

Analogies between hadron-in-jet and dihadron fragmentation

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We describe the formal analogies in the description of the inclusive production in hard processes of hadron pairs (based on dihadron fragmentation functions) and of a single hadron inside a jet (based on hadron-in-jet fragmentation functions). Since several observables involving dihadron fragmentation functions have been proposed in the past, we are able to suggest new interesting observables involving hadron-in-jet fragmentation functions, in lepton-hadron deep-inelastic scattering and hadronic collisions.

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I. INTRODUCTION

Investigation of the partonic structure of hadrons is based on the crucial method of factorization, which makes it possible to split the cross section of a given process in a perturbative calculable hard cross section (describing the underlying elementary process at the partonic level) and one or more nonperturbative functions (describing the distribution of partons inside hadrons and/or their fragmentation into detected hadronic final states). Although factorization has been established for many hard processes in the collinear framework, where transverse momenta of all partons are integrated, this is not the case for transverse-momentum dependent partonic functions (TMDs).

For certain processes involving two hadrons in the initial state and with observed hadronic final states, e.g., inclusive production of hadrons in hadronic collisions $A + B \rightarrow C + D + X$, TMD factorization can be explicitly broken because the strongly interacting particles are entangled by a complicated color flow [1,2]. Because of this, it is not possible to describe these processes in terms of the TMDs that appear in other processes, like the inclusive production of a hadron C in deep-inelastic scattering (DIS) [semi-inclusive DIS (SIDIS), denoted as $\ell + A \rightarrow \ell' + C + X$] [3,4] or the inclusive production of two hadrons C, D in electron-positron annihilations ($e^+ + e^- \rightarrow C + D + X$) [5] or the Drell-Yan process [6]. Even neglecting factorization-breaking contributions, the TMDs involved in

hadron-hadron collisions would be different from the ones in the other processes, an effect that has been referred to as generalized universality [7–10].

The most familiar example where this problem occurs is the study of the Collins effect [11]. The so-called “Collins function” can be used as an analyzer of the transverse polarization of the fragmenting quark. It can appear in SIDIS in combination with the chiral-odd TMD parton distribution function (PDF) h_1 , called “transversity,” and in the $e^+ + e^- \rightarrow C + D + X$ process [12–14]. However, because of TMD factorization breaking, it is not possible to rigorously study the Collins function in hadronic collisions.

An alternative option to the Collins effect is represented by the inclusive production of two hadrons coming from the fragmentation of a single parton. In this case, the analyzer of the transverse polarization of the fragmenting quark is represented by the transverse component of the relative momentum of the hadron pair [15]. The advantage is that this correlation survives the integration over parton transverse momenta and can be analyzed in the collinear framework. Hence, in the SIDIS process $\ell + A \rightarrow \ell' + (C_1 C_2) + X$ the transversity h_1 can be extracted as a collinear PDF through the chiral-odd partner $H_1^\perp$, the dihadron fragmentation function (DiFF) that describes the fragmentation of a transversely polarized quark into the hadron pair [16–18] and is also called an interference fragmentation function (IFF) [16,17]. As for the Collins function, the $H_1^\perp$ can be independently extracted from azimuthal asymmetries in the production of two opposite dihadron pairs in e^+e^- annihilations, in the collinear framework [19–22]. This last remark makes the crucial difference. First of all, it allows one to cross-check the universality of both h_1 and $H_1^\perp$ in hadronic collisions of the type $A + B \rightarrow (C_1 C_2) + X$ [23]. Secondly, it makes it possible to extract the chiral-odd PDF h_1 from a global fit of SIDIS, e^+e^- and hadronic collision data in the same theoretically rigorous way as it is usually done for the other unpolarized f_1 and helicity g_1 PDFs [24].

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Another intriguing option is represented by the inclusive production of a hadron inside a jet. In fact, for a collision process like $A + B \rightarrow (\text{Jet } C) + X$ the cross section can be factorized in a hybrid form [25]: it involves collinear PDFs in the initial collision, but the final state is represented by a new function, the jet TMDFF (jTMDFF) that depends on the jet kinematics. The formalism for the fragmentation of a hadron inside a jet was introduced in the collinear framework in Ref. [26] and later generalized to include the dependence on the transverse momentum of the hadron with respect to the jet axis [27].¹ The jTMDFF can be matched onto the same TMD FF of hadron C which appears in SIDIS and e^+e^- cross sections in the TMD framework [28].² It is then possible to access TMD FFs even for that class of processes where factorization in the TMD framework is not available. When one of the two colliding hadrons is transversely factorized, say $B^\uparrow$, the fragmentation of the transversely polarized quark is described by the polarized jTMDFF $\mathcal{H}_1^\perp$ that can be matched onto the Collins function $H_1^\perp$ [29]; this “Collins-in-jet” effect makes it possible to check the universality of the Collins function and gives an alternative option to access the transversity h_1 in a rigorously factorized framework.

The hybrid factorization for the hadron-in-jet inclusive production has been shown to work also for the SIDIS cross section [30–32]. Hence, it comes natural to consider the formal similarities between the inclusive production of dihadrons and of hadrons inside a jet, i.e. between DiFFs and jTMDFFs. In this way, we are able to transfer the knowledge acquired on one mechanism to the other one, and suggest new channels to investigate the partonic structure of hadrons. We stress that the correspondence we propose is derived at leading order (LO) in the strong coupling α_s . Inclusion of higher-order corrections will lead to different evolution equations for the two types of functions. Nevertheless, these differences will not affect the number and type of fragmentation functions nor the observables that can be constructed with them.

The paper is organized as follows. In Sec. II, we recall the formalism for describing the inclusive dihadron production in unpolarized proton-proton collisions. In Sec. III, we illustrate the formulas for the inclusive hadron-in-jet production in the same process. In Sec. IV, we generalize the formalism to the case of collisions with one transversely polarized hadron. In Sec. V, by comparing the cross sections for the two mechanisms we establish a general set of correspondence rules. In Sec. VI, we use these rules to extend the study of two processes: (a) the inclusive production of two back-to-back hadrons-in-jet in

unpolarized proton-proton collisions, which could give access to jets initiated by linearly polarized gluons; (b) the inclusive production of a hadron-in-jet in SIDIS up to subleading twist, which could give access to the chiral-odd PDF $e(x)$ related to the nucleon scalar charge. Finally, in Sec. VII we conclude and give some future perspectives.

II. FRAGMENTATION INTO A PAIR OF HADRONS

We consider the fragmentation of an unpolarized quark q , with 4-momentum p and mass m , into two unpolarized hadrons inside the same jet, with 4-momenta P_1, P_2 and masses M_1, M_2 , respectively. We define the total 4-momentum $P = P_1 + P_2$ and relative 4-momentum $R = (P_1 - P_2)/2$ of the pair, where $P^2 = M_{hh}^2$ is its invariant mass. We choose the $\hat{z}$ axis along the direction of the jet axis. At LO, we identify the jet axis with the direction of the 3-momentum $\mathbf{p}$. We choose the so-called “collinear” kinematics where the 3-momentum $\mathbf{P}$ is pointing along $\mathbf{p}$. The transverse components of R with respect to $\hat{z}$ are denoted by $R_\perp$, with $R_\perp^2 = -R_\perp^2$ (see Fig. 1).³

The hadron pair is inclusively produced from a hard process in a deep-inelastic regime. When specifying the kinematics on the light cone, the dominant components P_1^-, P_2^-, p^- can be used to define the following invariants [18,21,34,35],⁴

$$\begin{aligned} z_{hh} &= \frac{p^-}{P^-} = \frac{P_1^- + P_2^-}{P^-} = z_1 + z_2 \\ \zeta &= \frac{2R^-}{P^-} = \frac{z_1 - z_2}{z_{hh}}, \end{aligned} \quad (1)$$

which represent the fraction of the fragmenting quark momentum carried by the hadron pair and how this fraction is split inside the pair, respectively.

The fragmentation is described starting from the quark-quark correlator [18,34]

$$\begin{aligned} \Delta(p, P, R) &= \sum_X \int \frac{dx}{(2\pi)^4} e^{ip \cdot x} \langle 0 | \psi(x) | X, P, R \rangle \\ &\times \langle X, P, R | \bar{\psi}(0) | 0 \rangle, \end{aligned} \quad (2)$$

where ψ is the quark field operator and the sum runs over all possible final states $|X, P, R\rangle$ containing a hadron pair with total and relative momenta P, R , respectively. At leading twist, the fragmentation of an unpolarized quark

¹In Ref. [27], the jTMDFF is called generalized fragmenting jet function (GJF).

²In Ref. [28], the jTMDFF is called semi-inclusive TMD fragmenting jet function (siTMDJF).

³In the following, we adhere to the conventions adopted in Ref. [33]: the fragmenting quark momentum is denoted by p , the transverse component of hadron and quark vectors with respect to each other is denoted by the $\perp$ subscript, in all other cases with the τ subscript.

⁴In Refs. [17,19,20,36,37], the less symmetric definition $\xi = (\zeta + 1)/2 = z_1/z = 1 - z_2/z$ was adopted.

