Gauge fixing and gauge correlations in noncompact Abelian gauge models

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We investigate some general properties of linear gauge fixings and gauge-field correlators in lattice models with noncompact U(1) gauge symmetry. In particular, we show that, even in the presence of a gauge fixing, some gauge-field observables (like the photon-mass operator) are not well defined, depending on the specific gauge fixing adopted and on its implementation. Numerical tests carried out in the three-dimensional noncompact lattice Abelian-Higgs model fully support the analytical results and provide further insights. Apparently, only the hard Lorenz-gauge fixing provides a consistent definition of non-gauge-invariant quantities in three dimensions.

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I. INTRODUCTION

Some nonperturbative features of quantum field theories (QFTs) can be studied from first principles by using the lattice discretization. In this formulation the Euclidean version of the theory is regularized on a space-time lattice, and the QFT problem is mapped to a statistical-mechanics one. Continuum physics emerges as the correlation length of the statistical system diverges, i.e., close to a continuous phase transition (critical point) of the lattice system. For this strategy to be feasible, there should exist a stable fixed point of the QFT renormalization group (RG) flow, which encodes the universal properties of the critical point of the statistical system.

This approach has been extensively used to investigate for example four-dimensional non-Abelian gauge theories and QCD in particular [1,2], and the ϕ^4 QFTs associated with classical and quantum phase transitions in lower-dimensional systems [3–5]. Only in a few cases is it possible to carry out this strategy with full analytical control [6,7], so that one has to rely on numerical simulations of the discretized theory.

Four-dimensional non-Abelian gauge theories are peculiar, since the existence of a fixed point of the RG flow can be shown analytically by using one-loop perturbation theory [8–10]. For typical three-dimensional QFTs this is not the case, and the fixed point, if present, is generically in the strongly coupled regime. To analytically investigate its existence and extract universal information, nonperturbative approaches are required, like the expansion in the number of components [11] or the continuation in the number of dimensions obtained by resumming the ϵ-expansion series [3,12].

Three-dimensional gauge theories coupled to matter fields have features in common both with four-dimensional non-Abelian gauge theories and with three-dimensional scalar models. On the one hand, the gauge coupling of three-dimensional gauge theories has positive mass dimension (the theory is super-renormalizable); thus the energy scaling of the coupling is dictated by dimensional analysis, and asymptotic freedom is clear already at tree level. On the other hand, there is also the possibility that nontrivial fixed points exist, at which the gauge coupling does not vanish, and which are usually referred to as charged fixed points. While the asymptotically free fixed points of three-dimensional Abelian and non-Abelian gauge theories have been thoroughly investigated by numerical simulations (see, e.g., Refs. [13–18]), the case of the charged fixed points has attracted less attention until quite recently, when the existence of strongly coupled charged fixed points has been suggested to explain some peculiar critical phenomena [19–23].

The existence of these charged fixed points, and their critical properties, can be investigated using several complementary techniques: the ϵ expansion close to four dimensions [24–33], the expansion in the number of components [3,11,34], and numerical simulation of lattice models. Numerical studies have recently addressed this issue in the Abelian-Higgs (AH) model, i.e., in scalar quantum electrodynamics with N-component scalar fields, and there is by now compelling evidence that some lattice models undergo a continuous transition related to the AH QFT charged fixed point. This has been observed for $N \gtrsim 7$

using the noncompact discretization [35,36] and the higher-charge compact discretization [37,38], while only first-order phase transitions have been found in other cases [39–42] and for smaller N values [35,37,43,44]. For $N = 2$, continuous transitions were observed [39,41], where gauge fields play no role. Topological excitations likely play an important role for the existence of the charged fixed point; however, this point is not yet fully understood [45–49].

Analytical and numerical results thus support the fact that gauge-invariant correlators are well defined beyond perturbation theory in the Abelian-Higgs QFT, if a large enough number of scalar flavors is present. It is then natural to ask if gauge-dependent correlators, which play a fundamental role in the usual perturbative treatment of gauge QFTs, can be given a similar nonperturbative status. This may be important for several reasons: First of all in the gauge fixed continuum QFT formulation it is possible to obtain relations, valid to all orders of the perturbative expansion, which fix the anomalous dimension of the gauge field if a charged fixed point exists. If the gauge-field correlator is well defined in the nonperturbative setting, the QFT prediction provides a clear-cut test for the connection of the critical behavior observed in [35,36] with a charged fixed point. A second reason is that non-gauge-invariant observables defined by means of specific gauge-fixing procedures correspond in fact to quite involved (and generally nonlocal) gauge-invariant observables. The use of gauge fixings can thus be seen as a convenient tool for the investigation of some nonlocal aspects of the theory in a computationally simpler setting. Finally, gauge fixing can also be seen as a (very specific) form of explicit breaking of the gauge symmetry, and the investigation of gauge fixed models could be relevant to understand the continuous phase transitions in models with explicit gauge symmetry breaking [50,51].

In this work we aim to clarify how the large-distance behavior of the gauge-field correlators depends on the gauge-fixing procedure adopted. For this purpose, we study gauge correlations in the noncompact formulation of Abelian gauge models. The relations that will be derived are independent of the matter content of the theory. Moreover, they are valid in the whole phase diagram of the model, and not only on the critical lines associated with charged fixed points. In the present paper, we also add a numerical study of the behavior of the gauge-field correlations in generic points of the phase diagram of the three-dimensional Abelian-Higgs model, which provides further insights into the role of the different gauge fixings. A detailed analysis of the critical behavior is left to [52].

To make gauge correlation functions well defined, it is necessary to introduce a gauge-fixing term that completely breaks the gauge invariance of the model. In noncompact discretizations, the gauge fixing plays a crucial role, since, only in the presence of a gauge fixing, the partition function and the average values of non-gauge-invariant quantities are finite. This is at variance with what happens in compact formulations, in which a gauge fixing is not necessary. Also in the absence of it, the partition function is well defined and so are average values of non-gauge-invariant quantities. In particular, correlations of non-gauge-invariant quantities are either trivial or equivalent to gauge-invariant observables obtained by averaging the non-gauge-invariant quantity over the whole (compact) group of gauge transformations [53–55]. The latter equivalence does not hold in noncompact formulations, since the group of gauge transformations is not compact and therefore, averages over all gauge transformations are not defined.

Once a gauge fixing is introduced, the first point to be investigated is whether and how the results for non-gauge-invariant quantities depend on it. Here we consider two widely used gauge fixings, the axial and the Lorenz one. We derive general results and perform a complementary numerical study in the AH model. They both indicate that gauge correlations depend somehow on the gauge choice made. In particular, we show that the photon-mass operator is well defined only in what we call the hard Lorenz gauge (see Sec. II). Unphysical results are obtained when using the axial gauge and the soft Lorenz gauge. The conclusions of this work should be independent of the type of matter fields considered (fermions or bosons) as they only rely on some specific features of the gauge-fixing functions.

The paper is organized as follows. In Sec. II we introduce the lattice model, define the gauge fixings and the gauge observable that we will focus on. In Sec. III we derive general relations, which are independent of the nature of the matter fields, between the gauge-field correlation functions in the presence of different gauge fixings. In Sec. IV we present numerical results obtained in the scalar AH model, with the purpose of determining the behavior of gauge-field correlation functions in the different phases present in the model. In Sec. V we review some field-theory results for the gauge dependence of the gauge-field correlation functions. Finally, in Sec. VI we draw our conclusions. In Appendix A, we summarize some analytic results for the pure gauge model, while in Appendix B we derive some general relations for the gauge-dependent part of the gauge correlation functions.

II. THE LATTICE MODEL

We consider a noncompact Abelian gauge theory on a d-dimensional cubiclike lattice of size L, with fermionic and bosonic matter fields that we collectively indicate with Ψ and Φ, respectively. The gauge interaction is mediated by real fields $A_{\boldsymbol{x},\mu} \in \mathbb{R}$ defined on the lattice links, each link being labeled by a lattice site $\boldsymbol{x}$ and a positive lattice direction $\hat{\mu}$ ($\mu = 1, \ldots, d$). The action is given by

$$S = S_{\mathrm{matter}}(\Psi, \Phi, A) + S_{\mathrm{gauge}}(A), \tag{1}$$

where $S_{\rm matter}$ is the action for the matter fermionic and bosonic fields, and $S_{\rm gauge}(A)$ is the action for the gauge fields, which is given by

$$S_{\rm gauge}(A) = \frac{\kappa}{2}\sum_{\boldsymbol{x},\mu>\nu}(\Delta_\mu A_{\boldsymbol{x},\nu} - \Delta_\nu A_{\boldsymbol{x},\mu})^2. \tag{2}$$

Here κ is the inverse lattice gauge coupling, Δ_μ is a discrete derivative defined by $\Delta_\mu f_{\boldsymbol{x}} = f_{\boldsymbol{x}+\hat{\mu}} - f_{\boldsymbol{x}}$, and we have taken the lattice spacing equal to 1. We assume that the action is invariant under local gauge transformations, which act on the gauge field as

$$A_{\boldsymbol{x},\mu} \to A'_{\boldsymbol{x},\mu} = A_{\boldsymbol{x},\mu} - \Delta_\mu \phi_{\boldsymbol{x}}. \tag{3}$$

Matter fields do not couple with $A_{\boldsymbol{x},\mu}$ directly, but rather through $\lambda_{\boldsymbol{x},\mu} = \exp(iA_{\boldsymbol{x},\mu})$. This implies that, in a finite system with periodic boundary conditions, the action S is also invariant under the transformation $A_{\boldsymbol{x},\mu} \to A_{\boldsymbol{x},\mu} + 2\pi n_\mu$, where $n_\mu \in \mathbb{Z}$ depends on the direction μ but not on the point $\boldsymbol{x}$. This transformation makes the averages of some gauge-invariant quantities (for instance, of Polyakov loops, which, in noncompact formulations, are defined as the sum of the gauge fields along paths that wrap around the lattice), ill defined. To make the averages of all gauge-invariant observables well defined on a finite lattice, we adopt C^* boundary conditions [35,56,57] that correspond to considering antiperiodic boundary conditions for the gauge fields, i.e., to

$$A_{\boldsymbol{x}+L\hat{\nu},\mu} = -A_{\boldsymbol{x},\mu}, \tag{4}$$

for all lattice directions ν. When using C^* boundary conditions, the local U(1) gauge symmetry is preserved by using antiperiodic gauge transformations $\phi_{\boldsymbol{x}}$ in Eq. (3).

