitarity at one-loop order. To implement the generalization, we focus on two diagrams contributing to the one-loop Bhabha and Compton scattering processes. We take advantage and use several expressions that have been derived, including the dispersion relation, their mode, and eigenspinor solutions in [31].

The organization of this work is as follows. In Sec. II, we obtain the Myers-Pospelov timelike model starting from the generalized mass dimension fermion model of the SME. In Sec. III, we recall the dispersion relation, their mode, and spinor solutions that we have found previously in [31]. We discuss the interaction term and use it to compute the matrix elements of the S-matrix. Further, we provide a closed formula for the pseudo-unitarity relation in the presence of an indefinite metric. In Sec. IV, we focus on the one-loop Compton and Bhabha scattering diagrams to study the preservation of unitarity. We use the perturbative tool of the optical theorem and check that no ghost degrees of freedom are propagated through the cuts of amplitude diagrams. Section V contains some further comments and a summary of our results.

II. MODIFIED FERMION SECTOR

Our starting point is the Lagrangian density of the fermion sector of the SME [9,10]

$$\mathcal{L}_{\mathrm{SME}} = \bar{\psi}(i\widehat{\Gamma}^{\mu}\partial_{\mu} - \widehat{M})\psi, \tag{1}$$

where all possible minimal and nonminimal contributions that break CPT and Lorentz symmetry can be expanded in terms of the 16 Dirac matrices

$$\widehat{\Gamma}^{\mu} = \gamma^{\mu} + \widehat{c}^{\alpha\mu}\gamma_{\alpha} + \widehat{d}^{\alpha\mu}\gamma_5\gamma_{\alpha} + \widehat{e}^{\mu} + i\widehat{f}^{\mu}\gamma_5 + \frac{1}{2}\widehat{g}^{\kappa\lambda\mu}\sigma_{\kappa\lambda} \tag{2a}$$

and

$$\widehat{M} = m + \widehat{m} + i\widehat{m}_5\gamma_5 + \widehat{a}^{\mu}\gamma_{\mu} + \widehat{b}^{\mu}\gamma_5\gamma_{\mu} + \frac{1}{2}\widehat{H}^{\mu\nu}\sigma_{\mu\nu}, \tag{2b}$$

where the effective derivative operators $\widehat{c}^{\alpha\mu}$, $\widehat{d}^{\alpha\mu}$, $\widehat{m}$, $\widehat{m}_5$, $\widehat{H}^{\mu\nu}$ are CPT even, while $\widehat{e}^{\mu}$ $\widehat{f}^{\mu}$ $\widehat{g}^{\kappa\lambda\mu}$ $\widehat{a}^{\mu}$, $\widehat{b}^{\mu}$ are CPT odd.

We are interested in making the connection with the Myers and Pospelov (MP) model [30], which contain dimension-five operators. Hence, we turn off several effective terms and retain

$$\widehat{\Gamma}^{\mu} = \gamma^{\mu}, \tag{3a}$$

$$\widehat{M} = m + \widehat{a}^{(5)\mu}\gamma_{\mu} + \widehat{b}^{(5)\mu}\gamma_5\gamma_{\mu}, \tag{3b}$$

with

$$\widehat{a}^{(5)\mu} \coloneqq -\frac{\eta_1}{m_{\mathrm{Pl}}}(n\cdot\partial)^2 n^{\mu}, \tag{3c}$$

$$\widehat{b}^{(5)\mu} \coloneqq \frac{\eta_2}{m_{\mathrm{Pl}}}(n\cdot\partial)^2 n^{\mu}. \tag{3d}$$

Considering the operators (3a) and (3b) and replacing in Eq. (1), we arrive at the fermion MP Lagrangian density

$$\mathcal{L}_{\mathrm{MP}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{\bar{\psi}}{m_{\mathrm{Pl}}}(\eta_1 n\!\!\!/ + \eta_2 n\!\!\!/\gamma_5)(n\cdot\partial)^2\psi, \tag{4}$$

where n^{μ} is a preferred four-vector responsible to break Lorentz symmetry, m_{Pl} is the Planck mass, and η_1, η_2 are coupling constants. Also one can show that η_1 is charge conjugation odd and η_2 charge conjugation even.

The free equation of motion is

$$\left(i\partial\!\!\!/ - m + \frac{1}{m_{\mathrm{Pl}}}(\eta_1 n\!\!\!/ + \eta_2 n\!\!\!/\gamma_5)(\partial\cdot n)^2\right)\psi(x) = 0. \tag{5}$$

Using the plane wave ansatz $\psi(x) = \int d^3k\psi(k)e^{-ik\cdot x}$ the dispersion relation becomes

$$(p^2 - m^2 - 2g_1(n\cdot p)^3 + n^2(g_1^2 - g_2^2)(n\cdot p)^4)^2 - 4g_2^2(n\cdot p)^4 D(n,p) = 0, \tag{6}$$

where we define $g_1 := \frac{\eta_1}{m_{\mathrm{Pl}}}$, $g_2 := \frac{\eta_2}{m_{\mathrm{Pl}}}$, and $D(n,p) := (n\cdot p)^2 - p^2 n^2$.

In this work we utilize the chiral representation for Dirac matrices, i.e.,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \tag{7}$$

with $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$, $\bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$, and $\mathbb{1}_2$ the 2×2 identity matrix. For the metric signature in Minkowski spacetime we employ the mostly minus sign convention $(+,-,-,-)$.

III. THE TIMELIKE MODEL

In this section, we provide the basic properties of the fermion timelike MP model. We take advantage of the dispersion relations, their modes' solutions, and the eigenspinors presented in [31]. We additionally discuss the interaction term in our modified QED model and present a detailed derivation of the unitary equation satisfied by the S-matrix in an indefinite metric theory.

A. Dispersion relation and spinors

We set the background four-vector in the pure timelike direction $n^\mu = (1,0,0,0)$ and turn off the charge conjugation odd sector with $\eta_1 = 0$. With these choices, the Lagrangian (4) takes the form

$$\mathcal{L}_{\substack{\mathrm{MP}\\ \text{C-even}}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + g_2\bar{\psi}\gamma_0\gamma_5\ddot{\psi}. \tag{8}$$

The dispersion relation can be found from Eq. (6) to be

$$(p_0^2 - |\vec{p}|^2 - m^2 - g_2^2 p_0^4)^2 - 4g_2^2 p_0^4 |\vec{p}|^2 = 0. \tag{9}$$

Let us introduce the quantities

$$\Lambda_+^2(p) =: p_0^2 - |\vec{p}|^2 - m^2 - g_2^2 p_0^4 - 2g_2 p_0^2 |\vec{p}|, \tag{10a}$$

$$\Lambda_-^2(p) =: p_0^2 - |\vec{p}|^2 - m^2 - g_2^2 p_0^4 + 2g_2 p_0^2 |\vec{p}|. \tag{10b}$$

In terms of these quantities the dispersion relation can be written as

$$\Lambda_+^2(p)\Lambda_-^2(p) = (p_0^2 - |\vec{p}|^2 - m^2 - g_2^2 p_0^4)^2 - 4g_2^2 p_0^4 |\vec{p}|^2. \tag{11}$$

The solutions of the equation $\Lambda_+^2(p) = 0$ are

$$\omega_1 = \sqrt{\frac{1 - 2g_2|\vec{p}| - \sqrt{(1-2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}},$$

$$W_1 = \sqrt{\frac{1 - 2g_2|\vec{p}| + \sqrt{(1-2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}}, \tag{12a}$$

and the solutions of the equation $\Lambda_-^2(p) = 0$ are

$$\omega_2 = \sqrt{\frac{1 + 2g_2|\vec{p}| - \sqrt{(1+2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}},$$

$$W_2 = \sqrt{\frac{1 + 2g_2|\vec{p}| + \sqrt{(1+2g_2|\vec{p}|)^2 - 4g_2^2 E_p^2}}{2g_2^2}}. \tag{12b}$$

We also have the negative mode solutions $-\omega_1, -W_1, -\omega_2, -W_2$, where $E_p = \sqrt{|\vec{p}|^2 + m^2}$. In [31], we have shown that modes $\pm W_{1,2}$ correspond to heavy ghost modes while $\pm\omega_{1,2}$ can be associated with perturbative modes of standard particles.

In Sec. IV, we study the unitarity of the model and will need to examine the modes in the complex p_0-plane. The poles ω_1 and W_1 exhibit a peculiar momentum-dependent behavior. As the magnitude of spatial momenta $|\vec{p}|$ increases, ω_1 moves in the positive direction of the real axis, while W_1 moves in the opposite direction. The two modes coincide at

$$|\vec{p}|_{\max} = \frac{1 - 4g_2^2 m^2}{4g_2}, \tag{13}$$

where they both take the value $\frac{1}{2g_2}\sqrt{1 + 4g_2^2 m^2}$, and for higher momenta, they start to have an imaginary component. For complex ω_1 and W_1, the first solution moves downward in the imaginary axis, while the latter heavy mode W_1 moves upward (see Fig. 2). The solutions ω_2 and W_2 remain real for all momentum values. The negative modes behave similarly, with the only difference being that when they become complex, the solution $-\omega_1$ moves upward, while $-W_1$ moves downward.