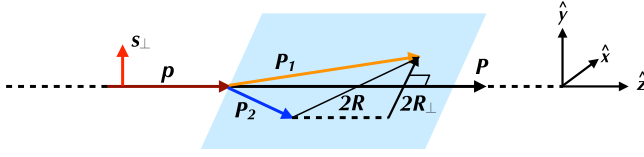


FIG. 1. Collinear kinematics for the fragmentation of a quark with 3-momentum $\mathbf{p}$ (and transverse polarization $\mathbf{s}_{\perp}$) into a pair of hadrons with total 3-momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ pointing along $\mathbf{p}$, namely with $\mathbf{p}_{\perp} = 0$. The $\hat{z}$ axis is along the same direction of $\mathbf{p}$ and $\mathbf{P}$.

into two unpolarized hadrons can be parametrized in terms of a single DiFF according to [34]

$$D_1(z_{hh}, \zeta, \mathbf{R}_{\perp}^2) = 4\pi \text{Tr}[\Delta(z_{hh}, \zeta, \mathbf{R}_{\perp}^2) \gamma^-], \quad (3)$$

where

$$\Delta(z_{hh}, \zeta, \mathbf{R}_{\perp}^2) = \frac{z_{hh}}{32} \int dp^+ \int d\mathbf{p}_{\perp} \Delta(p, P, R) \Big|_{p^- = P^-/z_{hh}}. \quad (4)$$

In fact, the full dependence of the correlator in Eq. (2) is reduced to the one in Eq. (4) by considering that [36]

- (i) in Eq. (4) we integrate over the light-cone suppressed variable p^+ and over $\mathbf{p}_{\perp}$ with the condition $p^- = P^-/z_{hh}$;
- (ii) our choice of frame and kinematics implies no dependence on $\mathbf{P}_{\perp}$;
- (iii) the following kinematical relations hold [18]:

$$\begin{aligned} P^2 &= M_{hh}^2, \\ R^2 &= \frac{M_1^2 + M_2^2}{2} - \frac{M_{hh}^2}{4} - \frac{(M_1^2 - M_2^2)^2}{4M_{hh}^2}, \\ R_{\perp}^2 &= \frac{1}{2} \left[\frac{(1-\zeta)(1+\zeta)}{2} M_{hh}^2 \right. \\ &\quad \left. - (1-\zeta)M_1^2 - (1+\zeta)M_2^2 \right]. \end{aligned} \quad (5)$$

It is useful to recall also that [18]

$$\mathbf{p} \cdot \mathbf{R} = \frac{M_1^2 - M_2^2 - \frac{\zeta}{2} M_{hh}^2}{2z_{hh}} + z_{hh} \zeta \frac{p^2 + p_{\perp}^2}{2} - \mathbf{p}_{\perp} \cdot \mathbf{R}_{\perp}, \quad (6)$$

from which we deduce that in general DiFFs depend only on the relative angle between $\mathbf{p}_{\perp}$ and $\mathbf{R}_{\perp}$.

A. Cross section for dihadron production in proton-proton collisions

If the hadron pair is inclusively produced from the collision of two unpolarized protons with momenta $\mathbf{P}_A$ and $\mathbf{P}_B$, we can identify the reaction plane as the plane

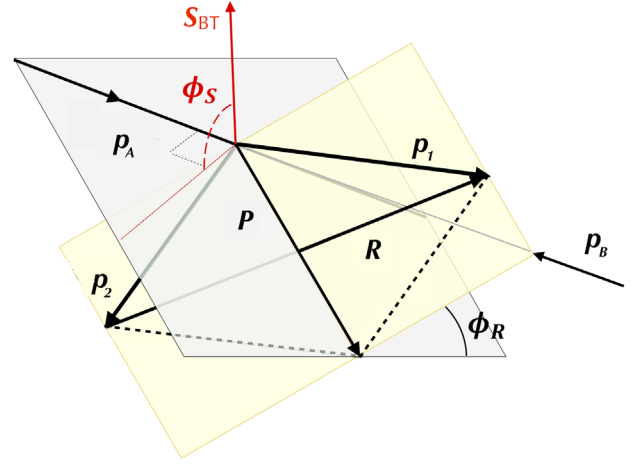


FIG. 2. Kinematics for the collision of a proton with 3-momentum $\mathbf{P}_A$ and a (transversely polarized) proton with momentum $\mathbf{P}_B$ (and polarization $\mathbf{S}_{BT}$), inclusively producing two unpolarized hadrons with total and relative momenta $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{R} = (\mathbf{p}_1 - \mathbf{p}_2)/2$. The plane formed by $\mathbf{P}$ and $\mathbf{R}$ is oriented by the azimuthal angle ϕ_R around $\mathbf{P}$ with respect to the reaction plane formed by $\mathbf{P}_A$ and $\mathbf{P}_B$.

formed by $\mathbf{P}_A$ and $\mathbf{P}_B$. The azimuthal orientation around $\mathbf{P}$ of the plane formed by $\mathbf{p}_1$ and $\mathbf{p}_2$ with respect to the reaction plane is described by the azimuthal angle ϕ_R (see Fig. 2 and Ref. [38] for a formal definition). The transverse component of $\mathbf{P}$ with respect to $\mathbf{P}_A$ is denoted by $\mathbf{P}_T$. Its modulus represents the hard scale of the process, namely we assume that $|\mathbf{P}_T| \gg M_{hh}, M_1, M_2$. For simplicity, in the following the dependence of DiFFs on $|\mathbf{P}_T|$ is understood.

At leading order in $1/|\mathbf{P}_T|$, the differential cross section for the process $A + B \rightarrow (C_1 C_2) + X$ reads (see Appendix A and Eq. (15) of Ref. [38])

$$\begin{aligned} \frac{d\sigma_{UU}}{d\eta d|\mathbf{P}_T| d\zeta d\mathbf{R}_{\perp}} &= \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{x_A x_B z_{hhC}^2} f_1^a(x_A) f_1^b(x_B) \\ &\times \frac{|\mathbf{P}_T| \hat{s}}{2\pi s} \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \\ &\times D_1^c(z_{hhC}, \zeta, \mathbf{R}_{\perp}^2), \end{aligned} \quad (7)$$

where f_1^a and f_1^b are the usual parton distribution functions in the proton for partons a, b with fractional momenta x_A, x_B , respectively, and η is the pseudorapidity of the hadron pair with respect to $\mathbf{P}_A$:

$$\eta = \frac{1}{2} \log \frac{P^0 + P_z}{P^0 - P_z}. \quad (8)$$

The elementary cross section $d\hat{\sigma}$ describes the scattering of partons a and b into partons c (with momentum $\mathbf{P}/z_{hhC}$) and d , which is not detected. The partonic Mandelstam variables $\hat{s}, \hat{t}, \hat{u}$ are related to the external ones by

$$\hat{s} = x_A x_B s, \quad \hat{t} = \frac{x_A}{z_{hhC}} t, \quad \hat{u} = \frac{x_B}{z_{hhC}} u. \quad (9)$$

The δ function in Eq. (7) expresses the momentum conservation in the partonic scattering, and it can be rewritten as [38]

$$\hat{s}\delta(\hat{s} + \hat{t} + \hat{u}) = z_{hhC}\delta(z_{hhC} - \bar{z}_{hh}), \quad (10)$$

where

$$\bar{z}_{hh} = \frac{|\mathbf{P}_T|}{\sqrt{s}} \frac{x_A e^{-\eta} + x_B e^{\eta}}{x_A x_B}. \quad (11)$$

In Eq. (7), the sum runs upon all possible combinations of parton flavors. The elementary cross sections $d\hat{\sigma}_{ab \rightarrow cd}$ for the independent combinations are listed in the Appendix of Ref. [38].

III. HADRON-IN-JET FRAGMENTATION

We now consider the distribution of a hadron with 4-momentum P_h and mass M_h inside a jet with radius r , initiated by a unpolarized quark q with 4-momentum p and mass m . Following Ref. [28], we denote by $\mathbf{j}_\perp$ the transverse momentum of the hadron inside the jet (see Fig. 3). The latter is defined with respect to the standard jet axis (rather than using a recoil-free algorithm) because only in this case a direct connection to the TMD FF can be made [28]. As in the dihadron case, the $\hat{z}$ axis is chosen along the standard jet axis and at LO it is identified with the direction of $\mathbf{p}$.

When the jet is produced in a hard process in a deep-inelastic kinematical regime, the large light-cone components of quark, jet, and hadron vectors are denoted by p^- , J^- , and P_h^- , respectively. They are used to define the following invariants:

$$z_J = \frac{J^-}{p^-}, \quad z_h = \frac{P_h^-}{J^-}, \quad (12)$$

which represent the fraction of the fragmenting quark momentum carried by the jet and the fraction of jet momentum carried by the hadron inside the jet,

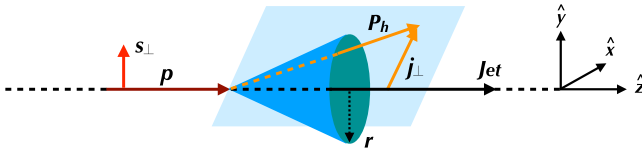


FIG. 3. Fragmentation of a unpolarized hadron with 3-momentum $\mathbf{P}_h$ inside a jet of radius r initiated by a quark with 3-momentum $\mathbf{p}$ (and transverse polarization $\mathbf{s}_\perp$). The transverse component of $\mathbf{P}_h$ with respect to the standard jet axis is denoted by $\mathbf{j}_\perp$ [28]. For the sake of simplicity, $\mathbf{p}$ and the jet axis are approximately taken along the same direction.

respectively. The J^- is related to the transverse momentum of the reconstructed jet in the hard process, whose size is denoted as $|\mathbf{P}_T|$ and represents the hard scale of the process itself.