To study correlation functions of the gauge fields, it is necessary to add a gauge fixing. We consider gauge fixings that are linear in the fields and that are translation invariant. We introduce a gauge-fixing function

$$F_{\boldsymbol{x}}(A) = \sum_{\boldsymbol{y}\mu} M_{\boldsymbol{x}-\boldsymbol{y},\mu} A_{\boldsymbol{y},\mu}, \tag{5}$$

where $M_{\boldsymbol{x},\mu}$ is a field-independent vector, and define the partition function as

$$Z_{\rm hard} = \int [{\rm d}\Phi {\rm d}\bar{\Phi}][{\rm d}\Psi {\rm d}\bar{\Psi}][dA]\left[\prod_{\boldsymbol{x}}\delta[F_{\boldsymbol{x}}(A)]\right]e^{-S}, \tag{6}$$

where the product extends to all lattice sites. Note that the insertion of the gauge-fixing term does not change the expectation values of gauge-invariant quantities. In perturbation theory, one usually replaces the partition function (6) with a different one (see, e.g., Refs. [1,3,58]) defined by adding a term of the form

$$S_{\rm GF}(A) = \frac{1}{2\zeta}\sum_{\boldsymbol{x}}[F_{\boldsymbol{x}}(A)]^2 \tag{7}$$

to the action. In this case one considers the partition function

$$Z_{\rm soft} = \int [{\rm d}\Phi {\rm d}\bar{\Phi}][{\rm d}\Psi {\rm d}\bar{\Psi}][dA]\, e^{-S-S_{\rm GF}(A)}. \tag{8}$$

Since the gauge-fixing function is linear in the gauge fields, no field-dependent Jacobian should be considered in the gauge-fixed model and, therefore, no Faddeev-Popov term should be added. The partition function $Z_{\rm soft}$ depends on the parameter ζ. For $\zeta \to 0$, the model with partition function (8) is equivalent to the one with partition function (6). We will call the gauge fixings appearing in Eqs. (6) and (8) hard- and soft-gauge fixing, respectively.

In this work we will mainly focus on two widely used gauge-fixing functions. We consider the axial-gauge fixing with

$$F_{A,\boldsymbol{x}}(A) = A_{\boldsymbol{x},d}, \tag{9}$$

and the Lorenz-gauge fixing with

$$F_{L,\boldsymbol{x}}(A) = \sum_{\mu=1}^{d}(A_{\boldsymbol{x},\mu} - A_{\boldsymbol{x}-\hat{\mu},\mu}). \tag{10}$$

Note that in a finite system with C^* boundary conditions, both gauge fixings completely fix the gauge (they are complete gauge fixings). Indeed, there are no distinct configurations $A_{\boldsymbol{x},\mu}$ and $A'_{\boldsymbol{x},\mu}$ related by a gauge transformation such that $F_{\boldsymbol{x}}(A) = F_{\boldsymbol{x}}(A') = 0$ for all lattice points $\boldsymbol{x}$.

We consider correlation functions of the gauge fields. We define the Fourier transform of the field as[1]

$$\tilde{A}_\mu(\boldsymbol{p}) = e^{ip_\mu/2}\sum_x A_{\boldsymbol{x},\mu} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{11}$$

Under C^* boundary conditions, $A_{\boldsymbol{x},\mu}$ is antiperiodic, so that the allowed momenta for $\tilde{A}_\mu(\boldsymbol{p})$ are $\boldsymbol{p} = (2n_1 + 1, \ldots, 2n_d + 1)\pi/L$ $(n_i = 0, \ldots L-1)$. In particular, $\boldsymbol{p} = 0$ is not an allowed momentum. The corresponding momentum-space two-point function is

$$\tilde{G}_{\mu\nu}(\boldsymbol{p}) = \frac{1}{L^d}\langle \tilde{A}_\mu(\boldsymbol{p})\tilde{A}_\nu(-\boldsymbol{p})\rangle. \tag{12}$$

[1] The added factor $e^{ip_\mu/2}$ is needed to guarantee that $\tilde{A}_\mu(\boldsymbol{p})$ is odd under reflections in momentum space, $\boldsymbol{p} \to (p_1, \ldots, -p_\mu, \ldots, p_d)$. Intuitively, it can be understood by noting that $A_{\boldsymbol{x},\mu}$ is associated with a lattice link and thus it would be more naturally considered as a function of the link midpoint; i.e., we should write it as $A_{\boldsymbol{x}+\hat{\mu}/2,\mu}$.

We assume that the matter action is invariant under charge conjugation. As this property is preserved by the C^* boundary conditions and by linear gauge fixings, the full theory is also invariant under charge conjugation, which guarantees $\langle A_{\boldsymbol{x},\mu}\rangle = 0$.

We also consider the composite operator

$$B_{\boldsymbol{x}} = \sum_{\mu} A_{\boldsymbol{x},\mu}^2, \tag{13}$$

which, in perturbative approaches, is included in the action to provide a mass to the photon and therefore an infrared regulator to the theory (see, e.g., Ref. [3]). We define its Fourier transform

$$\tilde{B}(\boldsymbol{p}) = \sum_{x} B_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}, \tag{14}$$

where $\boldsymbol{p} = (2n_1, \ldots, 2n_d)\pi/L$ ($B_{\boldsymbol{x}}$ is periodic) and the correlation function

$$\tilde{G}_B(\boldsymbol{p}) = \frac{1}{L^d}[\langle \tilde{B}(\boldsymbol{p})\tilde{B}(-\boldsymbol{p})\rangle - \langle \tilde{B}(\boldsymbol{p})\rangle\langle \tilde{B}(-\boldsymbol{p})\rangle]. \tag{15}$$

The long-distance properties of the correlators $\tilde{G}_{\mu\nu}(\boldsymbol{p})$ and $\tilde{G}_B(\boldsymbol{p})$ can be determined by studying the gauge susceptibilities

$$\chi_{\mu\nu} = \tilde{G}_{\mu\nu}(\boldsymbol{p}_a), \qquad \chi_B = \tilde{G}_B(\boldsymbol{0}), \tag{16}$$

where the momentum $\boldsymbol{p}_a$ is defined by

$$\boldsymbol{p}_a = (p_{\min}, \ldots, p_{\min}), \qquad p_{\min} = \frac{\pi}{L}. \tag{17}$$

Note that, since $A_{\boldsymbol{x},\mu}$ is antiperiodic, each component of the momentum can only take the values $(2n+1)p_{\min}$ and thus $\boldsymbol{p}_a$ is one of the acceptable momenta for which $|\boldsymbol{p}|$ is as small as possible.

III. CORRELATION FUNCTIONS IN DIFFERENT GAUGES

In this section we derive relations among correlation functions in different gauges. These relations will help us to understand the nonperturbative behavior of correlation functions that will be discussed in Sec. IV. We focus on the axial and Lorenz gauge, but it is easy to generalize the discussion to any arbitrary gauge-fixing function that is linear in the gauge field. Moreover, all results concerning the gauge-field two-point correlation functions can in principle be generalized to any correlation function of the gauge fields. Finally, note that all results are independent of the nature of the matter fields.

A. Hard Lorenz and axial gauges

To relate Lorenz-gauge and axial-gauge results, we first determine a gauge transformation that maps the Lorenz-gauge fixing onto the axial one. More precisely, given a field configuration $\{A_{\boldsymbol{x},\mu}\}$ we want to determine a gauge transformation (3), i.e. a function $\phi_{\boldsymbol{x}}$, such that

$$A'_{\boldsymbol{x},d} = \sum_{\mu}(A_{\boldsymbol{x},\mu} - A_{\boldsymbol{x}-\hat{\mu},\mu}). \tag{18}$$

Working in Fourier space, this corresponds to choosing

$$\tilde{\phi}(\boldsymbol{p}) = \frac{i}{\hat{p}_d}\left(ie^{ip_d/2}\sum_{\mu}\hat{p}_\mu\tilde{A}_\mu(\boldsymbol{p}) + \tilde{A}_d(\boldsymbol{p})\right), \tag{19}$$

where $\hat{p}_\mu = 2\sin(p_\mu/2)$. This transformation is well defined on a finite lattice with C^* boundary conditions as $\hat{p}_d$ never vanishes. It maps the action with a soft Lorenz-gauge fixing onto the axial-gauge action with the same parameter ζ. If we take the limit $\zeta \to 0$, it allows us to relate the two hard-gauge-fixed models.

To relate correlation functions we interpret the gauge transformation with gauge function (19) as a change of variables. Since the transformation is linear in the fields, the Jacobian is independent of the fields and plays no role. Therefore, if $O(A_{\boldsymbol{x},\mu})$ is a gauge-dependent operator, we have

$$\langle O(A_{\boldsymbol{x},\mu})\rangle_{A,\zeta} = \langle O(A_{\boldsymbol{x},\mu} - \Delta_\mu\phi_{\boldsymbol{x}})\rangle_{L,\zeta} \tag{20}$$

where $\phi_{\boldsymbol{x}}$ is the anti-Fourier transform of Eq. (19) and the two average values refer to the models with axial (A) and Lorenz (L) soft-gauge fixing, respectively, with the same parameter ζ.

We can use Eq. (20) to relate $\tilde{G}^{(A)}_{\mu\nu}(\boldsymbol{p})$ and $\tilde{G}^{(L)}_{\mu\nu}(\boldsymbol{p})$ (axial and Lorenz gauge, respectively). Considering only the hard case ($\zeta = 0$), using $\sum_\mu \hat{p}_\mu\tilde{G}^{(L)}_{\mu\nu}(\boldsymbol{p}) = 0$ (see Appendix B), we can express $\tilde{G}^{(L)}_{d\mu}(\boldsymbol{p})$ in terms of the components of the Lorenz function $\tilde{G}^{(L)}_{\mu\nu}(\boldsymbol{p})$ with $\mu, \nu \le (d-1)$. This allows us to prove the relation ($1 \le \mu, \nu \le d-1$),

$$\begin{aligned}\tilde{G}^{(A)}_{\mu\nu}(\boldsymbol{p}) = \tilde{G}^{(L)}_{\mu\nu}(\boldsymbol{p}) + \frac{\hat{p}_\mu\hat{p}_\nu}{\hat{p}_d^4}\sum_{\alpha\beta}\hat{p}_\alpha\hat{p}_\beta\tilde{G}^{(L)}_{\alpha\beta}(\boldsymbol{p}) \\ + \frac{\hat{p}_\mu}{\hat{p}_d^2}\sum_{\alpha}\hat{p}_\alpha\tilde{G}^{(L)}_{\alpha\nu}(\boldsymbol{p}) + \frac{\hat{p}_\nu}{\hat{p}_d^2}\sum_{\alpha}\hat{p}_\alpha\tilde{G}^{(L)}_{\alpha\mu}(\boldsymbol{p}),\end{aligned} \tag{21}$$

where α and β run from 1 to $(d-1)$ only. Obviously, as we are considering the hard-gauge fixing, $\tilde{G}^{(A)}_{\mu\nu}(\boldsymbol{p}) = 0$, if μ or ν are equal to d. We can use Eq. (21) to relate $\chi^{(A)}_{\mu\nu}$ with $\chi^{(L)}_{\mu\nu}$. Because of the cubic symmetry of the lattice and of the momentum $\boldsymbol{p}_{\rm a}$ [see Eq. (17)], only two components of $\tilde{G}^{(L)}_{\mu\nu}(\boldsymbol{p}_a)$ are independent. Therefore, we can write