We can write the dispersion relation (11) as

$$\Lambda_+^2(p)\Lambda_-^2(p) = g_2^4(p_0^2 - \omega_1^2)(p_0^2 - W_1^2)(p_0^2 - \omega_2^2) \times (p_0^2 - W_2^2) = 0. \tag{14}$$

The positive-energy eigenspinors for $\Lambda_+^2(p) = 0$ are

$$u^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\,\xi^{(+)}(\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\,\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0 = \omega_1}, \tag{15a}$$

$$U^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\,\xi^{(+)}(\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\,\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0 = W_1}, \tag{15b}$$

and for $\Lambda_-^2(p) = 0$ are

$$u^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\,\xi^{(-)}(-\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\,\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0 = \omega_2}, \tag{16a}$$

$$U^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\xi^{(-)}(-\vec{p}) \\ \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_2}. \quad (16b)$$

The negative-energy eigenspinors for $\Lambda_+^2(p) = 0$ are

$$v^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\xi^{(-)}(-\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=\omega_1}, \quad (17a)$$

$$V^{(1)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 + |\vec{p}|}\xi^{(-)}(-\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 - |\vec{p}|}\xi^{(-)}(-\vec{p}) \end{pmatrix}_{p_0=W_1}, \quad (17b)$$

and for $\Lambda_-^2(p) = 0$ are

$$v^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\xi^{(+)}(\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=\omega_2}, \quad (18a)$$

$$V^{(2)}(p) = \begin{pmatrix} \sqrt{p_0 + g_2 p_0^2 - |\vec{p}|}\xi^{(+)}(\vec{p}) \\ -\sqrt{p_0 - g_2 p_0^2 + |\vec{p}|}\xi^{(+)}(\vec{p}) \end{pmatrix}_{p_0=W_2}. \quad (18b)$$

Above we have introduced the bispinors $\xi^{(\pm)}(\vec{p})$, given by

$$\xi^{(+)}(\vec{p}) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p^3)}} \begin{pmatrix} |\vec{p}| + p^3 \\ p^1 + ip^2 \end{pmatrix}, \quad (19)$$

$$\xi^{(-)}(\vec{p}) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| - p^3)}} \begin{pmatrix} p^1 - ip^2 \\ |\vec{p}| - p^3 \end{pmatrix}, \quad (20)$$

which satisfies the properties

$$(\vec{p}\cdot\vec{\sigma})\xi^{(\pm)}(\vec{p}) = |\vec{p}|\xi^{(\pm)}(\vec{p}),$$
$$(\vec{p}\cdot\vec{\sigma})\xi^{(\pm)}(-\vec{p}) = -|\vec{p}|\xi^{(\pm)}(-\vec{p}), \quad (21)$$

and the orthogonality relations

$$\xi^{(+)\dagger}(\vec{p})\xi^{(+)}(\vec{p}) = \xi^{(-)\dagger}(\vec{p})\xi^{(-)}(\vec{p}) = 1,$$
$$\xi^{(+)\dagger}(\vec{p})\xi^{(-)}(-\vec{p}) = \xi^{(-)\dagger}(-\vec{p})\xi^{(+)}(\vec{p}) = 0, \quad (22)$$

together with

$$\xi^{(\pm)}(\vec{p})\xi^{(\pm)\dagger}(\vec{p}) = \frac{1}{2}\left(1 + \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right), \quad (23)$$

$$\xi^{(\pm)}(-\vec{p})\xi^{(\pm)\dagger}(-\vec{p}) = \frac{1}{2}\left(1 - \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\right). \quad (24)$$

The modified propagator is found to be

$$S_F(p) = \frac{iF(p)}{D_p}, \quad (25)$$

where

$$F(p) = \bar{M}(p)N(p)\bar{N}(p), \quad (26)$$

$$D_p = \Lambda_+^2(p + i\epsilon)\Lambda_-^2(p + i\epsilon), \quad (27)$$

with

$$M = \not{p} - m - g_2 p_0^2 \gamma_0\gamma_5, \quad (28a)$$

$$\bar{M} = \not{p} + m - g_2 p_0^2 \gamma_0\gamma_5, \quad (28b)$$

$$N = \not{p} + m + g_2 p_0^2 \gamma_0\gamma_5, \quad (28c)$$

$$\bar{N} = \not{p} - m + g_2 p_0^2 \gamma_0\gamma_5. \quad (28d)$$

By taking into consideration the $i\epsilon$ prescription in (10a) and (10b) the pole structure is described by

$$\Lambda_+^2(p + i\epsilon) = -g_2^2(p_0 + \omega_1 - i\varepsilon)(p_0 - \omega_1 + i\varepsilon) \times (p_0 + W_1 - i\varepsilon)(p_0 - W_1 + i\varepsilon), \quad (29a)$$

$$\Lambda_-^2(p + i\epsilon) = -g_2^2(p_0 + \omega_2 - i\varepsilon)(p_0 - \omega_2 + i\varepsilon) \times (p_0 + W_2 - i\varepsilon)(p_0 - W_2 + i\varepsilon). \quad (29b)$$

Note that negative and positive solutions are placed above and below the real axis, respectively, in the complex p_0-plane. We also have the identities

$$2(p^2 - m^2 - g_2^2 p_0^4)_{p_0=\omega_1} = -g_2^2(\omega_1^2 - \omega_2^2)(\omega_1^2 - W_2^2), \quad (30a)$$

$$2(p^2 - m^2 - g_2^2 p_0^4)_{p_0=\omega_2} = -g_2^2(\omega_2^2 - \omega_1^2)(\omega_2^2 - W_1^2). \quad (30b)$$

In Sec. IV we use the expressions

$$u^{(1)}(p)\bar{u}^{(1)}(p) = \frac{F(\omega_1, \vec{p})}{g_2^2(\omega_1^2 - \omega_2^2)(W_2^2 - \omega_1^2)}, \quad (31a)$$

$$u^{(2)}(p)\bar{u}^{(2)}(p) = -\frac{F(\omega_2, \vec{p})}{g_2^2(\omega_1^2 - \omega_2^2)(W_1^2 - \omega_2^2)}, \quad (31b)$$

$$v^{(1)}(p)\bar{v}^{(1)}(p) = -\frac{F(-\omega_1, -\vec{p})}{g_2^2(\omega_1^2 - \omega_2^2)(W_2^2 - \omega_1^2)}, \quad (31c)$$

$$v^{(2)}(p)\bar{v}^{(2)}(p) = \frac{F(-\omega_2, -\vec{p})}{g_2^2(\omega_1^2 - \omega_2^2)(W_1^2 - \omega_2^2)}. \quad (31d)$$

The demonstration of the above relations are not difficult; however, we proceed to prove the first one with the other ones following similarly. From (28c) and (28d) we find

$$N\bar{N} = p^2 - m^2 - g_2^2 p_0^4 + 2g_2 p_0^2 p_i \gamma^i \gamma_0 \gamma_5, \quad (32)$$

and evaluated in the mode ω_1

$$(N\bar{N})_{p_0=\omega_1} = 2g_2\omega_1^2|\vec{p}|\left(1 + \frac{p_i\gamma^i\gamma_0\gamma_5}{|\vec{p}|}\right). \tag{33}$$

From (23) we have

$$(N\bar{N})_{p_0=\omega_1} = 4g_2\omega_1^2|\vec{p}|\xi^{(+)}(\vec{p})\xi^{(+)\dagger}(\vec{p}). \tag{34}$$

On the other hand, working directly with the eigenspinors, we can prove that

$$u^{(1)}(p)\bar{u}^{(1)}(p) = (M)_{p_0=\omega_1}\xi^{(+)}(\vec{p})\xi^{(+)\dagger}(\vec{p}), \tag{35}$$

and hence replacing (34) above and using $(p^2 - m^2 - g_2^2 p_0^4)_{p_0=\omega_1} = 2g_2\omega_1^2|\vec{p}|$ and (30a), we have the relation

$$u^{(1)}(p)\bar{u}^{(1)}(p) = \frac{(MN\bar{N})_{p_0=\omega_1}}{g_2^2(\omega_1^2 - \omega_2^2)(W_2^2 - \omega_1^2)}. \tag{36}$$

A detailed derivation of the previous expressions, including the eigenspinors, the propagator, and the consistency of the pole prescription can be found in [31].

B. The QED model

Consider the higher-order Lorentz-violating QED that arises by performing the minimal coupling substitution in the modified MP fermion Lagrangian (4) and coupling to the Maxwell Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi + \frac{1}{m_{\text{Pl}}}\bar{\psi}(\eta_1\not{n} + \eta_2\not{n}\gamma_5)(D\cdot n)^2\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{37}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength tensor. The local gauge invariance of the Lagrangian (37) can be checked by using the gauge transformations of the fields

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x), \tag{38a}$$

$$\psi'(x) = e^{-ie\lambda(x)}\psi(x), \tag{38b}$$

and the induced transformations for the derivative terms

$$D_\mu\psi' = e^{-ie\lambda}D_\mu\psi, \tag{38c}$$

$$D_\alpha D_\mu\psi' = e^{-ie\lambda}D_\alpha D_\mu\psi. \tag{38d}$$

We can always choose to work in the axial gauge where the gauge field is imposed to fulfill the relation $A\cdot n = 0$. This choice is advantageous, since we arrive at the usual QED interaction, and so is the one we use throughout this work.

Considering the restricted C-even part and timelike fermion sector produces the modified QED Lagrangian

$$\mathcal{L}'_{\text{QED}} = \bar{\psi}(i\not{\partial} + e\not{A} - m)\psi + g_2\bar{\psi}\gamma_0\gamma_5\ddot{\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{39}$$

C. Indefinite metric theories

Here we briefly discuss the general aspects of the unitarity of theories that have regular and indefinite metric η.

