Entanglement entropy of the proton in coordinate space

Adrian Dumitru
Department of Natural Sciences, Baruch College, CUNY, 17 Lexington Avenue, New York, New York 10010, USA
and The Graduate School and University Center, The City University of New York, 365 Fifth Avenue, New York, New York 10016, USA

Alex Kovner
Physics Department, University of Connecticut, 2152 Hillside Road, Storrs, Connecticut 06269, USA

Vladimir V. Skokov
North Carolina State University, Raleigh, North Carolina 27695, USA
and RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA

(Received 24 April 2023; accepted 26 June 2023; published 14 July 2023)

We calculate the entanglement entropy of a model proton wave function in coordinate space by integrating out degrees of freedom outside a small circular region $\bar{A}$ of radius L, where L is much smaller than the size of the proton. The wave function provides a nonperturbative distribution of three valence quarks. In addition, we include the perturbative emission of a single gluon and calculate the entanglement entropy of gluons in $\bar{A}$. For both quarks and gluons, we obtain the same simple result: $S_E = -\int \frac{dx}{\Delta x} N_{L^2}(x) \log[N_{a^2}(x)]$, where a is the UV cutoff in coordinate space and Δx is the longitudinal resolution scale. Here $N_S(x)$ is the number of partons (of the appropriate species) with longitudinal momentum fraction x inside an area S. It is related to the standard parton distribution function by $N_S(x) = \frac{S}{A_p} \Delta x F(x)$, where A_p denotes the transverse area of the proton.

DOI: 10.1103/PhysRevD.108.014014

I. INTRODUCTION

Rapid advent of quantum science in recent years provides strong motivation for asking new types of questions in many areas of inquiry, including high energy nuclear and particle physics. In particular, there is an ongoing vigorous discussion about the relevance of entanglement (and the associated entanglement entropy) in the context of particle production in high energy hadronic collisions [1–25].

The initial discussion by Kharzeev and Levin [8] is framed in the context of entanglement of the degrees of freedom inside a small area of the proton actually probed in a deep inelastic scattering (DIS) experiment, with the rest of the degrees of freedom in the proton wave function and, in particular, with soft modes of the gluon field responsible for confinement. It was suggested that the entropy of this entanglement translates into the Boltzmann entropy of particles produced in the collision. Some model calculations have been performed to probe this picture [9,12–15], and it has also been subjected to an experimental test [26]. However, no direct calculation of entanglement entropy in coordinate space has so far been reported in the literature. The aim of this manuscript is to fill this gap.

Of course, such a calculation requires knowledge of the wave function of the proton, and needless to say, the exact proton wave function is not known. Nevertheless, several simple model wave functions that provide the distribution of valence quarks at large x and low resolution Q^2 have been used in QCD phenomenology over the years with reasonable success, e.g. Refs. [27–30]. These quark wave functions can be improved by including a perturbative gluon component, as described in Ref. [31], and used in Ref. [32] to compute DIS structure functions at high energy, and in Ref. [22] to study entanglement of momentum-space degrees of freedom over the whole area of the proton. In this paper our main goal is to derive expressions for the density matrix and entropy of a small "hole" in the proton in such a setup. For numerical estimates we will use one specific light-cone valence quark model wave function from Refs. [27,28].

The idea of our calculation is very straightforward. We divide the transverse area of the proton into a small disk $\bar{A}$ and its complement A, and integrate out all degrees of freedom in A. The result is the reduced density matrix $\rho_{\bar{A}}$ which contains complete information for the calculation of any observable localized in $\bar{A}$. We then calculate the von Neumann entropy of $\rho_{\bar{A}}$.

Even before including the perturbative gluon component, the result is nontrivial. The entropy in this case is associated with different numbers of quarks that can reside inside $\bar{A}$. Note that the total number of valence quarks in the model wave function is fixed (three), nevertheless the wave function carries finite probabilities of finding different numbers of quarks inside $\bar{A}$. Integrating over A therefore generates a reduced density matrix which spans Hilbert subspaces with different occupation numbers n. The von Neumann entropy arises precisely due to nonvanishing eigenvalues of $\rho_{\bar{A}}$ in subspaces with different n.

Note that, in the simple case when the total number of partons is fixed, the reduced density matrix is diagonal in the particle number basis *by fiat*. This follows immediately since in reducing the density matrix we trace over A, and thus calculate matrix elements between states which have equal numbers of partons in A. For wave functions that do not preserve the number of partons we expect, in general, that $\rho_{\bar{A}}$ would not be diagonal in the n basis. Thus, including a perturbative gluon emission may lead to such a nondiagonal $\rho_{\bar{A}}$. As it turns out, in the first order of perturbation theory this does not happen due to the fact that in the valence part of the wave function the color and spatial degrees of freedom are not entangled with each other.

We first perform the calculation in the way described above for the valence wave function that contains three quarks only. We next include a one gluon state which is generated by the first order perturbative correction. Here, for simplicity, we modify our procedure somewhat, i.e. we trace over the quark degrees of freedom in the whole wave function, and only then do we generate $\rho_{\bar{A}}$ by reducing over the gluon degrees of freedom in A. We then calculate the entanglement entropy of the resulting density matrix, which now has the meaning of entropy of gluons inside $\bar{A}$.

This paper is structured as follows. In Sec. II we prepare our tools for performing the calculation in coordinate space and describe the model wave function for valence quarks. In Sec. III we calculate the reduced density matrix $\rho_{\bar{A}}$ and the entropy for a small disk $\bar{A}$ in this model. Here "small" means small relative to the nonperturbative scale which determines the spatial size of the model wave function. We discuss the dependence of the entropy on the area of $\bar{A}$ in this regime. In Sec. IV we include an additional perturbatively emitted gluon in the wave function, and again calculate the reduced density matrix (in the way described above) and discuss its properties. The entanglement entropy is also calculated in Sec. V. In both cases (quarks and gluons) the entanglement entropy can be written in a very suggestive form in terms of the parton distribution function (PDF) of the appropriate parton species, Eq. (94). Finally, in Sec. VI we discuss our results and their possible relation to the suggestion of Ref. [8].

II. LAYING THE GROUNDWORK

In the following we denote any three vector p as $p = (p^+, \vec{p})$, where p^+ and $\vec{p}$ are longitudinal and transverse components of the vector, respectively. We will be using a mixed representation for the wave function where coordinate space is used to represent the transverse degrees of freedom, and momentum space is used for the longitudinal ones.

In this mixed representation we denote a state of the proton at c.m. position $\vec{R} = 0$ and longitudinal momentum P^+ by $|\vec{R} = 0, P^+\rangle$. This convoluted notation does not reference the wave function for the internal degrees of freedom, i.e. the coordinates, color and spin states of the constituents, which we will specify in a short while.

The coordinate space proton state vector is related to the momentum-space state vector through (see e.g. Ref. [33])

$$|\vec{R}, P^+\rangle = \mathcal{N} \int_{\vec{P}} e^{i\vec{P}\cdot\vec{R}} |\vec{P}, P^+\rangle, \tag{1}$$

where $\vec{P}$ is the transverse momentum of the proton, and the integration measure is

$$\int_{\vec{P}} \equiv \int \frac{\mathrm{d}^2 P}{(2\pi)^2}. \tag{2}$$

The normalization factor is determined from the condition $|\mathcal{N}|^2 \int_{\vec{P}} = 1$. A proton centered at $\vec{R} = 0$ is then

$$|\vec{R} = 0, P^+\rangle = \mathcal{N} \int_{\vec{P}} |\vec{P}, P^+\rangle. \tag{3}$$

We employ the standard normalization of the momentum-space states,

$$\langle K|P\rangle = 16\pi^3 P^+ \delta(P^+ - K^+)\delta^2(\vec{P} - \vec{K}), \tag{4}$$

which leads to the following normalization of the mixed space state vector:

$$\langle \vec{R} = 0, P'^+ | \vec{R} = 0, P^+\rangle = 4\pi P^+ \delta(P^+ - P'^+). \tag{5}$$

The density operator for this state is

$$\hat{\rho} = |\vec{R} = 0, P^+\rangle\langle\vec{R} = 0, P^+|. \tag{6}$$

In the following we will be calculating matrix elements of $\hat{\rho}$ between states of the partonic (Fock) Hilbert space

$$\rho_{\alpha\alpha'} = \langle\alpha'|\vec{R} = 0, P^+\rangle\langle\vec{R} = 0, P^+|\alpha\rangle, \tag{7}$$

where α denotes a collection of "labels" (such as the light-cone (LC) momentum fractions x_i, coordinates and color indices) assigned to the basis vectors of the Fock space.

A. The valence quark Fock state

We start with considering states that contain three valence quarks only. In the model described below the color and spatial degrees of freedom are not entangled, i.e. the wave function is a direct product of the color and spatial state vectors. In this case, for the spatial wave function $\alpha = \{x_i, \vec{r}_i\}$ refers to the quark LC momentum fractions and their transverse coordinates.

The state vector $|P^+, \vec{P}\rangle$ of a proton made of N_c "valence" quarks is written as

$$|P\rangle = \sum_{h_i} \int_{[0,1]^{N_c}} [\mathrm{d}x_i] \int [\mathrm{d}^2 k_i] \Psi(k_i, h_i) \frac{1}{\sqrt{N_c!}} \times \sum_{i_1 \ldots i_{N_c}} \epsilon_{i_1 \cdots i_{N_c}} |p_1, i_1, h_1; \cdots; p_{N_c}, i_{N_c}, h_{N_c}\rangle, \quad (8)$$

where

$$[\mathrm{d}x_i] = \delta\left(1 - \sum_i x_i\right) \prod_i \frac{\mathrm{d}x_i}{2x_i}, \quad (9)$$

$$[\mathrm{d}^2 k_i] = (2\pi)^3 \delta\left(\sum_i \vec{k}_i\right) \prod_i \frac{\mathrm{d}^2 k_i}{(2\pi)^3}. \quad (10)$$

Here $k_i = (k_i^+, \vec{k}_i)$ denote the momenta of the ith quark in the transverse rest frame of the proton, and $\vec{p}_i = \vec{k}_i + x_i \vec{P}$. The space-helicity wave function $\Psi(k_i, h_i)$ is symmetric under exchange of any two quarks while the state is antisymmetric in color space. In what follows we will mainly focus on the spatial wave function and trace out spin-flavor and color degrees of freedom.

We can now write the proton state in terms of the quark Fock space states

$$|\vec{R} = 0, P^+\rangle = \mathcal{N} \int_{\vec{P}} \int [\mathrm{d}x_i] \int [\mathrm{d}^2 k_i] \Psi(k_i) |p_1; p_2; p_3\rangle, \quad (11)$$

where we have omitted the quark (and proton) spins, for simplicity. Summing up, we integrate over the Galilean-invariant "internal" quark transverse momenta subject to the constraint that they add up to zero, and then over the c.m. transverse momentum $\vec{P}$, which is also the momentum of the proton.

Analogously, the three-quark coordinate space state with the quarks located at $\vec{r}_i$ and carrying LC momentum fractions x_i is constructed as

$$|x_1, \vec{r}_1; x_2, \vec{r}_2; x_3, \vec{r}_3\rangle = \mathcal{N} \int_Q \int [\mathrm{d}^2 q_i] e^{-i \sum (\vec{q}_i + x_i \vec{Q}) \cdot \vec{r}_i} |x_i, \vec{q}_i + x_i \vec{Q}\rangle. \quad (12)$$

Equation (12) can be extended to four (and more) particles simply by adding labels for momentum fraction and transverse position/momentum of the additional particle to the state vector and including the momentum of the additional particle in the integration measure Eq. (10).

The overlap of the proton state with the state of three quarks localized at fixed transverse coordinates is given by

$$\langle \vec{R} = 0, P^+ | x_i, \vec{r}_i\rangle = |\mathcal{N}|^2 \int_{P,Q} \int [\mathrm{d}y_i] \int [\mathrm{d}^2 k_i] \int [\mathrm{d}^2 q_i] e^{-i \sum (\vec{q}_i + x_i \vec{Q}) \cdot \vec{r}_i} \Psi^*(y_i, \vec{k}_i) \prod_i \langle y_i, \vec{k}_i + y_i \vec{P} | x_i, \vec{q}_i + x_i \vec{Q}\rangle$$
$$= |\mathcal{N}|^2 (2\pi)^3 \delta\left(1 - \sum x_i\right) \delta\left(\sum x_i \vec{r}_i\right) \int [\mathrm{d}^2 q_i] e^{-i \sum \vec{q}_i \cdot \vec{r}_i} \Psi^*(x_i, \vec{q}_i), \quad (13)$$

where we used Eqs. (4), (9), and (10). Note that the overlap does not vanish only for states with c.m. located at the origin, $\sum x_i \vec{r}_i = 0$, just as for the proton, c.f. Eq. (11) in [33]; or Ref. [34] for the analogous case of a $q\bar{q}$ dipole. Also, the LC momentum fractions of the quarks must sum up to one. Since only such states contribute to the proton density matrix, and we included the constraints on the longitudinal momentum fractions/the transverse momenta in the integration measure (9) and (10), a matrix element of the properly normalized density matrix is given by

$$\rho_{\alpha\alpha'} = \frac{\langle \vec{R} = 0, P^+ | \alpha' \rangle}{|\mathcal{N}|^2 (2\pi)^3 \delta(1 - \sum x_i') \delta(\sum x_i' \vec{r}_i')} \frac{\langle \alpha | \vec{R} = 0, P^+ \rangle}{|\mathcal{N}|^2 (2\pi)^3 \delta(1 - \sum x_i) \delta(\sum x_i \vec{r}_i)} \quad (14)$$

$$= \int [\mathrm{d}^2 q_i] e^{i \sum \vec{q}_i \cdot \vec{r}_i} \int [\mathrm{d}^2 q_i'] e^{-i \sum \vec{q}_i' \cdot \vec{r}_i'} \Psi^*(x_i', \vec{q}_i') \Psi(x_i, \vec{q}_i) \quad (15)$$

$$= \Psi^*(x_i', \vec{r}_i') \Psi(x_i, \vec{r}_i), \quad (16)$$

where $\alpha = \{x_i, \vec{r}_i | \sum x_i = 1, \sum x_i \vec{r}_i = 0\}$ and $\alpha' = \{x'_i, \vec{r}'_i | \sum x'_i = 1, \sum x'_i \vec{r}'_i = 0\}$ denote two sets of LC momentum fractions and transverse quark positions. Here in the last step we used the definition (B4) of Ref. [33] for the coordinate space LC wave functions,

$$\Psi(x_i, \vec{r}_i) = \int [\mathrm{d}^2 q_i] e^{i \sum \vec{q}_i \cdot \vec{r}_i} \Psi(x_i, \vec{q}_i). \tag{17}$$

The normalization of the coordinate space wave function will be obtained later in Eq. (31) from the requirement that the trace of the density matrix $\mathrm{tr}\,\hat{\rho} = 1$.

For the model wave function considered here (see below) the color degrees of freedom of the above density matrix could be restored simply by multiplying by the normalized color space matrix $\frac{1}{3!}\epsilon_{i_1 i_2 i_3}\epsilon_{i'_1 i'_2 i'_3}$.

B. A model wave function

Our main goal here is to obtain general expressions for the reduced density matrix in a transverse region $\bar{A}$ of the proton and to estimate the entropy associated with this density matrix (which we do in Sec. III B). For this we require an explicit expression for the three-quark wave function Ψ_{qqq}.

We employ a simple model due to Schlumpf and Brodsky [27,28],

$$\Psi(x_i, \vec{k}_i) \sim \sqrt{x_1 x_2 x_3}\, e^{-\mathcal{M}^2/2\beta^2}; \qquad \mathcal{M}^2 = \sum \frac{\vec{k}_i^2 + m_q^2}{x_i}. \tag{18}$$

Here $\mathcal{M}^2$ is the invariant mass squared of the noninteracting three-quark system [35], i.e. the sum of the quark LC energies multiplied by P^+. The nonperturbative parameters $m_q = 0.26$ and $\beta = 0.55$ GeV have been fixed in Refs. [27,28] to match empirical properties of the proton at low energy and low resolution. Note that β is of order $N_c = 3$ times the root-mean-square valence quark transverse momentum in the proton.