$$\chi_{\mu\nu}^{(L)} = \tilde{G}_{\mu\nu}^{(L)}(\boldsymbol{p}_a) = c_{L1}\delta_{\mu\nu} + c_{L2}(1-\delta_{\mu\nu}), \qquad (22)$$

where $c_{L2} = -c_{L1}/(d-1)$ because of the Lorenz condition (see Appendix B). Substituting in Eq. (21), we obtain (again $1 \le \mu, \nu \le d-1$)

$$\chi_{\mu\nu}^{(A)} = \tilde{G}_{\mu\nu}^{(A)}(\boldsymbol{p}_a) = c_{A1}\delta_{\mu\nu} + c_{A2}(1-\delta_{\mu\nu}), \qquad (23)$$

with

$$c_{A1} = \frac{2d}{d-1}c_{L1}, \qquad c_{A2} = -dc_{L2}. \qquad (24)$$

The simple relations (24) and (21) do not extend, however, to composite operators. Indeed, the transformation with function (19) that relates the two gauges is singular in the limit $L \to \infty$, because of the factor $1/\hat{p}_d$, which diverges as $L \to \infty$. This shows up in the presence of singular coefficients in Eq. (21). As a consequence, as we discuss in Sec. IV, the average

$$\langle B_{\boldsymbol{x}} \rangle = \frac{1}{V}\sum_{\mu\nu}\sum_{p}\tilde{G}_{\mu\nu}(\boldsymbol{p}) \qquad (25)$$

behaves differently in the axial and Lorenz gauges.

B. Hard and soft axial gauges

Let us now determine how correlation functions vary in soft axial gauges as the parameter ζ varies. As before, we consider changes of variables that are gauge transformations. For the case at hand, we consider the gauge function

$$\tilde{\phi}(\boldsymbol{p}) = \left(1 - \sqrt{\frac{\zeta_2}{\zeta_1}}\right)\frac{i}{\hat{p}_d}\tilde{A}_d(\boldsymbol{p}) \qquad (26)$$

that allows us to map the model with parameter ζ_1 onto the model with parameter ζ_2. It is immediate to relate correlation functions. Using Eq. (20) modified for the case at hand, we obtain ($\mu, \nu \le d-1$),

$$\begin{aligned} \tilde{G}_{\mu\nu}^{(A)}(\boldsymbol{p},\zeta_2) &= \tilde{G}_{\mu\nu}^{(A)}(\boldsymbol{p},\zeta_1) + r^2\frac{\hat{p}_\mu\hat{p}_\nu}{\hat{p}_d^2}\tilde{G}_{dd}^{(A)}(\boldsymbol{p},\zeta_1) \\ &\quad - r\frac{\hat{p}_\mu}{\hat{p}_d}\tilde{G}_{\nu d}^{(A)}(\boldsymbol{p},\zeta_1) - r\frac{\hat{p}_\nu}{\hat{p}_d}\tilde{G}_{\mu d}^{(A)}(\boldsymbol{p},\zeta_1), \\ r &= \left(1 - \sqrt{\frac{\zeta_2}{\zeta_1}}\right). \end{aligned} \qquad (27)$$

To simplify this expression, we can use the Ward identity (see Appendix B):

$$\hat{p}_d\tilde{G}_{d\mu}^{(A)}(\boldsymbol{p},\zeta) = \zeta\hat{p}_\mu. \qquad (28)$$

We end up with ($\mu, \nu \le d-1$),

$$\tilde{G}_{\mu\nu}^{(A)}(\boldsymbol{p},\zeta_2) = \tilde{G}_{\mu\nu}^{(A)}(\boldsymbol{p},\zeta_1) + (\zeta_2-\zeta_1)\frac{\hat{p}_\mu\hat{p}_\nu}{\hat{p}_d^2}. \qquad (29)$$

Taking the limit $\zeta_1 \to 0$ this relation allows us to relate the hard-gauge and soft-gauge susceptibilities. We find

$$\chi_{\mu\nu,\zeta} = \chi_{\mu\nu,HA} + \zeta, \qquad (30)$$

where the $\chi_{\mu\nu,\zeta}$ and $\chi_{\mu\nu,HA}$ are computed in the soft gauge with parameter ζ and in the hard gauge, respectively.

C. Hard and soft Lorenz gauges

The same calculation can be performed in the Lorenz case. We consider

$$\tilde{\phi}(\boldsymbol{p}) = \left(1 - \sqrt{\frac{\zeta_2}{\zeta_1}}\right)\frac{1}{\hat{p}^2}\sum_\mu i\hat{p}_\mu\tilde{A}_\mu(\boldsymbol{p}) \qquad (31)$$

that allows us to map the model with parameter ζ_1 onto the model with parameter ζ_2. Here $\hat{p}^2 = \sum_\mu \hat{p}_\mu^2$. The calculation is analogous to that performed before. If we parametrize the susceptibilities as in Eq. (22), we obtain

$$\begin{aligned} c_{L1}(\zeta_2) &= \frac{d-1}{d}[c_{L1}(\zeta_1) - c_{L2}(\zeta_1)] \\ &\quad + \frac{\zeta_2}{d\zeta_1}[c_{L1}(\zeta_1) + (d-1)c_{L2}(\zeta_1)], \\ c_{L2}(\zeta_2) &= -\frac{1}{d}[c_{L1}(\zeta_1) - c_{L2}(\zeta_1)] \\ &\quad + \frac{\zeta_2}{d\zeta_1}[c_{L1}(\zeta_1) + (d-1)c_{L2}(\zeta_1)]. \end{aligned} \qquad (32)$$

To simplify this expression, we use the Ward identity (see Appendix B)

$$\sum_\mu \hat{p}_\mu\tilde{G}_{\mu\nu}^{(L)}(\boldsymbol{p},\zeta) = \zeta\frac{\hat{p}_\nu}{\hat{p}^2}, \qquad (33)$$

which implies

$$\sum_\mu \tilde{G}_{\mu\nu}^{(L)}(\boldsymbol{p}_a,\zeta) = \frac{\zeta}{d\hat{p}_{\min}^2}, \qquad (34)$$

with $p_{\min} = \pi/L$. Substituting in Eq. (32) we obtain

$$\begin{aligned} c_{L1}(\zeta_2) &= c_{L1}(\zeta_1) + \frac{1}{d^2\hat{p}_{\min}^2}(\zeta_2-\zeta_1), \\ c_{L2}(\zeta_2) &= c_{L2}(\zeta_1) + \frac{1}{d^2\hat{p}_{\min}^2}(\zeta_2-\zeta_1). \end{aligned} \qquad (35)$$

IV. NUMERICAL RESULTS

To understand the role that the different gauge fixings play, we now discuss the behavior of the gauge correlations in the three-dimensional AH model. This lattice model has been extensively studied [35,43,44] and we will use it as a paradigmatic system to investigate how gauge correlations vary with the gauge fixing adopted.

We consider N-dimensional scalar fields $z_{\boldsymbol{x}}$, which are defined on the lattice sites and satisfy the unit-length constraint $\bar{z}\cdot z = 1$. The matter action is

$$S_{\rm matter} = -JN\sum_{\boldsymbol{x},\mu}{\rm Re}(\bar{z}_{\boldsymbol{x}}\cdot\lambda_{\boldsymbol{x},\mu}z_{\boldsymbol{x}+\hat{\mu}}), \qquad (36)$$

where the sum extends to all lattice sites and directions (μ runs from 1 to $d = 3$), and $\lambda_{\boldsymbol{x},\mu} = \exp(iA_{\boldsymbol{x},\mu})$.

The phase diagram is reported in Fig. 1. It displays three different phases characterized by the different behavior of the gauge field and by the possible breaking of the global SU(N) symmetry. For small J-values the gauge field is expected to have long-range correlations as it occurs for $J = 0$ and the SU(N) symmetry is realized in the spectrum (Coulomb phase). For large J two phases occur: The SU(N) symmetry is broken in both phases, while the gauge field is expected to be long ranged for small κ (molecular phase) and short ranged for large κ (Higgs phase). The properties of the Higgs phase are supposedly those that are usually associated, in the perturbative setting, with the spontaneous breaking of the U(1) gauge symmetry. The transition line separating the Coulomb and the Higgs phases is the one along which (for $N \gtrsim 7$) the continuum limit associated with the AH QFT emerges.

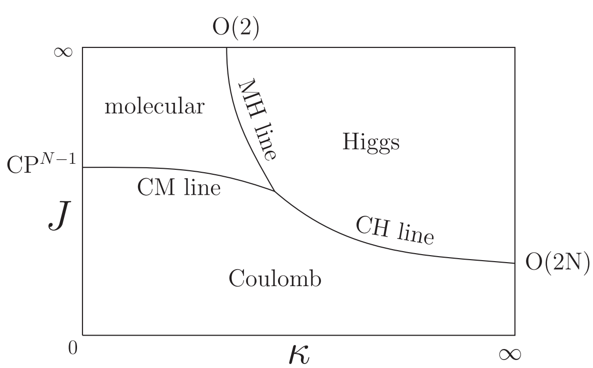

FIG. 1. Sketch of the phase diagram of the three-dimensional lattice AH model with noncompact gauge fields and unit-length N-component complex scalar fields, for generic $N \geq 2$. Three transition lines can be identified: the Coulomb-to-Higgs (CH) line between the Coulomb and Higgs phases, the Coulomb-to-molecular (CM) line, and the molecular-to-Higgs (MH) line. For $\kappa = 0$, the model is equivalent to the CP^{N-1} model, for $\kappa \to \infty$ to the O($2N$) vector, and for $J \to \infty$ to the inverted XY or O(2) model.

In this work we consider scalar fields with $N = 25$ components focusing on the large-size behavior of the gauge observables in the Higgs and Coulomb phases. We perform simulations for $(\kappa, J) = (0.4, 0.2)$ and (0.4,0.4) that lie in the Coulomb and Higgs phase, respectively [for $\kappa = 0.4$, the transition between the Coulomb and Higgs phases occurs [50] at $J = 0.295515(4)$]. We report results for four different gauge fixings. We consider the hard Lorenz- and axial-gauge fixings and the corresponding soft versions with $\zeta = 1$. We show that the long-distance behavior of the gauge observables defined before depends, to some extent, on the gauge fixing used. For the Coulomb case, the results are consistent with the ones that can be analytically obtained for $J = 0$, i.e., the noncompact Abelian lattice gauge theory without matter, which are summarized in Appendix A.