Consider a complex vector space $\mathcal{F}$ with vector basis $\{|\alpha\rangle\} \in \mathcal{B}$ and metric

$$\eta_{\alpha\beta} \coloneqq \langle\alpha|\beta\rangle. \tag{40}$$

We assume the metric not to be positive definite, so in principle each element $\eta_{\alpha\beta}$ may have a positive or negative value.

A representation of the identity operator is

$$\mathbb{I} = \sum_{\alpha,\beta\in\mathcal{B}} |\alpha\rangle\eta^{-1}_{\alpha\beta}\langle\beta|. \tag{41}$$

The matrix elements of an arbitrary linear operator $\mathcal{U}$ are defined as $\widetilde{U}_{\alpha\beta} \coloneqq \langle\alpha|\mathcal{U}|\beta\rangle$, and the unitarity condition for $\mathcal{U}$ is the requirement that the linear transformation leaves the inner product invariant, i.e.,

$$\langle\alpha|\beta\rangle = \langle\mathcal{U}\alpha|\mathcal{U}\beta\rangle = \langle\alpha|\mathcal{U}^\dagger\mathcal{U}|\beta\rangle, \tag{42}$$

or in terms of the matrix elements written as [21,22]

$$\eta_{\alpha\beta} = \sum_{\alpha',\beta'}\langle\alpha|\mathcal{U}^\dagger|\alpha'\rangle\eta^{-1}_{\alpha'\beta'}\langle\beta'|\mathcal{U}|\beta\rangle = \sum_{\alpha',\beta'}\widetilde{U}^\dagger_{\alpha\alpha'}\eta^{-1}_{\alpha'\beta'}\widetilde{U}_{\beta'\beta}, \tag{43}$$

where we have used (41). In matrix notation, we have the expression

$$\widetilde{U}^\dagger\eta^{-1}\widetilde{U} = \eta \tag{44}$$

for the pseudo-unitarity condition in the presence of an indefinite metric η. For a theory with an indefinite metric we should expect to have an evolution operator $\widetilde{U}$ to have this property. However, the inner product cannot be interpreted as a probability amplitude, as it can only have a meaningful interpretation in the positive metric sector. Then, if $\mathcal{U}$ stands for the time evolution, we can write

$$\mathcal{U} = \mathcal{I} + i\mathcal{T}, \tag{45}$$

or in terms of matrix elements, by projecting the previous equation on $\langle\alpha|$ and $|\beta\rangle$

$$\widetilde{U}_{\alpha\beta} = \eta_{\alpha\beta} + i\widetilde{T}_{\alpha\beta}, \tag{46}$$

where the $\widetilde{T}$ is the transition matrix (or S matrix) for a dynamically evolving state. So, the condition of the invariance of the inner product becomes, in terms of matrix elements,

$$-i(\widetilde{T} - \widetilde{T}^{\dagger}) = \widetilde{T}^{\dagger}\eta^{-1}\widetilde{T}. \tag{47}$$

The diagonal part of this matrix equation can be written as

$$2\mathrm{Im}(\widetilde{T})_{\alpha\alpha} = (\widetilde{T}^{\dagger}\eta^{-1}\widetilde{T})_{\alpha\alpha}, \tag{48}$$

which is the pseudo-unitarity version of the optical theorem.

Now, let us write all in terms of a basis buildup of physical particles and ghost states, where any state of the basis is of the form

$$|\alpha\rangle = |p_1, r_1\rangle \otimes |p_2, r_2\rangle \otimes \cdots \otimes |p_N, r_N\rangle \otimes |\widetilde{q}_1, s_1\rangle \otimes |\widetilde{q}_2, s_2\rangle \otimes \cdots \otimes |\widetilde{q}_M, s_M\rangle, \tag{49}$$

where p_i, r_i are momentum and other quantum numbers of physical particles and $\widetilde{q}_j$, s_j are momentum and other quantum numbers of ghost. Impose one-particle and ghost states being orthogonal,

$$\langle p_i, r_i | p_j, r_j\rangle = N_i \delta^3(p_i - p_j)\delta_{r_i r_j}, \tag{50}$$

$$\langle \widetilde{q}_i, s_i | \widetilde{q}_j, s_j\rangle = -\mathcal{N}_i \delta^3(\widetilde{q}_i - \widetilde{q}_j)\delta_{s_i s_j}, \tag{51}$$

and

$$\langle p_i, r_i | \widetilde{q}_j, s_j\rangle = 0, \tag{52}$$

with N_i and $\mathcal{N}_i$ positive normalization constants. From this basis, it is easy to build up the matrix elements of η. The choice of $N_i = \mathcal{N}_i = 1$ simplifies the expressions because in this case we have that $\eta_{\alpha\alpha} = \pm 1$ and then, $\eta^{-1} = \eta$. In the rest of this section, we use this choice for simplicity.

Following the Lee-Wick prescription, only states with particles will be considered as asymptotic states. So, it is convenient to split the space of states, $\mathcal{F}$, in two orthogonal subspaces,

$$\mathcal{F} = \mathcal{V}^{+} \oplus \mathcal{V}^{-}.$$

The first one, the sector spanned by the physical particle states, which we call the physical sector, $\mathcal{V}^{+}$, is generated by the basis $\mathcal{B}^{+}$,

$$\mathcal{B}^{+} = \{|p_1, r_1\rangle \otimes |p_2, r_2\rangle \otimes \cdots \otimes |p_N, r_N\rangle\}_{p_i, r_i, M}, \tag{53}$$

and the unphysical space, $\mathcal{V}^{-}$, its orthogonal complement, which is spanned by $\mathcal{B}^{-}$, is given by ghost particle states.

The interpretation of probability amplitudes of the inner product is meaningful only for the physical sector $\mathcal{V}^{+}$. So, pseudo-unitarity of time evolution is compatible with probability conservation if restricted to the physical sector where one has the standard unitarity relation

$$2\mathrm{Im}(\langle \mathrm{phys} \to \mathrm{phys}\rangle) = |\langle \mathrm{phys} \to \mathrm{any\ phys}\rangle|^2. \tag{54}$$

Our pseudo-unitarity condition, however, restricted to physical state $\alpha \in \mathcal{V}^{+}$ is

$$2\mathrm{Im}(T_{\alpha\to\alpha}) = \sum_{\beta\in\mathcal{B}^{+}} |T_{\alpha\to\beta}|^2 + \sum_{\gamma\in\mathcal{B}^{-}} \eta_{\gamma\gamma}|T_{\alpha\to\gamma}|^2, \tag{55}$$

which is just the standard optical theorem if the last term vanishes. So the conclusion is that probability is conserved if and only if the transitions between physical states and nonphysical states always vanish. In this case, the probability is well defined and unitarity can be realized in the theory.

IV. ONE-LOOP DIAGRAMS

As demonstrated in the previous section, the physical S-matrix evolves unitarily if the last term in Eq. (55) is zero. In the following section, we investigate whether this condition can be satisfied in our modified QED (39). We focus on the two relevant $2 \to 2$ scattering processes being the one-loop Bhabha and Compton reactions. In particular, we focus on the diagrams depicted in Figs. 1 and 3. It is worth noting that in the forward scattering processes under consideration, the corresponding diagrams in the t-channel have zero imaginary part. This is because the virtual momenta involved are proportional to the difference between incoming and outgoing momenta, which is zero in our forward scattering case. As a result, there is no branch cut singularity in the spatial integral. A similar result can be obtained in the self-interacting $\lambda\phi^4$ model in the u and t channels [32].

It is important to note, however, that there are additional contributions to the optical theorem from diagrams involving more photons and fermions internal lines in both the Bhabha and the Compton processes. For the present

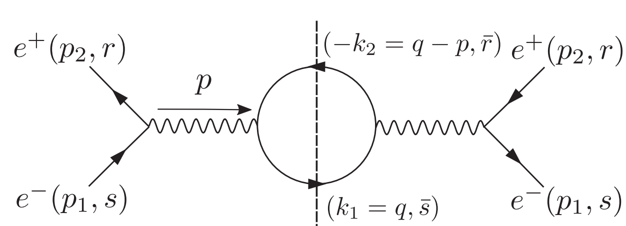

FIG. 1. The one-loop Bhabha scattering process $e^+e^- \to e^+e^-$ and the cut diagram of the right-hand side of Eq. (55) indicated with the vertical segmented line.

analysis, we will not consider these contributions and leave them for future work.

A. One-loop Bhabha scattering

We start with the electron and positron forward scattering reaction $e^-(p_1, s) + e^+(p_2, r) \to e^-(p_1, s) + e^+(p_2, r)$ (see Fig. 1).