This Gaussian wave function can be easily transformed to position space. One obtains (up to normalization)

$$\Psi(x_i, \vec{r}_i) \sim F(x_1, x_2, x_3) e^{-\frac{1}{2} a_{13} \beta^2 r_{13}^2} e^{-\frac{1}{2} a_{23} \beta^2 r_{23}^2} e^{b \beta^2 \vec{r}_{13} \cdot \vec{r}_{23}} \tag{19}$$

with

$$\begin{aligned} \vec{r}_{ij} &\equiv \vec{r}_i - \vec{r}_j, \\ F(x_1, x_2, x_3) &= (2\pi\beta^2)^2 \frac{(x_1 x_2 x_3)^{3/2}}{(2\pi)^6} e^{-\frac{m_q^2}{2\beta^2} \sum \frac{1}{x_i}}, \\ a_{23} &= x_2(1 - x_2), \\ a_{13} &= x_1(1 - x_1), \\ b &= x_1 x_2. \end{aligned} \tag{20}$$

One can easily verify that this is symmetric under the exchange of any two quarks, $(x_i, \vec{r}_i) \leftrightarrow (x_j, \vec{r}_j)$; $i, j = 1, 2, 3$.

III. THE REDUCED DENSITY MATRIX AND ENTANGLEMENT ENTROPY OF A THREE-QUARK SYSTEM

We can now construct a reduced density matrix by tracing over a subset of degrees of freedom. Here we are interested in the reduced density matrix that determines observables localized to a small circle in the center of the proton. To find this density matrix we have to trace over the region A of the proton which is the outside of the circle in question. In other words we have to integrate over the transverse positions and LC momentum fractions of all quarks located in A.

A. The density matrix for a small disk

First we note that the Hilbert space inside the disk $\bar{A}$ is a direct sum of Hilbert spaces of zero, one, two and three particles. In addition, it is obvious that since we are tracing over A, the reduced density matrix does not contain off diagonal elements that connect states with different particle numbers. The reduced density matrix therefore can be represented as a block diagonal matrix of the form

$$\rho_{\bar{A}} = \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & \rho_3 \end{pmatrix}. \tag{21}$$

Note that the various blocks along the diagonal are density matrices over Hilbert spaces of different dimensionality.

To calculate ρ_0 we place all quarks in A,

$$\rho_0 = \int [\mathrm{d}x_i] \int [\mathrm{d}^2 r_i] \Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)\Theta_A(\vec{r}_3) |\Psi(x_i, \vec{r}_i)|^2. \tag{22}$$

Here,

$$[\mathrm{d}^2 r_i] = \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 \mathrm{d}^2 r_3 \delta\Big(\sum x_i \vec{r}_i\Big), \tag{23}$$

and $\Theta_A(\vec{r}) = 1$ if $\vec{r} \in A$ and 0 otherwise. This is a pure dimensionless [by the normalization condition in Eq. (31) below] number giving the probability that in our wave function no quarks reside in $\bar{A}$.

The second block ρ_1 of (21) is the probability density that only one of the quarks is localized in $\bar{A}$ while the other two are localized in A. Tracing over A we have to set $\vec{r}_1 = \vec{r}'_1 \in A$ and $\vec{r}_2 = \vec{r}'_2 \in A$, so by virtue of the c.m. constraint we also have $\vec{r}_3 = \vec{r}'_3$, with $\vec{r}_3 \in \bar{A}$, so ρ_1 is diagonal in coordinate space,

$$(\rho_1)_{\alpha\alpha} = 3\int \frac{\mathrm{d}x_1\mathrm{d}x_2}{8x_1x_2x_3}\delta\Big(1-\sum x_i\Big) \times \int \mathrm{d}^2r_1\mathrm{d}^2r_2\delta\Big(\sum x_i\vec{r}_i\Big)\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)|\Psi(x_i,\vec{r}_i)|^2, \quad (\vec{r}_3\in\bar{A}). \tag{24}$$

The matrix indices here are $\alpha = \{x_3, \vec{r}_3\}$, defined over the domain $0 \le x_3 \le 1$ and $\vec{r}_3 \in \bar{A}$.

Clearly, the dimensionalities of ρ_1 and ρ_0 are different. While ρ_0 is dimensionless and has the meaning of probability, ρ_1 has dimension $1/r^2$ and has the meaning of probability density. To construct a probability from ρ_1 we would have to multiply it by the "lattice spacing" in the transverse coordinate space a^2 and in fact also by the elementary length in the longitudinal momentum space Δx. If we take this route, the integration over the coordinate $\vec{r}_3$ and the momentum fraction x_3 will have to be performed with the dimensionless measure $\mathrm{d}^2r_3/a^2\ \mathrm{d}x_3/\Delta x$.

For the discussion of the density matrix itself this is not crucial since a calculation of the average of any observable involves integration over x and r_i and the minimal area cancels in the product of the probability density and the integration measure. However, when we calculate von Neumann entropies S_E this becomes important, since we need to define a dimensionless *probability* in order to take its logarithm. In fact it is also crucial to work with a dimensionless density matrix when we calculate the trace of any nontrivial (not first) power of ρ. Since the index on ρ_1 is continuous, the density matrix is infinitely dimensional. We therefore expect its individual matrix elements to vanish in the strict continuum limit (for vanishing a^2 and Δx) as the first power of $a^2\Delta x$. When calculating $\mathrm{tr}\rho_1$ this smallness of the matrix elements is compensated by the integration over $\vec{r}_3, x_3$. However, when we calculate $\mathrm{tr}\rho_1^N$, the diagonal matrix elements now vanish as $(a^2\Delta x)^N$, while there is still only a single integral over $\vec{r}_3, x_3$ involved in calculating the trace. Therefore $\mathrm{tr}\rho_1^N \to_{a,\Delta x\to 0} (a^2\Delta x)^{N-1}$, and it is imperative to keep the lattice spacing finite in order to obtain any physical information about $\mathrm{tr}\rho_1^N$ beyond the trivial fact that it vanishes in the continuum limit. We will therefore introduce the lattice spacing in the definition of the density matrix and will forthwith work with

$$(\rho_1)_{\alpha\alpha} = 3\Delta x a^2\int \frac{\mathrm{d}x_1\mathrm{d}x_2}{8x_1x_2x_3}\delta\Big(1-\sum x_i\Big) \times \int \mathrm{d}^2r_1\mathrm{d}^2r_2\delta\Big(\sum x_i\vec{r}_i\Big)\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)|\Psi(x_i,\vec{r}_i)|^2, \quad (\vec{r}_3\in\bar{A}), \tag{25}$$

with the understanding that the trace is taken with respect to the measure $\frac{\mathrm{d}^2r_3}{a^2}\frac{\mathrm{d}x_3}{\Delta x}\Theta_{\bar{A}}(\vec{r}_3)$. Furthermore, we included the x_3-dependent part of the integration measure (9), i.e. the factor $1/x_3$, in the definition of ρ_1 so that the trace is given by the x_3-independent integration measure $\frac{\mathrm{d}x_3}{\Delta x}$. One can easily understand why this is necessary by considering the classical Shannon entropy of a probability density distribution, see Appendix A.

The third block ρ_2 corresponds to the configuration where two of the quarks are located in $\bar{A}$ while the third one is located in A,

$$(\rho_2)_{\alpha\alpha'} = 3(a^2\Delta x)^2\frac{\Psi^*(\alpha', x_3, \vec{r}_3)}{x_3\sqrt{8x_1'x_2'x_3}}\frac{\Psi(\alpha, x_3, \vec{r}_3)}{x_3\sqrt{8x_1x_2x_3}}, \tag{26}$$

with $\alpha = \{x_1, \vec{r}_1, x_2, \vec{r}_2\}$ and $\alpha' = \{x_1', \vec{r}_1', x_2', \vec{r}_2'\}$. Note that there is no integral over the coordinate of the third quark in this equation. This is due to the fact that the c.m. constraint rigidly determines $\vec{r}_3, x_3$ for given coordinates and longitudinal momenta of the first two quarks as

$$x_3 = 1 - x_1 - x_2 = 1 - x_1' - x_2', \quad \vec{r}_3 = -(x_1\vec{r}_1 + x_2\vec{r}_2)/x_3 = -(x_1'\vec{r}_1' + x_2'\vec{r}_2')/x_3. \tag{27}$$

The matrix indices α, α' are defined over the domain where these relations are satisfied with $0 \le x_3 \le 1$ and $\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2' \in \bar{A}$, $\vec{r}_3 \in A$. We have again introduced the lattice spacing into the definition of a matrix element of ρ_2 to make it dimensionless. The factor 3 in (26) arises since either one of the three quarks can reside in A.

The trace of ρ_2 on the subspace with two particles is defined as

$$\mathrm{tr}\rho_2 = \int \frac{\mathrm{d}x_1}{\Delta x}\frac{\mathrm{d}x_2}{\Delta x}\int \frac{\mathrm{d}^2r_1}{a^2}\frac{\mathrm{d}^2r_2}{a^2}\Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)\Theta(x_3) \times \Theta_A(\vec{r}_3)(\rho_2)_{\alpha\alpha} = 3\int[\mathrm{d}x_i]\int[\mathrm{d}^2r_i]\Theta_A(\vec{r}_3)\Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)|\Psi(x_i,\vec{r}_i)|^2, \tag{28}$$

where the lattice spacing cancels between the matrix element and the integration measure. Once again, the trace operation does not involve any Jacobians which depend on x_1 and x_2.

Finally, the fourth block corresponds to all three quarks in $\bar{A}$,

$$(\rho_3)_{\alpha\alpha'} = \frac{(a^2\Delta x)^2}{x_3'\sqrt{8x_1'x_2'x_3'}}\frac{1}{x_3\sqrt{8x_1x_2x_3}}\Psi^*(x_i', \vec{r}_i')\Psi(x_i, \vec{r}_i). \tag{29}$$

Here $\alpha = \{x_1, \vec{r}_1, x_2, \vec{r}_2\}$ and similarly for α'. These indices are defined over the domain $0 \le x_1, x_2, x_1', x_2' \le 1$ with $0 \le x_3 = 1 - x_1 - x_2 \le 1$, $0 \le x_3' = 1 - x_1' - x_2' \le 1$; and $\vec{r}_1, \vec{r}_1', \vec{r}_2, \vec{r}_2', \vec{r}_3, \vec{r}_3' \in \bar{A}$, with $\vec{r}_3 = -(x_1\vec{r}_1 + x_2\vec{r}_2)/x_3$, $\vec{r}_3' = -(x_1'\vec{r}_1' + x_2'\vec{r}_2')/x_3'$. We have again introduced the lattice spacing in this definition so that the elements of ρ_3 are dimensionless, although these factors cancel in the trace of an arbitrary power of ρ_3. To take the trace in this block we calculate

$$\begin{aligned}\mathrm{tr}\rho_3 &= \int \frac{\mathrm{d}x_1}{\Delta x}\frac{\mathrm{d}x_2}{\Delta x}\int \frac{\mathrm{d}^2 r_1}{a^2}\frac{\mathrm{d}r_2}{a^2}\Theta(x_3)\Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)\\ &\quad\times\Theta_{\bar{A}}(\vec{r}_3)(\rho_3)_{\alpha\alpha}\\ &= \int [\mathrm{d}x_i][\mathrm{d}^2 r_i]|\Psi(x_i,\vec{r}_i)|^2\prod\Theta_{\bar{A}}(\vec{r}_i). \end{aligned} \tag{30}$$

Putting this all together we obtain that the total trace of the density matrix is[1]

$$\begin{aligned}\mathrm{tr}\rho_{\bar{A}} &= \rho_0 + \mathrm{tr}\rho_1 + \mathrm{tr}\rho_2 + \mathrm{tr}\rho_3\\ &= \int [\mathrm{d}x_i][\mathrm{d}^2 r_i]|\Psi(x_i,\vec{r}_i)|^2 = 1.\end{aligned} \tag{31}$$

The normalization of the coordinate space wave function is determined from this relation.

In Appendix B we present expressions for calculating traces of powers of ρ which illustrate explicitly the need to introduce the lattice spacing in our calculation.

B. Entanglement entropy

We now discuss the von Neumann entropy associated with tracing the pure state $|\vec{R}=0,P^+\rangle\langle\vec{R}=0,P^+|$ over the area A,

$$S_{\mathrm{vN}} = -\lim_{\epsilon\to 0}\frac{\mathrm{tr}(\rho_{\bar{A}})^{1+\epsilon}-1}{\epsilon}. \tag{32}$$

Because we performed a partial trace over a *pure* state, this entropy represents a measure for the entanglement of the degrees of freedom remaining in $\bar{A}$ with those from region A, which have been traced out. We discuss the nature of entanglement in more detail in the following Sec. III C.

Using the expressions from Appendix B for $N = 1+\epsilon$ and expanding to linear order in ϵ this gives

$$\begin{aligned}-S_{\mathrm{vN}} &= \rho_0\log\rho_0 + \mathrm{tr}\rho_3\log\mathrm{tr}\rho_3 + 3\int[\mathrm{d}x_i][\mathrm{d}^2 r_i]\Theta_{\bar{A}}(\vec{r}_3)\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)|\Psi(x_i,\vec{r}_i)|^2\\ &\quad\times\log\left(3\Delta x a^2\int[\mathrm{d}y_i][\mathrm{d}^2 s_i]\delta(\vec{s}_3-\vec{r}_3)\delta(x_3-y_3)\Theta_A(\vec{s}_1)\Theta_A(\vec{s}_2)|\Psi(y_i,\vec{s}_i)|^2\right)\\ &\quad+3\int[\mathrm{d}x_i][\mathrm{d}^2 r_i]\Theta_A(\vec{r}_3)\Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)|\Psi(x_i,\vec{r}_i)|^2\\ &\quad\times\log\left(3\Delta x a^2\int[\mathrm{d}y_i][\mathrm{d}^2 s_i]\delta(\vec{s}_3-\vec{r}_3)\delta(x_3-y_3)\Theta_{\bar{A}}(\vec{s}_1)\Theta_{\bar{A}}(\vec{s}_2)|\Psi(y_i,\vec{s}_i)|^2\right),\end{aligned} \tag{33}$$

where we used $\mathrm{tr}\rho_{\bar{A}} = 1$.

This is a rather formal expression, and to understand some of its properties we will consider the dependence of the entropy on the area $\bar{A}$ of the cutout.

When the region $\bar{A}$ shrinks to a point we, of course, expect the entropy to vanish.[2] Indeed, all the terms in Eq. (33) vanish, except ρ_0 which approaches 1: for vanishingly small area the probability to find zero particles inside is unity. Taking the area as small but nonvanishing, for a circular cutout with radius L, we have

$$\frac{\partial S^{(0)}_{\mathrm{vN}}}{\partial L} = -\frac{\partial\rho_0}{\partial L} = 2\pi L\int \mathrm{d}x I(x) \quad (\text{for } L\to 0) \tag{34}$$

$$\begin{aligned}I(x) &= 3\int[\mathrm{d}y_i]\delta(y_3-x)\int \mathrm{d}^2 r_1\mathrm{d}^2 r_2\delta(x_1\vec{r}_1+x_2\vec{r}_2)\\ &\quad\times|\Psi(y_1,\vec{r}_1;y_2,\vec{r}_2;y_3,\vec{0})|^2.\end{aligned} \tag{35}$$

For the model wave function from Sec. II B we obtain the numerical estimate

$$\frac{1}{\beta}\frac{\partial S^{(0)}_{\mathrm{vN}}}{\partial L} = 2\pi L\beta\cdot 0.534(1). \tag{36}$$

There is an additional contribution of order $\sim L^2$ to the entropy. It is due to the third term in Eq. (33) which originates from ρ_1. Again, for a circular cutout $\bar{A}$ of radius L centered at the origin, we have

[1]Note that on account of the permutation symmetry of the wave function, the following equality holds when multiplied by $|\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)|^2$ under the integral: $\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)\Theta_A(\vec{r}_3) + 3\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)\Theta_{\bar{A}}(\vec{r}_3) + 3\Theta_A(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)\Theta_{\bar{A}}(\vec{r}_3) + \Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)\Theta_{\bar{A}}(\vec{r}_3) = 1$.

