Is $f_2(1950)$ the tensor glueball?

Arthur Vereijken,[1,*] Shahriyar Jafarzade,[1,†] Milena Piotrowska,[1,‡] and Francesco Giacosa[1,2,§]

[1]*Institute of Physics, Jan Kochanowski University, ul. Uniwersytecka 7, 25-406, Kielce, Poland*
[2]*Institute for Theoretical Physics, J. W. Goethe University, Max-von-Laue-Straße. 1, 60438 Frankfurt am Main, Germany*

(Received 17 April 2023; accepted 22 June 2023; published 20 July 2023)

Glueballs remain an experimentally undiscovered expectation of QCD. Lattice QCD (As well as other theoretical approaches) predicts a spectrum of glueballs, with the tensor ($J^{PC} = 2^{++}$) glueball being the second lightest, behind the scalar glueball. Here, using a chiral hadronic model, we compute decay ratios of the tensor glueball into various meson decay channels. We find the tensor glueball to primarily decay into two vector mesons, dominated by $\rho\rho$ and K^*K^* channels. These results are compared to experimental data of decay rates of isoscalar tensor mesons. Based on this comparison, we make statements on the eligibility of these mesons as potential tensor glueball candidates: the resonance $f_2(1950)$ turns out to be, at present, the best match as being predominantly a tensor glueball.

DOI: 10.1103/PhysRevD.108.014023

I. INTRODUCTION

Glueballs are mesons made up by gluons only. They are one of the earliest predictions of quantum chromodynamics (QCD) that follow from the non-Abelian nature of the $SU(3)$ gauge symmetry and confinement [1].

The existence of glueballs has been since then confirmed by various theoretical approaches [2–5], most notably lattice QCD (LQCD), according to which a whole tower of such states has been computed [6–12] in the Yang-Mills (YM) sector of QCD (that is, QCD with without dynamical quarks; for an unquenched study see Ref. [13]). Similar outcomes were found in recent functional approaches in, e.g., [14,15]. The lightest gluonic state is a scalar ($J^{PC} = 0^{++}$), the second lightest a tensor $J^{PC} = 2^{++}$ (for the tensor/scalar mass ratio, see [16]), and the third lightest a pseudoscalar ($J^{PC} = 0^{-+}$). Quite interestingly, as shown in Ref. [17], the glueball spectrum of the recent LQCD compilation of Ref. [8] generates a glueball resonance gas that is consistent with the pressure of pure YM below T_c as also evaluated in LQCD [18], thus being a further confirmation of the existence and accuracy of the LQCD masses.

Yet, although glueballs are such a long-standing prediction of QCD, their experimental status is still inconclusive. Admittedly, some notable candidates do exist, especially for the three lightest ones, e.g., Ref. [4]. Nevertheless, the problem of a definitive identification of glueballs in experiments is made difficult by the mixing with nearby conventional quark-antiquark mesons and by their poorly known decay strength(s).

In this work, we concentrate on the tensor glueball, which is is interesting because of various reasons. First, as already mentioned, is the second lightest gluonium. Second, the experimental observation of several isoscalar-tensor resonances such as $f_2(1430)$, $f_2(1565)$, $f_2(1640)$, $f_2(1810)$, $f_2(1910)$, $f_2(1950)$, $f_2(2010)$, $f_2(2150)$, $f_J(2220)$ etc. implies that one of them (especially close to 2 GeV, see below) could be the tensor glueball. The attempt to interpret one of those f_2 states as a tensor glueball has a long history, see, e.g., [1,19–21]. In particular the $f_J(2220)$ was historically considered as a good candidate for the tensor glueball [22], but, as we shall see, this is no longer the case.

In LQCD simulations various physical masses for the tensor glueball are predicted such as 2150(30)(100) MeV [9], 2324(42)(32) MeV [10], 2376(32) MeV [8], 2390(30)(120) MeV [6], and 2400(25)(120) MeV [7]. Moreover, QCD sum rules predict the tensor glueball mass in the range of 2000–2300 MeV [23,24]. A recent functional method result for the tensor glueball mass is around 2610 MeV [14]. Holographic methods are also used to study glueballs for instance in [25–33], with tensor glueball masses ranging from 1900 to 4300 MeV depending on the model [30,32]. Thus, besides differences, the results converge on the existence of a tensor glueball in the region near 2000 MeV.

[*]arthur.vereijken@gmail.com
[†]Shahriyar.jzade@gmail.com
[‡]milena.piotrowska@ujk.edu.pl
[§]francesco.giacosa@gmail.com

In [27,28] the decays of the tensor glueball are computed in the so-called Witten-Sakai-Sugimoto model and it is found that this state is broad if the mass is above the $\rho\rho$ threshold. Other hadronic approaches are studied the decays of the tensor glueball [34,35], where different decay ratios were presented. Yet, an explicit inclusion of chiral symmetry was then not considered. The tensor glueball is also speculated to be related to the occurrence of Okubo Zweig-Iizuka (OZI) forbidden process such as $\pi^- p \to \phi\phi n$ [36].

A novel data analysis on J/ψ decays is looking promising for the identification of the scalar and tensor glueballs [1,37–39], with a tensor glueball mass of around 2210 MeV. Further recent measurable production of tensor glueballs in experiments are predicted in [40,41].

In view of this revival of both theoretical and experimental interest on the enigmatic tensor glueball, a quite natural question is what a chiral hadronic approach can tell about this state, especially in connection to certain decay ratios. The present paper deals exactly with this question: we study the tensor glueballs via a suitable extension of the linear sigma model (LSM), called the extended linear sigma model (eLSM), e.g., [42,43], in which chiral symmetry is linearly realized and undergoes both explicit and spontaneous symmetry breaking (SSB) and where, besides (pseudo)scalar mesons, also (axial-)vector states and the dilaton are included from the very beginning. Previous applications of the eLSM to the scalar [43,44], pseudoscalar [45,46], and vector [47] glueballs were performed. It is then natural to apply the formalism to the tensor gluonium.

Quite recently, we have employed the eLSM to study the tensor and axial-tensor mesons in Ref. [48], which sets the formal framework to model the tensor glueball, as it has the same quantum numbers $J^{PC} = 2^{++}$ as the tensor mesons (details of the fields and Lagrangians in Sec. II). Within this framework, we evaluate (Sec. III) various decay rates, most importantly the $\rho\rho/\pi\pi$ decay ratio, which turns out to be a prediction of the chiral approach (and its SSB). We then compare the outcomes to the isoscalar-tensor states in the range close to 2000 MeV and find out that the resonance $f_2(1950)$ is, according to present data, our best candidate. We also speculate (Sec. IV) about the partial decay widths of the glueball and the overall assignment of excited tensor states. Finally, conclusions are outlined in Sec. V.

II. CHIRAL MODEL AND NONETS

The eLSM is an effective model based on chiral symmetry in the linear form, together with explicit and spontaneous breaking aimed to reproduce—at the hadronic level—basic QCD features.

In this section we shall describe the fields entering the Lagrangian as well as the interaction terms. While the eLSM can contain many different fields and interactions, in this paper we shall only highlight those relevant for the decays of a tensor glueball. For a more complete review of the eLSM we refer to [42,43,49,50] and references therein (applications at nonzero temperature and density can be found in, e.g., Refs. [51–53]).

A. Meson nonets

The mesonic $\bar{q}q$ fields are contained in nonets. The list of decay products of the tensor glueball is, in matrix form, the following:

(1) Vector mesons $\{\rho(770), K^*(892), \omega(782), \phi(1020)\}$ with the quantum number $J^{PC} = 1^{--}$ ($^3S_1$) can be classified to the following nonet

$$V^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu + \rho^{0\mu}}{\sqrt{2}} & \rho^{+\mu} & K^{*+\mu} \\ \rho^{-\mu} & \frac{\omega_N^\mu - \rho^{0\mu}}{\sqrt{2}} & K^{*0\mu} \\ K^{*-\mu} & \bar{K}^{*0\mu} & \omega_S^\mu \end{pmatrix}, \quad (1)$$

where ω_N and ω_S are purely nonstrange and strange, respectively. The physical fields arise upon mixing

$$\begin{pmatrix} \omega(782) \\ \phi(1020) \end{pmatrix} = \begin{pmatrix} \cos\beta_V & \sin\beta_V \\ -\sin\beta_V & \cos\beta_V \end{pmatrix} \begin{pmatrix} \omega_N \\ \omega_S \end{pmatrix}, \quad (2)$$

where the small isoscalar-vector mixing angle $\beta_V = -3.9°$ [54]. The physical states ϕ and ω are predominantly strange and nonstrange components, respectively. This is a reflection of the "homochiral" nature of these states [55].