The fragmentation is described starting from the quark-quark correlator

$$\Delta(p, J, P_h) = \sum_X \int \frac{dx}{(2\pi)^4} e^{ip \cdot x} \langle 0 | \psi(x) | X, J, P_h \rangle \times \langle X, J, P_h | \bar{\psi}(0) | 0 \rangle, \quad (13)$$

where, as before, ψ is the quark field operator and the sum runs over all possible final states $|X, J, P_h\rangle$ containing a hadron P_h inside a jet J . At leading twist, the object describing the observed hadron inside the produced jet can be parametrized in terms of a jTMDFF according to [28]

$$\mathcal{D}_1(z_J, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r) = \frac{z_J}{4N_c} \text{Tr}[\Delta(z_J, z_h, \mathbf{j}_\perp, |\mathbf{P}_T|, |\mathbf{P}_T|r) \gamma^-], \quad (14)$$

where N_c is the number of quark colors, $|\mathbf{P}_T|r$ is the typical momentum scale of the jet [28], and

$$\Delta(z_J, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r) = \int dp^+ \Delta(p, J, P_h) \Big|_{\substack{p^- = J^-/z_J \\ |\mathbf{P}_T| = |\mathbf{P}_T|/z_J \\ J^- = P_h^-/z_h}}. \quad (15)$$

Depending on the relative size of $|\mathbf{j}_\perp|$, $|\mathbf{P}_T|r$, and the QCD nonperturbative scale Λ_{QCD} , the jTMDFF of Eq. (14) can be expressed in different factorized forms. Here, we are interested in the kinematical region $\Lambda_{\text{QCD}} \lesssim |\mathbf{j}_\perp| \ll |\mathbf{P}_T|r$ where collinear radiation within the jet and soft radiation of order $|\mathbf{j}_\perp|$ are relevant, while harder radiation is allowed only outside the jet and it does not affect the distribution of the hadron transverse momentum $|\mathbf{j}_\perp|$. In this regime, a factorized form for $\mathcal{D}_1$ is given in Ref. [28] in terms of a hard matching function (related to the hard out-of-jet radiation) and a convolution of a usual TMD FF and a soft function (accounting for the soft radiation inside the jet). It is obtained by initially evolving the TMD FF in the usual Collins-Soper-Sterman (CSS) scheme up to the jet scale $|\mathbf{P}_T|r$, then matching to the calculable hard function describing the out-of-jet radiation, and finally evolving to the hard scale by using the standard timelike DGLAP equations. All calculations in Ref. [28] are performed at NLO. At LO, the direction of the quark momentum $\mathbf{p}$ coincides with the standard jet axis and its transverse component is equal to the transverse momentum of the reconstructed jet in the hard process, $|\mathbf{p}_T| \approx |\mathbf{P}_T|$. In this approximation, the jTMDFF $\mathcal{D}_1^q$ for the fragmentation of a quark q into a hadron inside the jet reduces to

$$\begin{aligned}
& \mathcal{D}_1^q(z_J, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r)|_{\text{LO}} \\
&= \sum_i \delta(1 - z_J) \delta_{qi} D_1^i(z_h, \mathbf{j}_\perp^2; |\mathbf{P}_T|) \\
&= \delta(1 - z_J) D_1^q(z_h, \mathbf{j}_\perp^2; |\mathbf{P}_T|), \quad (16)
\end{aligned}$$

where D_1^q is the standard single-hadron TMD FF that can be isolated also in e^+e^- annihilations or in semi-inclusive deep-inelastic scattering.

A. Cross section for hadron-in-jet fragmentation in proton-proton collisions

We consider the same situation as in Sec. II A, namely the collision of two unpolarized protons with momenta P_A and P_B . The final state is now described by the inclusive production of a jet where a hadron is identified inside it with transverse momentum $\mathbf{j}_\perp$ with respect to the standard jet axis. Following Ref. [29], the factorization theorem for the process $A + B \rightarrow (\text{Jet } C) + X$ can be written as

$$\frac{d\sigma_{UU}}{d\eta d|\mathbf{P}_T| dz_h d\mathbf{j}_\perp} = \frac{2\pi|\mathbf{P}_T|}{s} \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{JC}}{x_A x_B z_{JC}^2} f_1^a(x_A) f_1^b(x_B) H_{ab \rightarrow cd}^U \mathcal{D}_1^c(z_{JC}, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r) z_{JC}^2 \delta(z_{JC} - \bar{z}_J), \quad (17)$$

where $\bar{z}_J$ is given as in Eq. (11) and $H_{ab \rightarrow cd}^U$ describes the elementary hard process $a + b \rightarrow c + d$ from which the parton c initiates the reconstructed jet.

As detailed in Appendix B, the $H_{ab \rightarrow cd}^U$ of Ref. [29] can be reconnected to the $d\hat{\sigma}_{ab \rightarrow cd}$ of Eq. (7) by

$$H_{ab \rightarrow cd}^U = \frac{\hat{s}}{\pi z_{JC}} \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}}. \quad (18)$$

The cross section of Eq. (17) can then be cast in the form

$$\begin{aligned}
\frac{d\sigma}{d\eta d|\mathbf{P}_T| dz_h d\mathbf{j}_\perp} &= \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{JC}}{x_A x_B z_{JC}^2} f_1^a(x_A) f_1^b(x_B) 2 \frac{|\mathbf{P}_T| \hat{s}}{s} \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} z_{JC} \delta(z_{JC} - \bar{z}_J) \mathcal{D}_1^c(z_{JC}, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r) \\
&= \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{JC}}{x_A x_B z_{JC}^2} f_1^a(x_A) f_1^b(x_B) 2 \frac{|\mathbf{P}_T| \hat{s}}{s} \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \mathcal{D}_1^c(z_{JC}, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r), \quad (19)
\end{aligned}$$

where we used Eq. (10) adapted to the case of hadron-in-jet fragmentation, i.e., by replacing z_{hhC} with z_{JC} for the fragmenting parton c and using the pseudorapidity of the jet with respect to $\mathbf{P}_A$.

IV. FRAGMENTATION OF TRANSVERSELY POLARIZED QUARKS

We extend our study to the case of a fragmenting quark with transverse polarization $\mathbf{s}_\perp$. We first consider the fragmentation in a pair of unpolarized hadrons (see Fig. 1). In the kinematic conditions described in Sec. II, the leading-twist correlator of Eq. (4) can be expanded as [34]

$$\begin{aligned}
\Delta(z_{hh}, \zeta, \mathbf{R}_\perp^2) &= \frac{1}{16\pi} \left\{ D_1(z_{hh}, \zeta, \mathbf{R}_\perp^2) \not{\mathbf{p}}_- \right. \\
&\quad \left. + H_1^\triangleleft(z_{hh}, \zeta, \mathbf{R}_\perp^2) \frac{i}{2M_{hh}} [\not{\mathbf{R}}_\perp, \not{\mathbf{p}}_-] \right\}, \quad (20)
\end{aligned}$$

where $H_1^\triangleleft$ describes the probability density for a transversely polarized quark to fragment into a pair of unpolarized hadrons with total momentum collinear with the quark momentum. The $H_1^\triangleleft$ can be extracted by the following projection,

$$\frac{(\mathbf{s}_\perp \times \mathbf{R}_\perp) \cdot \mathbf{P}}{M_{hh}} H_1^\triangleleft(z_{hh}, \zeta, \mathbf{R}_\perp^2) = 4\pi \text{Tr}[\Delta(z_{hh}, \zeta, \mathbf{R}_\perp^2) i\sigma^{i-} \gamma_5], \quad (21)$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$ and its spatial index i points in the direction of $\mathbf{s}_\perp$.

Similarly, if the transversely polarized quark fragments into a hadron inside a jet in the kinematical conditions described in Sec. III (see Fig. 3), we can project out the ‘‘Collins-in-jet’’ function $\mathcal{H}_1^\perp$ from the correlator in Eq. (15) as

$$\begin{aligned}
& \frac{(\mathbf{s}_\perp \times \mathbf{j}_\perp) \cdot \mathbf{P}}{z_h M_h} \mathcal{H}_1^\perp(z_J, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r) \\
&= \frac{z_J}{4N_c} \text{Tr}[\Delta(z_J, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T|r) i\sigma^{i-} \gamma_5], \quad (22)
\end{aligned}$$

where again the spatial index i of $\sigma^{\mu\nu}$ points in the direction of $\mathbf{s}_\perp$.

In the following section, for the two fragmentation scenarios we analyze the contributions that arise in the cross section for proton-proton collisions when one of the two protons is transversely polarized.