[2]The same is true if $\bar{A}$ encompasses the entire transverse space.

$$\frac{\partial S_{\mathrm{vN}}^{(1)}}{\partial L} = 2\pi L \int \mathrm{d}x\, I(x) \log \frac{1}{a^2(\Delta x) I(x)}, \quad (\text{for } L \to 0). \tag{37}$$

For small lattice spacing a the logarithm in this expression is large, and this is in fact the dominant contribution to the derivative of the entropy with respect to L. For example, for the model wave function from Sec. II B, and for a fairly coarse resolution of transverse position and longitudinal momentum, $\Delta x (a\beta)^2 = 0.1$, we obtain the numerical estimate

$$\frac{1}{\beta} \frac{\partial S_{\mathrm{vN}}^{(1)}}{\partial L} = 2\pi L \beta \cdot 1.21(1). \tag{38}$$

Thus, for small L^2 the leading contribution to the entropy is

$$S_{\mathrm{vN}}^{(1)} = -\pi L^2 \int \mathrm{d}x\, I(x) \log[a^2 (\Delta x) I(x)]. \tag{39}$$

This can be rewritten in a more transparent way if we notice that $I(x)$ as defined in (35) is nothing but the density of quarks with longitudinal momentum x in the proton $I(x) = F(x)/A_p$, where A_p is the transverse area of the proton and $F(x)$ is the quark PDF. We then have

$$S_{\mathrm{vN}}^{(1)} = -\frac{\pi L^2}{A_p} \int \mathrm{d}x\, F(x) \log\left[\frac{a^2}{A_p} (\Delta x) F(x)\right]. \tag{40}$$

The dependence of the entropy on the lattice spacing is easily understood. Since ρ_1 is a matrix with continuous index, we expect its eigenvalues to be small, i.e. of order a^2, while the number of nonvanishing eigenvalues is large $O(1/a^2)$. For such a matrix with a large number of small eigenvalues, the entropy is indeed proportional to the logarithm of the inverse eigenvalue, and this is what we see in (37). The area scaling of the entropy is also quite natural, since at small L the number of degrees of freedom in the reduced density matrix is proportional to the area of the cutout.

C. What is entangled here?

We would now like to comment on the nature of entanglement that produces the entanglement entropy that we calculated. It is somewhat different from the naive picture of entanglement we are used to in a vacuum state of a quantum field theory (QFT). In the QFT setting one divides space into two regions A and $\bar{A}$ and considers the wave function of local field degrees of freedom in the two regions $\Phi(x \in \bar{A})$ and $\Phi(y \in A)$. The entanglement is then understood in terms of nonfactorizability of the wave function $\Psi[\Phi(x), \Phi(y)] \neq \Psi_1[\Phi(x)]\Psi_2[\Phi(y)]$, and the entanglement entropy is associated with this nonfactorizability.

In our case the nature of entanglement is somewhat different. It is not that some internal degree of freedom of quarks in A, like color or helicity, is entangled with quarks in $\bar{A}$. In fact, we do not have to consider several quarks with internal degrees of freedom at all to understand our result. Let us imagine having just one quark in the proton area. This quark can be either in A or in $\bar{A}$. We can write the total wave function of the quark in terms of the basis states in the Hilbert spaces H_A and $H_{\bar{A}}$. For simplicity, we will even forget about different transverse coordinates in A and $\bar{A}$. The wave function of our quark can then be written as

$$\Psi = a|0\rangle_A \times |1\rangle_{\bar{A}} + b|1\rangle_A \times |0\rangle_{\bar{A}}, \tag{41}$$

where $|a|^2$ is the probability that the quark is in $\bar{A}$ and $|b|^2 = 1 - |a|^2$ is the probability that it is in A. Tracing over A removes the relative phase of a and b and we generate the reduced density operator $\hat{\rho}_{\bar{A}} = [|a|^2|1\rangle\langle 1| + |b|^2|0\rangle\langle 0|]_{\bar{A}}$. This is a mixed state over $\bar{A}$ and carries the entanglement entropy. Thus, the entanglement in our calculation is between the quark being (or not being) in A and the *same* quark being (or not being) in $\bar{A}$. These states are maximally entangled since the total number of quarks is fixed to be exactly one. This is a quantum mechanical rather than QFT type entanglement, very similar to the "Schrödinger cat" thought experiment [36,37], where one should read one quark in A as "the cat is alive" and no quark in $\bar{A}$ as "radioactive nucleus intact"; also no quark in A should be read as "the cat is dead" and the quark is in $\bar{A}$ as "radioactive nucleus decayed."

IV. INCLUDING THE $|qqqg\rangle$ FOCK STATE

We now add the $|qqqg\rangle$ Fock states into our calculation. In perturbation theory, such states have nonvanishing probability at order g^2. We write the proton state schematically in the form

$$|P\rangle \sim \Psi_{qqq} \epsilon_{i_1 i_2 i_3} |q_{i_1} q_{i_2} q_{i_3}\rangle + \Psi_{qqqg} [(t^a)_{j i_1} \epsilon_{i_1 i_2 i_3} |q_j q_{i_2} q_{i_3} g_a\rangle - (i_1 \leftrightarrow i_2) - (i_1 \leftrightarrow i_3)]. \tag{42}$$

In the leading perturbative order the three-quark wave function Ψ_{qqq} includes the $O(g^2)$ virtual corrections, and Ψ_{qqqg} is the (3 quarks + 1 gluon) spatial wave function at order $O(g)$.

In the two components the quarks are in different representations of color SU(3): they are in the color singlet in the $|qqq\rangle$ state and in the color octet in $|qqqg\rangle$.

In the following we will calculate the entanglement entropy for the same geometry as in the previous section. To simplify the calculations, however, we will trace the density matrix over the colors of the quarks. The pure state (42) is described by a density operator which, in principle, contains off diagonal matrix elements in the particle number basis

$$
\begin{aligned}
|P\rangle\langle P| \sim\ & \Psi_{qqq}\epsilon_{i_1 i_2 i_3}\Psi^*_{qqq}\epsilon_{i i'_2 i'_3}|q_{i_1}q_{i_2}q_{i_3}\rangle\langle q_{i'_1}q_{i'_2}q_{i'_3}| \\
& + \Psi_{qqqg}(t^a)_{j i_1}\epsilon_{i_1 i_2 i_3}\Psi^*_{qqqg}(t^{a'})_{i'_1 j'}\epsilon_{i'_1 i'_2 i'_3}|q_j q_{i_2} q_{i_3} g_a\rangle\langle q_{j'} q_{i'_2} q_{i'_3} g_{a'}| \\
& + \Psi_{qqq}\epsilon_{i_1 i_2 i_3}\Psi^*_{qqqg}(t^{a'})_{i'_1 j'}\epsilon_{i'_1 i'_2 i'_3}|q_{i_1} q_{i_2} q_{i_3}\rangle\langle q_{j'} q_{i'_2} q_{i'_3} g_{a'}| \\
& + \Psi_{qqqg}(t^a)_{j i_1}\epsilon_{i_1 i_2 i_3}\Psi^*_{qqq}\epsilon_{i'_1 i'_2 i'_3}|q_j q_{i_2} q_{i_3} g_a\rangle\langle q_{i'_1} q_{i'_2} q_{i'_3}|. \qquad (43)
\end{aligned}
$$

However, the off diagonal matrix elements vanish after tracing over quark colors precisely due to the fact that the three quarks are in the color singlet state in Ψ_{qqq} and in the color octet state in Ψ_{qqqg}. The reduced (over the quark color) density operator is diagonal in particle number and has the form

$$
\begin{aligned}
\mathrm{tr}_{qqq-\mathrm{colors}}\hat{\rho} \sim\ & 3!\Psi_{qqq}\Psi^*_{qqq}|qqq\rangle\langle qqq| \\
& + \delta^{aa'}\Psi_{qqqg}\Psi^*_{qqqg}|qqqg\rangle\langle qqqg|. \qquad (44)
\end{aligned}
$$

A. Ψ_{qqqg} at order g and Ψ_{qqq} at order g^2

Our first order of business is to calculate the perturbative wave function. For simplicity, we restrict ourselves to the soft gluon approximation, i.e. we assume that the gluon longitudinal momentum is much smaller than the typical longitudinal momentum of a quark.

Let us begin with Ψ_{qqqg}. The emission of a gluon from one of the quarks generates the following $\mathcal{O}(g)$ correction[3] to the momentum-space proton state $|P\rangle$:

$$
\begin{aligned}
|P^+,\vec{P}\rangle_{\mathcal{O}(g)} =\ & g\int[\mathrm{d}x_i]\int[\mathrm{d}^2k_i]\Psi_{\mathrm{qqq}}(k_i)\frac{1}{\sqrt{6}}\sum_{j_1 j_2 j_3}\epsilon_{j_1 j_2 j_3}\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\sum_{\sigma m a} \\
& \times\Bigg[(t^a)_{m j_1}\frac{\Theta(x_1-x_g)}{x_1-x_g}\hat{\psi}_{q\to qg}(p_1;p_1-k_g,k_g)|m,p_1-k_g;j_2,p_2;j_3,p_3\rangle \\
& + (t^a)_{m j_2}\frac{\Theta(x_2-x_g)}{x_2-x_g}\hat{\psi}_{q\to qg}(p_2;p_2-k_g,k_g)|j_1,p_1;m,p_2-k_g;j_3,p_3\rangle \\
& + (t^a)_{m j_3}\frac{\Theta(x_3-x_g)}{x_3-x_g}\hat{\psi}_{q\to qg}(p_3;p_3-k_g,k_g)|j_1,p_1;j_2,p_2;m,p_3-k_g\rangle\Bigg]\otimes|a,k_g,x_g,\sigma\rangle. \qquad (45)
\end{aligned}
$$

The integration measures here, $[\mathrm{d}x_i]$ and $[\mathrm{d}^2k_i]$, pertain to coordinates of the *parent* quarks. We have cut off the integration over the light-cone momentum fraction of the gluon x_g by Δx to regularize the soft singularity in QCD. That is, we prohibit gluon emission into the lowest "bin" of x_g.

The light-cone gauge Fock space amplitude for the qg state of a quark in the soft gluon approximation in $D=4$ dimensions is

$$
\hat{\psi}_{q\to qg}(p;k_q,k_g) = 2\frac{x_p}{k_g^2+\Delta^2}\vec{k}_g\cdot\vec{\epsilon}^{\,*}_\sigma, \qquad (46)
$$

where $x_p = p^+/P^+$, and Δ^2 is a regulator for the collinear singularity. Physically, the regularization is provided by the finite size of the color singlet state which the emitter is a part of. Thus the magnitude of the regulator Δ is of order Λ_{QCD} or, in our case, of the order of the inverse size of the model proton wave function set by the parameter β. It is much smaller that the inverse radius squared of the cutout $\bar{A}$.

Projecting on the Fock space state $|\alpha\rangle$, where α denotes a set of four momentum fractions x_i, transverse positions $\vec{r}_i$ and colors i_1, i_2, i_3, a, we obtain

$$
\begin{aligned}
\langle\alpha|P^+,\vec{R}=0\rangle_{\mathcal{O}(g)} =\ & 2g\frac{|\mathcal{N}|^2}{(2\pi)^2}(2\pi)^3\delta\left(1-\sum x_i\right)\delta\left(\sum x_i\vec{r}_i\right)\frac{1}{\sqrt{6}}\int[\mathrm{d}^2k_i]e^{i\sum\vec{k}_i\cdot\vec{r}_i}\frac{\vec{k}_g\cdot\vec{\epsilon}^{\,*}_\sigma}{k_g^2+\Delta^2} \\
& \times\sum_j[\epsilon_{j i_2 i_3}(t^a)_{i_1 j}\Psi_{\mathrm{qqq}}(k_1+k_g;k_2;k_3)+\epsilon_{i_1 j i_3}(t^a)_{i_2 j}\Psi_{\mathrm{qqq}}(k_1;k_2+k_g;k_3) \\
& + \epsilon_{i_1 i_2 j}(t^a)_{i_3 j}\Psi_{\mathrm{qqq}}(k_1;k_2;k_3+k_g)]. \qquad (47)
\end{aligned}
$$

[3]We are following the notation and expressions from Refs. [22,31].

To properly account for probability conservation we also need to include the $\mathcal{O}(g^2)$ virtual corrections to $\Psi_{\rm qqq}$. There are two types of such corrections. The first one arises due to emission and reabsorption of a gluon by one of the quarks and amounts to multiplying the momentum-space quark state vectors in Eq. (11) by the wave function renormalization factor

$$
\begin{aligned}
&(Z_q(x_1)Z_q(x_2)Z_q(x_3))^{1/2} \\
&= 1-\frac{1}{2}(C_q(x_1)+C_q(x_2)+C_q(x_3)), \qquad (48)
\end{aligned}
$$

with

$$
\begin{aligned}
C_q(x_1) &= \frac{g^2 C_F}{4\pi^2}\int_{\Delta x/x_1}^{1}\frac{\mathrm{d}z}{z}A_0(\Delta^2), \\
A_0(\Delta^2) &= 4\pi\int\frac{\mathrm{d}^2 n}{(2\pi)^2}\frac{1}{\vec{n}^2+\Delta^2}. \qquad (49)
\end{aligned}
$$

Again, a cutoff Δx on the momentum fraction of the gluon was introduced here. We regulate $A_0(\Delta^2)$ in the UV by a Pauli-Villars type regulator

$$
\begin{aligned}
A_0^{\rm reg}(\Lambda^2/\Delta^2) &= A_0(\Delta^2)-A_0(\Lambda^2) \\
&= 4\pi\int\frac{\mathrm{d}^2 n}{(2\pi)^2}\left[\frac{1}{\vec{n}^2+\Delta^2}-\frac{1}{\vec{n}^2+\Lambda^2}\right] \\
&= \log\frac{\Lambda^2}{\Delta^2}, \qquad (50)
\end{aligned}
$$

where Λ^2 is a UV cutoff. Then,

$$
C_q^{\rm reg}(x_1) = \frac{g^2 C_F}{4\pi^2}\log\frac{x_1}{\Delta x}\log\frac{\Lambda^2}{\Delta^2}. \qquad (51)
$$

We were forced to introduce the momentum UV regulator in the present calculation in order to regulate gluon emissions at short transverse distances. Recall that earlier we had to introduce a similar (coordinate space) regulator a in order to define probabilities and entropy for a continuous system, e.g. in (25). The two regulators, of course, should not be considered independent. In the following we take them to be related as $\Lambda^2 = 1/a^2$, in the same way as we took the regulator of the soft divergence to be equal to the lattice spacing in the longitudinal momentum space Δx.