Simulations have been performed by using the same combination of Metropolis and microcanonical updates discussed in Ref. [35], which can be easily extended to the case of the soft gauges discussed in this paper. Hard axial simulations have been carried out by fixing $A_{\boldsymbol{x},d} = 0$ and updating only the $d-1$ nonvanishing components of $A_{\boldsymbol{x},\mu}$. To obtain the results in the hard Lorenz gauge, we have instead performed simulations with no gauge fixing and implemented the gauge fixing before each measure. Given the gauge configuration $\{A_{\boldsymbol{x},\mu}\}$ obtained in the simulation, we have determined a gauge transformation (3) so that the fields $\{A'_{\boldsymbol{x},\mu}\}$ satisfy the condition $F_{L,\boldsymbol{x}}(A') = 0$ for all $\boldsymbol{x}$ [see Eq. (10)]. Gauge correlations are then computed using the fields $\{A'_{\boldsymbol{x},\mu}\}$. The gauge transformation has been determined by using a conjugate-gradient solver.

A. Coulomb phase

We start by investigating the behavior of the gauge model in the Coulomb phase (simulations for $J = 0.2$). In the whole Coulomb phase the gauge field is expected to have long-range correlations, and thus $\chi_{\mu\nu}$ should diverge as L increases, in all gauges considered. Results for the two hard gauges are reported in Fig. 2. We observe that $\chi_{\mu\nu}$ diverges as L^2 in both cases, a fact that is consistent with the analytic results for $J = 0$ (in which case $1/L^2$ corrections are expected); see Appendix A. The relation Eq. (24) is fully confirmed by the data, see Fig. 2, and results in the soft gauges behave analogously and are in full agreement with relations (35) and (30).

Let us now consider the average of the photon-mass operator $B_{\boldsymbol{x}}$. Results in the Lorenz gauges are reported in Fig. 3. In both cases $\langle B_{\boldsymbol{x}}\rangle$ has a finite infinite-volume limit with corrections of order $1/L$. Again this is in agreement with the results for $J = 0$ reported in Appendix A. We have determined the same quantity in the axial gauges obtaining a different result. In this case $\langle B_{\boldsymbol{x}}\rangle$ diverges with the system size as L increases; see Fig. 4: $B_{\boldsymbol{x}}$ is not a well-defined operator in the infinite-volume limit. The different behavior

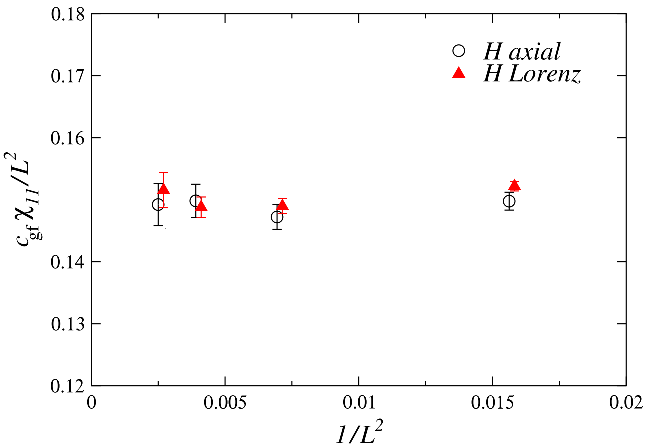

FIG. 2. Coulomb phase: estimates of $c_{\mathrm{gf}}\chi_{11}/L^2$ versus $1/L^2$ in the hard Lorenz and hard axial gauge, where c_{gf} is a gauge-fixing-dependent constant. We use $c_{\mathrm{gf}} = 3$ for the Lorenz-gauge fixing and $c_{\mathrm{gf}} = 1$ for the axial one. Lorenz-gauge data have been slightly shifted toward the right to improve readability. Results in the Coulomb phase, for $J = 0.2$.

can be understood by noting the completely different role the two gauge fixings play in infinite volume. In infinite volume, only the transformations $A'_{\boldsymbol{x},\mu} = A_{\boldsymbol{x},\mu} + c_\mu$, where c_μ is a constant, leave the Lorenz-gauge-fixed action invariant. Indeed, in the Lorenz gauge, a gauge transformation leaves $A_{\boldsymbol{x},\mu}$ invariant only if

$$\sum_\mu [\phi_{\boldsymbol{x}+\hat{\mu}} - 2\phi_{\boldsymbol{x}} + \phi_{\boldsymbol{x}-\hat{\mu}}] = 0 \tag{37}$$

for all points $\boldsymbol{x}$. By working in Fourier space, one can show that all solutions of this equation can be written as $\phi_{\boldsymbol{x}} = a + \sum_\mu c_\mu x_\mu$, so that, $\Delta_\mu \phi_{\boldsymbol{x}} = c_\mu$. Note that these gauge transformations are valid only in infinite volume. In a

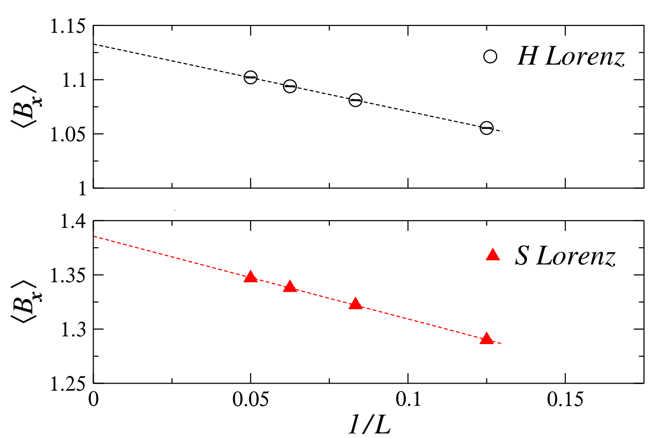

FIG. 3. Coulomb phase: estimates of $\langle B_{\boldsymbol{x}} \rangle$ versus $1/L$ in the hard Lorenz (top) and soft Lorenz gauge with $\zeta = 1$ (bottom). For $L \to \infty \langle B_{\boldsymbol{x}} \rangle \approx 1.1328$ and 1.3857 in the two cases, respectively. Results in the Coulomb phase, for $J = 0.2$.

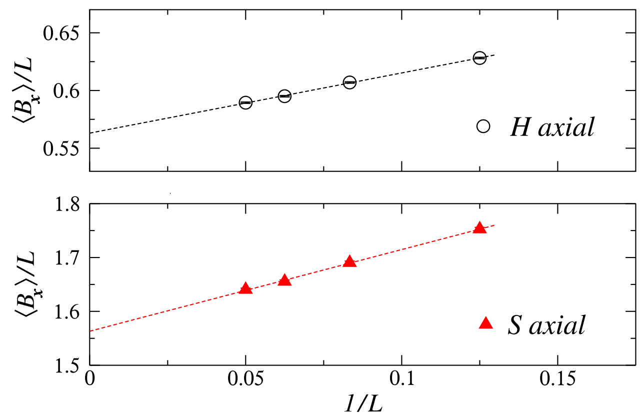

FIG. 4. Coulomb phase: estimates of $\langle B_{\boldsymbol{x}} \rangle / L$ versus $1/L$ in the hard axial (top) and soft axial gauge with $\zeta = 1$ (bottom). For $L \to \infty \langle B_{\boldsymbol{x}} \rangle / L \approx 0.5631$ and 1.563 in the two cases, respectively. Results in the Coulomb phase, for $J = 0.2$.

finite volume with C^* boundary conditions, the gauge fixing is complete and c_μ necessarily vanishes.

On the other hand, in the axial gauge, any gauge transformation with function $\phi_{\boldsymbol{x}} = \phi(x, y, z)$ that only depends on x and y leaves the action invariant. Thus, the axial-gauge action is invariant under a large set of space-dependent transformations and this causes the divergence of $\langle B_{\boldsymbol{x}} \rangle$. This result can also be understood by looking at the relation between the axial and Lorenz correlation functions; see Eq. (21). While the Lorenz correlation function is expected to be singular only for $p = 0$ (due to the presence of the zero modes discussed above), the axial correlation function is singular for $p_d = 0$, irrespective of the value of the other components of the momentum, i.e., on a $(d-1)$-dimensional momentum surface. These singularities make the sum appearing in Eq. (25) diverge as $L \to \infty$.

It is well known that perturbation theory in the axial gauges is problematic [3,59]. The results presented here show that the difficulties one encounters using axial gauges are not simply technical ones due to the infrared problems of the perturbative expansion. Also nonperturbatively, axial gauges do not allow a proper definition of some gauge-dependent quantities, for instance, the photon-mass operator, in the infinite-volume limit.

We have also determined the behavior of the susceptibility χ_B, obtaining results that are analogous to those that hold for $J = 0$. We find $\chi_B \sim L$ in Lorenz gauges and $\chi_B \sim L^3$ in axial gauges.

Finally, let us make a few comments on the apparently equivalent Lorenz-gauge fixing

$$F_{L'\boldsymbol{x}}(A) = \sum_{\mu=1}^{d} (A_{\boldsymbol{x}+\hat{\mu},\mu} - A_{\boldsymbol{x},\mu}), \tag{38}$$

which differs from the one reported in Eq. (10) in the choice of the lattice derivative (forward instead of backward).

This gauge-fixing function has several shortcomings. First of all, in even dimension, it does not represent a complete gauge fixing for some values of L. For instance, if $L = 4n + 2$ and $d = 4$, transformations with $[\boldsymbol{x} = (x_1, x_2, x_3, x_4)]$,

$$\phi_{\boldsymbol{x}} = A \cos\left[\frac{\pi}{2}(x_1 - x_2)\right] \cos\left[\frac{\pi}{2}(x_3 - x_4)\right] \qquad (39)$$

leave $F_{L',\boldsymbol{x}}(A)$ invariant and are consistent with the C^* boundary conditions ($\phi_{\boldsymbol{x}}$ is antiperiodic). In $d = 3$ the gauge fixing is complete in a finite volume. However, in infinite volume, $F_{L',\boldsymbol{x}}(A)$ is invariant under a large set of gauge transformations, as it occurs in the axial case. Thus, we do not expect $B_{\boldsymbol{x}}$ to be a well-defined operator if $F_{L',\boldsymbol{x}}(A)$ is used. For $J = 0$, the average value of $B_{\boldsymbol{x}}$ diverges in the infinite-volume limit; see Appendix A. We have also performed some simulations for $J = 0.2$, observing that also in this case $\langle B_{\boldsymbol{x}}\rangle$ increases as L is varied.

B. Higgs phase

Let us now discuss the behavior of gauge-dependent observables in the Higgs phase (numerical simulations have been performed for $J = 0.4$). In Fig. 5 we report the susceptibility χ_{11} for the hard and the soft axial gauge (with $\zeta = 1$), versus $1/L$. In both cases χ_{11} has a finite limit as $L \to \infty$ and satisfies relation (30). The finite value in the Higgs phase is consistent with the presence of a finite photon mass. However, the apparent presence of size corrections that decay as $1/L$ points to an unusual behavior of the system, since in a standard massive phase corrections are typically expected to scale as $e^{-L/\xi}$.