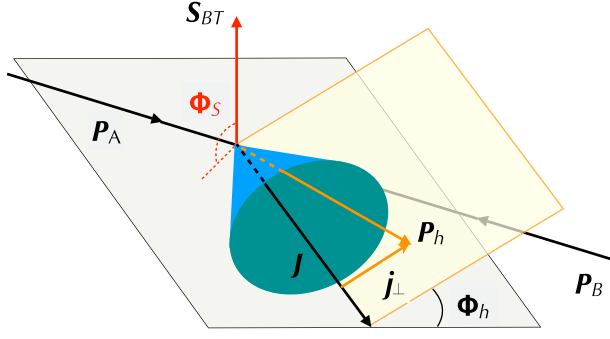


FIG. 4. Kinematics for the collision of a proton with 3-momentum $\mathbf{P}_A$ and a transversely polarized proton with momentum $\mathbf{P}_B$ (and polarization $\mathbf{S}_{BT}$), inclusively producing inside a jet a hadron with 3-momentum $\mathbf{P}_h$ and transverse component $\mathbf{j}_\perp$ with respect to the jet axis $\hat{\mathbf{J}}$. The plane formed by $\hat{\mathbf{J}}$ and $\mathbf{P}_h$ is oriented by the azimuthal angle ϕ_h around $\hat{\mathbf{J}}$ with respect to the reaction plane formed by $\mathbf{P}_A$ and $\hat{\mathbf{J}}$.

A. Transversely polarized proton-proton collisions

For the process $A + B^\uparrow \rightarrow (C_1 C_2) + X$ depicted in Fig. 2, the polarized part of the cross section reads (see Appendix A and Eq. (16) of Ref. [38])

$$\begin{aligned} \frac{d\sigma_{UT}}{d\eta d|\mathbf{P}_T| d\zeta d\mathbf{R}_\perp d\phi_{S_B}} &= \frac{|\mathbf{S}_{BT}|}{4\pi^2} \sin(\phi_{S_B} - \phi_R) \\ &\times \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{x_A x_B z_{hhC}^2} f_1^a(x_A) \\ &\times h_1^b(x_B) \frac{|\mathbf{P}_T| \hat{s} d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}}{s d\hat{t}} \\ &\times \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \frac{|\mathbf{R}_\perp|}{M_{hh}} \\ &\times H_1^{\leq c}(z_{hhC}, \zeta, \mathbf{R}_\perp^2), \end{aligned} \quad (23)$$

where $\mathbf{S}_{BT}$ is the transverse polarization of the colliding proton with orientation ϕ_{S_B} with respect to the reaction plane, and h_1^b is the transversity distribution for the

transversely polarized parton b with fractional momentum x_B . The elementary cross sections $d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}$ describe the scattering of parton a and b with transfer of the transverse polarization of the latter to parton c while summing on the undetected fragments from parton d . All the possible independent flavor combinations are listed in the appendix of Ref. [38].

The corresponding process $A + B^\uparrow \rightarrow (\text{Jet } C) + X$ is displayed in Fig. 4. A hadron with 3-momentum $\mathbf{P}_h$ is inclusively produced inside a jet with standard axis $\hat{\mathbf{J}}$ from the collisions of a proton with 3-momentum $\mathbf{P}_A$ and a transversely polarized proton with 3-momentum $\mathbf{P}_B$ and polarization $\mathbf{S}_{BT}$. The azimuthal angles ϕ_{S_B} and ϕ_h describe the orientation of $\mathbf{S}_{BT}$ and of the plane formed by $\mathbf{P}_h$ and $\hat{\mathbf{J}}$, respectively, with respect to the reaction plane formed by $\mathbf{P}_A$ and $\hat{\mathbf{J}}$. The polarized part of the cross section reads [29]⁵

$$\begin{aligned} \frac{d\sigma_{UT}}{d\eta d|\mathbf{P}_T| dz_h d\mathbf{j}_\perp d\phi_{S_B}} &= |\mathbf{S}_{BT}| \sin(\phi_{S_B} - \phi_h) \frac{|\mathbf{P}_T|}{s} \\ &\times \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{JC}}{x_A x_B z_{JC}^2} f_1^a(x_A) \\ &\times h_1^b(x_B) H_{ab^\uparrow \rightarrow c^\uparrow d}^{\text{Collins}} \frac{|\mathbf{j}_\perp|}{M_h} \\ &\times \mathcal{H}_1^{\perp c}(z_{JC}, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T| r) \\ &\times z_{JC}^2 \delta(z_{JC} - \bar{z}_J), \end{aligned} \quad (24)$$

where $H_{ab^\uparrow \rightarrow c^\uparrow d}^{\text{Collins}}$ is the cross section for the transfer of transverse polarization in the elementary hard process $a + b^\uparrow \rightarrow c^\uparrow + d$, and $\mathcal{H}_1^{\perp c}$ is the polarized jTMDFF describing the hadron inside the jet produced by the transversely polarized fragmenting parton c .

By extending the relation (18) to the polarized case involving $H_{ab^\uparrow \rightarrow c^\uparrow d}^{\text{Collins}}$ of Ref. [29] and $d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}$ of Ref. [38] (and exchanging $\hat{t} \leftrightarrow \hat{u}$ to account for the fact that the transversely polarized parton in Ref. [29] is $a^\uparrow$ while in Ref. [38] is $b^\uparrow$), we finally get

$$\begin{aligned} \frac{d\sigma_{UT}}{d\eta d|\mathbf{P}_T| dz_h d\mathbf{j}_\perp d\phi_{S_B}} &= |\mathbf{S}_{BT}| \sin(\phi_{S_B} - \phi_h) \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{JC}}{x_A x_B z_{JC}^2} f_1^a(x_A) h_1^b(x_B) \frac{|\mathbf{P}_T| \hat{s}}{\pi s} z_{JC} \delta(z_{JC} - \bar{z}_J) \\ &\times \frac{d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}}{d\hat{t}} \frac{|\mathbf{j}_\perp|}{M_h} \mathcal{H}_1^{\perp c}(z_{JC}, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T| r) \\ &= |\mathbf{S}_{BT}| \sin(\phi_{S_B} - \phi_h) \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{JC}}{x_A x_B z_{JC}^2} f_1^a(x_A) h_1^b(x_B) \frac{|\mathbf{P}_T| \hat{s}}{\pi s} \frac{d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}}{d\hat{t}} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \\ &\times \frac{|\mathbf{j}_\perp|}{M_h} \mathcal{H}_1^{\perp c}(z_{JC}, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T| r). \end{aligned} \quad (25)$$

⁵The expression in Eq. (24) differs from Ref. [29] by a $1/z_h$ term because of a definition of the Collins function inherited from Ref. [25] which does not adhere to the Trento conventions [39].

V. CORRESPONDENCE BETWEEN DIHADRON AND HADRON-IN-JET FRAGMENTATION

We are now in the position to compare the cross sections for the $A + B^{(\dagger)} \rightarrow (C_1 C_2) + X$ and $A + B^{(\dagger)} \rightarrow (\text{Jet } C) + X$ processes. We deduce that

- (i) from Eq. (12), the combination $z_J z_h = P_h^- / p^-$ describes the fraction of fragmenting quark momentum carried by the hadron inside the jet; hence, it can be mapped onto z_{hh} of Eq. (1);
- (ii) by comparing the same two equations, we can map z_h onto $\zeta z_{hh} = z_1 - z_2$, the relative fractional momentum carried by the hadron pair; thus, for both hadronic final states (dihadron and hadron inside jet) the light-cone kinematics can be described by a pair of invariants and we can establish a correspondence between these pairs, namely $(z_{hh}, \zeta) \leftrightarrow (z_J, z_h)$;
- (iii) along the same line, we can map the transverse momentum $\mathbf{j}_\perp$ of the hadron inside the jet with respect to the standard jet axis onto the transverse component $\mathbf{R}_\perp$ of the hadron pair relative momentum with respect to the direction of the pair total momentum, which in collinear kinematics coincides with the standard jet axis; obviously, the same mapping holds for their azimuthal angles, i.e., $\phi_h \leftrightarrow \phi_R$;
- (iv) by directly comparing Eqs. (7) with (19) and Eqs. (23) with (25), the jTMDFF can be mapped onto the corresponding DiFF according to

$$4\pi \mathcal{D}_1^q(z_J, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T| r) \longleftrightarrow D_1^q(z_{hh}, \zeta, \mathbf{R}_\perp^2; |\mathbf{P}_T|), \quad (26)$$

$$4\pi \mathcal{H}_1^{\perp q}(z_J, z_h, \mathbf{j}_\perp; |\mathbf{P}_T|, |\mathbf{P}_T| r) \longleftrightarrow H_1^{\perp q}(z_{hh}, \zeta, \mathbf{R}_\perp^2; |\mathbf{P}_T|). \quad (27)$$

We remark that the above correspondence is derived at LO in α_S . In fact, DiFFs have been extracted so far only

through a LO analysis of inclusive dihadron production in e^+e^- annihilation [21], in combined e^+e^- and SIDIS processes [22], and in a global fit of e^+e^- , SIDIS, and hadron-hadron collision data [24]. The formalism of jTMDFFs is available instead up to NLO, but only in the unpolarized case [28,29]. The inclusion of higher order corrections (NLO, NNLO, etc.) would expose sharp differences in the scale dependence of the two categories of functions; in particular, DiFFs would not incorporate any rapidity divergence, like all collinear objects following DGLAP evolution equations. However, the structure of the quark-quark correlators and, consequently, of the cross sections will remain independent of evolution effects.