The amplitude is given by

$$i\mathcal{M}_F^{(1)} = (-1)(\bar{v}^r(p_2)(-ie\gamma^\mu)u^s(p_1))\left(\frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}\right) \times \int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}[(-ie\gamma^\rho)S_F(q-p)(-ie\gamma^\nu)S_F(q)] \times \left(\frac{-i\eta_{\rho\sigma}}{p^2 + i\epsilon}\right)(\bar{u}^s(p_1)(-ie\gamma^\sigma)v^r(p_2)). \tag{56}$$

In terms of the propagator in (25), and the currents

$$J_1^\mu(p_1, p_2) \coloneqq \bar{v}^r(p_2)\gamma^\mu u^s(p_1), \tag{57a}$$

$$J_2^\mu(p_1, p_2) \coloneqq \bar{u}^s(p_1)\gamma^\mu v^r(p_2) = [J_1^\mu(p_1, p_2)]^*, \tag{57b}$$

we write

$$\mathcal{M}_F^{(1)} = \frac{ie^4}{(p^2 + i\epsilon)^2}J_1^\mu(p_1, p_2)J_2^\nu(p_1, p_2) \times \int \frac{d^3\vec{q}}{(2\pi)^4}\oint_{\mathcal{C}^{(1)}} dq_0 \mathrm{Tr}\left[\gamma_\nu \frac{F(q-p)}{D_{q-p}}\gamma_\mu \frac{F(q)}{D_q}\right]. \tag{58}$$

The poles in the complex q_0-plane are dominated by the quantities

$$\frac{1}{D_q} = \frac{1}{g_2^4}\prod_{s=1,2}\left[\frac{1}{(q_0 + \omega_s - i\epsilon)(q_0 - \omega_s + i\epsilon)} \times \frac{1}{(q_0 + W_s - i\epsilon)(q_0 - W_s + i\epsilon)}\right] \tag{59a}$$

and

$$\frac{1}{D_{q-p}} = \frac{1}{g_2^4}\prod_{s=1,2}\left[\frac{1}{(q_0 - p_0 + \overline{\omega}_s - i\epsilon)(q_0 - p_0 - \overline{\omega}_s + i\epsilon)} \times \frac{1}{(q_0 - p_0 + \overline{W}_s - i\epsilon)(q_0 - p_0 - \overline{W}_s + i\epsilon)}\right], \tag{59b}$$

where we introduce the notation $\overline{x} = x(\vec{q} - \vec{p})$. Let us focus on the last integral in (58)

$$I_{\mu\nu}^{(1)} = \oint_{\mathcal{C}^{(1)}} dq_0 \mathrm{Tr}[\gamma_\nu F(q-p)\gamma_\mu F(q)]\frac{1}{D_{q-p}D_q}. \tag{60}$$

To compute the integral we close the contour of integration in the lower half-plane with the curve $\mathcal{C}^{(1)}$ enclosing the eight poles

$$q_1 = \omega_1 - i\epsilon, \quad q_2 = W_1 - i\epsilon, \quad q_3 = \omega_2 - i\epsilon, \quad q_4 = W_2 - i\epsilon, \tag{61}$$

and the displaced ones

$$q_5 = p_0 + \overline{\omega}_1 - i\epsilon, \quad q_6 = p_0 + \overline{W}_1 - i\epsilon, \quad q_7 = p_0 + \overline{\omega}_2 - i\epsilon, \quad q_8 = p_0 + \overline{W}_2 - i\epsilon, \tag{62}$$

as indicated in Fig. 2.

The integral is

$$I_{\mu\nu}^{(1)} = -2\pi i\sum_{i=1}^{8}\mathrm{Tr}[\gamma_\nu F(q-p)\gamma_\mu F(q)]_{q_0=q_i}\mathrm{Res}(q_i), \tag{63}$$

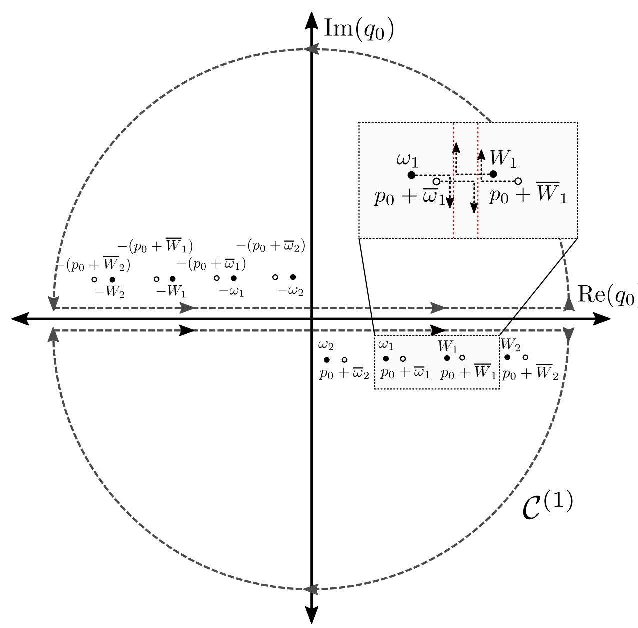

FIG. 2. The poles in the lower half-plane are displaced with the $-i\epsilon$ prescription, while those in the upper half-plane are displaced with the $i\epsilon$ prescription. According to the explanation as part of Eq. (13) the poles q_1, q_2 and q_5, q_6 take an imaginary component beyond the momentum value $|\vec{p}|_{\max}$ and $|\vec{q} - \vec{p}|_{\max}$, respectively. At these values, the poles q_1, q_5 move downwards and the poles q_2, q_6 move upwards parallel to the imaginary axis as indicated in the zoom region.

where we denote $\mathrm{Res}(q_i)$ the residue at q_i of the singular part $\frac{1}{D_q D_{q-p}}$. A direct calculation gives

$$\mathrm{Res}(q_1) = \frac{1}{2g_2^4\omega_1(\omega_1^2 - W_1^2)(\omega_1^2 - \omega_2^2)(\omega_1^2 - W_2^2)} \times \left(\frac{1}{D_{q-p}}\right)_{q_0=q_1}, \tag{64a}$$

$$\mathrm{Res}(q_2) = \frac{1}{2g_2^4 W_1(W_1^2 - \omega_1^2)(W_1^2 - \omega_2^2)(W_1^2 - W_2^2)} \times \left(\frac{1}{D_{q-p}}\right)_{q_0=q_2}, \tag{64b}$$

$$\mathrm{Res}(q_3) = \frac{1}{2g_2^4\omega_2(\omega_2^2 - \omega_1^2)(\omega_2^2 - W_1^2)(\omega_2^2 - W_2^2)} \times \left(\frac{1}{D_{q-p}}\right)_{q_0=q_3}, \tag{64c}$$

$$\mathrm{Res}(q_4) = \frac{1}{2g_2^4 W_2(W_2^2 - \omega_1^2)(W_2^2 - \omega_2^2)(W_2^2 - W_1^2)} \times \left(\frac{1}{D_{q-p}}\right)_{q_0=q_4}, \tag{64d}$$

and

$$\mathrm{Res}(q_5) = \frac{1}{2g_2^4\overline{\omega}_1(\overline{\omega}_1^2 - \overline{W}_1^2)(\overline{\omega}_1^2 - \overline{\omega}_2^2)(\overline{\omega}_1^2 - \overline{W}_2^2)} \times \left(\frac{1}{D_q}\right)_{q_0=q_5}, \tag{65a}$$

$$\mathrm{Res}(q_6) = \frac{1}{2g_2^4\overline{W}_1(\overline{W}_1^2 - \overline{\omega}_1^2)(\overline{W}_1^2 - \overline{\omega}_2^2)(\overline{W}_1^2 - \overline{W}_2^2)} \times \left(\frac{1}{D_q}\right)_{q_0=q_6}, \tag{65b}$$

$$\mathrm{Res}(q_7) = \frac{1}{2g_2^4\overline{\omega}_2(\overline{\omega}_2^2 - \overline{\omega}_1^2)(\overline{\omega}_2^2 - \overline{W}_1^2)(\overline{\omega}_2^2 - \overline{W}_2^2)} \times \left(\frac{1}{D_q}\right)_{q_0=q_7}, \tag{65c}$$

$$\mathrm{Res}(q_8) = \frac{1}{2g_2^4\overline{W}_2(\overline{W}_2^2 - \overline{\omega}_1^2)(\overline{W}_2^2 - \overline{\omega}_2^2)(\overline{W}_2^2 - \overline{W}_1^2)} \times \left(\frac{1}{D_q}\right)_{q_0=q_8}, \tag{65d}$$

where we have eliminated the $i\epsilon$ where it is not relevant.

We consider p_0 to be positive, and hence the last four terms above do not contribute to the amplitude's discontinuity or imaginary part. Decomposing in a partial fraction the relevant contributions come from

$$\left(\frac{1}{D_{q-p}}\right)_{q_0=q_1} = \frac{1}{g_2^4}\prod_{s=1,2}\frac{\zeta(\omega_1)}{(p_0 - \omega_1 + \overline{\omega}_s)(p_0 - \omega_1 + \overline{W}_s)}, \tag{66a}$$

$$\left(\frac{1}{D_{q-p}}\right)_{q_0=q_2} = \frac{1}{g_2^4}\prod_{s=1,2}\frac{\zeta(W_1)}{(p_0 - W_1 + \overline{\omega}_s)(p_0 - W_1 + \overline{W}_s)}, \tag{66b}$$

$$\left(\frac{1}{D_{q-p}}\right)_{q_0=q_3} = \frac{1}{g_2^4}\prod_{s=1,2}\frac{\zeta(\omega_2)}{(p_0 - \omega_2 + \overline{\omega}_s)(p_0 - \omega_2 + \overline{W}_s)}, \tag{66c}$$

$$\left(\frac{1}{D_{q-p}}\right)_{q_0=q_4} = \frac{1}{g_2^4}\prod_{s=1,2}\frac{\zeta(W_2)}{(p_0 - W_2 + \overline{\omega}_s)(p_0 - W_2 + \overline{W}_s)}, \tag{66d}$$

with

$$\begin{aligned}\zeta(x) = {} & \frac{1}{(\overline{\omega}_1 - \overline{W}_1)(\overline{\omega}_1 - \overline{\omega}_2)(\overline{\omega}_1 - \overline{W}_2)(p_0 - x - \overline{\omega}_1 + i\epsilon)} \\ & + \frac{1}{(\overline{W}_1 - \overline{\omega}_1)(\overline{W}_1 - \overline{\omega}_2)(\overline{W}_1 - \overline{W}_2)(p_0 - x - \overline{W}_1 + i\epsilon)} \\ & + \frac{1}{(\overline{\omega}_2 - \overline{\omega}_1)(\overline{\omega}_2 - \overline{W}_1)(\overline{\omega}_2 - \overline{W}_2)(p_0 - x - \overline{\omega}_2 + i\epsilon)} \\ & + \frac{1}{(\overline{W}_2 - \overline{\omega}_1)(\overline{W}_2 - \overline{W}_1)(\overline{W}_2 - \overline{\omega}_2)(p_0 - x - \overline{W}_2 + i\epsilon)}.\end{aligned} \tag{67}$$