(2) The chiral partners of the vector mesons [56] having the quantum number $J^{PC} = 1^{++}$ ($^3P_1$) forms the following nonet:

$$A_1^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{1,N}^\mu + a_1^{0\mu}}{\sqrt{2}} & a_1^{+\mu} & K_{1A}^{+\mu} \\ a_1^{-\mu} & \frac{f_{1,N}^\mu - a_1^{0\mu}}{\sqrt{2}} & K_{1A}^{0\mu} \\ K_{1A}^{-\mu} & \bar{K}_{1A}^{0\mu} & f_{1,S}^\mu \end{pmatrix}. \quad (3)$$

The isoscalar sector reads

$$\begin{pmatrix} f_1(1285) \\ f_1(1420) \end{pmatrix} = \begin{pmatrix} \cos\beta_{A_1} & \sin\beta_{A_1} \\ -\sin\beta_{A_1} & \cos\beta_{A_1} \end{pmatrix} \begin{pmatrix} f_{1,N}^\mu \\ f_{1,S}^\mu \end{pmatrix}, \quad (4)$$

where the mixing angle β_{A_1} is expected to be small because of the homochiral nature of the multiplet [55], see also Ref. [57]. In the computations we shall set this angle to zero for simplicity.

(3) Pseudoscalar mesons with $J^{PC} = 0^{-+}$ ($^1S_0$) consisting of three pions, four kaons, the $\eta(547)$ and the $\eta'(958)$. They are collected into the following nonet with light-quark elements $P_{ij} \equiv 2^{-1/2}\bar{q}_j i\gamma^5 q_i$:

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\eta_N+\pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_N-\pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \eta_S \end{pmatrix}, \tag{5}$$

where $\eta_N \equiv \sqrt{1/2}(\bar{u}u + \bar{d}d)$ stands for the purely nonstrange state and $\eta_S \equiv \bar{s}s$ for the purely strange one. The physical isoscalar fields appear upon mixing [58] as

$$\begin{pmatrix} \eta(547) \\ \eta' \equiv \eta(958) \end{pmatrix} = \begin{pmatrix} \cos\beta_P & \sin\beta_P \\ -\sin\beta_P & \cos\beta_P \end{pmatrix} \begin{pmatrix} \eta_N \\ \eta_S \end{pmatrix}, \tag{6}$$

with mixing angle $\beta_P = -43.4°$. This sizable mixing angle is a consequence of the $U_A(1)$ axial anomaly [59,60]. Namely, pseudoscalar mesons belong to a "heterochiral" multiplet [55].

(4) The axial-vector matrix is shifted due to spontaneous chiral symmetry breaking. This breaking induces a mixing of pseudoscalar and axial-vector fields, which allows the decay into pseudoscalar mesons. The shift is as follows:

$$A_{1\mu} \to A_{1\mu} + \partial_\mu \mathcal{P},$$
$$\mathcal{P} := \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{Z_\pi w_\pi(\eta_N+\pi^0)}{\sqrt{2}} & Z_\pi w_\pi \pi^+ & Z_K w_K K^+ \\ Z_\pi w_\pi \pi^- & \frac{Z_\pi w_\pi(\eta_N-\pi^0)}{\sqrt{2}} & Z_K w_K K^0 \\ Z_K w_K K^- & Z_K w_K \bar{K}^0 & Z_{\eta_S} w_{\eta_S} \eta_S \end{pmatrix}, \tag{7}$$

where the numerical values $Z_\pi = Z_{\eta_N} = 1.709$, $Z_K = 1.604$, $Z_{\eta_S} = 1.539$ and $w_\pi = w_{\eta_N} = 0.683\ \mathrm{GeV}^{-1}$, $w_K = 0.611\ \mathrm{GeV}^{-1}$, $w_{\eta_S} = 0.554\ \mathrm{GeV}^{-1}$ are taken from [42]. Note, this shift will be particularly important in this work since it allows to link different decay channels (such as $\rho\rho/\pi\pi$) that otherwise could not be connected by flavor symmetry alone.

(5) The $J^{PC} = 2^{++}$ ($^3P_2$) tensor states with elements $T_{ij}^{\mu\nu} = 2^{-1/2}\bar{q}_j(i\gamma^\mu\partial^\nu + \cdots)q_i$:

$$T^{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{2,N}^{\mu\nu}+a_2^{0\mu\nu}}{\sqrt{2}} & a_2^{+\mu\nu} & K_2^{*+\mu\nu} \\ a_2^{-\mu\nu} & \frac{f_{2,N}^{\mu\nu}-a_2^{0\mu\nu}}{\sqrt{2}} & K_2^{*0\mu\nu} \\ K_2^{*-\mu\nu} & \bar{K}_2^{*0\mu\nu} & f_{2,S}^{\mu\nu} \end{pmatrix}. \tag{8}$$

The physical isoscalar-tensor states are

$$\begin{pmatrix} f_2(1270) \\ f_2'(1525) \end{pmatrix} = \begin{pmatrix} \cos\beta_T & \sin\beta_T \\ -\sin\beta_T & \cos\beta_T \end{pmatrix} \begin{pmatrix} f_{2,N} \\ f_{2,S} \end{pmatrix}, \tag{9}$$

where $\beta_T \simeq 5.7°$ is the small mixing angle reported in the Particle Data Group (PDG). Tensor mesons belong to a homochiral multiplet, just as (axial-) vector states. We have no further experimental information about the chiral partner of this nonet $A_2^{\mu\nu}$ (see, e.g., [48] for recent phenomenological analyses of this nonet).

B. Interaction terms

In this subsection, we list the chiral interaction involving a tensor glueball field $G_{2,\mu\nu}$.

A possible way to understand how the searched terms emerge is to consider the interaction terms that describe the decays of (axial-)tensor mesons in the recent Ref. [48]. What one needs to apply is the following replacement into the Lagrangians with ordinary tensor meson nonet:

$$T_{\mu\nu} \to \frac{1}{\sqrt{6}} G_{2,\mu\nu} \cdot \mathbf{1}_3, \tag{10}$$

thus effectively realizing flavor blindness. Of course, the coupling constants must be renamed and are, at first, not known. Note, we follow the same convention that implies the normalization with respect to the flavor singlet mode. Yet, the chirally invariant interaction terms of the tensor glueball with other mesons that we are going to introduce stand on their own and can be formally introduced without resorting to (axial-)tensor mesons.

The first term in the eLSM leading to tensor glueball decays involves solely left- and right-handed chiral fields:

$$\mathcal{L}_\lambda = \frac{\lambda}{\sqrt{6}} G_{2,\mu\nu}(\mathrm{Tr}[\{L^\mu, L^\nu\}] + \mathrm{Tr}[\{R^\mu, R^\nu\}]), \tag{11}$$

where the chiral fields consist of the vector and axial-vector mesons that we shall define with nonets

$$L^\mu := V^\mu + A_1^\mu, \qquad R^\mu := V^\mu - A_1^\mu, \tag{12}$$

which transform as $L^\mu \to U_L L^\mu U_L^\dagger$, $R^\mu \to U_R R^\mu U_R^\dagger$ under the chiral transformations of $U_L(3) \times U_R(3)$. The Lagrangian (11) leads to three kinematically allowed decay channels with the following expressions for the decay rates with three momentum

$$|\vec{k}_{a,b}| := \frac{1}{2m_{G_2}} \sqrt{(m_{G_2}^2 - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2}. \tag{13}$$

TABLE I. Coefficients for the decay channel $G_2 \to A + B$. $\phi_N \approx 0.158$ GeV and $\phi_S \approx 0.138$ GeV are due to the chiral condensate. The $\eta\eta'$ coefficient is small because it only occurs due to the flavor symmetry breaking.