The correspondence expressed in Eqs. (26) and (27) represents the main result of this paper. It has been derived by considering the case of proton-proton collisions but it can be extended to all hard processes where collinear factorization holds, like inclusive dihadron production in e^+e^- annihilations [19–21] and SIDIS. In the following, we outline some interesting applications involving proton-proton collisions and the SIDIS process.

VI. OPPORTUNITIES WITH HADRON-IN-JET FRAGMENTATION

In this section, we mention two possible applications of the above correspondence where results known for dihadron inclusive production can be formally translated into the cross section for hadron-in-jet fragmentation, opening up new channels for investigating the partonic structure of hadrons.

A. Inclusive production of two dihadrons and back-to-back hadron-in-jets in proton-proton collisions

For the process $A + B \rightarrow (C_1 C_2)_C + (D_1 D_2)_D + X$ depicted in the left panel of Fig. 5, after summing over the polarizations of initial hadrons the leading-twist cross section reads (see Appendix A and Eqs. (20–22) in Ref. [38])

$$\begin{aligned} \frac{d\sigma_{UU}}{d\eta_C d|\mathbf{P}_{CT}| d\zeta_C d\mathbf{R}_{C\perp} d\eta_D d|\mathbf{P}_{DT}| d\zeta_D d\mathbf{R}_{D\perp}} &= \frac{|\mathbf{P}_{CT}| |\mathbf{P}_{DT}|}{8\pi^2} \sum_{a,b} \int \frac{dx_A dx_B dz_{hhC} dz_{hhD}}{z_{hhC}^2 z_{hhD}^2} f_1^a(x_A) f_1^b(x_B) \\ &\times \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \delta\left(\frac{|\mathbf{P}_{CT}|}{z_{hhC}} - \frac{|\mathbf{P}_{DT}|}{z_{hhD}}\right) \delta\left(x_A P_{Az} - x_B P_{Bz} - \frac{P_{Cz}}{z_{hhC}} - \frac{P_{Dz}}{z_{hhD}}\right) \\ &\times \left\{ \sum_{c,d} \left[\frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} D_1^c(z_{hhC}, \zeta_C, \mathbf{R}_{C\perp}^2) D_1^d(z_{hhD}, \zeta_D, \mathbf{R}_{D\perp}^2) + \cos(\phi_{R_C} - \phi_{R_D}) \right. \right. \\ &\times \frac{d\Delta\hat{\sigma}_{ab \rightarrow c^\dagger d^\dagger}}{d\hat{t}} \frac{|\mathbf{R}_{C\perp}|}{M_C} H_1^{\leftarrow c}(z_{hhC}, \zeta_C, \mathbf{R}_{C\perp}^2) \frac{|\mathbf{R}_{D\perp}|}{M_D} H_1^{\leftarrow d}(z_{hhD}, \zeta_D, \mathbf{R}_{D\perp}^2) \left. \right] \\ &+ \cos(2\phi_{R_C} - 2\phi_{R_D}) \frac{d\Delta\hat{\sigma}_{ab \rightarrow g^\dagger g^\dagger}}{d\hat{t}} \frac{|\mathbf{R}_{C\perp}|^2}{M_C^2} H_1^{\leftarrow g}(z_{hhC}, \zeta_C, \mathbf{R}_{C\perp}^2) \\ &\times \frac{|\mathbf{R}_{D\perp}|^2}{M_D^2} H_1^{\leftarrow g}(z_{hhD}, \zeta_D, \mathbf{R}_{D\perp}^2) \left. \right\}, \quad (28) \end{aligned}$$

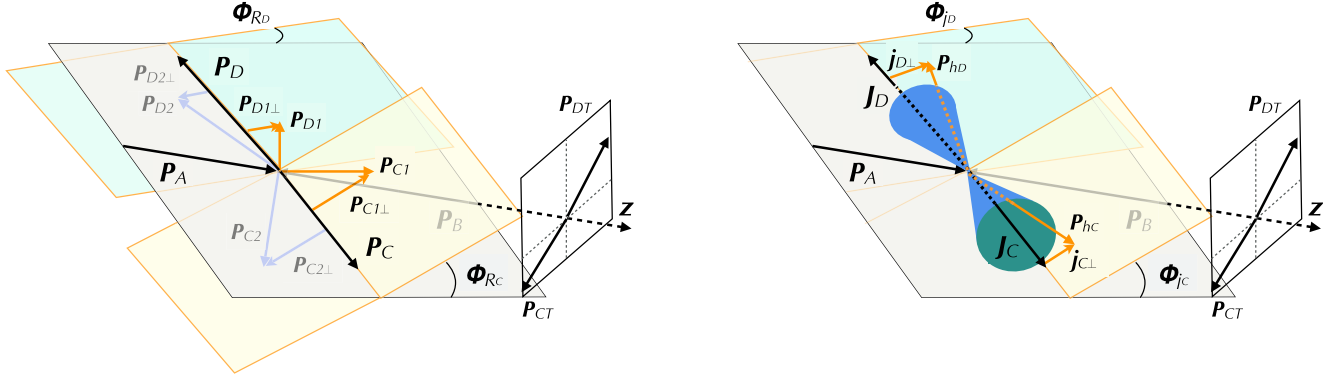


FIG. 5. Kinematics for the collision of two unpolarized protons with 3-momenta $\mathbf{P}_A$ and $\mathbf{P}_B$ along the $\hat{z}$ axis. Left panel: inclusive production of two back-to-back hadron pairs with total momenta $\mathbf{P}_C = \mathbf{P}_{C1} + \mathbf{P}_{C2}$ and $\mathbf{P}_D = \mathbf{P}_{D1} + \mathbf{P}_{D2}$, and back-to-back projections $\mathbf{P}_{CT}$ and $\mathbf{P}_{DT}$ on the transverse plane ($\phi_C = \phi_D + \pi$), respectively; the planes containing the momenta of each pair form the azimuthal angles ϕ_{Rc} and ϕ_{Rd} with the reaction plane containing $\mathbf{P}_A$ and $\mathbf{P}_C$. Right panel: inclusive production of two back-to-back jets with axis $\hat{\mathbf{J}}_C, \hat{\mathbf{J}}_D$, and back-to-back projected momenta $\mathbf{P}_{CT}$ and $\mathbf{P}_{DT}$ on the transverse plane ($\phi_C = \phi_D + \pi$), respectively; in each jet a hadron is detected with 3-momentum $\mathbf{P}_{hc}$ ($\mathbf{P}_{hd}$) and transverse component $\mathbf{j}_{C\perp}$ ($\mathbf{j}_{D\perp}$) with respect to the jet axis $\hat{\mathbf{J}}_C$ ($\hat{\mathbf{J}}_D$); the planes containing $\hat{\mathbf{J}}_C$, $\mathbf{P}_{hc}$ and $\hat{\mathbf{J}}_D$, $\mathbf{P}_{hd}$ form the azimuthal angles ϕ_{jc} , ϕ_{jd} with the reaction plane containing $\mathbf{P}_A$ and $\hat{\mathbf{J}}_C$.

where the momenta and the angles of the second hadron pair are defined in complete analogy with the first pair by replacing the labels c, C with d, D . The delta functions describe in the elementary process the conservation of energy and of momentum both along the longitudinal direction of the $\hat{z}$ axis (identified with $\mathbf{P}_A$, see Fig. 5) and in the transverse plane.

In Eq. (28), the elementary cross sections $d\Delta\hat{\sigma}_{ab \rightarrow c^\dagger d^\dagger}$ involve only quarks for the final partons c, d , while $d\Delta\hat{\sigma}_{ab \rightarrow g^\dagger g^\dagger}$ contain only final gluons linearly polarized in the transverse plane. Hence, the $H_1^{\leftarrow g}$ function describes the fragmentation of such linearly polarized gluons into pairs of unpolarized hadrons.⁶ For both cases of final

polarized quarks and gluons, all nonvanishing combinations are listed in the appendix of Ref. [38].

Therefore, by disentangling specific asymmetries in the azimuthal orientation of the planes containing the momenta of the two dihadrons one can access the DiFFs $H_1^{\leftarrow q}$ and $H_1^{\leftarrow g}$ for the fragmentation of transversely polarized quarks and linearly polarized gluons, respectively, without considering any polarization in the initial hadronic collision [38]. Because of the correspondence described in Sec. V, it is interesting to explore the same possibility for the process $A + B \rightarrow (\text{Jet } C) + (\text{Jet } D) + X$, as depicted in the right panel of Fig. 5. Using the same rules of correspondence as in the single hadron-pair production, we get