In Fig. 6 we show results for the susceptibility χ_{11} in the Lorenz gauges. In the hard case, χ_{11} is finite in the infinite-volume limit and satisfies the exact relation (24) with the corresponding quantity in the hard axial gauge. Instead, in the soft Lorenz gauge, we find $\chi_{11} \sim L^2$. This divergence might be, erroneously, interpreted as an indication of the presence of physical long-range gauge correlations in the Higgs phase—this would be in contrast with the idea that the photon is massive. The correct interpretation is instead, that in the soft Lorenz gauge there are unphysical gauge modes that are long ranged and contribute to $\chi_{\mu\nu}$, even though they do not have physical meaning. This interpretation is supported by Eq. (35) that we rewrite as

$$\chi_{11,L\zeta} = \chi_{11,HL} + \frac{\zeta}{d^2 \hat{p}^2_{\rm min}} \approx \chi_{11,HL} + \frac{\zeta}{d^2 \pi^2} L^2 \qquad (40)$$

where $\chi_{11,L\zeta}$ and $\chi_{11,HL}$ refer to the soft Lorenz gauge with parameter ζ and to the hard Lorenz gauge, respectively. Since $\chi_{11,HL}$ has a finite large-L limit, this relation shows that the divergence of $\chi_{11,L\zeta}$ is only due to the last term, which has no physical meaning, and is related to the presence of propagating longitudinal modes that are instead completely suppressed in the hard gauge ($\zeta = 0$).

Perturbation theory provides the recipe for the definition of a susceptibility that only couples the physical modes. We define

$$\tilde{G}_{\rm tr}(\boldsymbol{p}) = \sum_{\mu\nu} \left(1 - \frac{\hat{p}_\mu \hat{p}_\nu}{\hat{p}^2}\right) \tilde{G}_{\mu\nu}(\boldsymbol{p}). \qquad (41)$$

and a transverse susceptibility $\chi_{\rm tr} = \tilde{G}_{\rm tr}(\boldsymbol{p}_a)$. Using the parametrization (22) we obtain

$$\chi_{\rm tr} = (d-1)(c_{L1} - c_{L2}). \qquad (42)$$

Equation (35) then implies

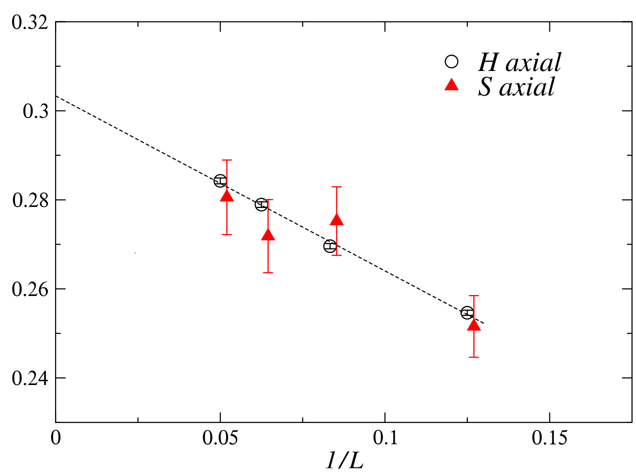

FIG. 5. Higgs phase: estimates of χ_{11} (hard axial gauge) and $\chi_{11} - 1$ (soft axial gauge with $\zeta = 1$), versus $1/L$, in the Higgs phase, $J = 0.4$. Soft axial-gauge data have been slightly moved to the right to improve readability. The line is only meant to guide the eye, since we have no theoretical understanding of the possible origin of the $1/L$ correction.

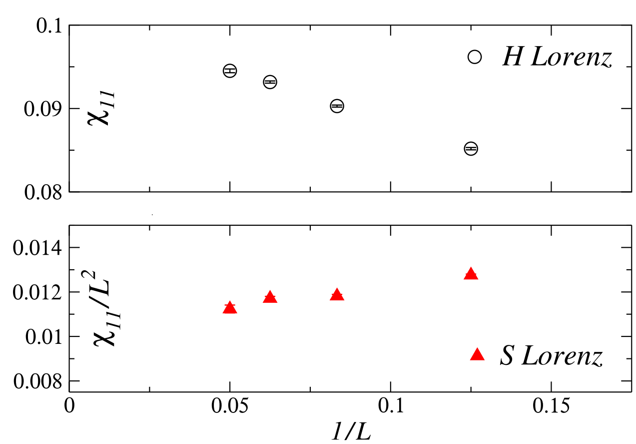

FIG. 6. Higgs phase. Top: estimates of χ_{11} in the hard Lorenz gauge. Bottom: estimates of χ_{11}/L^2 in the soft Lorenz gauge with $\zeta = 1$. Results in the Higgs phase for $J = 0.4$.

$$\chi_{\rm tr}(\zeta_1) = \chi_{\rm tr}(\zeta_2). \tag{43}$$

The transverse susceptibility is independent of ζ and therefore is the same in hard and soft gauges. In particular, $\chi_{\rm tr}(\zeta)$ is finite in the Higgs phase for all values of ζ, as expected.

The behavior of $\langle B_{\boldsymbol{x}}\rangle$ and of the corresponding susceptibility is analogous to that observed in the Coulomb phase; see Fig. 7. The average $\langle B_{\boldsymbol{x}}\rangle$ is well defined only for Lorenz gauges. In the axial gauge, we have instead $\langle B_{\boldsymbol{x}}\rangle \sim L$. This is not unexpected since the argument we have presented in the previous section, i.e., that the divergence of $\langle B_{\boldsymbol{x}}\rangle$ is related to the large number of quasizero modes present in the axial case, does not rely on any particular property of the two phases.

Finally, let us consider the susceptibility χ_B. Not surprisingly, in the axial-gauge data are consistent with a behavior $\chi_B \sim L^3$, as in the Coulomb phase. In the hard Lorenz case, we observe that χ_B is finite as L increases. This is the expected behavior in the Higgs phase, in which the photon is massive. In the soft Lorenz gauge instead, data are consistent with $\chi_B \sim L$. It is easy to realize that this divergence is due to the contributions of the nonphysical longitudinal modes present for nonzero values of ζ. The linear divergence with L can be predicted by a simple argument. Let us assume that the hard-gauge correlation function has the form (at least for small values of $\boldsymbol{p}$)

$$\tilde{G}_{\mu\nu}(\boldsymbol{p}, \zeta = 0) = \frac{Z}{\hat{p}^2 + M^2}\left(\delta_{\mu\nu} - \frac{\hat{p}_\mu \hat{p}_\nu}{\hat{p}^2}\right), \tag{44}$$

and, as predicted by the Ward identities, that

$$\tilde{G}_{\mu\nu}(\boldsymbol{p}, \zeta) = \tilde{G}_{\mu\nu}(\boldsymbol{p}, \zeta = 0) + \zeta \frac{\hat{p}_\mu \hat{p}_\nu}{(\hat{p}^2)^2}. \tag{45}$$

In a Gaussian approximation—we neglect irreducible four-field contributions—we have

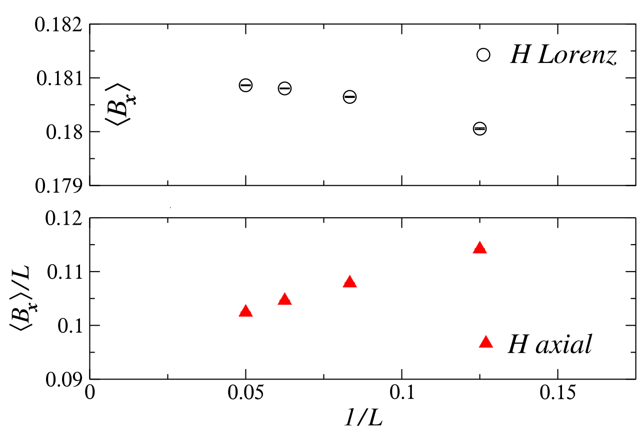

FIG. 7. Higgs phase. Top: estimates of $\langle B_{\boldsymbol{x}}\rangle$ in the hard Lorenz gauge. Bottom: estimates of $\langle B_{\boldsymbol{x}}\rangle/L$ in the hard axial gauge. Results in the Higgs phase for $J = 0.4$.

$$\chi_B = \frac{2}{L^d}\sum_{\boldsymbol{p}}\sum_{\mu\nu} \tilde{G}_{\mu\nu}(\boldsymbol{p}, \zeta)\tilde{G}_{\mu\nu}(-\boldsymbol{p}, \zeta), \tag{46}$$

and therefore,

$$\chi_B = 2(d-1)Z^2 \frac{1}{L^d}\sum_{\boldsymbol{p}} \frac{1}{(\hat{p}^2 + M^2)^2} + 2\zeta \frac{1}{L^d}\sum_{\boldsymbol{p}} \frac{1}{(\hat{p}^2)^2}. \tag{47}$$

The first sum has a finite limit as $L \to \infty$, while the second one, see Appendix A, diverges as L and $\ln L$ in $d = 3$ and $d = 4$, respectively. Thus, in three dimensions the longitudinal modes give rise to a contribution that increases as L, in agreement with the numerical results. We conclude that the photon-mass operator is not well defined nonperturbatively in the Lorenz soft gauge, because of the contributions of the nonphysical longitudinal modes. Apparently, only the hard Lorenz gauge is a consistent gauge fixing in which the operator is correctly defined.

V. SOME FIELD-THEORY RESULTS

The results of the previous sections can be combined with QFT results to obtain some general predictions of the behavior of Abelian gauge systems at charged fixed points.

First, let us note that our previous results also allow us to predict that the anomalous dimension of the gauge field is the same in the axial gauge as in the Lorenz gauge. Indeed, as we have discussed before, the large-scale behavior of the susceptibilities $\chi_{\mu\nu}$ (for $\mu, \nu < d$) is the same for all gauge fixings (although some caution should be exercised in the soft Lorenz case). Indeed, a summary of the results obtained is the following:

(i) the susceptibilities $\chi_{\mu\nu}$ ($\mu, \nu < d$) in the hard Lorenz and in the hard axial gauge differ only by a multiplicative constant: $2d/(d-1)$ for $\mu = \nu$ and $-d$ for $\mu \neq \nu$, see Eq. (24);

(ii) the susceptibilities in the hard and soft axial gauges differ by an additive constant, see Eq. (30);

(iii) the susceptibilities in the hard and soft Lorenz gauge behave differently, because of the coupling with the longitudinal modes. If one considers the transverse definition, see Eq. (41), results are independent of ζ, i.e., are the same in the hard and soft case.