The second virtual correction to $\Psi_{\rm qqq}$ is due to the exchange of a gluon between any pair of quarks. Let quark 1 emit and quark 2 absorb the gluon in $|P\rangle$; we then have (again for $x_g \to 0$)

$$
\begin{aligned}
|P^+,\vec{P}\rangle^{12}_{\mathcal{O}(g^2)} = &\int[\mathrm{d}x_i]\int[\mathrm{d}^2k_i]\Psi_{\rm qqq}(k_1;k_2;k_3)\frac{1}{\sqrt{6}}\sum_{j_1j_2j_3}\epsilon_{j_1j_2j_3} \\
&\times g^2\sum_{\sigma,a,n,m}(t^a)_{mj_1}(t^a)_{nj_2}\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\frac{1}{x_1}\hat{\psi}_{q\to qg}(p_1;p_1-k_g,k_g) \\
&\times\frac{1}{x_2}\hat{\psi}_{qg\to q}(p_2,k_g;p_2+k_g)|m,p_1-k_g;n,p_2+k_g;j_3,p_3\rangle. \qquad (52)
\end{aligned}
$$

Here, the amplitude for the absorption of a gluon by a quark is

$$
\hat{\psi}_{qg\to q}(k_q,k_g;p) = -2x_p\frac{\vec{k}_g\cdot\vec{\epsilon}_\sigma}{k_g^2+\Delta^2}. \qquad (53)
$$

We can now sum over gluon polarizations, $\sum_\sigma\vec{k}_g\cdot\vec{\epsilon}_\sigma{}^*\vec{k}_g\cdot\vec{\epsilon}_\sigma = k_g^2$. Changing variables, $\vec{k}_1\to\vec{k}_1+\vec{k}_g$ and $\vec{k}_2\to\vec{k}_2-\vec{k}_g$, we obtain

$$
\begin{aligned}
|P^+,\vec{P}\rangle^{12}_{\mathcal{O}(g^2)} = &-4g^2\int[\mathrm{d}x_i]\int[\mathrm{d}^2k_i]\frac{1}{\sqrt{6}}\sum_{j_1j_2j_3}\epsilon_{j_1j_2j_3}\sum_{a,n,m}(t^a)_{mj_1}(t^a)_{nj_2} \\
&\times\int_x\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\Psi_{\rm qqq}(k_1+k_g;k_2-k_g;k_3)\frac{k_g^2}{(k_g^2+\Delta^2)^2}|m,p_1;n,p_2;j_3,p_3\rangle. \qquad (54)
\end{aligned}
$$

Adding analogous contributions corresponding to gluon exchanges between quarks 1, 3, and 2, 3 we finally have

$$|P^+,\vec{R}=0\rangle_{\mathcal{O}(g^2)} = -4g^2\int[\mathrm{d}x_i]\int[\mathrm{d}^2r_i]\frac{1}{\sqrt{6}}\sum_{j_1j_2j_3}\epsilon_{j_1j_2j_3}\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\frac{k_g^2}{(k_g^2+\Delta^2)^2}\int[\mathrm{d}^2k_i]e^{i\sum\vec{k}_i\cdot\vec{r}_i}\sum_{a,n,m}$$
$$\times[(t^a)_{mj_1}(t^a)_{nj_2}\Psi_{\rm qqq}(k_1+k_g;k_2-k_g;k_3)|m,x_1,\vec{r}_1;n,x_2,\vec{r}_2;j_3,x_3,\vec{r}_3\rangle$$
$$+(t^a)_{mj_1}(t^a)_{nj_3}\Psi_{\rm qqq}(k_1+k_g;k_2;k_3-k_g)|m,x_1,\vec{r}_1;j_2,x_2,\vec{r}_2;n,x_3,\vec{r}_3\rangle$$
$$+(t^a)_{mj_2}(t^a)_{nj_3}\Psi_{\rm qqq}(k_1;k_2+k_g;k_3-k_g)|j_1,x_1,\vec{r}_1;m,x_2,\vec{r}_2;n,x_3,\vec{r}_3\rangle]. \quad (55)$$

Projecting this onto the three-quark Fock state $\langle j_i,x_i,\vec{r}_i|$ gives

$$\langle j_i,x_i,\vec{r}_i|P^+,\vec{R}=0\rangle_{\mathcal{O}(g^2)} = -4g^2\frac{|\mathcal{N}|^2}{(2\pi)^2}\delta\Big(1-\sum x_i\Big)(2\pi)^3\delta\Big(\sum x_i\vec{r}_i\Big)\int[\mathrm{d}^2q_i]e^{i\sum\vec{q}_i\cdot\vec{r}_i}\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\frac{k_g^2}{(k_g^2+\Delta^2)^2}$$
$$\times\sum_{i_1i_2i_3}\Bigg[\frac{1}{\sqrt{6}}\epsilon_{i_1i_2j_3}(t^a)_{j_1i_1}(t^a)_{j_2i_2}\Psi_{\rm qqq}(x_1,\vec{q}_1+\vec{k}_g;x_2,\vec{q}_2-\vec{k}_g;x_3,\vec{q}_3)$$
$$+\frac{1}{\sqrt{6}}\epsilon_{i_1j_2i_3}(t^a)_{j_1i_1}(t^a)_{j_3i_3}\Psi_{\rm qqq}(x_1,\vec{q}_1+\vec{k}_g;x_2,\vec{q}_2;x_3,\vec{q}_3-\vec{k}_g)$$
$$+\frac{1}{\sqrt{6}}\epsilon_{j_1i_2i_3}(t^a)_{j_2i_2}(t^a)_{j_3i_3}\Psi_{\rm qqq}(x_1,\vec{q}_1;x_2,\vec{q}_2+\vec{k}_g;x_3,\vec{q}_3-\vec{k}_g)\Bigg]. \quad (56)$$

Alternatively, in terms of the position space wave function,

$$\langle j_i,x_i,\vec{r}_i|P^+,\vec{R}=0\rangle_{\mathcal{O}(g^2)} = -4g^2\frac{|\mathcal{N}|^2}{(2\pi)^2}\delta\Big(1-\sum x_i\Big)(2\pi)^3\delta\Big(\sum x_i\vec{r}_i\Big)\Psi_{\rm qqq}(x_i,\vec{r}_i)\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\frac{k_g^2}{(k_g^2+\Delta^2)^2}$$
$$\times\sum_{i_1i_2i_3}\Bigg[\frac{1}{\sqrt{6}}\epsilon_{i_1i_2j_3}(t^a)_{j_1i_1}(t^a)_{j_2i_2}e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_2)}+\frac{1}{\sqrt{6}}\epsilon_{i_1j_2i_3}(t^a)_{j_1i_1}(t^a)_{j_3i_3}e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_3)}$$
$$+\frac{1}{\sqrt{6}}\epsilon_{j_1i_2i_3}(t^a)_{j_2i_2}(t^a)_{j_3i_3}e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}_3)}\Bigg]. \quad (57)$$

B. First perturbative correction to the density matrix

We are now in a position to calculate the perturbative correction to the density matrix.

As already mentioned at the beginning of this section, Eq. (44), after tracing over the colors of the quarks the density matrix takes the form

$$\rho=\begin{pmatrix}\rho^{qqq} & 0\\ 0 & \rho^{qqqg}\end{pmatrix}. \quad (58)$$

Note that since ρ^{qqq} and ρ^{qqqg} are probability densities on subspaces with different numbers of particles, they have different dimensions. The trace operations over the two entries are given by $\mathrm{d}x_1/(2x_1)\mathrm{d}x_2/(2x_2)\mathrm{d}x_3/(2x_3)$ $\delta(1-\sum_i x_i)\mathrm{d}^2r_1\mathrm{d}^2r_2\mathrm{d}^2r_3\delta(\sum x_i\vec{r}_i)$ and $\mathrm{d}x_1/(2x_1)\mathrm{d}x_2/(2x_2)\mathrm{d}x_3/(2x_3)\delta(1-\sum_i x_i)\mathrm{d}x_g/(2x_g)\mathrm{d}^2r_1\mathrm{d}^2r_2\mathrm{d}^2r_3 2\pi\mathrm{d}^2r_g$ $\delta(\sum x_i\vec{r}_i)$, for ρ^{qqq} and ρ^{qqqg}, respectively. Hence, if one is interested in probabilities given by ρ (or its purity, entropy, etc.) one must multiply ρ^{qqqg} by $2\pi a^2\Delta x/2x_g$, as we did in the previous sections.

Let us now compute the matrix (58). We begin with ρ^{qqq} which gives the probability density on the three-quark state Hilbert space and includes $\mathcal{O}(g^2)$ virtual corrections. The first correction is to multiply the $\mathcal{O}(1)$ nonperturbative density matrix from Eq. (14) by six wave function renormalization factors, $\prod Z^{1/2}(x_i)=1-\sum C_q^{\rm reg}(x_i)/2$.

Second, we add a term similar to Eq. (14) where we replace one of the three-quark states of the proton by the $\mathcal{O}(g^2)$ virtual correction due to the exchange of a gluon by two quarks, Eq. (56). We also trace over the quark colors. In all,

$$\rho^{qqq}_{\alpha\alpha'} = \left[1 - \frac{1}{2}(C_q^{\rm reg}(x_1) + C_q^{\rm reg}(x_2) + C_q^{\rm reg}(x_3) + C_q^{\rm reg}(x'_1) + C_q^{\rm reg}(x'_2) + C_q^{\rm reg}(x'_3))\right]\Psi^*_{\rm qqq}(x'_i,\vec{r}'_i)\Psi_{\rm qqq}(x_i,\vec{r}_i)$$
$$+ 2g^2 C_F \int [{\rm d}^2 q_i] \int_{\Delta x} \frac{{\rm d}x_g}{2x_g} \frac{{\rm d}^2 k_g}{(2\pi)^3} \frac{k_g^2}{(k_g^2+\Delta^2)^2} \Big\{ e^{i\sum \vec{q}_i\cdot\vec{r}_i} \Psi^*_{\rm qqq}(x'_i,\vec{r}'_i)\Big[\Psi_{\rm qqq}\Big(x_1,\vec{q}_1+\vec{k}_g;x_2,\vec{q}_2-\vec{k}_g;x_3,\vec{q}_3\Big)$$
$$+ \Psi_{\rm qqq}\Big(x_1,\vec{q}_1+\vec{k}_g;x_2,\vec{q}_2;x_3,\vec{q}_3-\vec{k}_g\Big) + \Psi_{\rm qqq}\Big(x_1,\vec{q}_1;x_2,\vec{q}_2+\vec{k}_g;x_3,\vec{q}_3-\vec{k}_g\Big)\Big]$$
$$+ e^{-i\sum \vec{q}_i\cdot\vec{r}'_i}\Psi_{\rm qqq}(x_i,\vec{r}_i)\Big[\Psi^*_{\rm qqq}\Big(x'_1,\vec{q}_1+\vec{k}_g;x'_2,\vec{q}_2-\vec{k}_g;x'_3,\vec{q}_3\Big) + \Psi^*_{\rm qqq}\Big(x'_1,\vec{q}_1+\vec{k}_g;x'_2,\vec{q}_2;x'_3,\vec{q}_3-\vec{k}_g\Big)$$
$$+ \Psi^*_{\rm qqq}\Big(x'_1,\vec{q}_1;x'_2,\vec{q}_2+\vec{k}_g;x'_3,\vec{q}_3-\vec{k}_g\Big)\Big]\Big\}. \tag{59}$$

$$= \left[1 - \frac{1}{2}(C_q^{\rm reg}(x_1) + C_q^{\rm reg}(x_2) + C_q^{\rm reg}(x_3) + C_q^{\rm reg}(x'_1) + C_q^{\rm reg}(x'_2) + C_q^{\rm reg}(x'_3))\right]\Psi^*_{\rm qqq}(x'_i,\vec{r}'_i)\Psi_{\rm qqq}(x_i,\vec{r}_i)$$
$$+ 2g^2 C_F \Psi^*_{\rm qqq}(x'_i,\vec{r}'_i)\Psi_{\rm qqq}(x_i,\vec{r}_i)\int_{\Delta x}\frac{{\rm d}x_g}{2x_g}\frac{{\rm d}^2 k_g}{(2\pi)^3}\frac{k_g^2}{(k_g^2+\Delta^2)^2}$$
$$\times\Big[e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_2)} + e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_3)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}_3)} + e^{i\vec{k}_g\cdot(\vec{r}'_1-\vec{r}'_2)} + e^{i\vec{k}_g\cdot(\vec{r}'_1-\vec{r}'_3)} + e^{i\vec{k}_g\cdot(\vec{r}'_2-\vec{r}'_3)}\Big]. \tag{60}$$

Here, as in Eq. (16), $\alpha = \{x_i, \vec{r}_i | \sum x_i = 1, \sum x_i\vec{r}_i = 0\}$ and $\alpha' = \{x'_i, \vec{r}'_i | \sum x'_i = 1, \sum x'_i\vec{r}'_i = 0\}$ denote two sets of quark LC momentum fractions and transverse positions.

Now we proceed to ρ^{qqqg}. We trace it over quark and gluon colors, and, in addition, for simplicity over the gluon polarizations. Using Eq. (47) in the definition (14) we obtain

$$\rho^{qqqg}_{\alpha\alpha'} = 2g^2 C_F \int [{\rm d}^2 k_i][{\rm d}^2 k'_i] e^{i\sum \vec{k}_i\cdot\vec{r}_i - i\sum \vec{k}'_i\cdot\vec{r}'_i} \frac{\vec{k}_g\cdot\vec{k}'_g}{(k_g^2+\Delta^2)(k_g'^2+\Delta^2)}$$
$$\times\{2(\Psi^*_{\rm qqq}(k'_1+k'_g;k'_2;k'_3)\Psi_{\rm qqq}(k_1+k_g;k_2;k_3) + \Psi^*_{\rm qqq}(k'_1;k'_2+k'_g;k'_3)\Psi_{\rm qqq}(k_1;k_2+k_g;k_3)$$
$$+ \Psi^*_{\rm qqq}(k'_1;k'_2;k'_3+k'_g)\Psi_{\rm qqq}(k_1;k_2;k_3+k_g)) - \Psi_{\rm qqq}(k_1+k_g;k_2;k_3)\Psi^*_{\rm qqq}(k'_1;k'_2+k'_g;k'_3)$$
$$- \Psi_{\rm qqq}(k_1+k_g;k_2;k_3)\Psi^*_{\rm qqq}(k'_1;k'_2;k'_3+k'_g) - \Psi_{\rm qqq}(k_1;k_2+k_g;k_3)\Psi^*_{\rm qqq}(k'_1+k'_g;k'_2;k'_3)$$
$$- \Psi_{\rm qqq}(k_1;k_2+k_g;k_3)\Psi^*_{\rm qqq}(k'_1;k'_2;k'_3+k'_g) - \Psi_{\rm qqq}(k_1;k_2;k_3+k_g)\Psi^*_{\rm qqq}(k'_1+k'_g;k'_2;k'_3)$$
$$- \Psi_{\rm qqq}(k_1;k_2;k_3+k_g)\Psi^*_{\rm qqq}(k'_1;k'_2+k'_g;k'_3)\}. \tag{61}$$

Here, $\alpha = \{x_i, \vec{r}_i | \sum x_i = 1, \sum x_i\vec{r}_i = 0\}$ and $\alpha' = \{x'_i, \vec{r}'_i | \sum x'_i = 1, \sum x'_i\vec{r}'_i = 0\}$ denote two sets of quark and gluon momentum fractions and transverse positions.

In Appendix C we show that the density matrix is indeed properly normalized.

V. ENTANGLEMENT ENTROPY OF THE PERTURBATIVE DENSITY MATRIX

In this section we calculate the entanglement entropy of the density matrix which includes one perturbatively emitted gluon. We will change our strategy somewhat to simplify the calculation. Integrating all degrees of freedom in A and calculating the entanglement entropy turns out to be rather awkward as there are many degrees of freedom in $\bar{A}$. Instead we choose to reduce the density matrix to a partial set of degrees of freedom in the whole proton wave function, and only then do we integrate over A. We will follow two different routes.

In Sec. V A we reduce the density matrix calculated above by tracing over the gluon degrees of freedom in the whole space. The resulting quark density matrix is then traced over A and the associated entanglement entropy is calculated. Note that already after integrating over the gluon degrees of freedom the quark density matrix does not describe a pure state and therefore in all probability carries a nonvanishing entropy (which we do not calculate here). Thus the entropy we calculate is not exactly the entanglement entropy between the two spatial regions A and $\bar{A}$, but instead measures entanglement of quarks in $\bar{A}$ with the rest of the proton wave function[4] (quarks in A and gluon anywhere).

[4]Quantum correlations of regions A and $\bar{A}$ could be analyzed using entanglement measures other than the von Neumann entropy, which apply also to mixed states. One such example is entanglement negativity [38,39] which has been used recently to study two-quark azimuthal correlations in the light-cone wave function of the proton [23].

In Sec. V B we perform a complementary procedure: we integrate over the quark degrees of freedom in the whole space, and then reduce the resulting gluon density matrix over A and calculate the entanglement entropy. Again, this entropy measures entanglement of gluons in $\bar{A}$ with the rest of the proton wave function.