$$\begin{aligned}
 \frac{d\sigma_{UU}}{d\eta_C d|\mathbf{P}_{CT}| dz_{hC} d\mathbf{j}_{C\perp} d\eta_D d|\mathbf{P}_{DT}| dz_{hD} d\mathbf{j}_{D\perp}} &= 2|\mathbf{P}_{CT}||\mathbf{P}_{DT}| \sum_{a,b} \int \frac{dx_A dx_B dz_{JC} dz_{JD}}{z_{JC}^2 z_{JD}^2} f_1^a(x_A) f_1^b(x_B) \\
 &\times \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \delta\left(\frac{|\mathbf{P}_{CT}|}{z_{JC}} - \frac{|\mathbf{P}_{DT}|}{z_{JD}}\right) \delta\left(x_A P_{Az} - x_B P_{Bz} - \frac{P_{Cz}}{z_{JC}} - \frac{P_{Dz}}{z_{JD}}\right) \\
 &\times \left\{ \sum_{c,d} \left[\frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} \mathcal{D}_1^c(z_{JC}, z_{hC}, \mathbf{j}_{C\perp}^2; |\mathbf{P}_{CT}|, |\mathbf{P}_{CT}| r_C) \mathcal{D}_1^d(z_{JD}, z_{hD}, \mathbf{j}_{D\perp}^2; |\mathbf{P}_{DT}|, |\mathbf{P}_{DT}| r_D) \right. \right. \\
 &+ \cos(\phi_{jc} - \phi_{jd}) \frac{d\Delta\hat{\sigma}_{ab \rightarrow c^\dagger d^\dagger}}{d\hat{t}} \frac{|\mathbf{j}_{C\perp}|}{M_{hc}} \mathcal{H}_1^{\perp c}(z_{JC}, z_{hC}, \mathbf{j}_{C\perp}^2; |\mathbf{P}_{CT}|, |\mathbf{P}_{CT}| r_C) \frac{|\mathbf{j}_{D\perp}|}{M_{hd}} \mathcal{H}_1^{\perp d}(z_{JD}, z_{hD}, \mathbf{j}_{D\perp}^2; |\mathbf{P}_{DT}|, |\mathbf{P}_{DT}| r_D) \\
 &\left. \left. + \cos(2\phi_{jc} - 2\phi_{jd}) \frac{d\Delta\hat{\sigma}_{ab \rightarrow g^\dagger g^\dagger}}{d\hat{t}} \frac{|\mathbf{j}_{C\perp}|^2}{M_{hc}^2} \mathcal{H}_1^{\perp g}(z_{JC}, z_{hC}, \mathbf{j}_{C\perp}^2; |\mathbf{P}_{CT}|, |\mathbf{P}_{CT}| r_C) \frac{|\mathbf{j}_{D\perp}|^2}{M_{hd}^2} \mathcal{H}_1^{\perp g}(z_{JD}, z_{hD}, \mathbf{j}_{D\perp}^2; |\mathbf{P}_{DT}|, |\mathbf{P}_{DT}| r_D) \right] \right\}, \quad (29)
 \end{aligned}$$

⁶The $H_1^{\leftarrow g}$ corresponds to the notation $\delta\hat{G}^{\leftarrow}$ of Ref. [38].

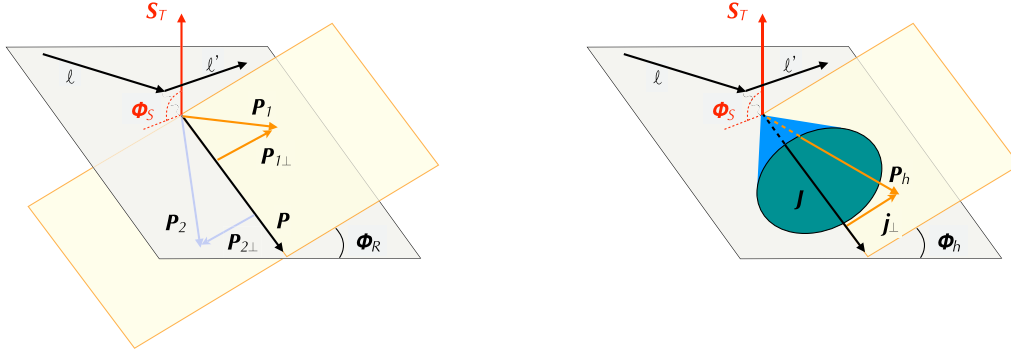


FIG. 6. Kinematics for the semi-inclusive deep-inelastic scattering of a lepton with initial momentum ℓ and final momentum ℓ' on a hadronic target with transverse polarization S_T oriented along ϕ_S with respect to the scattering plane formed by ℓ and ℓ' . The final state can be either a pair of unpolarized hadrons with momenta P_1 and P_2 with azimuthal orientation ϕ_R and total momentum $P = P_1 + P_2$ (left panel), or a hadron with momentum P_h and azimuthal orientation ϕ_h inside a jet with standard axis $\hat{J}$ (right panel). In both cases, the parallel kinematics is considered where P and $\hat{J}$ are along the momentum transfer $q = \ell - \ell'$, which identifies the $\hat{z}$ axis.

where ϕ_{j_c} and ϕ_{j_d} are the azimuthal angles with respect to the reaction plane of the transverse momenta $j_{c\perp}$, $j_{d\perp}$ of hadrons inside the jets with radius r_C and r_D , respectively. All other variables are defined in complete analogy with the single hadron-in-jet case, identifying each corresponding jet by using the labels c , C and d , D .

From Eq. (29), we deduce that the $\cos(\phi_{j_c} - \phi_{j_d})$ asymmetry in the azimuthal distribution of the two hadrons inside the two back-to-back jets is generated by two ‘‘Collins-in-jet’’ effects, one per each jet. This asymmetry allows one to isolate back-to-back jets produced by the fragmentation of back-to-back transversely polarized quarks, giving an alternative option to access the Collins function of each hadron inside the corresponding jet. Similarly and even more interestingly, extracting the $\cos(2\phi_{j_c} - 2\phi_{j_d})$ Fourier component in the azimuthal distribution allows one to isolate back-to-back jets produced by the fragmentation of back-to-back linearly polarized gluons, giving access to a new class of fragmentation functions: the $\mathcal{H}_1^{\perp g}$ describe the inclusive production of hadrons inside jets by the fragmentation of linearly polarized gluons, where the hadron transverse momentum $j_\perp$ with respect to the jet axis becomes the spin analyzer of the gluon linear polarization in the transverse plane.

B. Semi-inclusive deep-inelastic scattering up to subleading twist

The left panel of Fig. 6 describes the kinematics for the $\ell + A \rightarrow \ell' + (C_1 C_2) + X$ process, namely for the inclusive production of a hadron pair with momenta P_1 and P_2 by the scattering of a lepton with 4-momentum ℓ off a hadronic target with momentum P_A , mass M , and polarization S , leading to a final lepton with 4-momentum ℓ' . The azimuthal orientations of the transverse polarization S_T and of the hadron pair plane [represented by $R_\perp = (P_{1\perp} - P_{2\perp})/2$] are given by ϕ_S and ϕ_R , respectively, and they are all measured with respect to the scattering plane identified by ℓ and ℓ' . The hard scale of the process is given by $Q^2 = -q^2 = -(\ell - \ell')^2 \gg M^2 \geq 0$. The collinear kinematics is realized by integrating over the transverse components of the hadron-pair total 3-momentum $P = P_1 + P_2$ or, equivalently, by taking P collinear with $\hat{q}$, which identifies the $\hat{z}$ axis.

The expressions of the LO cross section for various combinations of polarization of lepton probe and proton target are listed in Eqs. (44)–(49) of Ref. [34] up to subleading twist. A slightly different notation was employed in Ref. [40], where the cross section was described in terms of structure functions, based on the analogous expression in Ref. [41]. Here, we limit ourselves to reproducing the terms that are more interesting for our discussion:

$$\begin{aligned} \frac{d\sigma}{dx dy dz_{hh} d\phi_S d\mathbf{R}_\perp d\zeta} &= \frac{2\alpha^2}{xyQ^2} \frac{y^2}{2(1-\epsilon)} \{ F_{UU,T} + \dots + |S_T| [\epsilon \sin(\phi_R + \phi_S) F_{UT}^{\sin(\phi_R + \phi_S)} + \dots] \\ &\quad + S_L [\sqrt{2\epsilon(1+\epsilon)} \sin \phi_R F_{UL}^{\sin \phi_R} + \dots] \\ &\quad + \lambda \sqrt{2\epsilon(1-\epsilon)} \sin \phi_R F_{LU}^{\sin \phi_R} + \dots + |S_T| \lambda [\dots] \}, \end{aligned} \quad (30)$$

where α is the fine structure constant, $x = Q^2/2P_A \cdot q \approx k^+/P_A^+$ is the fraction of target momentum carried by a parton with 4-momentum k and fractional charge e_q , $y = P_A \cdot q/P_A \cdot \ell \approx (E_\ell - E'_\ell)/E_\ell$ is the fraction of beam energy transferred to

the hadronic system, λ is the beam helicity, S_L is the target longitudinal polarization, ε is the ratio of longitudinal and transverse photon flux, and

$$\frac{y^2}{2(1-\varepsilon)} \approx \left(1 - y + \frac{1}{2}y^2\right), \quad \frac{y^2}{2(1-\varepsilon)} \varepsilon \approx (1-y), \quad (31)$$

$$\frac{y^2}{2(1-\varepsilon)} \sqrt{2\varepsilon(1+\varepsilon)} \approx (2-y)\sqrt{1-y}, \quad \frac{y^2}{2(1-\varepsilon)} \sqrt{2\varepsilon(1-\varepsilon)} \approx y\sqrt{1-y}. \quad (32)$$