Decay process	$\kappa_{g\circ\circ,i}$
$G_2 \to \rho(770)\rho(770)$	1
$G_2 \to \bar{K}^\star(892)K^\star(892)$	$\frac{4}{3}$
$G_2 \to \omega(782)\omega(782)$	$\frac{1}{3}$
$G_2 \to \omega(782)\phi(1020)$	0
$G_2 \to \phi(1020)\phi(1020)$	$\frac{1}{3}$
$G_2 \to \pi\pi$	$(Z_\pi^2 w_\pi^2)^2$
$G_2 \to \bar{K}K$	$\frac{4}{3} \times (Z_k^2 w_k^2)^2$
$G_2 \to \eta\eta$	$\frac{1}{3} \times (Z_{\eta_N}^2 w_{\eta_N}^2 \cos\beta_P^2 + Z_{\eta_S}^2 w_{\eta_S}^2 \sin\beta_P^2)^2$
$G_2 \to \eta\eta'(958)$	$\frac{1}{3} \times ((Z_{\eta_N}^2 w_{\eta_N}^2 - Z_{\eta_S}^2 w_{\eta_S}^2)\cos\beta_P \sin\beta_P)^2$
$G_2 \to \eta'(958)\eta'(958)$	$\frac{1}{18} \times (Z_{\eta_S}^2 w_{\eta_S}^2 \cos\beta_P^2 + Z_{\eta_N}^2 w_{\eta_N}^2 \sin\beta_P^2)^2$
$G_2 \to a_1(1260)\pi$	$\frac{1}{2} \times (Z_\pi w_\pi)^2$
$G_2 \to f_1(1285)\eta$	$\frac{1}{6}(Z_{\eta_S} w_{\eta_S} \sin\beta_{A_1} \sin\beta_P + Z_{\eta_N} w_{\eta_N} \cos\beta_{A_1} \cos\beta_P)^2$
$G_2 \to K_{1,A}K$	$\frac{2}{3} \times (Z_k w_k)^2$
$G_2 \to f_1(1420)\eta'(958)$	$\frac{1}{6}(Z_{\eta_N} w_{\eta_N} \sin\beta_{A_1} \sin\beta_P + Z_{\eta_S} w_{\eta_S} \cos\beta_{A_1} \cos\beta_P)^2$
$G_2 \to f_1(1285)\eta'(958)$	$\frac{1}{6}(Z_{\eta_S} w_{\eta_S} \sin\beta_{A_1} \cos\beta_P - Z_{\eta_N} w_{\eta_N} \cos\beta_{A_1} \sin\beta_P)^2$
$G_2 \to f_2(1270)\eta$	$\frac{1}{24}(\phi_N \cos\beta_p \cos\beta_T + \phi_S \sin\beta_p \sin\beta_T)^2$
$G_2 \to a_2(1320)\pi$	$\frac{\phi_N^2}{8}$
$G_2 \to K^*(892)K + \text{c.c.}$	$\frac{1}{48}(\sqrt{2}\phi_N - 2\phi_S)^2$

(1) Decays of the tensor glueball into two vector mesons has the following decay rate formula:

$$\begin{aligned}\Gamma_{G_2\to V^{(1)}V^{(2)}}(m_{G_2}, m_{v^{(1)}}, m_{v^{(2)}}) &= \frac{\kappa_{gvv,i}\lambda^2|\vec{k}_{v^{(1)},v^{(2)}}|}{120\pi m_{G_2}^2}\left(15 + \frac{5|\vec{k}_{v^{(1)},v^{(2)}}|^2}{m_{v^{(1)}}^2} + \frac{5|\vec{k}_{v^{(1)},v^{(2)}}|^2}{m_{v^{(2)}}^2}\right.\\ &\quad\left. + \frac{2|\vec{k}_{v^{(1)},v^{(2)}}|^4}{m_{v^{(1)}}^2 m_{v^{(2)}}^2}\right)\Theta(m_{G_2} - m_{v^{(1)}} - m_{v^{(2)}}); \qquad (14)\end{aligned}$$

(2) while into two pseudoscalar mesons [SSB-driven shift of Eq. (7) applied twice]

$$\Gamma^{tl}_{G_2\to P^{(1)}P^{(2)}}(m_{G_2}, m_{p^{(1)}}, m_{p^{(2)}}) = \frac{\kappa_{gpp,i}\lambda^2|\vec{k}_{p^{(1)},p^{(2)}}|^5}{60\pi m_{G_2}^2}\Theta(m_{G_2} - m_{p^{(1)}} - m_{p^{(2)}}); \qquad (15)$$

(3) and into the axial-vector and pseudoscalar mesons [SSB-driven shift of Eq. (7) applied once]

$$\begin{aligned}\Gamma^{tl}_{G_2\to A_1P}(m_{G_2}, m_{a_1}, m_p) &= \frac{\kappa_{gap,i}\lambda^2|\vec{k}_{a_1,p}|^3}{120\pi m_{G_2}^2}\left(5 + \frac{2|\vec{k}_{a_1,p}|^2}{m_{a_1}^2}\right)\\ &\quad\times\Theta(m_{G_2} - m_{a_1} - m_p). \qquad (16)\end{aligned}$$

The coupling constant λ is not known and thus we are limited to computing branching ratios rather than decay rates. The (sometimes dimensionful) coefficients $\kappa_{g\circ\circ,i}$ are shown in Table I. For the two-pseudoscalar channel, κ has a mass dimension of -4, in the vector-vector channel it is dimensionless, in the axial-vector plus pseudoscalar channel it has a mass dimension of -2, for the tensor and pseudoscalar channel it has a mass dimension of 2. The second chirally invariant term we will use for tensor glueball decays is of the form

$$\mathcal{L}_\alpha = \frac{\alpha}{\sqrt{6}}G_{2,\mu\nu}(\mathrm{Tr}[\Phi\mathbf{R}^{\mu\nu}\Phi^\dagger] + \mathrm{Tr}[\Phi^\dagger\mathbf{L}^{\mu\nu}\Phi]), \qquad (17)$$

where $\Phi = S + iP$ is the linear combination of scalar[1] and pseudoscalar nonets, and $\mathbf{L}^{\mu\nu} = T^{\mu\nu} + A_2^{\mu\nu}$, $\mathbf{R}^{\mu\nu} = T^{\mu\nu} - A_2^{\mu\nu}$ combine the tensor and axial-tensor nonets. Their chiral transformation rules are $\Phi \to U_L\Phi U_R^\dagger$, $\mathbf{R}^{\mu\nu} \to U_R\mathbf{R}^{\mu\nu}U_R^\dagger$, $\mathbf{L}^{\mu\nu} \to U_L\mathbf{L}^{\mu\nu}U_L^\dagger$. The linear combination of the scalar and pseudoscalar contains the chiral condensate $\Phi = \Phi + \Phi_0$ and so this term leads to the decay of tensor glueball into a tensor meson and pseudoscalar meson. The decay rate for this process is given by

[1]We do not study the decay to scalar mesons and so do not discuss the scalar nonet in this work.

TABLE II. Branching ratios of G_2 with respect to $\pi\pi$. The columns are sorted as PP on the left, VV in the middle, and AP on the right. The VV-$\pi\pi$ and AP-$\pi\pi$ ratios should be regarded as approximate due to large uncertainties (see text).