A. Entanglement entropy of quarks

Let us construct the three-quark density matrix by tracing out the gluon degrees of freedom in the whole space. Integrating over the gluon leads to the density matrix

$$\rho = \rho^{qqq} + \mathrm{tr}_g \rho^{qqqg}. \tag{62}$$

The first term ρ^{qqq} is given in Eq. (60). To trace $\rho^{qqqg}_{\alpha\alpha'}$ over the gluon we set $\vec{r}'_g = \vec{r}_g$ in Eq. (61) and integrate with the measure $\mathrm{d}x_g/(2x_g)2\pi \mathrm{d}^2 r_g$. In principle, the upper limit of x_g in each term of (61) is different. However, in the small-x_g approximation which we are employing here, only the leading $\log 1/x$ contribution is important and we may replace the upper limits by a typical quark momentum fraction $\langle x_q\rangle$. We then obtain

$$\begin{aligned}\mathrm{tr}_g \rho^{qqqg}_{\alpha\alpha'} = 2g^2 C_F \int_{\Delta x} \frac{\mathrm{d}x_g}{2x_g} \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \frac{1}{k_g^2+\Delta^2} \int [\mathrm{d}^2 k_i][\mathrm{d}^2 k'_i] e^{i\sum \vec{k}_i\cdot\vec{r}_i - i\sum \vec{k}'_i\cdot\vec{r}'_i} \Psi^*_{\mathrm{qqq}}(k'_i)\Psi_{\mathrm{qqq}}(k_i) \\ \times \Big\{ 2\big(e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_1)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_2)} + e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_3)}\big) - e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_2)} - e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_3)} \\ - e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_1)} - e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_3)} - e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_1)} - e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_2)} \Big\}.\end{aligned} \tag{63}$$

This can be written in terms of the position space wave functions (17),

$$\begin{aligned}\mathrm{tr}_g \rho^{qqqg}_{\alpha\alpha'} = 2g^2 C_F \Psi^*_{\mathrm{qqq}}(x'_i,\vec{r}'_i)\Psi_{\mathrm{qqq}}(x_i,\vec{r}_i) \int_{\Delta x} \frac{\mathrm{d}x_g}{2x_g} \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \Big\{ \Big(\frac{1}{k_g^2+\Delta^2} - \frac{1}{k_g^2+\Lambda^2}\Big) 2\Big(e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_1)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_2)} + e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_3)}\Big) \\ - \frac{1}{k_g^2+\Delta^2}\Big[e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_2)} + e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_3)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_1)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_3)} + e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_1)} + e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_2)}\Big] \Big\},\end{aligned} \tag{64}$$

where we have reinstated the UV regulator Λ.[5]

Let us now discuss the entropy of the density matrix (62). Both terms in (62) are proportional to the leading order (LO) density matrix $\rho^{\mathrm{LO}}_{\alpha\alpha'} = \Psi^*_{\mathrm{qqq}}(x'_i,\vec{r}'_i)\Psi_{\mathrm{qqq}}(x_i,\vec{r}_i)$ discussed in Sec. II A,

$$\rho^{\mathrm{qqq}}_{\alpha\alpha'} = F(\alpha,\alpha')\rho^{\mathrm{LO}}_{\alpha\alpha'}, \qquad \mathrm{tr}_g \rho^{\mathrm{qqqg}}_{\alpha\alpha'} = G(\alpha,\alpha')\rho^{\mathrm{LO}}_{\alpha\alpha'}, \tag{65}$$

with

$$\begin{aligned}F(\alpha,\alpha') &= 1 - 3C_q^{\mathrm{reg}}(\langle x_q\rangle) + 2g^2 C_F \int_{\Delta x} \frac{\mathrm{d}x_g}{2x_g} \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \frac{1}{k_g^2+\Delta^2} \\ &\quad \times \Big\{ e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_2)} + e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_3)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}_3)} + e^{i\vec{k}_g\cdot(\vec{r}'_1-\vec{r}'_2)} + e^{i\vec{k}_g\cdot(\vec{r}'_1-\vec{r}'_3)} + e^{i\vec{k}_g\cdot(\vec{r}'_2-\vec{r}'_3)} \Big\} \\ &= 1 - 3C_q^{\mathrm{reg}}(\langle x_q\rangle) + \frac{2g^2 C_F}{4\pi^2} \int_{\Delta x} \frac{\mathrm{d}x_g}{2x_g} \{K_0(|\vec{r}_1-\vec{r}_2|\Delta) + K_0(|\vec{r}_1-\vec{r}_3|\Delta) + K_0(|\vec{r}_2-\vec{r}_3|\Delta) \\ &\quad + K_0(|\vec{r}'_1-\vec{r}'_2|\Delta) + K_0(|\vec{r}'_1-\vec{r}'_3|\Delta) + K_0(|\vec{r}'_2-\vec{r}'_3|\Delta)\} \\ G(\alpha,\alpha') &= 2g^2 C_F \int_{\Delta x} \frac{\mathrm{d}x_g}{2x_g} \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \Big\{ \Big(\frac{1}{k_g^2+\Delta^2} - \frac{1}{k_g^2+\Lambda^2}\Big) 2\big(e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_1)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_2)} + e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_3)}\big) \\ &\quad - \frac{1}{k_g^2+\Delta^2}\big[e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_2)} + e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}'_3)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_1)} + e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}'_3)} + e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_1)} + e^{-i\vec{k}_g\cdot(\vec{r}_3-\vec{r}'_2)}\big] \Big\}\end{aligned}$$

[5]The dependence on the IR cutoffs Δx and Δ^2, and on the UV regulator Λ^2, cancels when Eq. (62) is traced over the quark degrees of freedom, as shown in Appendix C.

$$
\begin{aligned}
&= \frac{2g^2 C_F}{4\pi^2}\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\{2[K_0(|\vec{r}_1-\vec{r}_1'|\Delta)-K_0(|\vec{r}_1-\vec{r}_1'|\Lambda)]+2[K_0(|\vec{r}_2-\vec{r}_2'|\Delta)-K_0(|\vec{r}_2-\vec{r}_2'|\Lambda)]\\
&\quad +2[K_0(|\vec{r}_3-\vec{r}_3'|\Delta)-K_0(|\vec{r}_3-\vec{r}_3'|\Lambda)]-K_0(|\vec{r}_1-\vec{r}_2'|\Delta)-K_0(|\vec{r}_1-\vec{r}_3'|\Delta)-K_0(|\vec{r}_2-\vec{r}_1'|\Delta)\\
&\quad -K_0(|\vec{r}_2-\vec{r}_3'|\Delta)-K_0(|\vec{r}_3-\vec{r}_1'|\Delta)-K_0(|\vec{r}_3-\vec{r}_2'|\Delta)\}. \qquad (66)
\end{aligned}
$$

Interestingly, the diagonal matrix elements are unaffected by the presence of the gluon in the wave function, since $F(\alpha,\alpha)+G(\alpha,\alpha)=1$ due to real-virtual cancellations. Also note that integration over the gluon reinstates the center of mass constraint for the coordinates of the three quarks.

After tracing over region A both terms become block diagonal in the quark number basis, since the integration over the gluon results in a reduced density matrix with fixed number of particles. The sub-blocks correspond to 0, 1, 2, 3 quarks in $\bar{A}$, like in Sec. III,

$$
\mathrm{tr}_A\,\rho^{qqq}=\begin{pmatrix}\rho_0^{(F)} & 0 & 0 & 0\\ 0 & \rho_1^{(F)} & 0 & 0\\ 0 & 0 & \rho_2^{(F)} & 0\\ 0 & 0 & 0 & \rho_3^{(F)}\end{pmatrix},\qquad \mathrm{tr}_A\,\mathrm{tr}_g\,\rho^{qqqg}=\begin{pmatrix}\rho_0^{(G)} & 0 & 0 & 0\\ 0 & \rho_1^{(G)} & 0 & 0\\ 0 & 0 & \rho_2^{(G)} & 0\\ 0 & 0 & 0 & \rho_3^{(G)}\end{pmatrix}. \qquad (67)
$$

Recall from Sec. III that the ρ_2 and ρ_3 matrices are not diagonal in coordinate space [and their off diagonal elements do get modified at $\mathcal{O}(g^2)$] but that ρ_1 is, due to the c.m. constraint. In the limit of small L, ρ_0, and ρ_1 give the leading contribution $\sim L^2$ to the entropy.

For ρ_0 all quarks are in the region A that we trace over, so $\vec{r}_i=\vec{r}_i'$ and only the diagonal matrix elements of the density matrix (62) contribute. Since $F(\alpha,\alpha)+G(\alpha,\alpha)=1$, the constant ρ_0 remains equal to its value at LO, for any L. Hence, the derivative of $S^{(0)}$ for $L\to 0$ remains $\partial S^{(0)}/\partial L\sim L$ with the same numerical coefficient as in Eq. (36).

Now we consider ρ_1. Since this block is diagonal in the quark indices, we only need to consider

$$
(\rho_1)_{\alpha\alpha}=3\Delta x a^2\int\frac{\mathrm{d}x_1\mathrm{d}x_2}{8x_1x_2x_3}\delta\Big(1-\sum x_i\Big)\int \mathrm{d}^2r_1\mathrm{d}^2r_2\delta\Big(\sum x_i\vec{r}_i\Big)\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)[F(\vec{r}_i)+G(\vec{r}_i)]|\Psi(x_i,\vec{r}_i)|^2. \qquad (68)
$$

Here $\alpha=\{x_3,\vec{r}_3\in\bar{A}\}$. The perturbative correction again cancels as the sum $F(\vec{r}_i)+G(\vec{r}_i)=1$, and we return to the expression from Sec. III. The trace $[\int(\mathrm{d}x_3/\Delta x)(\mathrm{d}^2r_3/a^2)\Theta_{\bar{A}}(\vec{r}_3)]$ vanishes at $L=0$ so there is no contribution to $S(L=0)$. For $L>0$,

$$
\begin{aligned}
S^{(1)}=-3\int[\mathrm{d}x_i][\mathrm{d}^2r_i]\Theta_{\bar{A}}(\vec{r}_3)\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)[F(\vec{r}_i)+G(\vec{r}_i)]|\Psi(x_i,\vec{r}_i)|^2\\
\times\log\left(3\Delta x\,a^2\int[\mathrm{d}y_i][\mathrm{d}^2s_i]\delta(\vec{s}_3-\vec{r}_3)\delta(x_3-y_3)\Theta_A(\vec{s}_1)\Theta_A(\vec{s}_2)|\Psi(y_i,\vec{s}_i)|^2\right). \qquad (69)
\end{aligned}
$$

The derivative of $S^{(1)}$ with respect to L for $L\to 0$ is proportional to L with the coefficient given in Eq. (38).

To summarize, we find that, due to real-virtual cancellations in gluon emission, the leading (at small L) term in the entanglement entropy of quarks is identical to that for the initial nonperturbative three-quark wave function.

Let us now take a look at ρ_2 (two quarks inside the circle separated by a typical distance of order L). It is given by the LO expression, Eq. (26), times $F(\vec{r}_1,\vec{r}_2,\vec{r}_3;\vec{r}_1',\vec{r}_2',\vec{r}_3)+G(\vec{r}_1,\vec{r}_2,\vec{r}_3;\vec{r}_1',\vec{r}_2',\vec{r}_3)$. Due to the c.m. constraint $x_1\vec{r}_1+x_2\vec{r}_2=x_1'\vec{r}_1'+x_2'\vec{r}_2'=-x_3\vec{r}_3$, and $x_1+x_2=x_1'+x_2'=1-x_3$, so that there is in fact no integral over the coordinates of the quark in A ($\vec{r}_3$ or x_3) as those are completely determined by the coordinates and momentum fractions of the two quarks in $\bar{A}$.

We need $F+G$ for $\vec{r}_3=\vec{r}_3'$, and we shall use the position space (Bessel function) form of these functions from Eq. (66). In the limit $\vec{r}_3'\to\vec{r}_3$ we have that $2K_0(|\vec{r}_3-\vec{r}_3'|\Delta)-2K_0(|\vec{r}_3-\vec{r}_3'|\Lambda)\to\log\frac{\Lambda^2}{\Delta^2}$. Furthermore, we consider $K_0(|\vec{r}_1-\vec{r}_1'|\Lambda)$ and $K_0(|\vec{r}_2-\vec{r}_2'|\Lambda)$ to be exponentially small since generically $|\vec{r}_1-\vec{r}_1'|,|\vec{r}_2-\vec{r}_2'|\sim L$ and $L\Lambda\gg 1$. With that, and noting that $L\Delta\ll 1$, we can write

$$F+G\simeq 1-\frac{2g^2C_F}{4\pi^2}\log\frac{x_q}{\Delta x}\log\frac{\Lambda^2}{\Delta^2}+\frac{g^2C_F}{8\pi^2}\int_{\Delta x}\frac{\mathrm{d}x_g}{x_g}\left[\log\frac{1}{(\vec r_1-\vec r_2)^2\Delta^2}+\log\frac{1}{(\vec r_1'-\vec r_2')^2\Delta^2}+2\log\frac{1}{(\vec r_1-\vec r_1')^2\Delta^2}\right.$$
$$\left.+2\log\frac{1}{(\vec r_2-\vec r_2')^2\Delta^2}-\log\frac{1}{(\vec r_1-\vec r_2')^2\Delta^2}-\log\frac{1}{(\vec r_2-\vec r_1')^2\Delta^2}\right] \tag{70}$$

$$\simeq 1-\frac{2g^2C_F}{4\pi^2}\log\frac{x_q}{\Delta x}\log L^2\Lambda^2+\frac{g^2C_F}{8\pi^2}\log\frac{x_q}{\Delta x}\log\frac{(\vec r_1-\vec r_2')^2(\vec r_2-\vec r_1')^2}{(\vec r_1-\vec r_2)^2(\vec r_1'-\vec r_2')^2} \tag{71}$$

$$\simeq 1-\frac{2g^2C_F}{4\pi^2}\log\frac{x_q}{\Delta x}\log L^2\Lambda^2. \tag{72}$$

The equality here is valid with leading logarithmic accuracy, since in the second step, $\log\frac{1}{(\vec r_1-\vec r_1')^2\Delta^2}$ and $\log\frac{1}{(\vec r_2-\vec r_2')^2\Delta^2}$ were replaced by $\log\frac{1}{L^2\Delta^2}$; and in the last step an $O(1)$ (nonlogarithmic) term was dropped.

With these simplifications $F+G$ is just a number, i.e. it does not modify the matrix structure of ρ_2 relative to the leading order, but only multiplies the entire matrix by a numerical prefactor. Its contribution to the entropy $S^{(2)}$ is given by the last term in Eq. (33) with the substitution $|\Psi(x_i,\vec r_i)|^2\to(F+G)|\Psi(x_i,\vec r_i)|^2$,

$$S^{(2)}=-3\left(1-\frac{2g^2C_F}{4\pi^2}\log\frac{x_q}{\Delta x}\log L^2\Lambda^2\right)\int[\mathrm{d}x_i][\mathrm{d}^2r_i]\Theta_A(\vec r_3)\Theta_{\bar A}(\vec r_1)\Theta_{\bar A}(\vec r_2)|\Psi(x_i,\vec r_i)|^2$$
$$\times\log\left(3\Delta x a^2\int[\mathrm{d}y_i][\mathrm{d}^2s_i]\delta(\vec s_3-\vec r_3)\delta(x_3-y_3)\Theta_{\bar A}(\vec s_1)\Theta_{\bar A}(\vec s_2)|\Psi(y_i,\vec s_i)|^2\right). \tag{73}$$

Here, with leading logarithmic accuracy we have omitted the factor $F+G$ under the logarithm.