Let us consider the identity

$$\frac{1}{x \pm i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x), \tag{68}$$

where $\mathcal{P}$ denotes the principal value. The contributions to the imaginary part of the scattering amplitude are

$$\begin{aligned}\mathrm{Im}\left(\frac{1}{D_{q-p}}\right)_{q_0=q_1} = -\frac{\pi}{g_2^4}\Bigg[& \frac{\delta(p_0 - \omega_1 - \overline{\omega}_1)}{2\overline{\omega}_1(\overline{\omega}_1^2 - \overline{W}_1^2)(\overline{\omega}_1^2 - \overline{\omega}_2^2)(\overline{\omega}_1^2 - \overline{W}_2^2)} \\ & + \frac{\delta(p_0 - \omega_1 - \overline{W}_1)}{2\overline{W}_1(\overline{W}_1^2 - \overline{\omega}_1^2)(\overline{W}_1^2 - \overline{\omega}_2^2)(\overline{W}_1^2 - \overline{W}_2^2)} \\ & + \frac{\delta(p_0 - \omega_1 - \overline{\omega}_2)}{2\overline{\omega}_2(\overline{\omega}_2^2 - \overline{\omega}_1^2)(\overline{\omega}_2^2 - \overline{W}_1^2)(\overline{\omega}_2^2 - \overline{W}_2^2)} \\ & + \frac{\delta(p_0 - \omega_1 - \overline{W}_2)}{2\overline{W}_2^2(\overline{W}_2^2 - \overline{\omega}_1^2)(\overline{W}_2^2 - \overline{W}_1^2)(\overline{W}_2^2 - \overline{\omega}_2^2)}\Bigg],\end{aligned} \tag{69a}$$

$$\mathrm{Im}\left(\frac{1}{D_{q-p}}\right)_{q_0=q_2} = -\frac{\pi}{g_2^4}\left[\frac{\delta(p_0-W_1-\overline{\omega}_1)}{2\overline{\omega}_1(\overline{\omega}_1^2-\bar{W}_1^2)(\overline{\omega}_1^2-\bar{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\delta(p_0-W_1-\overline{W}_1)}{2\overline{W}_1(\overline{W}_1^2-\overline{\omega}_1^2)(\overline{W}_1^2-\overline{\omega}_2^2)(\overline{W}_1^2-\overline{W}_2^2)} + \frac{\delta(p_0-W_1-\overline{\omega}_2)}{2\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)} + \frac{\delta(p_0-W_1-\overline{W}_2)}{2\overline{W}_2(\overline{W}_2^2-\overline{\omega}_1^2)(\overline{W}_2^2-\overline{W}_1^2)(\overline{W}_2^2-\overline{\omega}_2^2)}\right], \tag{69b}$$

$$\mathrm{Im}\left(\frac{1}{D_{q-p}}\right)_{q_0=q_3} = -\frac{\pi}{g_2^4}\left[\frac{\delta(p_0-\omega_2-\overline{\omega}_1)}{2\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\delta(p_0-\omega_2-\overline{W}_1)}{2\overline{W}_1(\overline{W}_1^2-\overline{\omega}_1^2)(\overline{W}_1^2-\overline{\omega}_2^2)(\overline{W}_1^2-\overline{W}_2^2)} + \frac{\delta(p_0-\omega_2-\overline{\omega}_2)}{2\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)} + \frac{\delta(p_0-\omega_2-\overline{W}_2)}{2\overline{W}_2(\overline{W}_2^2-\overline{\omega}_1^2)(\overline{W}_2^2-\overline{W}_1^2)(\overline{W}_2^2-\overline{\omega}_2^2)}\right], \tag{69c}$$

$$\mathrm{Im}\left(\frac{1}{D_{q-p}}\right)_{q_0=q_4} = -\frac{\pi}{g_2^4}\left[\frac{\delta(p_0-W_2-\overline{\omega}_1)}{2\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\delta(p_0-W_2-\overline{W}_1)}{2\overline{W}_1(\overline{W}_1^2-\overline{\omega}_1^2)(\overline{W}_1^2-\overline{\omega}_2^2)(\overline{W}_1^2-\overline{W}_2^2)} + \frac{\delta(p_0-W_2-\overline{\omega}_2)}{2\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)} + \frac{\delta(p_0-W_2-\overline{W}_2)}{2\overline{W}_2(\overline{W}_2^2-\overline{\omega}_1^2)(\overline{W}_2^2-\overline{W}_1^2)(\overline{W}_2^2-\overline{\omega}_2^2)}\right]. \tag{69d}$$

In principle, some contributions depend on deltas involving ghost modes, as seen in (69a)–(69d). However, these contributions demand an energy of the order $1/g_2$, which lies far beyond the region of validity of the effective theory. Hence, the deltas involving a $W_{1,2}$ or $\overline{W}_{1,2}$ mode vanish; in other words, the initial $p_0 = \omega_s + \omega_r$ cannot balance the energetic restriction given by these deltas so we disregard them. In this way, we are left with the contributions

$$\begin{aligned} 2\mathrm{Im}(\mathcal{M}_F^{(1)}) = \frac{-e^4}{p^4} J_1^\mu(p_1,p_2) J_2^\nu(p_1,p_2) \int \frac{d^3\vec{q}}{(2\pi)^4}(2\pi)^2 \Bigg[&\frac{\mathrm{Tr}[\gamma_\nu F(\omega_1-p_0,\vec{q}-\vec{p})\gamma_\mu F(\omega_1,\vec{q})]}{2g_2^4\omega_1(\omega_1^2-W_1^2)(\omega_1^2-\omega_2^2)(\omega_1^2-W_2^2)} \\ &\times\left(\frac{\delta(p_0-\omega_1-\overline{\omega}_1)}{2g_2^4\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\delta(p_0-\omega_1-\overline{\omega}_2)}{2g_2^4\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)}\right) \\ &+ \frac{\mathrm{Tr}[\gamma_\nu F(\omega_2-p_0,\vec{q}-\vec{p})\gamma_\mu F(\omega_2,\vec{q})]}{2g_2^4\omega_2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2)(\omega_2^2-W_2^2)}\left(\frac{\delta(p_0-\omega_2-\overline{\omega}_1)}{2g_2^4\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} \right. \\ &\left. + \frac{\delta(p_0-\omega_2-\overline{\omega}_2)}{2g_2^4\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)}\right)\Bigg]. \end{aligned} \tag{70}$$

We introduce the variables k_1^0, k_2^0 followed by delta functions as follows:

$$\begin{aligned} 2\mathrm{Im}(\mathcal{M}_F^{(1)}) = \frac{-e^4}{p^4} J_1^\mu(p_1,p_2) J_2^\nu(p_1,p_2) \int \frac{d^3q}{(2\pi)^4}(2\pi)^2 \int dk_1^0 \int dk_2^0 \delta(p_0-k_1^0-k_2^0) \Bigg[&\frac{\mathrm{Tr}[\gamma_\nu F(-\overline{\omega}_1,\vec{q}-\vec{p})\gamma_\mu F(\omega_1,\vec{q})]}{2g_2^4\omega_1(\omega_1^2-W_1^2)(\omega_1^2-\omega_2^2)(\omega_1^2-W_2^2)} \\ \times \frac{\delta(k_1^0-\omega_1)\delta(k_2^0-\overline{\omega}_1)}{2g_2^4\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\mathrm{Tr}[\gamma_\nu F(-\overline{\omega}_2,\vec{q}-\vec{p})\gamma_\mu F(\omega_1,\vec{q})]}{2g_2^4\omega_1(\omega_1^2-W_1^2)(\omega_1^2-\omega_2^2)(\omega_1^2-W_2^2)} \\ \times \frac{\delta(k_1^0-\omega_1)\delta(k_2^0-\overline{\omega}_2)}{2g_2^4\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)} + \frac{\mathrm{Tr}[\gamma_\nu F(-\overline{\omega}_1,\vec{q}-\vec{p})\gamma_\mu F(\omega_2,\vec{q})]}{2g_2^4\omega_2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2)(\omega_2^2-W_2^2)} \\ \times \frac{\delta(k_1^0-\omega_2)\delta(k_2^0-\overline{\omega_1})}{2g_2^4\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\mathrm{Tr}[\gamma_\nu F(-\overline{\omega}_2,\vec{q}-\vec{p})\gamma_\mu F(\omega_2,\vec{q})]}{2g_2^4\omega_2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2)(\omega_2^2-W_2^2)} \\ \times \frac{\delta(k_1^0-\omega_2)\delta(k_2^0-\overline{\omega}_2)}{2g_2^4\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)}\Bigg], \end{aligned} \tag{71}$$