Branching ratio	Theory	Branching ratio	Theory	Branching ratio	Theory
$\frac{G_2(2210)\to\bar{K}K}{G_2(2210)\to\pi\pi}$	0.4	$\frac{G_2(2210)\to\rho(770)\rho(770)}{G_2(2210)\to\pi\pi}$	55	$\frac{G_2(2210)\to a_1(1260)\pi}{G_2(2210)\to\pi\pi}$	0.24
$\frac{G_2(2210)\to\eta\eta}{G_2(2210)\to\pi\pi}$	0.1	$\frac{G_2(2210)\to\bar{K}^*(892)\bar{K}^*(892)}{G_2(2210)\to\pi\pi}$	46	$\frac{G_2(2210)\to K_{1,A}K}{G_2(2210)\to\pi\pi}$	0.08
$\frac{G_2(2210)\to\eta\eta'}{G_2(2210)\to\pi\pi}$	0.004	$\frac{G_2(2210)\to\omega(782)\omega(782)}{G_2(2210)\to\pi\pi}$	18	$\frac{G_2(2210)\to f_1(1285)\eta}{G_2(2210)\to\pi\pi}$	0.02
$\frac{G_2(2210)\to\eta'\eta'}{G_2(2210)\to\pi\pi}$	0.006	$\frac{G_2(2210)\to\phi(1020)\phi(1020)}{G_2(2210)\to\pi\pi}$	6	$\frac{G_2(2210)\to f_1(1285)\eta}{G_2(2210)\to\pi\pi}$	0.02
				$\frac{G_2(2210)\to f_1(1420)\eta}{G_2(2210)\to\pi\pi}$	0.01

$$\Gamma^{tl}_{G_2\to TP}(m_{G_2}, m_t, m_p) = \frac{\alpha^2|\vec{k}_{t,p}|}{2m^2_{G_2}\pi}\left(1+\frac{4|\vec{k}_{t,p}|^4}{45m_t^4}+\frac{2|\vec{k}_{t,p}|^2}{3m_t^2}\right) \times \kappa_{g_2tp,i}\Theta(m_{G_2}-m_t-m_p). \quad (18)$$

The coupling α is not fixed, so branching ratios are only calculated for decays in this channel.

The third chiral Lagrangian describes the decay of a tensor glueball into a vector and pseudoscalar meson:

$$\mathcal{L}_{c^{\text{ten}}} = \frac{c_1}{\sqrt{6}}\partial^\mu G_2^{\nu\alpha}\text{Tr}[\tilde{L}_{\mu\nu}\partial_\alpha\Phi\Phi^\dagger - \Phi^\dagger\partial_\alpha\Phi\tilde{R}_{\mu\nu} - \tilde{R}_{\mu\nu}\partial_\alpha\Phi^\dagger\Phi + \Phi\partial_\alpha\Phi^\dagger\tilde{L}_{\mu\nu}] + \frac{c_2}{\sqrt{6}}\partial^\mu G_2^{\nu\alpha}\text{Tr}[\partial_\alpha\Phi\tilde{R}_{\mu\nu}\Phi^\dagger - \Phi^\dagger\tilde{L}_{\mu\nu}\partial_\alpha\Phi - \partial_\alpha\Phi^\dagger\tilde{L}_{\mu\nu}\Phi + \Phi\tilde{R}_{\mu\nu}\partial_\alpha\Phi^\dagger], \quad (19)$$

where $\tilde{L}_{\mu\nu} \coloneqq \frac{\varepsilon_{\mu\nu\rho\sigma}}{2}(\partial^\rho L^\sigma - \partial^\sigma L^\rho)$ and similarly for $\tilde{R}_{\mu\nu}$. Defining $c \coloneqq c_1 + c_2$, the tree-level decay rate formula reads

$$\Gamma^{tl}_{G\to VP}(m_{G_2}, m_v, m_p) = \frac{c^2|\vec{k}_{v,p}|^5}{40\pi}\kappa_{gvp,i}\Theta(m_{G_2}-m_v-m_p), \quad (20)$$

The decay of G_2 into a vector and pseudoscalar is suppressed; namely, the only nonzero κ is for KK^* decay products, which is suppressed by a factor of $(\phi_N - \sqrt{2}\phi_S)$, and vanishes in the chiral limit.

III. RESULTS FOR THE DECAY RATIOS

Recent investigation devoted to the search of the tensor glueball in the BESIII data obtained an enhancement on the mass around 2210 MeV [37]. Thus, for illustrative purposes, we first present our decay ratios for this mass value. Using the $\kappa_{g\circ\circ,i}$ in Table I and Eqs. (14)–(16), we obtain the decay ratios shown in Table II. The decays are primarily to two vector mesons, with $\rho\rho$ and K^*K^* being the two largest. The two ρ mesons would further decay into four pions before reaching the detector in an experiment. Likewise the K^*K^* pair decays to two $K\pi$ pairs.

It should be stressed that we expect a quite large indeterminacy of our results, in particular in connection to the $\rho\rho-\pi\pi$ ratio, that is at least of 50 percent. In fact, as discussed in Ref. [48], the $\rho\rho$ mode involves (because of the SSB shift) the factor Z_π^4, thus even a quite small indeterminacy of Z_π may generate large changes. The error on Z_π is of the order of 20%, that translates into a factor of 2 for $\rho\rho$. Yet, the main point of our study is that the ratio $\rho\rho$-$\pi\pi$ is expected to be large, as shown in Fig. 1. A similar dominance of $\rho\rho$ and K^*K^* decay products was found in [27] in a holographic model but to a somewhat lesser extent (with a ratio of around 10, depending on the parameters). Potential glueball candidates and some relevant

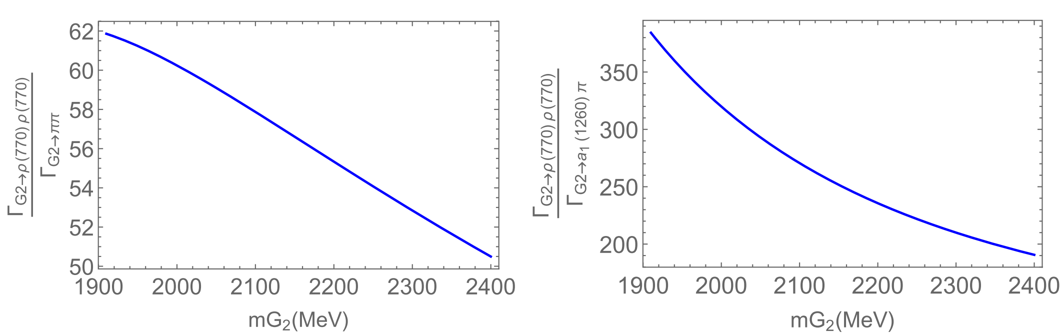

FIG. 1. Estimates for the ratios $\rho\rho/\pi\pi$ (left) and $\rho\rho/a1(1260)\pi$ (right) as function of the tensor glueball mass.

TABLE III. Spin-2 resonances heavier than 1.9 GeV listed in PDG [54].

Resonances	Masses (MeV)	Decay widths (MeV)	Decay channels
$f_2(1910)$	1900 ± 9	167 ± 21	$\pi\pi$, KK, $\eta\eta$, $\omega\omega$, $\eta\eta'$, $\eta'\eta'$, $\rho\rho$, $a_2(1320)\pi$, $f_2(1270)\eta$
$f_2(1950)$	1936 ± 12	464 ± 24	$\pi\pi$, KK, $\eta\eta$, K^*K^*, 4π
$f_2(2010)$	2011^{+60}_{-80}	202 ± 60	KK, $\phi\phi$
$f_2(2150)$	2157 ± 12	152 ± 30	$\pi\pi$, $\eta\eta$, KK, $a_2(1320)\pi$, $f_2(1270)\eta$
$f_J(2220)$	2231.1 ± 3.5	23^{+8}_{-7}	$\eta\eta'$
$f_2(2300)$	2297 ± 28	149 ± 41	KK, $\phi\phi$
$f_2(2340)$	2345^{+50}_{-40}	322^{+70}_{-60}	$\eta\eta$, $\phi\phi$

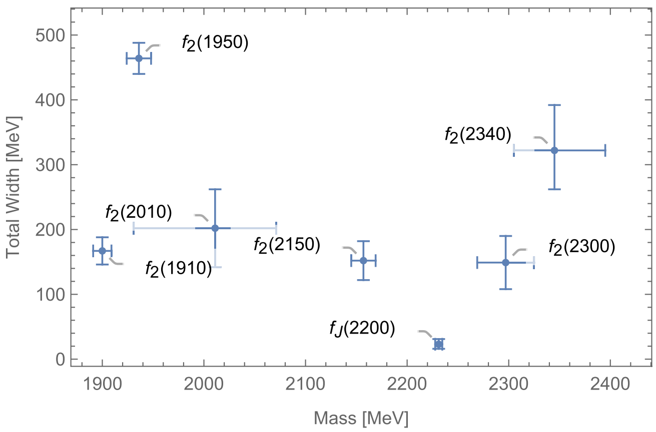

FIG. 2. Masses and widths of isoscalar-tensor resonances heavier than 1.9 GeV [54].

data is given in Table III and Fig. 2. While the mass value in Table II is not in line with every glueball candidate, it gives an overall picture on the decay ratios. One should also keep in mind that lattice determinations still have sizable errors (of the order of 100–200 MeV) and do not include the role of meson-meson loops, which can be quite relevant if the tensor glueball turns out to be broad (this being our favored scenario).