Note that, as opposed to the entropy associated with ρ_1, this contribution to the entropy does receive corrections from gluon emission. This arises because the two quarks may be at different positions in the amplitude (1, 2) and the conjugate amplitude (1', 2'). The mismatch between these positions is generically of order L, cf. Fig. 1. For such configurations the contribution of the "real" diagrams, i.e. the diagrams where a quark exchanges a gluon with itself across the cut, is proportional to $\log\frac{1}{L^2\Delta^2}$ rather than $\log\frac{\Lambda^2}{\Delta^2}$ as is the case for the "virtual" diagrams (where the gluon is exchanged in the amplitude or the complex conjugate amplitude). Thus the real-virtual cancellation is incomplete and leads to the logarithmic correction in (73). The real-virtual cancellation essentially only occurs when the gluon is emitted outside $\bar A$ but not inside $\bar A$. Also note that the perturbative correction to $S^{(2)}$ is negative, suggesting stronger correlations between quarks when perturbative gluon emission is included.

We note again that $S^{(2)}$ is subleading for small L^2 and thus only provides a small correction to the quark entanglement entropy.

B. Entanglement entropy of the gluon

We now integrate over the quark degrees of freedom in the whole space. The resulting density matrix for the gluon has the general structure

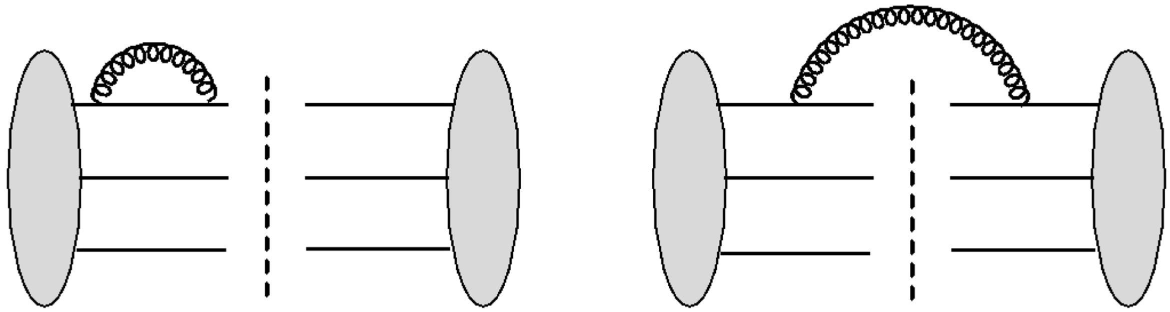

FIG. 1. Left (virtual correction): the transverse position of the gluon emitting and gluon absorbing quark is the same; hence here the gluon transverse momentum is integrated up to Λ, and we obtain a contribution $\sim\log\frac{\Lambda^2}{\Delta^2}$. Right (real emission): here there is a mismatch of order L in the transverse positions of quarks 1, 1' across the cut, and the diagram is only $\sim\log\frac{1}{L^2\Delta^2}$.

$$\rho = \begin{pmatrix} \mathrm{tr}_{\mathrm{qqq}}\rho^{qqq} & 0 \\ 0 & \rho^{g} \end{pmatrix}. \tag{74}$$

The first block is just a number which is equal to the probability that no gluons are present in the wave function. It is given by the integral of the diagonal of Eq. (60) over $[\mathrm{d}x_i]$ and $[\mathrm{d}^2 r_i]$,

$$\mathrm{tr}_{\mathrm{qqq}}\rho^{qqq} = 1 - 3C_q^{\mathrm{reg}}(\langle x_q\rangle) + 4g^2 C_F \int [\mathrm{d}x_i] \int [\mathrm{d}^2 r_i] |\Psi_{\mathrm{qqq}}(x_i, \vec{r}_i)|^2 \int_{\Delta x} \frac{\mathrm{d}x_g}{2x_g} \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \frac{1}{k_g^2 + \Delta^2} \times \left[\cos \vec{k}_g \cdot (\vec{r}_1 - \vec{r}_2) + \cos \vec{k}_g \cdot (\vec{r}_1 - \vec{r}_3) + \cos \vec{k}_g \cdot (\vec{r}_2 - \vec{r}_3)\right]. \tag{75}$$

For the second block we return to Eq. (61) and integrate the diagonal in the three-quark space ($x_i' = x_i$ and $\vec{r}_i' = \vec{r}_i$ for the three quarks) over $[\mathrm{d}x_i]$ and $[\mathrm{d}^2 r_i]$.[6] Note that since we have traced out the quarks in the whole space, the c.m. constraint forces $\vec{r}_g = \vec{r}_g'$, and the gluon density matrix is diagonal,

$$\rho^g_{\alpha\alpha} = 2g^2 C_F \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \int \frac{\mathrm{d}^2 k_g'}{(2\pi)^3} \frac{\vec{k}_g \cdot \vec{k}_g'}{(k_g^2 + \Delta^2)(k_g'^2 + \Delta^2)} e^{i\vec{k}_g\cdot\vec{r}_g - i\vec{k}_g'\cdot\vec{r}_g} \int [\mathrm{d}x_i][\mathrm{d}^2 r_i] |\Psi_{\mathrm{qqq}}(x_i, \vec{r}_i)|^2 \times \Big\{ 2\Big(e^{i(\vec{k}_g' - \vec{k}_g)\cdot\vec{r}_1} + e^{i(\vec{k}_g' - \vec{k}_g)\cdot\vec{r}_2} + e^{i(\vec{k}_g' - \vec{k}_g)\cdot\vec{r}_3}\Big) - e^{i\vec{k}_g'\cdot\vec{r}_2 - i\vec{k}_g\cdot\vec{r}_1} - e^{i\vec{k}_g'\cdot\vec{r}_3 - i\vec{k}_g\cdot\vec{r}_1} - e^{i\vec{k}_g'\cdot\vec{r}_1 - i\vec{k}_g\cdot\vec{r}_2} - e^{i\vec{k}_g'\cdot\vec{r}_3 - i\vec{k}_g\cdot\vec{r}_2} - e^{i\vec{k}_g'\cdot\vec{r}_1 - i\vec{k}_g\cdot\vec{r}_3} - e^{i\vec{k}_g'\cdot\vec{r}_2 - i\vec{k}_g\cdot\vec{r}_3} \Big\}. \tag{76}$$

Here, $\alpha = \{x_g, \vec{r}_g\}$ and $\alpha' = \{x_g', \vec{r}_g'\prime\}$. As before, the gluon "propagators" have to be regularized in the UV by the Pauli-Villars regulator. This entails substituting $\frac{1}{k_g^2+\Delta^2} \to \frac{1}{k_g^2+\Delta^2} - \frac{1}{k_g^2+\Lambda^2}$ and the same for k_g'. We can simplify the resulting expression somewhat, noting that the UV divergence only resides in the first term in the curly brackets in (76), since in the second term both integrations, over $\vec{k}_g$ and $\vec{k}_g'$, are already regulated by the phase factors, while in the first term the phase factors only regulate the integration over $k_g - k_g'$. Also, it is sufficient to regulate only one of the propagators in the product to eliminate the UV divergence, but this regularization has to be done symmetrically between $\vec{k}_g$ and $\vec{k}_g'$. Thus, we substitute

$$\frac{1}{(k_g^2 + \Delta^2)(k_g'^2 + \Delta^2)} \to \frac{1}{(k_g^2 + \Delta^2)(k_g'^2 + \Delta^2)} - \frac{1}{2}\left[\frac{1}{(k_g^2 + \Delta^2)(k_g'^2 + \Lambda^2)} + \frac{1}{(k_g^2 + \Lambda^2)(k_g'^2 + \Delta^2)}\right] \tag{77}$$

in the *first* term in the curly brackets of (76). We also use the symmetry of the quark wave function and the integration measure to rename coordinates in some terms. The resulting UV regular density matrix then is

$$\rho^g_{\alpha\alpha} = 12g^2 C_F \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \int \frac{\mathrm{d}^2 k_g'}{(2\pi)^3} \frac{\vec{k}_g \cdot \vec{k}_g'}{(k_g^2 + \Delta^2)(k_g'^2 + \Delta^2)} \int [\mathrm{d}x_i][\mathrm{d}^2 r_i] |\Psi_{\mathrm{qqq}}(x_i, \vec{r}_i)|^2 \Big\{ e^{i(\vec{k}_g' - \vec{k}_g)\cdot(\vec{r}_1 - \vec{r}_g)} - e^{i(\vec{k}_g - \vec{k}_g')\cdot\vec{r}_g + i\vec{k}_g'\cdot\vec{r}_2 - i\vec{k}_g\cdot\vec{r}_1} \Big\} - 6g^2 C_F \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \int \frac{\mathrm{d}^2 k_g'}{(2\pi)^3} \left[\frac{\vec{k}_g \cdot \vec{k}_g'}{(k_g^2 + \Delta^2)(k_g'^2 + \Lambda^2)} + \frac{\vec{k}_g \cdot \vec{k}_g'}{(k_g^2 + \Lambda^2)(k_g'^2 + \Delta^2)}\right] \int [\mathrm{d}x_i][\mathrm{d}^2 r_i] |\Psi_{\mathrm{qqq}}(x_i, \vec{r}_i)|^2 e^{i(\vec{k}_g' - \vec{k}_g)\cdot(\vec{r}_1 - \vec{r}_g)}. \tag{78}$$

Note that, when calculating trace of ρ, the regulator simply adds the term

$$-12g^2 C_F \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \frac{1}{\vec{k}_g^2 + \Lambda^2}, \tag{79}$$

[6] In principle, these are integrations over the LC momentum fractions and transverse coordinates of the three quarks with c.m. constraints which include the gluon, since the density matrix in Eq. (61) was defined over the domain $\sum_{i=1}^4 x_i = 1$, $\sum_{i=1}^4 x_i \vec{r}_i = 0$. However, when x_g is very small, the presence of the gluon will not significantly restrict the integrations over quark x_i, $\vec{r}_i$, and we can approximate $x_g \approx 0$ in the δ-functions for the c.m. constraints.

which (up to powers of Δ^2/Λ^2) cancels the similar term that arises from the regulator in $C_q^{\rm reg}$ in (75). Thus, our Pauli-Villars regularization preserves the trace of the density matrix.

On the other hand, $\mathrm{tr}\rho^g$ by itself has the meaning of the probability to find one gluon in the proton wave function. The trace $[2\pi \int d^2 r_g \int dx_g/(2x_g)]$ is given by

$$\mathrm{tr}\rho^g = \frac{6g^2 C_F}{8\pi^2} \log\frac{\langle x_q\rangle}{\Delta x}\left[\log\frac{\Lambda^2}{\Delta^2} - 2 \times \int [dx_i][d^2 r_i] |\Psi_{\rm qqq}(x_i, \vec{r}_i)|^2 K_0(|\vec{r}_2 - \vec{r}_1|\Delta)\right], \quad (80)$$

where Δx as before is the IR cutoff on possible longitudinal momenta and the integral over x_g is cut off at $\langle x_q\rangle$ consistently with the soft gluon approximation. In the second term the integral is dominated by $|\vec{r}_2 - \vec{r}_1|$ of order of the collinear regulator Δ^{-1}, so the second term is negligible.

Equation (80) can be related to the gluon PDF of the proton. To leading order in perturbation theory [see for example Eq. (65a) in Kovchegov and Mueller [40]],

$$\frac{\alpha_s C_F}{\pi}\frac{1}{\ell^2} = \frac{\partial}{\partial \ell^2} x G_q(x, \ell^2) \quad (81)$$

is the gluon PDF of a quark. Thus, for a proton consisting of three quarks we identify the gluon PDF as

$$\frac{3g^2 C_F}{4\pi^2}\int dk^2\left[\frac{1}{k^2+\Delta^2} - \frac{1}{k^2+\Lambda^2}\right] = \frac{3g^2 C_F}{4\pi^2}\log\frac{\Lambda^2}{\Delta^2} \to xG(x, \Lambda^2). \quad (82)$$

Hence, we have

$$\mathrm{tr}\rho^g = \int_{\Delta x} \frac{dx_g}{x_g} x_g G(x_g, \Lambda^2) = \int_{\Delta x} dx_g G(x_g, \Lambda^2). \quad (83)$$

Indeed this is just the total number of gluons at the resolution scale of the UV cutoff Λ. The fact that the UV cutoff appears in this quantity is not surprising, since here we are dealing with the density matrix of the entire proton wave function rather than the part of it probed by a DIS probe. If we were to calculate the density matrix of only those degrees of freedom that participate in a DIS process, we expect that the UV cutoff would be substituted by the external resolution scale $\Lambda^2 \to Q^2$ provided by the virtual photon.

Let us now construct the reduced density matrix after tracing over A. It is of the form

$$\rho = \begin{pmatrix} I + \rho_0^g & 0 \\ 0 & \rho_1^g \end{pmatrix}, \quad (84)$$

where $I \equiv \mathrm{tr}_{\rm qqq}\rho^{qqq}$ for brevity. The first entry is the probability that there are no gluons in $\bar{A}$ and is, of course, a pure number,

$$\rho_0^g = \int_{\Delta x} \frac{dx_g}{2x_g} 2\pi \int d^2 r_g \Theta_A(\vec{r}_g)\rho^g_{\alpha\alpha}. \quad (85)$$

The lower block is a diagonal (in coordinate space) matrix

$$\rho_1^g(\vec{r}) = \rho^g(\vec{r})\Theta_{\bar{A}}(\vec{r}), \quad (86)$$

with ρ^g from Eq. (78). Like for quarks, we need to scale ρ_1^g with the transverse-longitudinal lattice spacing and with the factor $2\pi/2x_g$ that accompanies the integration measure $dx_g d^2 r_g$. We will not do it explicitly here, but instead restore these factors directly in the expression for the entropy.

For $L = 0$, as already mentioned, $I + \rho_0^g = 1$. We are interested in the nontrivial small-L regime, $\Delta^{-1} \gg L \gg \Lambda^{-1}$ or $L\Delta \ll 1 \ll \Lambda L$. In this regime we expect $I + \rho_0^g \sim 1 - \mathcal{O}(Ł^2)$. The contribution to the entropy associated with this single eigenvalue of the density matrix should be $S^{(0)} \sim \mathcal{O}(L^2)$. The matrix ρ_1^g on the other hand, has small eigenvalues, for the same reason discussed previously. All the eigenvalues should be of order $(\Delta x)a^2\Delta^2$, due to the dimensionality of ρ_1^g. Thus, we expect the contribution from ρ_1^g to the entropy to contain an additional enhancement by a logarithm of $(\Delta x)a^2$,

$$S^{(1)} = -\int_{\Delta x} \frac{dx_g}{2x_g} 2\pi \int d^2 r_g \Theta_{\bar{A}}(\vec{r}_g)(\rho^g)_{\alpha\alpha} \times \log\left(2\pi a^2 \frac{\Delta x}{2x_g}(\rho^g)_{\alpha\alpha}\right). \quad (87)$$

This therefore is the leading contribution to the entropy and we will calculate it first.

Let us examine $(\rho^g)_{\alpha\alpha}$ for $|\vec{r}_g| < L \ll \Delta^{-1}$. The first term in (78) is

$$12g^2 C_F \int \frac{d^2 k_g}{(2\pi)^3}\int \frac{d^2 q}{(2\pi)^3}\left[\frac{1}{(\vec{k}_g + \vec{q})^2 + \Delta^2} + \frac{\vec{k}_g\cdot\vec{q}}{(\vec{k}_g^2 + \Delta^2)((\vec{k}_g+\vec{q})^2 + \Delta^2)}\right] e^{-i\vec{q}\cdot\vec{r}_g} \times \int [dx_i][d^2 r_i] |\Psi_{\rm qqq}(x_i, \vec{r}_i)|^2 \{e^{i\vec{q}\cdot\vec{r}_1} - e^{i\vec{k}_g\cdot(\vec{r}_2 - \vec{r}_1) + i\vec{q}\cdot\vec{r}_2}\}. \quad (88)$$

The integrals over the quark positions basically result in a "smeared δ-function" in $\vec{q}$ with width Δ, so $|\vec{q}| \sim \Delta$. That means that the phase $e^{-i\vec{q}\cdot\vec{r}_g} \sim 1$ since $L\Delta \ll 1$. Furthermore, the denominator of the second rational factor is essentially constant (independent of the direction of $\vec{q}$) both for small and large $\vec{k}_g$; hence, it gives zero after integration over the directions of $\vec{q}$. Finally, the second phase factor in the curly braces would restrict $|\vec{k}_g| \sim \Delta$ which results in a subleading contribution. In all, we simplify the above to

$$
\begin{aligned}
&12g^2 C_F \int \frac{\mathrm{d}^2 k_g}{(2\pi)^3} \frac{1}{\vec{k}_g^2 + \Delta^2} \int \frac{\mathrm{d}^2 q}{(2\pi)^3} \\
&\quad \times \int [\mathrm{d}x_i][\mathrm{d}^2 r_i] |\Psi_{\mathrm{qqq}}(x_i, \vec{r}_i)|^2 e^{i\vec{q}\cdot\vec{r}_1} \\
&\simeq 12g^2 C_F \frac{\Delta^2}{\pi} \int \frac{\mathrm{d}^2 k_g}{(2\pi)^4} \frac{1}{\vec{k}_g^2 + \Delta^2}. \qquad (89)
\end{aligned}
$$

We have assumed in these estimates that the nonperturbative scale entering the gluon propagator (Δ^2) and the nonperturbative scale appearing in the quark wave function (the average quark transverse momentum squared $\sim\beta^2$) are of the same order.