and then by using

$$\int \frac{d^3q}{(2\pi)^3} = \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k}_1-\vec{k}_2) \tag{72}$$

and defining $\vec{k}_1 = \vec{q}$, $\vec{k}_2 = \vec{p}-\vec{q}$ we write

$$\begin{aligned}2\mathrm{Im}(\mathcal{M}_F^{(1)}) = \frac{-e^4}{p^4}J_1^\mu(p_1,p_2)J_2^\nu(p_1,p_2)\int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-k_1-k_2)\Bigg[\frac{\mathrm{Tr}[\gamma_\nu F(-k_2^0,-\vec{k}_2)\gamma_\mu F(k_1^0,\vec{k}_1)]}{2g_2^4\omega_1(\omega_1^2-W_1^2)(\omega_1^2-\omega_2^2)(\omega_1^2-W_2^2)} \\ \times\frac{(2\pi)^2\delta(k_1^0-\omega_1)\delta(k_2^0-\overline{\omega}_1)}{2g_2^4\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\mathrm{Tr}[\gamma_\nu F(-k_2^0,-\vec{k}_2)\gamma_\mu F(k_1^0,\vec{k}_1)]}{2g_2^4\omega_1(\omega_1^2-W_1^2)(\omega_1^2-\omega_2^2)(\omega_1^2-W_2^2)} \\ \times\frac{(2\pi)^2\delta(k_1^0-\omega_1)\delta(k_2^0-\overline{\omega}_2)}{2g_2^4\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)} + \frac{\mathrm{Tr}[\gamma_\nu F(-k_2^0,-\vec{k}_2)\gamma_\mu F(k_1^0,\vec{k}_1)]}{2g_2^4\omega_2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2)(\omega_2^2-W_2^2)} \\ \times\frac{(2\pi)^2\delta(k_1^0-\omega_2)\delta(k_2^0-\overline{\omega}_1)}{2g_2^4\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)(\overline{\omega}_1^2-\overline{\omega}_2^2)(\overline{\omega}_1^2-\overline{W}_2^2)} + \frac{\mathrm{Tr}[\gamma_\nu F(-k_2^0,-\vec{k}_2)\gamma_\mu F(k_1^0,\vec{k}_1)]}{2g_2^4\omega_2(\omega_2^2-\omega_1^2)(\omega_2^2-W_1^2)(\omega_2^2-W_2^2)} \\ \times\frac{(2\pi)^2\delta(k_1^0-\omega_2)\delta(k_2^0-\overline{\omega}_2)}{2g_2^4\overline{\omega}_2(\overline{\omega}_2^2-\overline{\omega}_1^2)(\overline{\omega}_2^2-\overline{W}_1^2)(\overline{\omega}_2^2-\overline{W}_2^2)}\Bigg].\end{aligned} \tag{73}$$

Now, we will relate the amplitude with the total cross section of the cutting diagram. To archive this connection, we recall the relations (31a)–(31d) and arrive at

$$\begin{aligned}2\mathrm{Im}(\mathcal{M}_F^{(1)}) = \frac{e^4}{p^4}J_1^\mu(p_1,p_2)J_2^\nu(p_1,p_2)\int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-k_1-k_2)\Bigg[\frac{\mathrm{Tr}[\gamma_\nu v^{(1)}(k_2)\overline{v}^{(1)}(k_2)\gamma_\mu u^{(1)}(k_1)\overline{u}^{(1)}(k_1)]}{2g_2^2\omega_1(\omega_1^2-W_1^2)} \\ \times\frac{(2\pi)^2\delta(k_1^0-\omega_1)\delta(k_2^0-\overline{\omega}_1)}{2g_2^2\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)} + \frac{\mathrm{Tr}[\gamma_\nu v^{(2)}(k_2)\overline{v}^{(2)}(k_2)\gamma_\mu u^{(1)}(k_1)\overline{u}^{(1)}(k_1)]}{2g_2^2\omega_1(\omega_1^2-W_1^2)}\frac{(2\pi)^2\delta(k_1^0-\omega_1)\delta(k_2^0-\overline{\omega}_2)}{2g_2^2\overline{\omega}_2(\overline{\omega}_2^2-\overline{W}_2^2)} \\ + \frac{\mathrm{Tr}[\gamma_\nu v^{(1)}(k_2)\overline{v}^{(1)}(k_2)\gamma_\mu u^{(2)}(k_1)\overline{u}^{(2)}(k_1)]}{2g_2^2\omega_2(\omega_2^2-W_2^2)}\frac{(2\pi)^2\delta(k_1^0-\omega_2)\delta(k_2^0-\overline{\omega}_1)}{2g_2^2\overline{\omega}_1(\overline{\omega}_1^2-\overline{W}_1^2)} \\ + \frac{\mathrm{Tr}[\gamma_\nu v^{(2)}(k_2)\overline{v}^{(2)}(k_2)\gamma_\mu u^{(2)}(k_1)\overline{u}^{(2)}(k_1)]}{2g_2^2\omega_2(\omega_2^2-W_2^2)}\frac{(2\pi)^2\delta(k_1^0-\omega_2)\delta(k_2^0-\overline{\omega}_2)}{2g_2^2\overline{\omega}_2(\overline{\omega}_2^2-\overline{W}_2^2)}\Bigg].\end{aligned} \tag{74}$$

At this point, it is convenient to define a physical delta where we will exclude the ghost frequencies

$$\delta^{(\mathrm{phys})}(\Lambda_s^2(p_0)) = \sum_{\mathrm{phys,a}}\frac{\delta(p_0-p_a)}{|(\Lambda_s^2)'(p_a)|}, \tag{75}$$

for $s = 1, 2$ with the new notation $\Lambda_1^2 \equiv \Lambda_+^2$ and $\Lambda_2^2 \equiv \Lambda_-^2$ and where p_a are the zeros of the function $\Lambda_s^2(p_0)$. In our case, we have

$$\delta^{(\mathrm{phys})}(\Lambda_s^2(p_0)) = \frac{\delta(p_0-\omega_s)-\delta(p_0+\omega_s)}{2\omega_s g_2^2(W_s^2-\omega_s^2)}. \tag{76}$$

This allows us to write the left-hand side of the cutting equation as

$$\begin{aligned}2\mathrm{Im}(\mathcal{M}_F^{(1)}) &= \frac{e^4}{p^4}J_1^\mu(p_1,p_2)J_2^\nu(p_1,p_2) \\ &\times\int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-k_1-k_2) \\ &\times\sum_{\bar{s},\bar{r}=1,2}\Big[\mathrm{Tr}[\gamma_\nu v^{\bar{r}}(k_2)\overline{v}^{\bar{r}}(k_2)\gamma_\mu u^{\bar{s}}(k_1)\overline{u}^{\bar{s}}(k_1)] \\ &\times(2\pi)^2\delta^{(\mathrm{phys})}(\Lambda_{\bar{s}}^2(k_1^0))\delta^{(\mathrm{phys})} \\ &\times(\Lambda_{\bar{r}}^2(k_2^0))\theta(k_1^0)\theta(k_2^0)\Big]\end{aligned} \tag{77}$$

and

$$\begin{aligned}2\mathrm{Im}(\mathcal{M}_F^{(1)}) &= \frac{e^4}{p^4}J_1^\mu(p_1,p_2)J_2^\nu(p_1,p_2) \\ &\times\int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-k_1-k_2) \\ &\times\sum_{\bar{s},\bar{r}=1,2}\Big[\overline{u}^{\bar{s}}(k_1)\gamma_\nu v^{\bar{r}}(k_2)\overline{v}^{\bar{r}}(k_2)\gamma_\mu u^{\bar{s}}(k_1)\Big] \\ &\times(2\pi)^2\delta^{(\mathrm{phys})}(\Lambda_{\bar{s}}^2(k_1^0))\delta^{(\mathrm{phys})} \\ &\times(\Lambda_{\bar{r}}^2(k_2^0))\theta(k_1^0)\theta(k_2^0)\Big].\end{aligned} \tag{78}$$

Finally, we can recognize the scattering amplitude for the cut diagram using the symmetry of the (μ,ν) indices

$$i\widehat{\mathcal{M}}^{(1)} \coloneqq [\overline{v}^r(p_2)(-ie\gamma^\mu)u^s(p_1)]\left(\frac{-i\eta_{\mu\nu}}{p^2+i\epsilon}\right) \times [\bar{u}^{\bar{s}}(k_1)(-ie\gamma^\nu)v^{\bar{r}}(-k_2)]. \tag{79}$$

This proves the optical theorem and the unitary evolution of the S-matrix for the one-loop Bhabha scattering diagram

$$2\mathrm{Im}(\mathcal{M}_F^{(1)}) = \sum_{\bar{s},\bar{r}=1,2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-k_1-k_2) \times |\widehat{\mathcal{M}}^{(1)}|^2(2\pi)^2\delta^{(\mathrm{phys})}(\Lambda_{\bar{s}}^2(k_1^0))\delta^{(\mathrm{phys})}(\Lambda_{\bar{r}}^2(k_2^0)) \times \theta(k_1^0)\theta(k_2^0). \tag{80}$$

B. One-loop Compton scattering

We continue with the one-loop Compton scattering process $e^-(p,s)+\gamma(k)\to e^-(p,s)+\gamma(k)$ of Fig. 3.