Below we discuss each tensor-glueball candidate separately. The experimental data and theoretical predictions used for this discussion are shown in Table IV.

(1) The meson $f_2(1910)$ has a width of 167 ± 21 MeV and it decays into (among others) $\eta\eta$ and $K\bar{K}$. The decay ratio $\rho(770)\rho(770)/\omega(782)\omega(782)$ of about 2.6 ± 0.4 is not far from the theoretical result of 3.1, and the data on the decay ratio of $f_2(1270)\eta/a_2(1320)\pi$ also agrees with the theory. Yet, this state cannot be mainly gluonic since the experimental ratio $\eta\eta/\eta\eta'(958)$ is less than 0.05, while the theoretical result is much larger (about 8), and the ratio of $\omega\omega$ to $\eta\eta\prime$ is very large in theory but only 2.6 ± 0.6 in data. Summarizing, the $\eta\eta'(958)$ mode is a clear drawback for $f_2(1910)$ being predominantly gluonic.

(2) The meson $f_2(1950)$ decays into $\eta\eta$, $\pi\pi$, $K\bar{K}$, as well as K^*K^* and 4π modes, the latter likely to emerge from $\rho\rho$. The experimental ratio $\eta\eta/\pi\pi$ of 0.14 ± 0.05 agrees with theory, as well as the $K\bar{K}/\pi\pi$. Both in experiment and theory we have only lower bounds of the ratio $4\pi/\eta\eta$—where we assume $\rho\rho$ decays into four pions—but both agree that this ratio is large. While the theory fits the data well, its large total decay width of 460 MeV would imply a broad tensor glueball candidate. Quite interestingly, the decay $J/\psi \to \gamma K^*(892)\bar{K}^*(892)$ shows a relatively large

TABLE IV. Decay ratios for the decay channels with available data.

Resonances	Branching ratios	PDG	Model prediction
$f_2(1910)$	$\rho(770)\rho(770)/\omega(782)\omega(782)$	2.6 ± 0.4	3.1
$f_2(1910)$	$f_2(1270)\eta/a_2(1320)\pi$	0.09 ± 0.05	0.07
$f_2(1910)$	$\eta\eta/\eta\eta'(958)$	< 0.05	~8
$f_2(1910)$	$\omega(782)\omega(782)/\eta\eta\prime(958)$	2.6 ± 0.6	~200
$f_2(1950)$	$\eta\eta/\pi\pi$	0.14 ± 0.05	0.081
$f_2(1950)$	$K\bar{K}/\pi\pi$	~0.8	0.32
$f_2(1950)$	$4\pi/\eta\eta$	> 200	> 700
$f_2(2150)$	$f_2(1270)\eta/a_2(1320)\pi$	0.79 ± 0.11	0.1
$f_2(2150)$	$K\bar{K}/\eta\eta$	1.28 ± 0.23	~4
$f_2(2150)$	$\pi\pi/\eta\eta$	< 0.33	~10
$f_J(2220)$	$\pi\pi/K\bar{K}$	1.0 ± 0.5	~2.5

branching ratio of $(7.0 \pm 2.2)10^{-4}$. A strong coupling to J/ψ is an expected feature of a tensor glueball. Recent T-matrix pole analyses found compatible but slightly lower pole decay widths of 350 ± 114 MeV [61] and $237 \pm {\sim}60$ MeV [62], which are combined in the novel PDG estimate for the pole width of 330 ± 110 MeV (2023 update of the [54]). These values do not change the qualitative outcomes of our discussion and further confirm the existence (and the broadness) of the resonance $f_2(1950)$.

(3) The resonance $f_2(2010)$ has a total decay width of 202 ± 60 MeV. Yet, only $K\bar{K}$ and $\phi(1020)\phi(1020)$ decays have been seen. This fact suggests a large strange-antistrange content for this resonance, rather than a predominantly gluonic state, see also the discussions in Sec. IV.

(4) In view of the LQCD prediction for the tensor glueball mass around 2.2 GeV, one of the closest resonances is $f_2(2150)$. Yet, the ratio $K\bar{K}/\eta\eta$ is 1.28 ± 0.23, while the theoretical prediction is about 4. Similarly, the ratio of $\pi\pi/\eta\eta$ is experimentally less than 0.33, while the theoretical estimate is about 10. The predicted ratio of 0.1 for $f_2(1270)\eta/a_2(1320)\pi$ also does not fit the data of 0.79 ± 0.11.

(5) The meson $f_J(2220)$ (with $J = 2$ or 4) was historically treated as a good candidate [21]. However, in light of the new PDG compilation, most of the decays that the theory predicts are not seen experimentally. Moreover, the only channel denoted as "seen" in the decay modes table of the PDG [54] is the $\eta\eta'(958)$ one, which in turn is expected to be extremely suppressed for a glueball state (in the order of 10^{-3} times the $\pi\pi$ mode in our calculation). PDG data list also the decay ratio $\pi\pi/K\bar{K}$ of 1.0 ± 0.5 (even though in the "decay modes" table these channels are marked as not seen), while the theoretical prediction is approximately 2.5. We also note that while in [37] a potential tensor glueball resonance is found at 2210 MeV, the width of the enhancement is an order of magnitude larger than one listed in PDG for $f_J(2220)$. Also, the uncertainty in the mass allows us to fit the glueball structure to other resonances. For these reasons, we conclude that, on the basis of the present evidence, $f_J(2220)$ should not be considered as a good candidate for being a tensor glueball.

(6) The resonance $f_2(2300)$, with a total width of 149 ± 41 MeV, decays only into $K\bar{K}$ and $\phi\phi$, thus suggesting that it is predominantly a strange-antistrange object.

(7) Finally, the resonance $f_2(2340)$ decays into $\eta\eta$ and $\phi(1020)\phi(1020)$ that may also imply a large strange-antistrange component. Yet, both latter resonances should be investigated in more details in the future, especially for what concerns other possible decay modes of these resonances. We also refer to the discussion in the next section for additional discussions.

From the considerations above, it turns out that the resonances $f_2(1950)$ seems to be the best fit, although this would imply a broad tensor glueball state. Namely, all other states seem disfavored for various reasons, such as specific branching or decay ratios with available data [$f_2(1910)$ and the $f_2(2150)$] or decays in strange states only [$f_2(2010)$, $f_2(2300)$, and $f_2(2340)$].

It is of primary importance to monitor the experimental status of the states above in the future. In particular, the analysis for the states $f_J(2220)$, $f_2(2300)$, and $f_2(2340)$ would benefit from more experimental data, with special attention to the latter broad one.

IV. DISCUSSIONS

In this section, we discuss two additional important points. The first one addresses the actual partial decay widths of the tensor glueball; while a rigorous treatment is not possible within our framework, a "guess" is achieved by using large-N_c arguments and the decays of the conventional ground-state tensor mesons. The second point discusses the assignment of various tensor states as radially and orbitally excited conventional tensor mesons. In this framework, the tensor glueball should be a "supernumerary" state that does not fit into the quark-antiquark nonet picture.