The second term in (78) gives a similar expression, but with Δ replaced by Λ, and with the opposite sign. In all, for $|\vec{r}_g| < L$,

$$
(\rho^g)_{\alpha\alpha} \simeq \frac{12g^2 C_F}{(2\pi)^4} \Delta^2 \log\frac{\Lambda^2}{\Delta^2}. \qquad (90)
$$

This has a simple interpretation. We are calculating the probability density of a gluon to be emitted at point $\vec{r}$ inside $\bar{A}$. Since the region inside $\bar{A}$ is small compared to the proton ($L \ll 1/\Delta$) the emission probability does not depend on $\vec{r}$. It is given (with the appropriate prefactor) by the integral of the intensity of the Weizsäcker-Williams field of a quark, integrated over the coordinate of the quark weighted with the square of the quark wave function. With logarithmic accuracy this is simply $\int_{\Lambda^{-2}<r^2<\Delta^{-2}} \mathrm{d}^2 r \frac{1}{r^2}$, which is precisely the logarithm in (90).

Using this in Eq. (87) we obtain

$$
\begin{aligned}
S^{(1)} &= -L^2\Delta^2 \int_{\Delta x} \mathrm{d}x_g G(x_g, \Lambda^2/\Delta^2) \\
&\quad \times \log\left(\frac{a^2\Delta^2}{\pi}(\Delta x) G(x_g, \Lambda^2/\Delta^2)\right) \\
&= -L^2\Delta^2 \int_{\Delta x} \mathrm{d}x_g G(x_g, \Lambda^2/\Delta^2) \\
&\quad \times \log\left(\frac{\Delta^2}{\pi\Lambda^2}(\Delta x) G(x_g, \Lambda^2/\Delta^2)\right). \qquad (91)
\end{aligned}
$$

Once again, $(\Delta^2/\pi)G(x_g, \Lambda^2/\Delta^2)$ is the density of gluons per unit transverse area.

Since the density matrix (84) is normalized, we infer from (90)

$$
I + \rho_0^g = 1 - \frac{3g^2 C_F}{4\pi^2} L^2\Delta^2 \int_{\Delta x} \frac{\mathrm{d}x_g}{x_g} \log \Lambda^2/\Delta^2, \qquad (92)
$$

and the associated entropy is

$$
\begin{aligned}
S^{(0)} &= \frac{3g^2 C_F}{4\pi^2} L^2\Delta^2 \int_{\Delta x} \frac{\mathrm{d}x_g}{x_g} \log \Lambda^2/\Delta^2 \\
&= L^2\Delta^2 \int_{\Delta x} \mathrm{d}x_g G(x_g, \Lambda^2/\Delta^2). \qquad (93)
\end{aligned}
$$

This is the gluon density per unit transverse area multiplied by the area of the cutout. As expected, this is a subleading correction to (91) and can be neglected.

Thus our final result for the gluon entanglement entropy in the limit of small area of the cutout is given in Eq. (91).

VI. DISCUSSION

To summarize, we calculated the entanglement entropy of subsets (in several variations) of partonic modes in the model proton wave function inside a small disk of radius L by integrating all the other modes in the rest of the wave function. The area was taken small relative to the total area of the proton (a soft, nonperturbative scale) $L^2 \ll \pi/\Delta^2$, but greater than the inverse UV cutoff $L^2 \gg 1/\Lambda^2$.

We now want to comment on these results. Let us consider the two expressions (40) and (91). Equation (40) gives the entanglement entropy of quarks at leading order in the model wave function, while Eq. (91) is the entanglement entropy of gluons at next to leading order. They have almost identical structure and are reminiscent of the form of Boltzmann entropy of a system of noninteracting particles. The PDFs that enter (40) and (91) (F in the former and G in the latter) are the total numbers of quarks and gluons in the proton. Defining the number of partons (at a given x) inside an area S, in the longitudinal momentum interval Δx, as $N_S(x) = \frac{S}{A_p}(\Delta x)F(x)$ for quarks and $N_S(x) = (S\Delta^2/\pi)(\Delta x)G(x)$ for gluons, both equations can be written as

$$
S_E = -\int \frac{dx}{\Delta x} N_{L^2}(x) \log[N_{a^2}(x)]. \qquad (94)
$$

This expression is quite natural. For small a^2 and Δx, one can only have either one or no partons inside the elementary cell $a^2\Delta x$. The average number of partons $N_{a^2}(x)$ is then just the probability that the cell contains a single parton. Equation (94) then is just (the leading term of) the Shannon entropy of this distribution multiplied by the total area (or rather L^2/a^2—the number of independent elementary cells in the area of the cutout) and integrated over x with the appropriate measure. The fact that the entropy is proportional to the area L^2 is a trademark property of an extensive quantity. Of course, the entanglement entropy is not, strictly speaking, extensive—the proportionality to area only holds when the area of the cutout is small. Were we to take the area of the cutout to be equal to the area of the proton we would have to obtain vanishing entropy as we would not be integrating out any degrees of freedom. So, the dependence of entropy on area should follow some sort of a Page curve which could be obtained numerically from Eq. (33), for example.

The one significant difference between Eqs. (40) and (91) is that in the latter the number of particles is defined with the resolution scale Λ^2, as is appropriate in the QCD improved parton model, while in the former there is no need to specify a resolution scale.

Does the entropy calculated here have direct physical meaning? One should remember, of course, that the calculation presented here does not refer to any particular physical process, but rather to the properties of the proton wave function *per se*. As such, it is not observable directly. We can try, however, to interpret this result from the point of view of a DIS or jet production process. In this type of process there is a physical resolution scale, the momentum transfer Q^2 to the electron or the transverse momentum of a produced jet. A naive physical picture is then that this scale should determine the size of the area of the proton measured by the probe, as well as the resolution with which one measures the parton number. Taking $L^2 \sim a^2 \sim 1/Q^2$, $\Delta^2 = \pi\Lambda^2_{\rm QCD}$ and fixing the value of x as appropriate for DIS, we then may hope to define a more physical quantity. For gluons that would be

$$\begin{aligned} S_E(Q^2,x) &= -N_{Q^{-2}}(x)\log[N_{Q^{-2}}(x)] \\ &= -\frac{\Lambda^2_{\rm QCD}}{Q^2}(\Delta x)G(x,Q^2) \\ &\quad\times \log\left(\frac{\Lambda^2_{\rm QCD}}{Q^2}(\Delta x)G(x,Q^2)\right). \end{aligned} \tag{95}$$

It is not entirely clear to us what should be taken as the "longitudinal resolution scale" Δx. The inclusive DIS cross section does not provide for a scale of this sort. However, if one measures the spectrum of produced particles, perhaps Δx should be related to the width of the rapidity bin in which the particles are measured.

Finally, it would be interesting to compare our results with those of Ref. [8]. This may not be entirely straightforward for the following reason. Our expressions apply to the "dilute regime" when the entropy is dominated by states with one parton within the cutout area, $\frac{\Lambda^2_{\rm QCD}}{Q^2}(\Delta x)G(x,Q^2) \ll 1$. On the other hand, Ref. [8] focused on the saturation regime where the number of particles in the cutout is assumed to be $\mathcal{O}(1/\alpha_s)$. Still, the actual derivation of Ref. [8] only requires that the rapidity is large enough so that the exponential growth of the gluon density in rapidity has taken hold. This in itself does not imply saturation, but rather the presaturation Balitsky-Fadin-Kuraev-Lipatov-like regime, so that the gluon density is still small but low-x evolution already has to be resummed.

At any rate, one expects the same elements to appear in the expression for entropy both in Ref. [8] and in our calculation. Indeed, the parton density is the basic physical quantity that appears, and in this respect the two results are similar. However, there are some significant differences between the two. In particular, according to Ref. [8] the entropy is given by the logarithm of $xG(x)$. This is somewhat perplexing since $xG(x)$ has the meaning of the longitudinal momentum carried by the partons and not the parton number. Equation (95), on the other hand, contains $\frac{\Lambda^2_{\rm QCD}}{Q^2}(\Delta x)G(x,Q^2)$ which is precisely the number of partons in the area of the cutout (and in the rapidity interval Δx), which appears to be the natural basic element for quantifying the entropy. Whether the number of partons at high energy is somehow supplanted by the longitudinal momentum fraction carried by the partons is an interesting question which should be answered by an explicit calculation.

ACKNOWLEDGMENTS

A. D. acknowledges support by the DOE Office of Nuclear Physics through Award No. DE-SC0002307. A. K. is supported by the NSF Nuclear Theory Grant No. 2208387. V. S. is supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics through Award No. DE-SC002008. We also acknowledge support by the Saturated Glue (SURGE) Topical Collaboration. We thank M. Lublinsky (Ben-Gurion University of the Negev) and the organizers of the workshop "Physics of Saturation—precision and quasicollectivity" for their support and hospitality in May 2022; we are also grateful to the participants of the workshop for stimulating discussions which initiated this work.

APPENDIX A: SHANNON ENTROPY OF A PROBABILITY DENSITY FUNCTION FOR A CONTINUOUS DEGREE OF FREEDOM

In this appendix we review the definition of the entropy associated with a classical probability density function over a continuous degree of freedom. We discuss the extension to a quantum mechanical density matrix at the end.

First, recall the expression for the classical Shannon entropy for a set of *discrete* outcomes of a random draw, with probabilities P_i,

$$H(\{P_i\}) = -\sum_i P_i \log P_i. \tag{A1}$$

If the set of possible outcomes is *continuous*, e.g. $x \in \mathbb{R}^+$, their distribution is given by a normalized, integrable (including possibly the δ-function measure) probability density function $p(x) > 0$,

$$\int \mathrm{d}x\, p(x) = 1. \tag{A2}$$

Note that if x is dimensional then $\dim(p) = [\dim(x)]^{-1}$. Also, that the integration measure does not involve *any* x-dependent Jacobians, all of which must be absorbed into $p(x)$ for it to be a valid probability density with respect to the integration measure $\mathrm{d}x$.

To apply Shannon's formula here, we first discretize the continuous set of outcomes by introducing (equal size) bins $\Delta x > 0$. An outcome x falls into bin $i = \lfloor x/\Delta x \rfloor$. The probability P_i for an event in bin i is

$$P_i = \int_{i\Delta x}^{(i+1)\Delta x} \mathrm{d}x\, p(x) \equiv p_i \Delta x, \qquad (i \in \mathbb{N}_0). \tag{A3}$$

In the last step we defined the binned density p_i as the average of the probability density $p(x)$ over bin i.

We now have

$$H[p] = -\sum_i \int_{i\Delta x}^{(i+1)\Delta x} \mathrm{d}x\, p(x)\, \log(p_i \Delta x). \tag{A4}$$

If $p(x)$ is a continuous function then

$$H[p] = -\int \mathrm{d}x\, p(x)\, \log(p(x)\Delta x), \tag{A5}$$

and the entropy does not have a finite $\Delta x \to 0$ limit. (Even so, the relative entropy for two such probability densities does converge.) If, on the other hand, $p(x)$ is given by a sum of Dirac δ-functions then Eq. (A4) does converge since this basically recovers the case of discrete outcomes.

Along similar lines, let $\rho_{xx'}$ denote a density matrix describing a continuous degree of freedom. By postulate, its trace is normalized,

$$\mathrm{tr}\rho = \int \mathrm{d}x\, \rho_{xx} = \int \frac{\mathrm{d}x}{\Delta x} (\Delta x)\, \rho_{xx} = 1. \tag{A6}$$

Once again we stress that the trace measure must be $\mathrm{d}x$, with any x-dependent Jacobians absorbed into ρ. The von Neumann entropy S is given by the Shannon entropy of the vector of dimensionless eigenvalues of $(\Delta x)\rho$. Upon binning the eigenvalue distribution, it is given by

$$S = -\sum_i \int_{i\Delta\lambda}^{(i+1)\Delta\lambda} \mathrm{d}\lambda\, p(\lambda)\, \log(p_i \Delta\lambda). \tag{A7}$$

APPENDIX B: CALCULATING TRACES OF POWERS OF ρ

Gearing up for the calculation of the von Neumann entropy, we write expressions for the trace of powers of the density matrix, $\mathrm{tr}(\rho_{\bar{A}})^N$ (which, if desired can be used to determine the Rényi entropy).

Since our density matrix is block diagonal in the particle number basis, the different blocks do not talk to each other and can be considered separately.

In the zero particle subspace ρ_0 is a number and thus

$$\mathrm{tr}\rho_0^N = \rho_0^N. \tag{B1}$$

It is easy to see that in the three particle subspace we simply have

$$\mathrm{tr}\rho_3^N = (\mathrm{tr}\,\rho_3)^N = \left[\int [\mathrm{d}x_i][\mathrm{d}^2 r_i] \Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)\Theta_{\bar{A}}(\vec{r}_3)|\Psi(x_i, \vec{r}_i)|^2\right]^N. \tag{B2}$$

Note that in both (B1) and (B2) the lattice spacing a does not appear.

Consider now the single particle subspace,

$$\begin{aligned}(\rho_1^2)_{\alpha\alpha} &= (\rho_1)_{\alpha\alpha}(\rho_1)_{\alpha\alpha} \\ &= 3\Delta x\, a^2 \int \frac{\mathrm{d}x_1 \mathrm{d}x_2}{8x_1x_2x_3} \delta(1-x_1-x_2-x_3) \int \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 \delta(x_1\vec{r}_1 + x_2\vec{r}_2 + x_3\vec{r}_3)\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)|\Psi(x_1,\vec{r}_1;x_2,\vec{r}_2;x_3,\vec{r}_3)|^2 \\ &\quad \times 3\Delta x\, a^2 \int \frac{\mathrm{d}y_1 \mathrm{d}y_2}{8y_1y_2x_3} \delta(1-y_1-y_2-x_3) \int \mathrm{d}^2 s_1\, \mathrm{d}^2 s_2\, \delta(y_1\vec{s}_1 + y_2\vec{s}_2 + x_3\vec{r}_3)\Theta_A(\vec{s}_1)\Theta_A(\vec{s}_2)|\Psi(y_1,\vec{s}_1;y_2,\vec{s}_2;x_3,\vec{r}_3)|^2.\end{aligned} \tag{B3}$$

Taking the trace,

$$\begin{aligned}
\mathrm{tr}\rho_1^2 = &\int \frac{\mathrm{d}x_3}{\Delta x}\int \frac{\mathrm{d}^2 r_3}{a^2}\Theta_{\bar{A}}(\vec{r}_3) \\
&\times 3\Delta x a^2 \int \frac{\mathrm{d}x_1 \mathrm{d}x_2}{8x_1x_2x_3}\delta(1-x_1-x_2-x_3)\int \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 \delta(x_1\vec{r}_1 + x_2\vec{r}_2 + x_3\vec{r}_3)\Theta_A(\vec{r}_1)\Theta_A(\vec{r}_2)|\Psi(x_1,\vec{r}_1;x_2,\vec{r}_2;x_3,\vec{r}_3)|^2 \\
&\times 3\Delta x a^2 \int \frac{\mathrm{d}y_1 \mathrm{d}y_2}{8y_1y_2x_3}\delta(1-y_1-y_2-x_3)\int \mathrm{d}^2 s_1 \mathrm{d}^2 s_2 \delta(y_1\vec{s}_1 + y_2\vec{s}_2 + x_3\vec{r}_3)\Theta_A(\vec{s}_1)\Theta_A(\vec{s}_2)|\Psi(y_1,\vec{s}_1;y_2,\vec{s}_2;x_3,\vec{r}_3)|^2 \\
= &\int \frac{\mathrm{d}x_3}{\Delta x}\int \frac{\mathrm{d}^2 r_3}{a^2}\Theta_{\bar{A}}(\vec{r}_3)\left[3\Delta x a^2 \int [\mathrm{d}y_i]\delta(y_3 - x_3)\int [\mathrm{d}^2 s_i]\delta(\vec{s}_3 - \vec{r}_3)\Theta_A(\vec{s}_1)\Theta_A(\vec{s}_2)|\Psi(y_i,\vec{s}_i)|^2\right]^2 .
\end{aligned} \tag{B4}$$

For an arbitrary N we obtain

$$\mathrm{tr}\rho_1^N = \int \frac{\mathrm{d}x_3}{\Delta x}\int \frac{\mathrm{d}^2 r_3}{a^2}\Theta_{\bar{A}}(\vec{r}_3)\left[3\Delta x\, a^2 \int [\mathrm{d}y_i]\delta(y_3 - x_3)\int [\mathrm{d}^2 s_i]\delta(\vec{s}_3 - \vec{r}_3)\Theta_A(\vec{s}_1)\Theta_A(\vec{s}_2)|\Psi(x_i,\vec{s}_i)|^2\right]^N . \tag{B5}$$

As noted in the main text of this paper, the lattice spacing does not cancel in this expression, and formally this expression vanishes, for $N > 1$, in the "continuum limit" $\Delta x, a \to 0$.