We have the amplitude

$$i\mathcal{M}_F^{(2)} = (-1)[\varepsilon_\rho^*(k)\bar{u}^s(p)(-ie\gamma^\rho)]\left(\frac{iF(p')}{D_{p'}}\right) \times \int\frac{d^4q}{(2\pi)^4}\left[(-ie\gamma^\mu)\frac{iF(p'-q)}{D_{p'-q}}(-ie\gamma^\nu)\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}\right] \times \left(\frac{iF(p')}{D_{p'}}\right)[(-ie\gamma^\sigma)u^s(p)\varepsilon_\sigma(k)]. \tag{81}$$

By defining the quantities

$$J(p,k) \coloneqq \frac{F(p')\gamma^\sigma u^s(p)\varepsilon_\sigma(k)}{D_{p'}}, \tag{82}$$

$$J^*(p,k) \coloneqq \frac{\varepsilon_\rho^*(k)\bar{u}^s(p)\gamma^\rho F(p')}{D_{p'}}, \tag{83}$$

we rewrite as

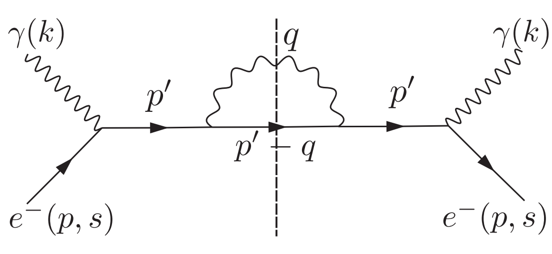

FIG. 3. The diagram representing the scattering process $\gamma e^-\to\gamma e^-$ and the cut diagram produced by the vertical segmented line.

$$\mathcal{M}_F^{(2)} = -ie^4J^*(p,k)\int\frac{d^3\vec{q}}{(2\pi)^4} \times \int dq_0\frac{\gamma^\mu F(p'-q)\gamma_\mu}{(q^2+i\epsilon)D_{p'-q}}J(p,k). \tag{84}$$

Consider the last integral

$$I^{(2)} = \int_{\mathcal{C}} dq_0\frac{F(p'-q)}{(q^2+i\epsilon)D_{p'-q}}, \tag{85}$$

where for the singular part of the fermion propagator $\frac{1}{D_{p'-q}}$ we have eight poles and for the photon part $\frac{1}{q^2+i\epsilon}$ we have two more (see Fig. 4).

To compute the integral, we employ the Cauchy residue theorem and consider the contour of integration $\mathcal{C}^{(2)}$ that closes from below, as shown in Fig. 4. The contour encloses five poles, and we obtain

$$I^{(2)} = -2\pi i\sum_{i=1}^{5}[F(p'-q)]_{q_0=q_i}\mathrm{Res}(q_i), \tag{86}$$

with

$$q_1 = |\vec{q}|-i\epsilon, \tag{87}$$

$$q_2 = p_0'+\widetilde{\omega}_1-i\epsilon, \tag{88}$$

$$q_3 = p_0'+\widetilde{\omega}_2-i\epsilon, \tag{89}$$

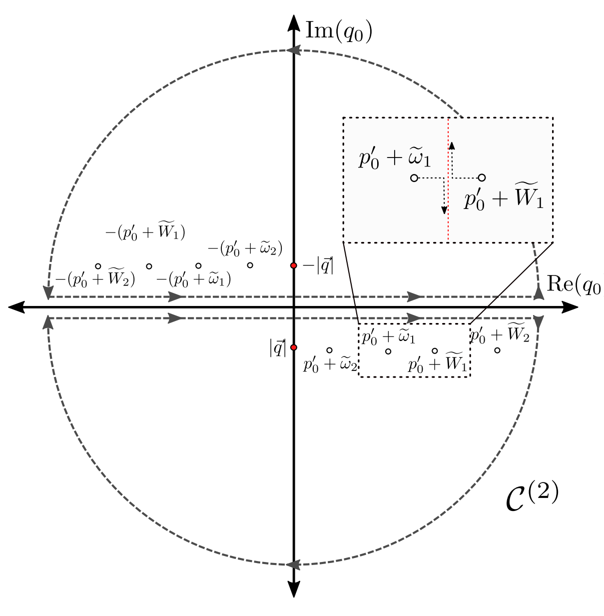

FIG. 4. The contour $\mathcal{C}^{(2)}$ encloses the poles q_1, q_2, q_3, q_4, q_5, and at the critical energy the two poles q_2 and q_4 become complex, as indicated in the figure.

$$q_4 = p'_0 + \widetilde{W}_1 - i\epsilon, \tag{90}$$

$$q_5 = p'_0 + \widetilde{W}_2 - i\epsilon, \tag{91}$$

with the residues of the singular part $\frac{1}{(q^2+i\epsilon)D_{p'-q}}$ at x denoted by $\mathrm{Res}(x)$.

A calculation gives

$$\begin{aligned}\mathrm{Res}(q_1) &= \frac{1}{2g_2^4|\vec{q}|(p'_0-|\vec{q}|+\widetilde{\omega}_1)(p'_0-|\vec{q}|+\widetilde{W}_1)} \\ &\times \frac{1}{(p'_0-|\vec{q}|+\widetilde{\omega}_2)(p'_0-|\vec{q}|+\widetilde{W}_2)(p'_0-|\vec{q}|-\widetilde{\omega}_1+2i\epsilon)} \\ &\times \frac{1}{(p'_0-|\vec{q}|-\widetilde{W}_1+2i\epsilon)(p'_0-|\vec{q}|-\widetilde{\omega}_2+2i\epsilon)} \\ &\times \frac{1}{(p'_0-|\vec{q}|-\widetilde{W}_2+2i\epsilon)},\end{aligned} \tag{92}$$

$$\mathrm{Res}(q_2) = \frac{1}{2g_2^4\widetilde{\omega}_1(\widetilde{W}_1^2-\widetilde{\omega}_1^2)(\widetilde{\omega}_2^2-\widetilde{\omega}_1^2)(\widetilde{W}_2^2-\widetilde{\omega}_1^2)} \times \frac{1}{(p'_0+\widetilde{\omega}_1-|\vec{q}|)(p'_0+\widetilde{\omega}_1+|\vec{q}|-2i\epsilon)}, \tag{93}$$

$$\mathrm{Res}(q_3) = \frac{1}{2g_2^4\widetilde{\omega}_2(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{W}_1^2-\widetilde{\omega}_2^2)(\widetilde{W}_2^2-\widetilde{\omega}_2^2)} \times \frac{1}{(p'_0+\widetilde{\omega}_2-|\vec{q}|)(p'_0+\widetilde{\omega}_2+|\vec{q}|-2i\epsilon)}, \tag{94}$$

$$\mathrm{Res}(q_4) = \frac{1}{2g_2^4\widetilde{W}_1(\widetilde{\omega}_1^2-\widetilde{W}_1^2)(\widetilde{\omega}_2^2-\widetilde{W}_1^2)(\widetilde{W}_2^2-\widetilde{W}_1^2)} \times \frac{1}{(p'_0+\widetilde{W}_1-|\vec{q}|)(p'_0+\widetilde{W}_1+|\vec{q}|-2i\epsilon)}, \tag{95}$$

$$\mathrm{Res}(q_5) = \frac{1}{2g_2^4\widetilde{W}_2(\widetilde{\omega}_1^2-\widetilde{W}_2^2)(\widetilde{W}_1^2-\widetilde{W}_2^2)(\widetilde{\omega}_2^2-\widetilde{W}_2^2)} \times \frac{1}{(p'_0+\widetilde{W}_2-|\vec{q}|)(p'_0+\widetilde{W}_2+|\vec{q}|-2i\epsilon)}. \tag{96}$$

Considering that $p'_0 \geq 0$ only $\mathrm{Res}(q_1)$ has poles that contribute to the discontinuity. We use the partial fraction decomposition for the relevant poles obtaining

$$\begin{aligned}\mathrm{Res}(q_1) = \frac{1}{2g_2^4|\vec{q}|}\prod_{i=1,2}\frac{1}{(p'_0-|\vec{q}|+\widetilde{\omega}_i)(p'_0-|\vec{q}|+\widetilde{W}_i)}\Bigg[&\frac{1}{(\widetilde{\omega}_1-\widetilde{\omega}_2)(\widetilde{\omega}_1-\widetilde{W}_1)(\widetilde{\omega}_1-\widetilde{W}_2)(p'_0-|\vec{q}|-\widetilde{\omega}_1+2i\epsilon)} \\ &-\frac{1}{(\widetilde{\omega}_1-\widetilde{\omega}_2)(\widetilde{\omega}_2-\widetilde{W}_1)(\widetilde{\omega}_2-\widetilde{W}_2)(p'_0-|\vec{q}|-\widetilde{\omega}_2+2i\epsilon)}+\frac{1}{(\widetilde{\omega}_1-\widetilde{W}_1)(\widetilde{\omega}_2-\widetilde{W}_1)(\widetilde{W}_1-\widetilde{W}_2)(p'_0-|\vec{q}|-\widetilde{W}_1+2i\epsilon)} \\ &-\frac{1}{(\widetilde{\omega}_1-\widetilde{W}_2)(\widetilde{\omega}_2-\widetilde{W}_2)(\widetilde{W}_1-\widetilde{W}_2)(p'_0-|\vec{q}|-\widetilde{W}_2+2i\epsilon)}\Bigg].\end{aligned} \tag{97}$$