For the soft Lorenz gauge, one can prove to all orders of perturbation theory that [3,60] $\eta_A = 4 - d$, independent of the nature of the matter fields. Indeed, the proof only relies on the relation $Z_A Z_e = 1$ between the renormalization constants of the gauge field and of the electric charge e. This implies [3]

$$\beta_{e^2} = e_r^2(d - 4 + \eta_A), \tag{48}$$

which connects the anomalous dimension $\eta_A = \frac{\mathrm{d}\log Z_A}{\mathrm{d}\log \mu}$ of the gauge field (μ is the RG energy scale) and the β function of the gauge coupling $\beta_{e^2} = \frac{\mathrm{d}e_r^2}{\mathrm{d}\log \mu}$, where $e_r^2 = e_B^2 \mu^{d-4}/Z_e$. In this expression the beta function β_{e^2} and the anomalous dimension η_A of the field are functions of the renormalized Lagrangian parameters; in particular they can be computed in perturbation theory, as an expansion in powers of e_r^2 and the other Lagrangian parameters. At a transition which is associated with a charged fixed point, i.e., where the gauge theory provides the effective critical behavior, we have $e_r^2 \neq 0$. Therefore, the fixed-point condition $\beta_{e^2} = 0$ implies [20,60]

$$\eta_A = 4 - d. \tag{49}$$

Numerical results [52] for the three-dimensional Abelian-Higgs model are in full agreement with this prediction.

A second interesting result concerns the parameter ζ that parametrizes the soft gauges. As a consequence of the Ward identities discussed in Sec. B, in the soft Lorenz gauge we have $\zeta = \zeta_r Z_A$, which implies

$$\beta_\zeta = -\zeta_r \eta_A. \tag{50}$$

The value $\zeta = 0$ is a fixed point of this equation, as expected. Indeed, if we start from a model with a purely transverse gauge field, no longitudinal contributions are generated by the RG flow. Instead, if we start the flow from a value $\zeta \neq 0$, ζ flows toward $+\infty$, indicating that the hard-gauge fixing is an unstable fixed point, at least for $d < 4$. Moreover, for $\zeta \neq 0$ the large-scale behavior is singular, as the non-gauge-invariant modes become unbounded under the RG transformations. Therefore, also QFT (which describes the critical behavior at charged transitions) predicts that only the hard Lorenz-gauge fixing provides a consistent definition of non-gauge-invariant quantities at the critical point in three dimensions.

Equations (49) and (50) allow us to predict the crossover behavior of $\chi_{\mu\nu}$ at a critical charged transition point in the soft Lorenz gauges. For $d < 4$ we predict

$$\chi_{\mu\nu}(\zeta) = L^{2-\eta_A} f_{\mu\nu}(\zeta L^{\eta_A}) = L^{d-2} f_{\mu\nu}(\zeta L^{4-d}). \tag{51}$$

This relation should hold for $L \to \infty$, $\zeta \to 0$ at fixed ζL^{4-d}. The function $f_{\mu\nu}(x)$ can be computed using Eq. (35). If $\chi_{\mu\nu}(\zeta = 0) \approx a_{\mu\nu} L^{d-2}$ for $L \to \infty$, Eq. (35) implies

$$\begin{aligned}\chi_{\mu\nu}(\zeta) &\approx a_{\mu\nu} L^{d-2} + \frac{\zeta L^2}{d^2\pi^2} \\ &= L^{d-2}\left(a_{\mu\nu} + \frac{1}{d^2\pi^2}\zeta L^{4-d}\right),\end{aligned} \tag{52}$$

so that $f_{\mu\nu}(x) = a_{\mu\nu} + x/(d^2\pi^2)$.

VI. CONCLUSIONS

In this work we investigate the behavior of gauge correlations in Abelian gauge theories with noncompact gauge fields. Because of the unbounded nature of the fluctuations of the gauge fields, a rigorous definition of the model requires the introduction of a gauge-fixing term. This is at variance with compact formulations (for instance, models with Wilson action), in which a gauge fixing is not required to make the model well defined. Here we consider two widely used gauge fixings, the axial and Lorenz one. We also distinguish between hard-gauge fixings—in this case the partition function is given in Eq. (6)—and soft ones depending on a parameter ζ—the corresponding partition function is given in Eq. (8).

Gauge-invariant correlations are obviously independent of the gauge-fixing procedure. On the other hand, the large-scale behavior of gauge-dependent quantities may have a nontrivial dependence. Here we first consider correlations of the gauge field $A_{\boldsymbol{x},\mu}$ and we derive general relations, independent of the nature of the matter couplings, between these correlations computed in the presence of different gauge fixings. Second, we consider the photon-mass composite operator $A_{\boldsymbol{x},\mu}^2$, which is usually introduced in the action, in perturbative calculations, as an infrared regulator of the theory.

As a specific example, we analyze the behavior of these correlation functions in the three-dimensional Abelian-Higgs model, in which an N-component complex scalar field is coupled with a noncompact real Abelian gauge field. In particular, we study their behavior in the so-called Coulomb and Higgs phases (see Fig. 1 for a sketch of the phase diagram). In the Coulomb phase, the correlation function $\tilde{G}_{\mu\nu}(\boldsymbol{p})$ of the gauge fields has the same small-momentum behavior as in the absence of matter fields, for all gauge fixings considered. In particular, the susceptibility $\chi_{\mu\nu}$ defined in Eq. (16) diverges as L^2 in the infinite-volume limit. In the Higgs phase, we expect the photon to be massive and therefore $\chi_{\mu\nu}$ should be finite as $L \to \infty$. This turns out to be true for the axial soft and hard gauges and for the hard Lorenz gauge. On the other hand, $\chi_{\mu\nu} \sim L^2$ in the soft Lorenz gauge. This divergence is caused by the unphysical contributions due to the longitudinal modes that propagate in the soft Lorenz gauge.

While the behavior of $\tilde{G}_{\mu\nu}(\boldsymbol{p})$ in all gauges is consistent with the general picture that the photon is massless/massive in the Coulomb/Higgs phase, the interpretation of the results for the photon-mass operator $B_{\boldsymbol{x}} = \sum_\mu A_{\boldsymbol{x},\mu}^2$ is more complicated. If we consider the soft and hard axial gauges, we find $\langle B_{\boldsymbol{x}} \rangle \sim L$ in both phases. The operator does not have a well-defined infinite-volume limit. The divergence is due to the presence of a $(d-1)$-dimensional family of quasizero modes, so that $A_{\boldsymbol{x},\mu}$ develops infinite-range fluctuations in the infinite-volume limit. Therefore, if an axial-gauge fixing is used, $B_{\boldsymbol{x}}$ cannot be defined nonperturbatively. In the soft and hard Lorenz gauge, the average $\langle B_{\boldsymbol{x}} \rangle$ is finite as $L \to \infty$ in both phases, and thus the operator is well defined. However, in the Higgs phase, the susceptibility χ_B defined in Eq. (16) behaves differently in the hard and soft case. In the hard case, χ_B has a finite infinite-volume limit, as expected—the photon mass is finite. Instead, χ_B diverges as L in the soft gauge. This divergence is due to the longitudinal modes that are not fully suppressed.

The results presented here show that neither the axial gauge nor the soft Lorenz gauge are appropriate for the study of generic gauge-dependent correlation functions. More precisely, we show that in these gauges some correlators of the gauge field do not have a smooth thermodynamic limit and cannot be used to characterize the large-distance behavior of the gauge field. For instance, in the axial and soft Lorenz-gauge A_μ^2 correlators cannot be used to distinguish the Higgs phase in which photons are massive from the Coulomb phase in which photons are massless, since they diverge in both the phases in the thermodynamic limit. Of course, this does not mean that these gauge fixings cannot be used *tout court*, since other gauge-dependent correlators may in principle be well defined and encode physically relevant information. Note, however, that also scalar correlators have analogous issues; see Appendix A of Ref. [52].

Axial gauges suffer from the existence of an infinite family of quasizero modes, giving rise to spurious divergences, unrelated to the presence of long-range physical correlations. Soft Lorenz gauges suffer instead from the presence of propagating unphysical longitudinal modes, that, at least for $d < 4$ and therefore in three dimensions, may hide the physical signal. Apparently, only the hard Lorenz-gauge fixing provides a consistent model in which gauge-dependent correlations have the expected large-scale (small-momentum) behavior. It is interesting to observe that also QFT singles out the hard Lorenz gauge as the gauge of choice for the study of gauge correlations. Note that the shortcomings of the axial gauge and of the soft Lorenz gauge are not related to the nature of the matter fields but are due to intrinsic properties of the gauge fixings. Therefore, our conclusions should be relevant also for systems in which fermions are present.

ACKNOWLEDGMENTS

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APPENDIX A: CRITICAL BEHAVIOR IN THE U(1) ABELIAN GAUGE THEORY

In this appendix we summarize the expressions of the observables defined in Sec. II for the free U(1) gauge theory, i.e., in the absence of matter fields. The susceptibilities $\chi_{\mu\nu}$ can be trivially derived from the small-momentum behavior of $\tilde{G}_{\mu\nu}(\boldsymbol{p})$ defined in Sec. II. Moreover, we have

$$\langle B_{\boldsymbol{x}} \rangle = \frac{1}{V} \sum_{\boldsymbol{p}} \sum_{\mu\nu} \tilde{G}_{\mu\nu}(\boldsymbol{p}), \tag{A1}$$

$$\chi_B = \frac{2}{V} \sum_{\boldsymbol{p}} \sum_{\mu\nu} [\tilde{G}_{\mu\nu}(\boldsymbol{p})]^2. \tag{A2}$$

Because of the C^* boundary conditions the sums go over the momenta

$$\boldsymbol{p} = \frac{\pi}{L}(2n_1 + 1, 2n_2 + 1, 2n_3 + 1), \tag{A3}$$

with $0 \le n_i < L$.

1. Lorenz gauge

In the Lorenz gauge, the propagator $\tilde{G}_{\mu\nu}(\boldsymbol{p})$ is given by

$$\tilde{G}_{\mu\nu}(\boldsymbol{p}) = \frac{1}{\kappa} \frac{\delta_{\mu\nu}}{\hat{p}^2} + \frac{\zeta\kappa - 1}{\kappa} \frac{\hat{p}_\mu \hat{p}_\nu}{(\hat{p}^2)^2}, \tag{A4}$$

where $\hat{p}_\mu = 2 \sin p_\mu/2$ and $\hat{p}^2 = \sum_\mu \hat{p}_\mu^2$. It follows that

$$\begin{aligned} \chi_{\mu\nu} &= (d\delta_{\mu\nu} + \zeta\kappa - 1)\frac{1}{\kappa d^2 \hat{p}_{\min}^2}, \\ \langle B_{\boldsymbol{x}} \rangle &= \frac{d - 1 + \zeta\kappa}{\kappa} I_{d,1}(L), \\ \chi_B &= \frac{2(d - 1 + \zeta^2\kappa^2)}{\kappa^2} I_{d,2}(L), \end{aligned} \tag{A5}$$

where $p_{\min} = \pi/L$ and

$$I_{d,n}(L) = \frac{1}{L^d} \sum_{\boldsymbol{p}} \frac{1}{(\hat{p}^2)^n}. \tag{A6}$$

The behavior of the sums $I_{d,n}$ depends on the dimension d. For $d > 2$, $I_{d,1}$ has a finite limit for $L \to \infty$, while it diverges logarithmically in $d = 2$. In particular, in $d = 3$ we have [61,62]

$$\begin{aligned} I_{3,1}(L) &\approx \int_{[-\pi,\pi]^3} \frac{d^3 p}{(2\pi)^3} \frac{1}{\hat{p}} \\ &= \frac{1}{192\pi^3} (\sqrt{3} - 1) \Gamma\left(\frac{1}{24}\right)^2 \Gamma\left(\frac{11}{24}\right)^2 \\ &\approx 0.252731. \end{aligned} \tag{A7}$$

Instead, the sum $I_{d,2}$ diverges for $L \to \infty$ in dimension $d \le 4$, as L^{4-d} (as $\ln L$ in $d = 4$). We find

$$I_{3,2}(L) \approx a_2 L[1 + O(L^{-1})], \qquad a_2 \approx 0.015216,$$
$$I_{4,2}(L) \approx a_2 \ln L + O(1), \qquad a_2 \approx \frac{1}{8\pi^2}. \tag{A8}$$

Thus, in three dimensions the susceptibilities $\chi_{\mu\nu}$ and χ_B diverge as L^2 and L, respectively, while $\langle B_{\boldsymbol{x}} \rangle$ is finite.