For the first point, let us consider the conventional mesons $f_2 \equiv f_2(1270) \simeq \sqrt{1/2}(\bar{u}u + \bar{d}d)$ and $f_2' \equiv f_2'(1525) \simeq \bar{s}s$, whose decays into $\pi\pi$ are well known: $\Gamma_{f_2\to\pi\pi} = 157.2$ MeV and $\Gamma_{f_2'\to\pi\pi} = 0.71$ MeV (for our qualitative purposes, we neglect the anyhow small errors). The amplitude for the decay $A_{f_2\to\pi\pi}$ requires the creation of a single $\bar{q}q$ pair from the vacuum and scales as $1/\sqrt{N_c}$, where N_c is the number of colors. On the other hand, the amplitude $A_{f_2'\to\pi\pi}$ implies that $\bar{s}s$ first converts into two gluons (gg) that subsequently transforms into $\sqrt{1/2}(\bar{u}u + \bar{d}d)$ (the very same mechanism is responsible for a small (about 3°) but nonzero mixing angle of the physical states in the strange-nonstrange basis [48]). Schematically:

$$\bar{s}s \to gg \to \sqrt{1/2}(\bar{u}u + \bar{d}d). \tag{21}$$

The amplitude $A_{f_2'\to\pi\pi}$ scales as $1/N_c^{3/2}$ and is OZI [63–65] suppressed with respect to the previous one. In order to be more specific, let us consider the transition Hamiltonian $\mathrm{H}_{\mathrm{int}} = \lambda(|\bar{u}u\rangle\langle gg| + |\bar{d}d\rangle\langle gg| + |\bar{s}s\rangle\langle gg| + \mathrm{H.c.})$, where λ controls the mixing and therefore scales as $1/\sqrt{N_c}$. Then $A_{f_2'\to\pi\pi} \simeq \sqrt{2}\lambda^2 A_{f_2\to\pi\pi}$, hence $\Gamma_{f_2'\to\pi\pi} \simeq 2\lambda^4\Gamma_{f_2\to\pi\pi}$, implying $\lambda \simeq 0.22$.

TABLE V. Estimations of the decay channel $G_2 \to PP$.

Decay process	Γ_i (MeV)
$G_2 \to \pi\pi$	~15
$G_2 \to \bar{K}K$	~6
$G_2 \to \eta\eta$	~1.6
$G_2 \to \eta\eta'(958)$	~0.06
$G_2 \to \eta'(958)\eta'(958)$	~0.08

Next, let us consider the tensor glueball decay into $\pi\pi$. Intuitively speaking, it is at an "intermediate stage," since it starts with a gg pair. One has $A_{G_2\to\pi\pi} \simeq \sqrt{2}\lambda A_{f_2\to\pi\pi}$, then $\Gamma_{G_2\to\pi\pi} \simeq 2\lambda^2\Gamma_{f_2\to\pi\pi} \simeq \sqrt{2}\sqrt{\Gamma_{f_2\to\pi\pi}\Gamma_{f_2'\to\pi\pi}} \simeq 15$ MeV. (Note, for a similar idea for the estimate of the coupling of glueballs to mesons, see Refs. [66,67].) $\Gamma_{G_2\to\pi\pi}$ scales as $1/N_c^2$ (as expected for glueballs), thus realizing the expectation $\Gamma_{f_2'\to\pi\pi} < \Gamma_{G_2\to\pi\pi} < \Gamma_{f_2\to\pi\pi}$.

Of course, the estimate $\Gamma_{G_2\to\pi\pi} \simeq 15$ MeV is only a very rough approximation for various reasons: it does not take into account phase space (that would increase the glueball width) or form factors (that would decrease it). It also avoids a microscopic evaluation (which is a difficult task). Yet, it gives an idea on how large the $\pi\pi$ mode (and as a consequence other PP channels) could be, see Table V.

It is interesting to point out that our $\pi\pi$ decay rate is comparable to the one obtained within so-called Witten-Sakai-Sugimoto model [27], in which $\Gamma_{G_2\to\pi\pi}/m_{G(2000)} \simeq 0.014$, thus implying $\Gamma_{G_2\to\pi\pi} \simeq 28$ MeV.

As a consequence of such decay width into $\pi\pi$, a large decay width into $\rho\rho \to \pi\pi\pi\pi$ is expected due to the evaluated large $\rho\rho/\pi\pi$ ratio.

Next, the second point to discuss relates to the assignment of quark-antiquark tensor states. Namely, up to now, we have considered all isoscalar f_2 states in the energy region about 2 GeV "democratically" as putative tensor glueball candidates. Yet, it is clear that many (if not all) of them should be rather interpreted as standard $\bar{q}q$ objects.

For this reason, it is useful to classify them accordingly. While the ground-state tensor mesons (with spectroscopic notation 1^3P_2 are well established as $a_2(1320)$, $K_2^*(1430)$, $f_2(1270)$, and $f_2'(1525)$, one expects a nonet of radially excited tensor states with 2^3P_2 as well as a nonet of orbitally excited tensor states with 1^3F_2. The former are predicted to be lighter than the latter [68], which is in agreement with the excited vector mesons [69].

The nonet of radially excited tensor mesons contains the isotriplet $a_2(1700)$, the isodoublet states $K_2^*(1980)$, the isoscalar (mostly nonstrange) $f_2(1640)$. The $\bar{s}s$ member of the nonet may be assigned to $f_2(1910)$, or $f_2(1950)$, or $f_2(2010)$.

Indeed, the quark model review of the PDG [54] considers $f_2(1950)$ as a possible $\bar{s}s$ state. However, the decay of that state does not fit quite well with a predominantly $\bar{s}s$ object. For instance, the experimental $\pi\pi/KK$ ratio is of order unity, while a $\bar{s}s$ state should imply a small ratio. In fact, the radially excited tensor states should display a small isoscalar mixing angle, just as the ground state (see discussion above). Moreover, the mass is too close to the tensor member $K_2^*(1980)$. Note that resonance "$f_2(2000)$" included in the further states list of PDG is proposed as a tensor glueball candidate in Ref. [21] due to its poorly fitted location in the Regge trajectory of spin-2 states. Similar to $f_2(1950)$, it has also a broad decay width.

The state $f_2(1910)$ is presently omitted from the summary table and it is not clear if it corresponds to an independent pole [for instance, the PDG notes that the $\bar{K}K$ mode could be well correspond to $f_2(1950)$]. Its mass is even lighter than $K_2^*(1980)$, what disfavors this state as being an $\bar{s}s$. The decays—which are quite uncertain—confirm this view: the modes $\rho\rho$, and $a_2(1320)\pi$ should be suppressed, and the latter should be smaller then $f_2(1270)\eta$, contrary to data (see above). On the other hand, the state $f_2(2010)$ is well established and decays into $\phi\phi$ and KK, which indicates a strange content.

Summarizing the discussion above, we tentatively identify the nonet of radially excited tensor mesons as

$$a_2(1700), \quad K_2^*(1980), \quad f_2(1640), \quad f_2(2010) \quad \text{with} \quad 2^3P_2 \quad \text{and} \quad J^{PC} = 2^{++}. \tag{22}$$

Within this assignment and upon neglecting the unsettled $f_2(1910)$, the state $f_2(1950)$ may be seen as supernumerary. This argumentation can be an additional hint toward the exotic nature of $f_2(1950)$. In [70] it is also concluded that the $f_2(1950)$ does not fit the 3^3P_2 nonet as the radial excitation of $f_2(1270)$, and the mass is too low to otherwise fit in that nonet according to the Regge trajectory.

Next, what about the orbitally excited states? Here the situation is much more unclear. There are no isotriplet or isodoublet states that could be used to identify the nonet. In the listing of the PDG one has $f_2(2150)$ (status unclear, omitted from the summary table) and $f_2(2300)$ and $f_2(2340)$. The latter two states have a prominent decay into $\phi\phi$, then, due to the vicinity of mass, one may regard them as a unique state corresponding to $\bar{s}s$ resonance. On the other hand, $f_2(2150)$ could be tentatively correspond to a nonstrange isoscalar state belonging to the next radially excited nonet 3^3P_2. Definitely, more data and studies are needed for these excited tensor states.

V. CONCLUSIONS

In this work, we have applied the eLSM to the study of the tensor glueball, constructing chirally invariant Lagrangians describing the tensor glueball decays. From this we computed tensor glueball decay ratios, with dominant decay channels being the vector-vector decay products, especially $\rho\rho$ and $K^*\bar{K}^*$. A quite large tensor glueball

TABLE VI. Spin-2 resonances and the status as the tensor glueball.