Considering the expression (68) we have

$$\begin{aligned}\mathrm{Im}(\mathrm{Res}(q_1)) = \frac{-\pi}{2g_2^4|\vec{q}|}\Bigg[&\frac{\delta(p'_0-|\vec{q}|-\widetilde{\omega}_1)}{2\widetilde{\omega}_1(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_1^2-\widetilde{W}_1^2)(\widetilde{\omega}_1^2-\widetilde{W}_2^2)} \\ &-\frac{\delta(p'_0-|\vec{q}|-\widetilde{\omega}_2)}{2\widetilde{\omega}_2(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_2^2-\widetilde{W}_1^2)(\widetilde{\omega}_2^2-\widetilde{W}_2^2)} \\ &+\frac{\delta(p'_0-|\vec{q}|-\widetilde{W}_1)}{2\widetilde{W}_1(\widetilde{\omega}_1^2-\widetilde{W}_1^2)(\widetilde{\omega}_2^2-\widetilde{W}_1^2)(\widetilde{W}_1^2-\widetilde{W}_2^2)} \\ &-\frac{\delta(p'_0-|\vec{q}|-\widetilde{W}_2)}{2\widetilde{W}_2(\widetilde{\omega}_1^2-\widetilde{W}_2^2)(\widetilde{\omega}_2^2-\widetilde{W}_2^2)(\widetilde{W}_1^2-\widetilde{W}_2^2)}\Bigg].\end{aligned} \tag{98}$$

We apply effective theory again to consider the possible contributions that involve intermediate states of ghost modes. This leads us to write the left-hand side of the optical theorem as follows:

$$\begin{aligned}&2\mathrm{Im}(\mathcal{M}_F^{(2)}) \\ &= e^4[J^*(p,k)]\int\frac{d^3q}{(2\pi)^4}(2\pi)^2\frac{1}{2|\vec{q}|} \\ &\times\Bigg[\frac{\gamma^\mu F(\widetilde{\omega}_1,\vec{p'}-\vec{q})\gamma_\mu\delta(p'_0-|\vec{q}|-\widetilde{\omega}_1)}{2g_2^4\widetilde{\omega}_1(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_1^2-\widetilde{W}_1^2)(\widetilde{\omega}_1^2-\widetilde{W}_2^2)} \\ &-\frac{\gamma^\mu F(\widetilde{\omega}_2,\vec{p'}-\vec{q})\gamma_\mu\delta(p'_0-|\vec{q}|-\widetilde{\omega}_2)}{2g_2^4\widetilde{\omega}_2(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_2^2-\widetilde{W}_1^2)(\widetilde{\omega}_2^2-\widetilde{W}_2^2)}\Bigg][J(p,k)].\end{aligned} \tag{99}$$

We introduce the variables k_1^0 and k_2^0 followed by the delta functions as follows:

$$2\mathrm{Im}(\mathcal{M}_F^{(2)}) = e^4[J^*(p,k)]\int\frac{d^3q}{(2\pi)^4}(2\pi)^2\frac{1}{2|\vec{q}|}\times\int dk_1^0\int dk_2^0\delta(p_0{}'-k_1^0-k_2^0)\times\left[\frac{\gamma^\mu F(\widetilde{\omega}_1,\vec{p}'-\vec{q})\gamma_\mu\delta(k_1^0-|\vec{q}|)\delta(k_2^0-\widetilde{\omega}_1)}{2g_2^4\widetilde{\omega}_1(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_1^2-\widetilde{W}_1^2)(\widetilde{\omega}_1^2-\widetilde{W}_2^2)}-\frac{\gamma^\mu F(\widetilde{\omega}_2,\vec{p}'-\vec{q})\gamma_\mu\delta(k_1^0-|\vec{q}|)\delta(k_2^0-\widetilde{\omega}_2)}{2g_2^4\widetilde{\omega}_2(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_2^2-\widetilde{W}_1^2)(\widetilde{\omega}_2^2-\widetilde{W}_2^2)}\right]\times[J(p,k)], \tag{100}$$

and then by using the same previous identity (72) and defining $\vec{k}_1=\vec{q},\vec{k}_2=\vec{p}'-\vec{q}$ we write

$$2\mathrm{Im}(\mathcal{M}_F^{(2)}) = e^4[J^*(p,k)][J(p,k)]\times\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\frac{(2\pi)^4}{2|\vec{k}_1|}\delta^{(4)}(p'-k_1-k_2)\times\left[\frac{\gamma^\mu F(k_2^0,\vec{k}_2)\gamma_\mu(2\pi)^2\delta(k_1^0-|\vec{k}_1|)\delta(k_2^0-\widetilde{\omega}_1)}{2g_2^4\widetilde{\omega}_1(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_1^2-\widetilde{W}_1^2)(\widetilde{\omega}_1^2-\widetilde{W}_2^2)}-\frac{\gamma^\mu F(k_2^0,\vec{k}_2)\gamma_\mu(2\pi)^2\delta(k_1^0-|\vec{k}_1|)\delta(k_2^0-\widetilde{\omega}_2)}{2g_2^4\widetilde{\omega}_2(\widetilde{\omega}_1^2-\widetilde{\omega}_2^2)(\widetilde{\omega}_2^2-\widetilde{W}_1^2)(\widetilde{\omega}_2^2-\widetilde{W}_2^2)}\right]. \tag{101}$$

Now using the identities (31a) and (31b) we find

$$2\mathrm{Im}(\mathcal{M}_F^{(2)}) = e^4[J^*(p,k)][J(p,k)]\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\frac{(2\pi)^4\delta^{(4)}(p'-k_1-k_2)}{2|\vec{k}_1|}\times\left[\frac{\gamma^\mu u^{(1)}(k_2)\bar{u}^{(1)}(k_2)\gamma_\mu(2\pi)^2\delta(k_1^0-|\vec{k}_1|)\delta(k_2^0-\widetilde{\omega}_1)}{2g_2^2\widetilde{\omega}_1(\widetilde{W}_1^2-\widetilde{\omega}_1^2)}\times\frac{\gamma^\mu u^{(2)}(k_2)\bar{u}^{(2)}(k_2)\gamma_\mu(2\pi)^2\delta(k_1^0-|\vec{k}_1|)\delta(k_2^0-\widetilde{\omega}_2)}{2g_2^2\widetilde{\omega}_2(\widetilde{W}_2^2-\widetilde{\omega}_2^2)}\right]. \tag{102}$$

Let us recall the physical delta definition and use the fact that

$$\delta(k_1^2)\theta(k_1^0) = \frac{\delta(k_1^0-|\vec{k}_1|)}{2|\vec{k}_1|}, \tag{103}$$

and the identity

$$\sum_{\mathrm{Pol}}\varepsilon_\mu(p)\varepsilon_\nu^*(p) = \eta_{\mu\nu}. \tag{104}$$

We obtain

$$2\mathrm{Im}(\mathcal{M}_F^{(2)}) = e^4[J^*(p,k)]\sum_{\mathrm{Pol}}\sum_{r=1,2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\times(2\pi)^4\delta^{(4)}(p'-k_1-k_2)\times[\gamma^\mu u^{(r)}(k_2)\varepsilon_\mu(k_1)\varepsilon_\nu^*(k_1)\bar{u}^{(r)}(k_2)\gamma^\nu]\times(2\pi)^2\delta(k_1^2)\delta^{(\mathrm{phys})}(\Lambda_r^2(k_2^0))\times\theta(k_1^0)\theta(k_2^0)[J(p,k)]. \tag{105}$$

Finally, we recognize the amplitude for the cut diagram as

$$i\mathcal{M}^{(2)} = [\varepsilon_\nu^*(k_1)\bar{u}^r(k_2)(-ie\gamma^\nu)]\left(\frac{iF(p')}{\Lambda_+^2(p')\Lambda_-^2(p')}\right)\times[(-ie\gamma^\mu)u^s(p)\varepsilon_\mu(k)]. \tag{106}$$

We have the optical theorem satisfied for the one-loop Compton scattering diagram

$$2\mathrm{Im}(\mathcal{M}_F^{(2)}) = \sum_{\mathrm{Pol}}\sum_{r=1,2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\times\delta^{(4)}(p'-k_1-k_2)|\mathcal{M}^{(2)}|^2\times(2\pi)^2\delta(k_1^2)\delta^{(\mathrm{phys})}(\Lambda_r^2(k_2^0))\theta(k_1^0)\theta(k_2^0). \tag{107}$$

V. CONCLUSIONS

We have studied the unitarity of the S-matrix in a Lorentz-violating theory of modified QED with higher-order operators. As is well known, higher-order operators in the Lagrangian density can lead to a potential loss of unitarity, especially in loop diagrams. The reason is that loop diagrams involve several off-shell virtual particles, which under combination may respect momentum conservation allowing high-energy modes associated with ghost states to be propagated through the cuts in the perturbative unitarity equation. This highly contrasts with the situation where conservation of momentum selects only particles to be propagated in tree-level diagrams.

In our particular model, selecting the preferred background in the pure timelike direction leads to higher time derivatives and implies a negative metric sector. We have seen that the effective approach and the Lee-Wick prescription can provide a unitarity theory. The Lee-Wick prescription is implemented by imposing stable particles to have a positive metric. A highlight of this work has been to provide a decoupled unitarity equation restricted to the positive metric sector. We have proved that the diagrams under consideration are unitary since no ghost modes are propagated through the cuts in the unitarity equation. A generalization of the Cutkosky rule, at the one-loop order considered, is possible by introducing physical Dirac deltas defined to select only positive-metric solutions. This extension can be useful in analyzing unitarity in higher-order diagrams.

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