2. Axial gauge

In the axial gauge

$$\tilde{G}_{\mu\nu}(\boldsymbol{p}) = \frac{1}{\kappa}\frac{\delta_{\mu\nu}}{\hat{p}^2} + \frac{\hat{p}_\mu \hat{p}_\nu}{\hat{p}_d^2}\left(\frac{1}{\kappa \hat{p}^2} + \zeta\right), \tag{A9}$$

if both μ and ν are not equal to d. Otherwise, we have

$$\tilde{G}_{\mu\nu}(\boldsymbol{p}) = \zeta \frac{\hat{p}_\mu \hat{p}_\nu}{\hat{p}_d^2}. \tag{A10}$$

As for the susceptibilities, we find $\chi_{d\mu} = \zeta$ and, for $\mu, \nu < d$,

$$\chi_{\mu\nu} \frac{\delta_{\mu\nu} + 1}{d\kappa} \frac{1}{\hat{p}^2_{\min}} + \zeta \approx \frac{\delta_{\mu\nu} + 1}{d\kappa\pi^2} L^2, \tag{A11}$$

where $p_{\min} = \pi/L$. As expected, $\chi_{d\mu}$ is finite (it vanishes in the hard axial gauge for which $\zeta = 0$), while the other susceptibility components diverge as L^2. Although the large-L behavior is the same as in the Lorenz case, here the asymptotic behavior is ζ independent: The susceptibilities behave identically in the hard and soft axial case, a result that does not hold in the Lorenz case.

As for $\langle B_{\boldsymbol{x}} \rangle$ and χ_B we find

$$\langle B_{\boldsymbol{x}} \rangle = \frac{1}{\kappa}((d-2)I_{d,1} + J_1) + \zeta(1 + (d-1)J_1 J_{-1}),$$
$$\chi_B = 2\zeta^2 + 2\zeta^2(d-1)(2J_1 J_{-1} + (d-2)J_2 J_{-1}^2 + J_2 J_{-2}) + \frac{2}{\kappa^2}((d-2)I_{d,2} + J_2 + 2(d-1)\zeta\kappa J_2 J_{-1}), \tag{A12}$$

where the quantities $J_n(L)$ correspond to the one-dimensional sums $[p = (2n+1)\pi/L$ with $n = 0, \ldots, L-1]$,

$$J_n(L) = \frac{1}{L}\sum_p \hat{p}^{-2n}. \tag{A13}$$

Since we have (these expressions can be derived as in Appendix B. 1. d of Ref. [63])

$$J_2 = \frac{1}{48}L(L^2 + 2),$$
$$J_1 = \frac{L}{4},$$
$$J_{-1} = 2,$$
$$J_{-2} = 6, \tag{A14}$$

we obtain for large values of L for $d > 2$:

$$\langle B_{\boldsymbol{x}} \rangle \approx \frac{1 + 2(d-1)\zeta\kappa}{4\kappa} L,$$
$$\chi_B \approx \frac{1}{24\kappa^2}[1 + 4(d-1)\zeta\kappa + 2(d-1)(2d-1)\zeta^2\kappa^2]L^3. \tag{A15}$$

Note that $\langle B_{\boldsymbol{x}} \rangle$ diverges, at variance with what happens in the Lorenz case. From a technical point of view this is due to the fact that the axial-gauge propagator is more divergent than the Lorenz one: Indeed, in the axial gauge $\tilde{G}_{\mu\nu}(\boldsymbol{p})$ diverges as $p_d \to 0$, for any value of the other momentum components, while in the Lorenz gauge a divergence is only observed as $|\boldsymbol{p}| \to 0$. More intuitively, note that, in infinite volume, the axial-gauge-fixed Hamiltonian is still invariant under the gauge transformations (3) if the function $\phi_{\boldsymbol{x}} = \phi_{(x_1,\ldots,x_d)}$ depends on x_i with $i < d$ only. This should be compared with the Lorenz case, in which only gauge transformations with $\Delta_\mu \phi_{\boldsymbol{x}} = c_\mu$, where c_μ is x independent, leave the infinite-volume gauge-fixed Hamiltonian invariant. The presence of this large family of quasizero modes is responsible for the divergence of the variance of $A_{\boldsymbol{x},\mu}$ for $\mu < d$.

3. Some other gauge fixings

It is interesting to note that the results for the Lorenz gauge apply only to the discretization (10). If instead the discretization (38) is used, different results are obtained. Indeed, in the latter case, in the infinite-volume limit, the gauge-fixed Hamiltonian is invariant under a large family of gauge transformations. For instance, one can consider transformations like those reported in Eq. (39). To determine the full set of transformations that leave $F_{L',\boldsymbol{x}}(A)$ invariant in infinite volume, we work in Fourier space and consider a function $\phi_{\boldsymbol{x}}$ of the form

$$\phi_{\boldsymbol{x}} = a e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \bar{a} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}, \tag{A16}$$

where a is an arbitrary complex constant. These transformations leave $F_{L',\boldsymbol{x}}(A)$ invariant, if at least one of these two conditions is satisfied:

$$\sum_\mu \cos p_\mu (1-\cos p_\mu) = 0,$$
$$\sum_\mu \sin p_\mu (1-\cos p_\mu) = 0. \tag{A17}$$

If only the first (the second) equation is satisfied, then a is necessarily real (purely imaginary). The transformation (39) corresponds to taking $\boldsymbol{p} = (\pi/2, -\pi/2, \pi/2, -\pi/2)$ and a real constant a. We have studied numerically the equations (A17) in three dimensions, finding that both equations are satisfied on a two-dimensional surface in momentum space. The presence of this family of gauge transformations that leave the Hamiltonian invariant, implies that the correlation function $\tilde{G}_{\mu\nu}(\boldsymbol{p})$ is singular in $\boldsymbol{p}$ space. In turn, this implies (we have performed a numerical check) the divergence of the variance of $A_{\boldsymbol{x}}$, as it also occurs in the axial gauge.

Finally, we would like to make some comments on the Coulomb gauge that we can define as

$$F_{L,\boldsymbol{x}}(A) = \sum_{\mu=1}^{d-1} (A_{\boldsymbol{x},\mu} - A_{\boldsymbol{x}-\hat{\mu},\mu}). \tag{A18}$$

In the hard case $\zeta = 0$, the correlation function is given by

$$\tilde{G}_{\mu\nu}(\boldsymbol{p}) = \frac{1}{\kappa}\frac{\delta_{\mu\nu}}{\hat{p}^2} - \frac{1}{\kappa}\frac{p_\mu p_\nu}{\hat{p}^2 \hat{p}_T^2}, \qquad \mu,\nu < d,$$
$$\tilde{G}_{d\mu} = 0, \qquad \mu < d,$$
$$\tilde{G}_{dd} = \frac{1}{\kappa}\frac{1}{\hat{p}_T^2} \tag{A19}$$

where $\hat{p}_T^2 = \sum_{\mu=1}^{d-1} \hat{p}_\mu^2$. The susceptibilities diverge as L^2 while $\langle B_{\boldsymbol{x}} \rangle$ is given by

$$\langle B_{\boldsymbol{x}} \rangle = \frac{1}{\kappa}(2I_{d,1} + I_{d-1,1}). \tag{A20}$$

In four dimensions, both sums are finite; therefore $\langle B_{\boldsymbol{x}} \rangle$ is well defined. In three dimensions, however, the result depends on the two-dimensional sum $I_{2,1}$, which diverges logarithmically. Therefore, for $d = 3$, the photon-mass operator is not well defined in the Coulomb gauge.

APPENDIX B: WARD IDENTITIES IN DIFFERENT GAUGES

A crucial ingredient in the derivations presented in Sec. III is the Ward identities satisfied by the correlation functions. We derive them here for the generic gauge-fixing function introduced in Sec. III; see Eq. (5). The corresponding function $S_{\rm GF}(A)$ defined in Eq. (7) is given by

$$S_M(A) = \frac{1}{2\zeta V}\sum_{\boldsymbol{p}\alpha\beta} M_\alpha(\boldsymbol{p}) M_\beta(-\boldsymbol{p}) e^{-i(p_\alpha - p_\beta)/2} \times \tilde{A}_\alpha(\boldsymbol{p})\tilde{A}_\beta(-\boldsymbol{p}). \tag{B1}$$

Under an infinitesimal gauge transformation, we have

$$\delta S_M = \frac{1}{V}\sum_{\boldsymbol{p}} \delta_M(\boldsymbol{p})\tilde{\phi}(\boldsymbol{p}),$$
$$\delta_M(\boldsymbol{p}) = \frac{1}{\zeta}\sum_{\alpha\beta} M_\alpha(\boldsymbol{p}) M_\beta(-\boldsymbol{p}) e^{-i(p_\alpha - p_\beta)/2} \times (i\hat{p}_\alpha)\tilde{A}_\beta(-\boldsymbol{p}). \tag{B2}$$

If we now consider $\langle A_{\boldsymbol{x},\gamma} \rangle$ and require its invariance under changes of variable represented by infinitesimal gauge transformations, we obtain

$$\langle \Delta_\gamma \phi_{\boldsymbol{x}} + A_{\boldsymbol{x},\gamma}\delta S_M \rangle = 0. \tag{B3}$$

In Fourier space, this implies the relation

$$\frac{1}{\zeta}\sum_{\alpha\beta} M_\alpha(\boldsymbol{p}) M_\beta(-\boldsymbol{p}) e^{-i(p_\alpha - p_\beta)/2}\hat{p}_\alpha \tilde{G}_{\gamma\beta}(\boldsymbol{p}) = \hat{p}_\gamma. \tag{B4}$$

In the axial gauge we have $M_\alpha(\boldsymbol{p}) = \delta_{\alpha d}$, while in the Lorenz gauge we have $M_\alpha(\boldsymbol{p}) = -e^{ip_\alpha/2} i\hat{p}_\alpha$. Substituting these relations in Eq. (B4), we obtain Eqs. (28) and (33).