Resonances	Interpretation status
$f_2(1910)$	Agreement with some data, but excluded by $\eta\eta/\eta\eta'$ and $\omega\omega/\eta\eta\prime$ ratios
$f_2(1950)$	$\eta\eta/\pi\pi$ agrees with data, no contradictions with data, but implies broad tensor glueball Best fit as predominantly glueball among considered resonances
$f_2(2010)$	Likely primarily strange-antistrange content
$f_2(2150)$	All available data contradicts theoretical prediction
$f_J(2220)$	Data on $\pi\pi/K\bar{K}$ disagrees with theory, largest predicted decay channels are not seen
$f_2(2300)$	Likely primarily strange-antistrange content
$f_2(2340)$	Likely primarily strange-antistrange content, would also imply a broad glueball

follows from the large predicted $\rho\rho/\pi\pi$ ratio, with the $\pi\pi$ mode being of the order of 10 MeV. Moreover, we also predict a very small decay width of the tensor glueball into $K^*(892)K$, which render this mode potentially interesting to exclude eventual glueball candidates in the future.

We compared the theoretical predictions to the experimental data. The interpretation of states based on the comparison is shown in Table VI. At present, the best match is for the resonance $f_2(1950)$, implying that the tensor glueball is a relatively broad state. The $f_2(1950)$ might be thought of as too light to be the tensor glueball, which—according to most lattice studies—has its mass in the range 2.2–2.4 GeV. However, unquenching effects included in additional meson-meson loop corrections are expected to bring the glueball mass down. The large width $f_2(1950)$ of the glueball is indeed in agreement with this view.

Here, the tensor glueball mixing with other conventional meson states was not taken into account. While mixing with the ground state tensor meson is expected to be small due to the large mass difference (recently, a small mixing of the pseudoscalar glueball and the η was studied on the lattice in [71]), this could be not the case for the nearby excited tensor states. The generalization to the eLSM is in principle possible and can be undertaken once better data, both from experiments and from lattice, will be available.

In the future, more information for decays of all tensor states into vector-vector, pseudoscalar-pseudoscalar as well as ground-state tensor-pseudoscalar would be very helpful to constrain models and falsify different scenarios. Moreover, also the decay of J/ψ into tensors as well as radiative (such as photon-photon) decays of tensor states could be of great use. In particular, more information about the broad state $f_2(2340)$ could shed light on its nature.

Another interesting future line of research is the study of glueball molecules [72,73]. While two scalar glueballs may interact and form a bound state (which is stable in pure YM), the question for the analogous tensor-tensor case (and also tensor-scalar and heavier glueballs, such as the pseudoscalar one) is unsettled.

ACKNOWLEDGMENTS

A. V. and F. G. acknowledge financial support from the Polish National Science Centre (NCN) via the OPUS Project No. 2019/33/B/ST2/00613. S. J. acknowledges the financial support through the project "Development Accelerator of the Jan Kochanowski University of Kielce," cofinanced by the European Union under the European Social Fund, with Grant No. POWR.03.05.00-00-Z212/18.

[1] F. Gross *et al.*, 50 years of quantum chromodynamics, arXiv:2212.11107.
[2] V. Mathieu, N. Kochelev, and V. Vento, The physics of glueballs, Int. J. Mod. Phys. E **18**, 1 (2009).
[3] V. Crede and C. A. Meyer, The experimental status of glueballs, Prog. Part. Nucl. Phys. **63**, 74 (2009).
[4] F. J. Llanes-Estrada, Glueballs as the Ithaca of meson spectroscopy: From simple theory to challenging detection, Eur. Phys. J. Spec. Top. **230**, 1575 (2021).
[5] H.-X. Chen, W. Chen, X. Liu, Y.-R. Liu, and S.-L. Zhu, An updated review of the new hadron states, Rep. Prog. Phys. **86**, 026201 (2023).
[6] Y. Chen *et al.*, Glueball spectrum and matrix elements on anisotropic lattices, Phys. Rev. D **73**, 014516 (2006).
[7] C. J. Morningstar and M. J. Peardon, The glueball spectrum from an anisotropic lattice study, Phys. Rev. D **60**, 034509 (1999).
[8] A. Athenodorou and M. Teper, The glueball spectrum of SU(3) gauge theory in 3+1 dimensions, J. High Energy Phys. 11 (2020) 172.
[9] H. B. Meyer, Glueball Regge trajectories, PhD thesis, University of Oxford, 2004.
[10] K. Sakai and S. Sasaki, Glueball spectroscopy in lattice QCD using gradient flow, Phys. Rev. D **107**, 034510 (2023).

[11] F. Chen, X. Jiang, Y. Chen, K.-F. Liu, W. Sun, and Y.-B. Yang, Glueballs at physical pion mass, Chin. Phys. C **47**, 063108 (2023).

[12] C. Bonanno, M. D'Elia, B. Lucini, and D. Vadacchino, Towards glueball masses of large-N SU(N) pure-gauge theories without topological freezing, Phys. Lett. B **833**, 137281 (2022).

[13] E. B. Gregory, A. C. Irving, B. Lucini, C. C. McNeile, A. Rago, C. Richards, and E. Rinaldi, Towards the glueball spectrum from unquenched lattice QCD, J. High Energy Phys. 10 (2012) 170.

[14] M. Q. Huber, C. S. Fischer, and H. Sanchis-Alepuz, Higher spin glueballs from functional methods, Eur. Phys. J. C **81**, 1083 (2021).

[15] J. M. Pawlowski, C. S. Schneider, J. Turnwald, J. M. Urban, and N. Wink, Yang-Mills glueball masses from spectral reconstruction, arXiv:2212.01113.

[16] E. Bennett, J. Holligan, D. K. Hong, J.-W. Lee, C. J. D. Lin, B. Lucini, M. Piai, and D. Vadacchino, Color dependence of tensor and scalar glueball masses in Yang-Mills theories, Phys. Rev. D **102**, 011501 (2020).

[17] E. Trotti, S. Jafarzade, and F. Giacosa, Thermodynamics of the glueball resonance gas, Eur. Phys. J. C **83**, 390 (2023).

[18] S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, and K. K. Szabo, Full result for the QCD equation of state with $2+1$ flavors, Phys. Lett. B **730**, 99 (2014).

[19] J. Turnau, Is there a 2^{++} glueball at 1410-MeV?, Z. Phys. C **25**, 299 (1984).

[20] E. Klempt, Meson spectroscopy: Glueballs, hybrids and $Q\bar{Q}$ mesons, in *Zuoz Summer School on Phenomenology of Gauge Interactions* (Cornell University, 2000), pp. 61–126, arXiv:hep-ex/0101031.

[21] V. V. Anisovich and A. V. Sarantsev, Observation of tensor glueball in the reactions $p\bar{p}\to \pi\pi$, $\eta\eta, \eta\eta'$, JETP Lett. **81**, 417 (2005).

[22] V. V. Anisovich, Systematization of tensor mesons and the determination of the 2^{++} glueball, JETP Lett. **80**, 715 (2004).

[23] S. Narison, Masses, decays and mixings of gluonia in QCD, Nucl. Phys. **B509**, 312 (1998).

[24] S. Narison, QCD spectral sum rules 2022, Nucl. Part. Phys. Proc. **324–329**, 94 (2023).

[25] Y. Chen, D. Li, and M. Huang, The dynamical holographic QCD method for hadron physics and QCD matter, Commun. Theor. Phys. **74**, 097201 (2022).

[26] F. Brünner and A. Rebhan, Nonchiral Enhancement of Scalar Glueball Decay in the Witten-Sakai-Sugimoto Model, Phys. Rev. Lett. **115**, 131601 (2015).

[27] F. Brünner, D. Parganlija, and A. Rebhan, Glueball decay rates in the Witten-Sakai-Sugimoto model, Phys. Rev. D **91**, 106002 (2015); Phys. Rev. D **93**, 109903(E) (2016).

[28] F. Hechenberger, J. Leutgeb, and A. Rebhan, Radiative meson and glueball decays in the Witten-Sakai-Sugimoto model, Phys. Rev. D **107**, 114020 (2023).

[29] R. C. Brower, S. D. Mathur, and C.-I. Tan, Glueball spectrum for QCD from AdS supergravity duality, Nucl. Phys. **B587**, 249 (2000).

[30] Y. Chen and M. Huang, Two-gluon and trigluon glueballs from dynamical holography QCD, Chin. Phys. C **40**, 123101 (2016).