Now consider traces of powers of ρ_2. We have

$$(\rho_2^2)_{\alpha\alpha'} = 9(\Delta x\, a^2)^3 \int [\mathrm{d}y_i][\mathrm{d}^2 s_i]\Theta_{\bar{A}}(\vec{s}_1)\Theta_{\bar{A}}(\vec{s}_2)\delta(x_3 - y_3)\delta(\vec{s}_3 - \vec{r}_3)|\Psi(y_i,\vec{s}_i)|^2 \frac{\Psi(x_i,\vec{r}_i)}{x_3\sqrt{x_1x_2x_3}}\frac{\Psi^*(x_i',\vec{r}_i')}{x_3\sqrt{x_1'x_2'x_3}} . \tag{B6}$$

In this expression $\vec{r}_3 = -(x_1\vec{r}_1 + x_2\vec{r}_2)/x_3 = \vec{r}_3' = -(x_1'\vec{r}_1' + x_2'\vec{r}_2')/x_3'$, while $x_3 = 1 - x_1 - x_2 = x_3' = 1 - x_1' - x_2'$. Recall, also, that here the indices $\alpha = \{x_1, \vec{r}_1, x_2, \vec{r}_2\}$, $\alpha' = \{x_1', \vec{r}_1', x_2', \vec{r}_2'\}$, are defined over the domain $\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2' \in \bar{A}$, $\vec{r}_3 \in A$.

The trace, defined with the measure $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}^2r_1\mathrm{d}^2r_2\Theta(x_3)\Theta_A(\vec{r}_3)\Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)/(a^2\Delta x)^2$, is

$$\begin{aligned}
\mathrm{tr}\rho_2^2 = 3&\int [\mathrm{d}x_i][\mathrm{d}^2 r_i]\Theta_A(\vec{r}_3)\Theta_{\bar{A}}(\vec{r}_1)\Theta_{\bar{A}}(\vec{r}_2)|\Psi(x_i,\vec{r}_i)|^2 \\
&\times 3\Delta x a^2 \int [\mathrm{d}y_i][\mathrm{d}^2 s_i]\Theta_{\bar{A}}(\vec{s}_1)\Theta_{\bar{A}}(\vec{s}_2)\delta(x_3 - y_3)\delta(\vec{s}_3 - \vec{r}_3)|\Psi(y_i,\vec{s}_i)|^2 .
\end{aligned} \tag{B7}$$

Note that this is actually identical to Eq. (B4) for $\mathrm{tr}\rho_1^2$ with $A \leftrightarrow \bar{A}$, as it should be.

For general power N,

$$\mathrm{tr}\rho_2^N = \int \frac{\mathrm{d}x_3}{\Delta x}\int \frac{\mathrm{d}^2 r_3}{a^2}\Theta_A(\vec{r}_3)\left[3\Delta x a^2 \int [\mathrm{d}y_i][\mathrm{d}^2 s_i]\delta(x_3 - y_3)\delta(\vec{s}_3 - \vec{r}_3)\Theta_{\bar{A}}(\vec{s}_1)\Theta_{\bar{A}}(\vec{s}_2)|\Psi(y_i,\vec{s}_i)|^2\right]^N . \tag{B8}$$

Again we see that the lattice spacing does not disappear in this expression and formally leads to its vanishing for $a \to 0$ and $N > 1$.

APPENDIX C: CHECKING TRACES

Here we check the normalization of the density matrix (58).

First let us take the trace of ρ^{qqq}, which amounts to setting $x_i' = x_i$, $\vec{r}_i' = \vec{r}_i$ and integrating over $[\mathrm{d}x_i]$ and $[\mathrm{d}^2 r_i]$. The first line gives

$$\int [\mathrm{d}x_i][\mathrm{d}^2 r_i](1 - 3C_q^{\mathrm{reg}}(x_1))|\Psi_{\mathrm{qqq}}(x_i,\vec{r}_i)|^2 = 1 - 3\int [\mathrm{d}x_i][\mathrm{d}^2 r_i]C_q^{\mathrm{reg}}(x_1)|\Psi_{\mathrm{qqq}}(x_i,\vec{r}_i)|^2 , \tag{C1}$$

where we used the normalization condition (31) for the three-quark wave function. From the rest of Eq. (59) we get

$$2g^2C_F\int[\mathrm{d}x_i][\mathrm{d}^2r_i]\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\frac{k_g^2}{(k_g^2+\Delta^2)^2}|\Psi_{\mathrm{qqq}}(x_i,\vec{r}_i)|^2 \times\left[e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_2)}+e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_3)}+e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}_3)}+\mathrm{c.c.}\right]. \quad (C2)$$

Now let us take the trace of the matrix (61). To do this set $x_i'=x_i$, $\vec{r}_i'=\vec{r}_i$, and integrate over all degrees of freedom, including the momentum fraction of the gluon with the measure $\mathrm{d}x_g/2x_g$, and its transverse position with the measure $2\pi\mathrm{d}^2r_g$. This is done by performing the following steps: (i) extract the integrations over $\mathrm{d}x_g/2x_g$, $\mathrm{d}^2k_g/(2\pi)^3$, and $\mathrm{d}^2k_g'/(2\pi)^3$ from $[\mathrm{d}x_i]$, $[\mathrm{d}^2k_i]$, and $[\mathrm{d}^2k_i']$; (ii) perform the integration over d^2r_g which produces a $(2\pi)^2\delta(\vec{k}_g-\vec{k}_g')$; and (iii) shift the quark momenta (as needed) by $-\vec{k}_g$ so that the arguments of the Ψ_{qqq} functions no longer involve $\vec{k}_g$; note that this also changes $\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_g)\to\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)$, and similar for the primed momenta. The first three terms of Eq. (61) then give (taking $\Delta^2\to 0$ where possible)

$$3\cdot 4g^2C_F\int[\mathrm{d}x_i][\mathrm{d}^2r_i]|\Psi_{\mathrm{qqq}}(x_i,\vec{r}_i)|^2\int_{\Delta x}^{x_1}\frac{\mathrm{d}x_g}{2x_g} \times\int\frac{\mathrm{d}^2k_g}{(2\pi)^3}\frac{1}{k_g^2+\Delta^2}. \quad (C3)$$

This cancels against the $\mathcal{O}(g^2)$ correction in Eq. (C1), after regularization of the UV divergence, $(k_g^2+\Delta^2)^{-1}\to(k_g^2+\Delta^2)^{-1}-(k_g^2+\Lambda^2)^{-1}$.

The remaining terms of (61) give

$$-2g^2C_F\int[\mathrm{d}x_i][\mathrm{d}^2r_i]|\Psi_{\mathrm{qqq}}(x_i,\vec{r}_i)|^2\int_{\Delta x}\frac{\mathrm{d}x_g}{2x_g}\frac{\mathrm{d}^2k_g}{(2\pi)^3}\frac{k_g^2}{(k_g^2+\Delta^2)^2} \times\left[e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_2)}+e^{-i\vec{k}_g\cdot(\vec{r}_1-\vec{r}_3)}+e^{-i\vec{k}_g\cdot(\vec{r}_2-\vec{r}_3)}+\mathrm{c.c.}\right]. \quad (C4)$$

This cancels against (C2). The cancellations of the perturbative corrections in the trace ensure that it remains $=1$, and independent of the coupling g^2 and the IR (collinear and soft) and UV cutoffs.

[1] A. Kovner and M. Lublinsky, Entanglement entropy and entropy production in the color glass condensate framework, Phys. Rev. D **92**, 034016 (2015).
[2] A. Kovner, M. Lublinsky, and M. Serino, Entanglement entropy, entropy production and time evolution in high energy QCD, Phys. Lett. B **792**, 4 (2019).
[3] N. Armesto, F. Dominguez, A. Kovner, M. Lublinsky, and V. Skokov, The color glass condensate density matrix: Lindblad evolution, entanglement entropy and Wigner functional, J. High Energy Phys. 05 (2019) 025.
[4] H. Duan, C. Akkaya, A. Kovner, and V. V. Skokov, Entanglement, partial set of measurements, and diagonality of the density matrix in the parton model, Phys. Rev. D **101**, 036017 (2020).
[5] H. Duan, A. Kovner, and V. V. Skokov, Gluon quasiparticles and the CGC density matrix, Phys. Rev. D **105**, 056009 (2022).
[6] H. Duan, A. Kovner, and V. V. Skokov, Classical entanglement and entropy, arXiv:2301.05735.
[7] Y. Hagiwara, Y. Hatta, B.-W. Xiao, and F. Yuan, Classical and quantum entropy of parton distributions, Phys. Rev. D **97**, 094029 (2018).
[8] D. E. Kharzeev and E. M. Levin, Deep inelastic scattering as a probe of entanglement, Phys. Rev. D **95**, 114008 (2017).
[9] D. E. Kharzeev and E. Levin, Deep inelastic scattering as a probe of entanglement: Confronting experimental data, Phys. Rev. D **104**, L031503 (2021).
[10] D. E. Kharzeev, Quantum information approach to high energy interactions, Phil. Trans. R. Soc. A **380**, 20210063 (2021).
[11] G. Dvali and R. Venugopalan, Classicalization and unitarization of wee partons in QCD and gravity: The CGC-black hole correspondence, Phys. Rev. D **105**, 056026 (2022).
[12] Z. Tu, D. E. Kharzeev, and T. Ullrich, Einstein-Podolsky-Rosen Paradox and Quantum Entanglement at Subnucleonic Scales, Phys. Rev. Lett. **124**, 062001 (2020).
[13] G. S. Ramos and M. V. T. Machado, Investigating entanglement entropy at small-x in DIS off protons and nuclei, Phys. Rev. D **101**, 074040 (2020).
[14] M. Hentschinski and K. Kutak, Evidence for the maximally entangled low x proton in deep inelastic scattering from H1 data, Eur. Phys. J. C **82**, 111 (2022).
[15] M. Hentschinski, K. Kutak, and R. Straka, Maximally entangled proton and charged hadron multiplicity in deep inelastic scattering, Eur. Phys. J. C **82**, 1147 (2022).
[16] B. Müller and A. Schäfer, Quark-hadron transition and entanglement, arXiv:2211.16265.
[17] P. J. Ehlers, Entanglement between valence and sea quarks in hadrons of 1 + 1 dimensional QCD, Ann. Phys. (Amsterdam) **452**, 169290 (2023).
[18] W. Kou, X. Wang, and X. Chen, Page entropy of a proton system in deep inelastic scattering at small x, Phys. Rev. D **106**, 096027 (2022).
[19] G. S. Ramos and M. V. T. Machado, Investigating the QCD dynamical entropy in high-energy hadronic collisions, Phys. Rev. D **105**, 094009 (2022).
[20] Y. Liu, M. A. Nowak, and I. Zahed, Rapidity evolution of the entanglement entropy in quarkonium: Parton and string duality, Phys. Rev. D **105**, 114028 (2022).

[21] Y. Liu, M. A. Nowak, and I. Zahed, Mueller's dipole wave function in QCD: Emergent KNO scaling in the double logarithm limit, arXiv:2211.05169.

[22] A. Dumitru and E. Kolbusz, Quark and gluon entanglement in the proton on the light cone at intermediate x, Phys. Rev. D **105**, 074030 (2022).

[23] A. Dumitru and E. Kolbusz, Quark pair angular correlations in the proton: Entropy versus entanglement negativity, arXiv:2303.07408.

[24] P. Asadi and V. Vaidya, Quantum entanglement and the thermal hadron, Phys. Rev. D **107**, 054028 (2023).

[25] P. Asadi and V. Vaidya, $1+1$D hadrons minimize their biparton Renyi free energy, arXiv:2301.03611.

[26] V. Andreev *et al.* (H1 Collaboration), Measurement of charged particle multiplicity distributions in DIS at HERA and its implication to entanglement entropy of partons, Eur. Phys. J. C **81**, 212 (2021).

[27] F. Schlumpf, Relativistic constituent quark model of electroweak properties of baryons, Phys. Rev. D **47**, 4114 (1993); **49**, 6246(E) (1994).

[28] S. J. Brodsky and F. Schlumpf, Wave function independent relations between the nucleon axial coupling g_A and the nucleon magnetic moments, Phys. Lett. B **329**, 111 (1994).

[29] S. Xu, C. Mondal, J. Lan, X. Zhao, Y. Li, and J. P. Vary (BLFQ Collaboration), Nucleon structure from basis light-front quantization, Phys. Rev. D **104**, 094036 (2021).

[30] E. Shuryak and I. Zahed, Hadronic structure on the light front. IV. Heavy and light baryons, Phys. Rev. D **107**, 034026 (2023).

[31] A. Dumitru and R. Paatelainen, Sub-femtometer scale color charge fluctuations in a proton made of three quarks and a gluon, Phys. Rev. D **103**, 034026 (2021).

[32] A. Dumitru, H. Mäntysaari, and R. Paatelainen, High-energy dipole scattering amplitude from evolution of low-energy proton light-cone wave functions, Phys. Rev. D **107**, 114024 (2023).

[33] M. Burkardt, Impact parameter space interpretation for generalized parton distributions, Int. J. Mod. Phys. A **18**, 173 (2003).

[34] Y. Hatta, B.-W. Xiao, and F. Yuan, Gluon tomography from deeply virtual Compton scattering at small-x, Phys. Rev. D **95**, 114026 (2017).

[35] B. L. G. Bakker, L. A. Kondratyuk, and M. V. Terentev, On the formulation of two-body and three-body relativistic equations employing light front dynamics, Nucl. Phys. **B158**, 497 (1979).

[36] E. Schrödinger, Discussion of probability relations between separated systems, Math. Proc. Cambridge Philos. Soc. **31**, 555 (1935).

[37] E. Schrödinger, Probability relations between separated systems, Math. Proc. Cambridge Philos. Soc. **32**, 446 (1936).

[38] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[39] M. B. Plenio, Logarithmic Negativity: A Full Entanglement Monotone That is Not Convex, Phys. Rev. Lett. **95**, 090503 (2005).

[40] Y. V. Kovchegov and A. H. Mueller, Gluon production in current-nucleus and nucleon-nucleus collisions in a quasi-classical approximation, Nucl. Phys. **B529**, 451 (1998).