[1] I. Montvay and G. Münster, *Quantum Fields on a Lattice* (Cambridge University Press, Cambridge, England, 1994).
[2] T. DeGrand and C. DeTar, *Lattice Methods for Quantum Chromodynamics* (World Scientific, Singapore, 2006).
[3] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 2002).
[4] A. Pelissetto and E. Vicari, Critical phenomena and renormalization group theory, Phys. Rep. **368**, 549 (2002).
[5] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 2011).
[6] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lect. Notes Phys. Vol. 159 (Springer-Verlag, Berlin, 1982).
[7] J. Glimm and A. Jaffe, *Quantum Physics* (Springer-Verlag, Berlin, 1987).

[8] D. J. Gross and F. Wilczek, Ultraviolet Behavior of Nonabelian Gauge Theories, Phys. Rev. Lett. **30**, 1343 (1973).
[9] H. D. Politzer, Reliable Perturbative Results for Strong Interactions?, Phys. Rev. Lett. **30**, 1346 (1973).
[10] S. R. Coleman and D. J. Gross, Price of Asymptotic Freedom, Phys. Rev. Lett. **31**, 851 (1973).
[11] M. Moshe and J. Zinn-Justin, Quantum field theory in the large N limit: A review, Phys. Rep. **385**, 69 (2003).
[12] K. G. Wilson and J. B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. **12**, 75 (1974).
[13] G. Bhanot and M. Creutz, The phase diagram of $Z(n)$ and $U(1)$ gauge theories in three-dimensions, Phys. Rev. D **21**, 2892 (1980).
[14] M. Caselle, M. Panero, R. Pellegrini, and D. Vadacchino, A different kind of string, J. High Energy Phys. 01 (2015) 105.
[15] A. Athenodorou and M. Teper, On the spectrum and string tension of U(1) lattice gauge theory in $2+1$ dimensions, J. High Energy Phys. 01 (**2019**) 063.
[16] M. J. Teper, SU(N) gauge theories in $(2+1)$-dimensions, Phys. Rev. D **59**, 014512 (1998).
[17] A. Athenodorou and M. Teper, SU(N) gauge theories in $2+1$ dimensions: Glueball spectra and k-string tensions, J. High Energy Phys. 02 (**2017**) 015.
[18] C. Bonati and S. Morlacchi, Flux tubes and string breaking in three dimensional SU(2) Yang-Mills theory, Phys. Rev. D **101**, 094506 (2020).
[19] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, Quantum criticality beyond the Landau-Ginzburg-Wilson paradigm, Phys. Rev. B **70**, 144407 (2004).
[20] I. Herbut, *A Modern Approach to Critical Phenomena* (Cambridge University Press, Cambridge, England, 2007).
[21] E. Fradkin, *Field Theories of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 2013).
[22] S. Sachdev, Topological order, emergent gauge fields, and Fermi surface reconstruction, Rep. Prog. Phys. **82**, 014001 (2019).
[23] R. Moessner and J. E. Moore, *Topological Phases of Matter* (Cambridge University Press, Cambridge, England, 2021).
[24] B. I. Halperin, T. C. Lubensky, and S. K. Ma, First-Order Phase Transitions in Superconductors and Smectic-A Liquid Crystals, Phys. Rev. Lett. **32**, 292 (1974).
[25] R. Folk and Y. Holovatch, On the critical fluctuations in superconductors, J. Phys. A **29**, 3409 (1996).
[26] B. Ihrig, N. Zerf, P. Marquard, I. F. Herbut, and M. M. Scherer, Abelian Higgs model at four loops, fixed-point collision and deconfined criticality, Phys. Rev. B **100**, 134507 (2019).
[27] A. Das, Phase transition in $SU(N)\times U(1)$ gauge theory with many fundamental bosons, Phys. Rev. B **97**, 214429 (2018).
[28] S. Sachdev, H. D. Scammell, M. S. Scheurer, and G. Tarnopolsky, Gauge theory for the cuprates near optimal doping, Phys. Rev. B **99**, 054516 (2019).
[29] C. Bonati, A. Franchi, A. Pelissetto, and E. Vicari, Three-dimensional lattice SU(N_c) gauge theories with multiflavor scalar fields in the adjoint representation, Phys. Rev. B **104**, 115166 (2021).
[30] C. Bonati, A. Franchi, A. Pelissetto, and E. Vicari, Phase diagram and Higgs phases of three-dimensional lattice SU(N_c) gauge theories with multiparameter scalar potentials, Phys. Rev. E **104**, 064111 (2021).
[31] J. Braun, H. Gies, L. Janssen, and D. Roscher, Phase structure of many-flavor QED_3, Phys. Rev. D **90**, 036002 (2014).
[32] I. F. Herbut, Chiral symmetry breaking in three-dimensional quantum electrodynamics as fixed point annihilation, Phys. Rev. D **94**, 025036 (2016).
[33] B. Ihrig, L. Janssen, L. N. Mihaila, and M. M. Scherer, Deconfined criticality from the QED_3-Gross-Neveu model at three loops, Phys. Rev. B **98**, 115163 (2018).
[34] J. A. Gracey, Fermion bilinear operator critical exponents at $O(1/N^2)$ in the QED-Gross-Neveu universality class, Phys. Rev. D **98**, 085012 (2018).
[35] C. Bonati, A. Pelissetto, and E. Vicari, Lattice Abelian-Higgs model with noncompact gauge fields, Phys. Rev. B **103**, 085104 (2021).
[36] C. Bonati and N. Francini, Noncompact lattice Higgs model with Abelian discrete gauge groups: Phase diagram and gauge symmetry enlargement, Phys. Rev. B **107**, 035106 (2023).
[37] C. Bonati, A. Pelissetto, and E. Vicari, Higher-charge three-dimensional compact lattice Abelian-Higgs models, Phys. Rev. E **102**, 062151 (2020).
[38] C. Bonati, A. Pelissetto, and E. Vicari, Critical behaviors of lattice U(1) gauge models and three-dimensional Abelian-Higgs gauge field theory, Phys. Rev. B **105**, 085112 (2022).
[39] A. Pelissetto and E. Vicari, Three-dimensional ferromagnetic CP(N-1) models, Phys. Rev. E **100**, 022122 (2019).
[40] A. Pelissetto and E. Vicari, Large-N behavior of three-dimensional lattice CP^{N-1} models, J. Stat. Mech. (2020) 033209.
[41] A. Pelissetto and E. Vicari, Multicomponent compact Abelian-Higgs lattice models, Phys. Rev. E **100**, 042134 (2019).
[42] G. Bracci-Testasecca and A. Pelissetto, Multicomponent gauge-Higgs models with discrete Abelian gauge groups, J. Stat. Mech. (**2023**) 043101.
[43] O. I. Motrunich and A. Vishwanath, Comparative study of Higgs transition in one-component and two-component lattice superconductor models, arXiv:0805.1494.
[44] A. B. Kuklov, M. Matsumoto, N. V. Prokof'ev, B. V. Svistunov, and M. Troyer, Deconfined Criticality: Generic First-Order Transition in the SU(2) Symmetry Case, Phys. Rev. Lett. **101**, 050405 (2008).
[45] O. I. Motrunich and A. Vishwanath, Emergent photons and transitions in the O(3) σ-model with hedgehog suppression, Phys. Rev. B **70**, 075104 (2004).
[46] G. Murthy and S. Sachdev, Actions of hedgehogs instantons in the disordered phase of $2+1$ dimensional CP^{N-1} model, Nucl. Phys. **B344**, 557 (1990).
[47] G. J. Sreejith and S. Powell, Scaling dimensions of higher-charge monopoles at deconfined critical points, Phys. Rev. B **92**, 184413 (2015).
[48] A. Pelissetto and E. Vicari, Three-dimensional monopole-free CP^{N-1} models, Phys. Rev. E **101**, 062136 (2020).

[49] C. Bonati, A. Pelissetto, and E. Vicari, Three-dimensional monopole-free CP^{N-1} models: Behavior in the presence of a quartic potential, J. Stat. Mech. (2022) 063206.

[50] C. Bonati, A. Pelissetto, and E. Vicari, Breaking of Gauge Symmetry in Lattice Gauge Theories, Phys. Rev. Lett. **127**, 091601 (2021).

[51] C. Bonati, A. Pelissetto, and E. Vicari, Lattice gauge theories in the presence of a linear gauge-symmetry breaking, Phys. Rev. E **104**, 014140 (2021).

[52] C. Bonati, A. Pelissetto, and E. Vicari, The Coulomb-Higgs phase transition of three-dimensional lattice Abelian-Higgs gauge models with noncompact gauge variables and gauge fixing, arXiv:2305.15236.

[53] S. Elitzur, Impossibility of spontaneously breaking local symmetries, Phys. Rev. D **12**, 3978 (1975).

[54] G. F. De Angelis, D. de Falco, and F. Guerra, A note on the Abelian Higgs-Kibble model on a lattice: Absence of spontaneous magnetization, Phys. Rev. D **17**, 1624 (1978).

[55] C. Itzykson and J. M. Drouffe, *Statistical Field Theory. Vol. 1: From Brownian Motion to Renormalization and Lattice Gauge Theory* (Cambridge University Press, Cambridge, England, 1989).

[56] A. S. Kronfeld and U. J. Wiese, SU(N) gauge theories with C periodic boundary conditions. 1. Topological structure, Nucl. Phys. **B357**, 521 (1991).

[57] B. Lucini, A. Patella, A. Ramos, and N. Tantalo, Charged hadrons in local finite-volume QED + QCD with C^* boundary conditions, J. High Energy Phys. 02 (2016) 076.

[58] H. J. Rothe, *Lattice Gauge Theories. An Introduction* (World Scientific, Singapore, 2005).

[59] G. Leibbrandt, Introduction to noncovariant gauges, Rev. Mod. Phys. **59**, 1067 (1987).

[60] I. F. Herbut and Z. Tešanović, Critical Fluctuations in Superconductors and the Magnetic Field Penetration Depth, Phys. Rev. Lett. **76**, 4588 (1996).

[61] M. L. Glasser and I. J. Zucker, Extended Watson integrals for the cubic lattices, Proc. Natl. Acad. Sci. U.S.A. **74**, 1800 (1977).

[62] G. S. Joyce and I. J. Zucker, Evaluation of the Watson integral and associated logarithmic integral for the d-dimensional hypercubic lattice, J. Phys. A **34**, 7349 (2001).

[63] S. Caracciolo and A. Pelissetto, Corrections to finite-size scaling in the lattice N-vector model for $N = \infty$, Phys. Rev. D **58**, 105007 (1998).