The structure functions of interest can be written in terms of PDFs and DiFFs in the following way:

$$F_{UT}^{\sin(\phi_R+\phi_S)} = \frac{1}{4\pi} x \sum_q e_q^2 \frac{|\mathbf{R}_\perp|}{M_{hh}} h_1^q(x) H_1^{\triangleleft q}(z_{hh}, \zeta, \mathbf{R}_\perp^2), \quad (33)$$

$$F_{LU}^{\sin\phi_R} = \frac{2M}{Q} \frac{1}{4\pi} x \sum_q e_q^2 \frac{|\mathbf{R}_\perp|}{M_{hh}} \left[e^q(x) H_1^{\triangleleft q}(z_{hh}, \zeta, \mathbf{R}_\perp^2) + \frac{M_{hh}}{M} f_1^q(x) \frac{\tilde{G}^{\triangleleft q}(z_{hh}, \zeta, \mathbf{R}_\perp^2)}{z_{hh}} \right], \quad (34)$$

$$F_{UL}^{\sin\phi_R} = \frac{2M}{Q} \frac{1}{4\pi} x \sum_q e_q^2 \frac{|\mathbf{R}_\perp|}{M_{hh}} \left[h_L^q(x) H_1^{\triangleleft q}(z_{hh}, \zeta, \mathbf{R}_\perp^2) + \frac{M_{hh}}{M} g_1^q(x) \frac{\tilde{G}^{\triangleleft q}(z_{hh}, \zeta, \mathbf{R}_\perp^2)}{z_{hh}} \right]. \quad (35)$$

Equation (33) represents the standard way to address in a collinear framework the chiral-odd transversity PDF $h_1(x)$. The integral of $h_1(x)$ is the tensor charge, which might represent a possible portal to new physics beyond the Standard Model [42] since it is relevant for explorations of new possible CP -violating couplings [43] or effects induced by tensor operators not included in the Standard Model Lagrangian [44]. The tensor charge can be computed in lattice QCD with very high precision [45]. Future facilities will have a large impact on the current uncertainty on the tensor charge extracted from phenomenological studies [46–48].

Equation (34) is particularly interesting because it contains the contribution of the twist-3 chiral-odd PDF $e(x)$, which contains crucial information on quark-gluon-quark correlations (see, e.g., [49]). The integral of $e(x)$ is the scalar charge of the nucleon and is related to the so-called σ term, which plays an important role in understanding the emergence of nucleon mass from chiral symmetry breaking [50] and its decomposition in terms of contributions from quarks and gluons [51–54]. The nucleon scalar charge can be important also for the search of physics beyond the Standard Model, since it probes scalar interactions and can be relevant for dark matter searches (see, e.g., Refs. [44,55]). The nucleon scalar charge and the σ term have been computed in lattice QCD (for a review, see Ref. [56] and references therein). The $e(x)$ has been studied in several nonperturbative models of hadron structure [57–65]. It can be extracted from the TMD framework by considering the beam spin asymmetry that isolates the $d\sigma_{LU}$ cross section for inclusive single-hadron production, where the chiral-odd partner is represented by the Collins function $H_1^\perp$ [66–68]. However, this observable contains three other contributions [69–71]. Moreover, each

term is represented by an intricate convolution upon transverse momenta. Therefore, it may be more convenient to work in the collinear framework and isolate the $e(x)$ through the simple product with its chiral-odd partner represented by the DiFF $H_1^{\triangleleft}$, as shown in Eq. (34).

In order to reach this goal, we need to deal with the second contribution in Eq. (34), which depends on the unknown twist-3 DiFF $\tilde{G}^{\triangleleft}$. Calculations in the spectator model show that $\tilde{G}^{\triangleleft}$ turns out to be small, and possibly with opposite sign to $H_1^{\triangleleft}$ [72]. The extraction of $e(x)$ from CLAS and CLAS12 data projected onto the x dependence was performed assuming that the M_{hh} dependence of $\tilde{G}^{\triangleleft}$ is the same as $H_1^{\triangleleft}$ but rescaled by a constant factor [73]. A possible strategy to overcome this problem could be to study the ratio $d\sigma_{LU}/d\sigma_{UL}$ [35]. In fact, if the term proportional to $\tilde{G}^{\triangleleft}$ would be negligible, using the flavor symmetries of $H_1^{\triangleleft}$ [21–24,37,74–77] the ratio should not exhibit any dependence on (z_{hh}, M_{hh}) , since the latter should cancel out between numerator and denominator. On the contrary, any observed dependence would hint at a non-negligible contribution from twist-3 DiFF, making the extraction of $e(x)$ more challenging.

In this perspective, collecting more information on these observables from other channels should give more insight. Hence, it might be useful to consider the $\ell + A \rightarrow \ell' + (\text{Jet } C) + X$ process depicted in the right panel of Fig. 6, namely the SIDIS on a (polarized) hadron target where a hadron with momentum P_h is inclusively produced inside a jet with transverse momentum $\mathbf{j}_\perp$ with respect to the jet axis $\hat{\mathbf{j}}$ taken parallel to the $\hat{\mathbf{z}} = \hat{\mathbf{q}}$ axis. The cross section of this process has the same structure as Eq. (30). By using the correspondence of Sec. V, the structure functions in Eqs. (33)–(35) become

$$F_{UT}^{\sin(\phi_j+\phi_s)} = x \sum_q e_q^2 \frac{|\mathbf{j}_\perp|}{M_h} h_1^q(x) \mathcal{H}_1^{\perp q}(z_J, z_h, \mathbf{j}_\perp^2; Q, Qr), \quad (36)$$

$$F_{LU}^{\sin \phi_j} = \frac{2M}{Q} x \sum_q e_q^2 \frac{|\mathbf{j}_\perp|}{M_h} \left[e^q(x) \mathcal{H}_1^{\perp q}(z_J, z_h, \mathbf{j}_\perp^2; Q, Qr) + \frac{M_h}{M} f_1^q(x) \frac{\tilde{\mathcal{G}}^{\perp q}(z_J, z_h, \mathbf{j}_\perp^2; Q, Qr)}{z_h} \right], \quad (37)$$

$$F_{UL}^{\sin \phi_j} = \frac{2M}{Q} x \sum_q e_q^2 \frac{|\mathbf{j}_\perp|}{M_h} \left[h_L^q(x) \mathcal{H}_1^{\perp q}(z_J, z_h, \mathbf{j}_\perp^2; Q, Qr) + \frac{M_h}{M} g_1^q(x) \frac{\tilde{\mathcal{G}}^{\perp q}(z_J, z_h, \mathbf{j}_\perp^2; Q, Qr)}{z_h} \right], \quad (38)$$

where Qr is the typical scale of the jet with radius r . Equation (36) indicates a new way to address the collinear transversity PDF $h_1(x)$, namely through the ‘‘Collins-in-jet’’ effect in the SIDIS process. Equations (37) and (38) show that in the same framework of the SIDIS ‘‘Collins-in-jet’’ effect one can address also the twist-3 collinear PDFs $e(x)$ and $h_L(x)$, provided that the remaining contribution given by $\tilde{\mathcal{G}}^\perp$ is small. The $\tilde{\mathcal{G}}^\perp$ is a new twist-3 jTMDFF that corresponds to the above twist-3 DiFF $\tilde{\mathcal{G}}^\triangleleft$. As in the dihadron case, by using the current knowledge on the ‘‘Collins-in-jet’’ effect one could predict the dependence of the ratio $d\sigma_{LU}/d\sigma_{UL}$ on the kinematic variables of the final state. The analysis of any possible deviation of data from these predictions would indicate if the contribution of the twist-3 $\tilde{\mathcal{G}}^\perp$ would or would not be negligible.

VII. CONCLUSIONS

Transverse-momentum-dependent factorization gives the possibility of measuring many interesting signals and accessing many intriguing features of the structure of hadrons. However, one of its shortcomings is that it cannot be applied to hadronic collisions with observed hadronic final states, like, e.g., the process $A + B \rightarrow C + D + X$.

Two alternative mechanisms have been proposed to recover part of the versatility of TMDs while preserving the applicability to hadronic processes: the inclusive production of dihadrons, or of a hadron inside a jet. The inclusive production of dihadrons, namely of two hadrons originating from the fragmentation of the same parton, can be usefully studied in the collinear framework, where transverse momenta of all partons are integrated; it involves universal collinear dihadron fragmentation functions. The inclusive production of a hadron inside a jet, namely the inclusive production of a jet with a detected substructure, can be studied in a hybrid factorization approach involving collinear partonic functions in the initial state and TMD hadron-in-jet fragmentation functions in the final state.

In this paper, we have explored similarities between the two formalisms of dihadron and hadron-in-jet production, and we have established a set of correspondence rules between DiFFs and jTMDFFs. We have used this correspondence to transfer to the jTMDFF case

some interesting results obtained with DiFFs, in particular for inclusive production of two back-to-back dihadrons in unpolarized proton-proton collisions, and for inclusive production of a dihadron in semi-inclusive deep-inelastic scattering.

In unpolarized proton-proton collisions with the inclusive production of two back-to-back jets where one hadron is detected inside each jet, the cross section contains specific modulations that can distinguish if the hadron is detected inside a jet generated by a quark or by a gluon. Moreover, one of the two modulations is sensitive to a polarized jTMDFF directly linked to the TMD fragmentation function of a linearly polarized gluon.

In semi-inclusive deep-inelastic scattering, the cross section for an unpolarized lepton probe and transversely polarized proton target offers a new channel to extract the chiral-odd transversity collinear parton distribution function $h_1(x)$, which is connected to the puzzling proton tensor charge. The cross section for a longitudinally polarized lepton and unpolarized proton contains a term proportional to the chiral-odd subleading-twist collinear parton distribution function $e(x)$, which is connected to the well-known nucleon σ term and to the physics of QCD chiral symmetry breaking.