[31] M. Rinaldi and V. Vento, Glueballs at high temperature within the hard-wall holographic model, Eur. Phys. J. C **82**, 140 (2022).

[32] M. Rinaldi, Meson and glueball spectroscopy within the graviton soft-wall model, Rev. Mex. Fis. Suppl. **3**, 0308019 (2022).

[33] D. Elander, M. Piai, and J. Roughley, Holographic glueballs from the circle reduction of Romans supergravity, J. High Energy Phys. 02 (2019) 101.

[34] F. Giacosa, T. Gutsche, V. E. Lyubovitskij, and A. Faessler, Decays of tensor mesons and the tensor glueball in an effective field approach, Phys. Rev. D **72**, 114021 (2005).

[35] B. A. Li, 2^{++} glueball, Phys. Rev. D **86**, 017501 (2012).

[36] R. S. Longacre and S. J. Lindenbaum, Evidence for a 4th state related to the three $J^{PC}=2^{++}, \pi^- p\to \phi\phi n$ states explainable by 2^{++} glueball production, Phys. Rev. D **70**, 094041 (2004).

[37] E. Klempt, K. V. Nikonov, A. V. Sarantsev, and I. Denisenko, Search for the tensor glueball, Phys. Lett. B **830**, 137171 (2022).

[38] A. V. Sarantsev and E. Klempt, Scalar and tensor mesons in $d\bar{d}$, $s\bar{s}$ and $gg\to f_0, f_2$, arXiv:2211.08791.

[39] E. Klempt, Glueballs, a fulfilled promise of QCD?, arXiv:2211.12901.

[40] N. Kivel and M. Vanderhaeghen, Production of a tensor glueball in the reaction $\gamma\gamma\to G_2\pi^0$ at large momentum transfer, Phys. Lett. B **781**, 169 (2018).

[41] W. Kou and X. Chen, Production of tensor glueball in ultraperipheral pp, $AuAu$ and $PbPb$ collisions, Eur. Phys. J. C **83**, 152 (2023).

[42] D. Parganlija, P. Kovacs, G. Wolf, F. Giacosa, and D. H. Rischke, Meson vacuum phenomenology in a three-flavor linear sigma model with (axial-)vector mesons, Phys. Rev. D **87**, 014011 (2013).

[43] S. Janowski, F. Giacosa, and D. H. Rischke, Is $f_0(1710)$ a glueball?, Phys. Rev. D **90**, 114005 (2014).

[44] S. Janowski, D. Parganlija, F. Giacosa, and D. H. Rischke, The glueball in a chiral linear sigma model with vector mesons, Phys. Rev. D **84**, 054007 (2011).

[45] W. I. Eshraim, S. Janowski, F. Giacosa, and D. H. Rischke, Decay of the pseudoscalar glueball into scalar and pseudoscalar mesons, Phys. Rev. D **87**, 054036 (2013).

[46] W. I. Eshraim, Decay of the pseudoscalar glueball into vector, axial-vector, scalar and pseudoscalar mesons, Eur. Phys. J. C **83**, 262 (2023).

[47] F. Giacosa, J. Sammet, and S. Janowski, Decays of the vector glueball, Phys. Rev. D **95**, 114004 (2017).

[48] S. Jafarzade, A. Vereijken, M. Piotrowska, and F. Giacosa, companion paper, From well-known tensor mesons to yet unknown axial-tensor mesons, Phys. Rev. D **106**, 036008 (2022).

[49] D. Parganlija, F. Giacosa, and D. H. Rischke, Vacuum properties of mesons in a linear sigma model with vector mesons and global chiral invariance, Phys. Rev. D **82**, 054024 (2010).

[50] W. I. Eshraim, C. S. Fischer, F. Giacosa, and D. Parganlija, Hybrid phenomenology in a chiral approach, Eur. Phys. J. Plus **135**, 945 (2020).
[51] P. Kovács, Z. Szép, and G. Wolf, Existence of the critical endpoint in the vector meson extended linear sigma model, Phys. Rev. D **93**, 114014 (2016).
[52] P. Kovács, G. Kovács, and F. Giacosa, Fate of the critical endpoint at large N_c, Phys. Rev. D **106**, 116016 (2022).
[53] A. Heinz, F. Giacosa, and D. H. Rischke, Chiral density wave in nuclear matter, Nucl. Phys. **A933**, 34 (2015).
[54] R. L. Workman *et al.* (Particle Data Group), Review of particle physics, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022 and 2023 update).
[55] F. Giacosa, A. Koenigstein, and R. D. Pisarski, How the axial anomaly controls flavor mixing among mesons, Phys. Rev. D **97**, 091901 (2018).
[56] H. Hatanaka and K.-C. Yang, $K_1(1270)$—$K_1(1400)$ mixing angle and new-physics effects in $B \to K_1 l^+ l^-$ decays, Phys. Rev. D **78**, 074007 (2008).
[57] F. Divotgey, L. Olbrich, and F. Giacosa, Phenomenology of axial-vector and pseudovector mesons: Decays and mixing in the kaonic sector, Eur. Phys. J. A **49**, 135 (2013).
[58] G. Amelino-Camelia *et al.*, Physics with the KLOE-2 experiment at the upgraded DAϕNE, Eur. Phys. J. C **68**, 619 (2010).
[59] T. Feldmann, P. Kroll, and B. Stech, Mixing and decay constants of pseudoscalar mesons, Phys. Rev. D **58**, 114006 (1998).
[60] G. 't Hooft, How instantons solve the $U(1)$ problem, Phys. Rep. **142**, 357 (1986).
[61] A. Rodas, A. Pilloni, M. Albaladejo, C. Fernandez-Ramirez, V. Mathieu, and A. P. Szczepaniak (Joint Physics Analysis Center Collaboration), Scalar and tensor resonances in J/ψ radiative decays, Eur. Phys. J. C **82**, 80 (2022).
[62] M. Albrecht *et al.* (Crystal Barrel Collaboration), Coupled channel analysis of $\bar{p}p \to \pi^0\pi^0\eta$, $\pi^0\eta\eta$ and $K^+K^-\pi^0$ at 900 MeV/c and of $\pi\pi$-scattering data, Eur. Phys. J. C **80**, 453 (2020).
[63] S. Okubo, Phi meson and unitary symmetry model, Phys. Lett. **5**, 165 (1963).
[64] G. Zweig, An SU(3) model for strong interaction symmetry and its breaking. Version 2, in *Developments in the Quark Theory of Hadrons, Vol. 1. 1964–1978*, edited by D. B. Lichtenberg and S. P. Rosen (CERN, Geneva, 1964), pp. 22–101.
[65] J. Iizuka, K. Okada, and O. Shito, Systematics and phenomenology of boson mass levels. 3, Prog. Theor. Phys. **35**, 1061 (1966).
[66] W. Toki, Search for glueballs, in *24th Annual SLAC Summer Institute on Particle Physics: The Strong Interaction, From Hadrons to Protons* (CERN, Geneva, 1996), pp. 141–169.
[67] F. Giacosa, Heavy glueballs: Status and large-N_c widths estimate, Acta Phys. Pol. B Proc. Suppl. **10**, 1021 (2017).
[68] S. Godfrey and N. Isgur, Mesons in a relativized quark model with chromodynamics, Phys. Rev. D **32**, 189 (1985).
[69] M. Piotrowska, C. Reisinger, and F. Giacosa, Strong and radiative decays of excited vector mesons and predictions for a new $\phi(1930)$ resonance, Phys. Rev. D **96**, 054033 (2017).
[70] Z.-C. Ye, X. Wang, X. Liu, and Q. Zhao, The mass spectrum and strong decays of isoscalar tensor mesons, Phys. Rev. D **86**, 054025 (2012).
[71] X. Jiang, W. Sun, F. Chen, Y. Chen, M. Gong, Z. Liu, and R. Zhang, η-glueball mixing from $N_f = 2$ lattice QCD, Phys. Rev. D **107**, 094510 (2023).
[72] F. Giacosa, A. Pilloni, and E. Trotti, Glueball–glueball scattering and the glueballonium, Eur. Phys. J. C **82**, 487 (2022).
[73] A. A. Petrov, Glueball molecules, Phys. Lett. B **843**, 138030 (2023).