The above examples illustrate how useful the formal comparison between DiFFs and jTMDFFs can be. The inclusive production of dihadrons has already been measured in hadronic colliders [78–80], e^+e^- colliders [81], and fixed-target experiments [82–85]. The inclusive production of hadrons in a jet has been measured only in hadronic colliders [86,87]. Both channels will be (abundantly) available at the future Electron-Ion Collider [47,48]. Therefore, we think it is worth exploring the above-mentioned possibilities and to push further the analysis of the consequences of the correspondence rules set in this paper.

APPENDIX A: CROSS SECTION FOR INCLUSIVE DIHADRON PRODUCTION

In Ref. [38], Eq. (15) shows the unpolarized cross section for the process $A + B \rightarrow (C_1 C_2) + X$. After integrating over the azimuthal orientation ϕ_{S_B} of the polarization of hadron B , it reads

$$\begin{aligned}
& \frac{d\sigma_{UU}}{d\eta d|\mathbf{P}_T| d\cos\theta_C dM_{hh}^2 d\phi_R} \\
&= \frac{|\mathbf{P}_T|}{2\pi} \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{z_{hhC}^2} f_1^a(x_A) f_1^b(x_B) z_{hhC} \delta(z_{hhC} - \bar{z}_{hh}) \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} D_1^c(z_{hhC}, \cos\theta_C, M_{hh}^2) \\
&= \frac{|\mathbf{P}_T|}{2\pi s} \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{x_A x_B z_{hhC}^2} f_1^a(x_A) f_1^b(x_B) \hat{s}^2 \delta(\hat{s} + \hat{t} + \hat{u}) \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} D_1^c(z_{hhC}, \cos\theta_C, M_{hh}^2), \tag{A1}
\end{aligned}$$

where η , $|\mathbf{P}_T|$, M_{hh}^2 , and ϕ_R are defined in Sec. II A, and θ_C is the polar angle between $\mathbf{P}$ and the direction of the back-to-back emission of the two hadrons in their center-of-mass (c.m.) frame (see Fig. 3 of Ref. [18]).

It turns out that $\zeta = a + b \cos\theta_C$, with a, b functions of only the invariant mass M_{hh} [18]. Therefore, the Jacobian of the transformation is $d\zeta = 2|\mathbf{R}|/M_{hh} d\cos\theta_C$ with [38]

$$|\mathbf{R}| = \frac{1}{2} \sqrt{M_{hh}^2 - 2(M_1^2 + M_2^2) + (M_1^2 - M_2^2)^2/M_{hh}^2}. \tag{A2}$$

Using the kinematic relations in Eq. (5) and the obvious definition $\mathbf{R}_\perp = (|\mathbf{R}_\perp| \cos\phi_R, |\mathbf{R}_\perp| \sin\phi_R)$, we can compute the Jacobian of the transformation $dM_{hh}^2 d\phi_R = d\mathbf{R}_\perp 8/(1 - \zeta^2)$. The cross section in Eq. (A1) can be conveniently rewritten as

$$\frac{d\sigma_{UU}}{d\eta d|\mathbf{P}_T| d\zeta d\mathbf{R}_\perp} = \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{x_A x_B z_{hhC}^2} f_1^a(x_A) f_1^b(x_B) \frac{|\mathbf{P}_T| \hat{s}}{2\pi s} \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) D_1^c(z_{hhC}, \zeta, \mathbf{R}_\perp^2), \tag{A3}$$

where

$$D_1^c(z_{hhC}, \cos\theta_C, M_{hh}^2) = 2 \frac{|\mathbf{R}|}{M_{hh}} \frac{1 - \zeta^2}{8} D_1^c(z_{hhC}, \zeta, \mathbf{R}_\perp^2) \tag{A4}$$

takes into account the above Jacobians.

In a similar way, Eq. (16) of Ref. [38] describes the polarized cross section for the process $A + B^\uparrow \rightarrow (C_1 C_2) + X$:

$$\begin{aligned}
\frac{d\sigma_{UT}}{d\eta d|\mathbf{P}_T| d\cos\theta_C dM_{hh}^2 d\phi_R d\phi_{S_B}} &= \frac{|\mathbf{P}_T|}{4\pi^2} |S_{BT}| \sin(\phi_{S_B} - \phi_R) \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{z_{hhC}^2} f_1^a(x_A) h_1^b(x_B) z_{hhC} \delta(z_{hhC} - \bar{z}_{hh}) \\
&\times \frac{d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}}{d\hat{t}} \frac{|\mathbf{R}|}{M_{hh}} \sin\theta_C H_1^{\leftarrow c}(z_{hhC}, \cos\theta_C, M_{hh}^2) \\
&= \frac{|\mathbf{P}_T|}{4\pi^2 s} |S_{BT}| \sin(\phi_{S_B} - \phi_R) \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{x_A x_B z_{hhC}^2} f_1^a(x_A) h_1^b(x_B) \hat{s}^2 \delta(\hat{s} + \hat{t} + \hat{u}) \\
&\times \frac{d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}}{d\hat{t}} \frac{|\mathbf{R}_\perp|}{M_{hh}} H_1^{\leftarrow c}(z_{hhC}, \cos\theta_C, M_{hh}^2), \tag{A5}
\end{aligned}$$

where S_{BT} is the transverse polarization of the colliding proton with orientation ϕ_{S_B} with respect to the reaction plane, and h_1^b is the transversity distribution for the transversely polarized parton b with fractional momentum x_B . The elementary cross sections $d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}$ describe the annihilation of parton a and b with transfer of the transverse polarization of the latter to parton c while summing on the undetected fragments from parton d . All the possible independent flavor combinations are listed in the appendix of Ref. [38].

By applying the same transformation of variables from $d\cos\theta_C dM_{hh}^2 d\phi_R$ to $d\zeta d\mathbf{R}_\perp$, we get

$$\begin{aligned}
\frac{d\sigma_{UT}}{d\eta d|\mathbf{P}_T| d\zeta d\mathbf{R}_\perp d\phi_{S_B}} &= \frac{|S_{BT}|}{4\pi^2} \sin(\phi_{S_B} - \phi_R) \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_{hhC}}{x_A x_B z_{hhC}^2} f_1^a(x_A) h_1^b(x_B) \frac{|\mathbf{P}_T| \hat{s}}{s} \frac{d\Delta\hat{\sigma}_{ab^\uparrow \rightarrow c^\uparrow d}}{d\hat{t}} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \\
&\times \frac{|\mathbf{R}_\perp|}{M_{hh}} H_1^{\leftarrow c}(z_{hhC}, \zeta, \mathbf{R}_\perp^2), \tag{A6}
\end{aligned}$$

where

$$H_1^{\leq c}(z_{hhC}, \cos \theta_C, M_{hh}^2) = 2 \frac{|\mathbf{R}|}{M_{hh}} \frac{1 - \zeta^2}{8} H_1^{\leq c}(z_{hhC}, \zeta, \mathbf{R}_\perp^2). \quad (\text{A7})$$

We generalize the above formulas to the case of the inclusive production of two dihadrons. In Ref. [38], from Eqs. (20)–(22) the unpolarized cross section for the process $A + B \rightarrow (C_1 C_2)_C + (D_1 D_2)_D + X$ reads (after integrating on the polarizations of initial hadrons)

$$\begin{aligned} & \frac{d\sigma_{UU}}{d\eta_C d|\mathbf{P}_{CT}| d\cos\theta_C dM_C^2 d\phi_{R_C} d\eta_D d|\mathbf{P}_{DT}| d\cos\theta_D dM_D^2 d\phi_{R_D}} \\ &= \frac{|\mathbf{P}_{CT}||\mathbf{P}_{DT}|}{8\pi^2} \sum_{a,b} \int \frac{dx_A dx_B dz_{hhC} dz_{hhD}}{z_{hhC}^2 z_{hhD}^2} f_1^a(x_A) f_1^b(x_B) x_B \delta(x_B - \bar{x}_B) z_{hhC} \delta(z_{hhC} - \bar{z}_{hhC}) \frac{z_{hhC} z_{hhD}^2}{|\mathbf{P}_{CT}||\mathbf{P}_{DT}|} \delta(z_{hhD} - \bar{z}_{hhD}) \\ &\times \left\{ \sum_{c,d} \left[\frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} D_1^c(z_{hhC}, \cos \theta_C, M_C^2) D_1^d(z_{hhD}, \cos \theta_D, M_D^2) \right. \right. \\ &+ \cos(\phi_{R_C} - \phi_{R_D}) \frac{d\Delta\hat{\sigma}_{ab \rightarrow c^\dagger d^\dagger}}{d\hat{t}} \frac{|\mathbf{R}_C|}{M_C} \sin \theta_C H_1^{\leq c}(z_{hhC}, \cos \theta_C, M_C^2) \frac{|\mathbf{R}_D|}{M_D} \sin \theta_D H_1^{\leq d}(z_{hhD}, \cos \theta_D, M_D^2) \Big] \\ &+ \left. \cos(2\phi_{R_C} - 2\phi_{R_D}) \frac{d\Delta\hat{\sigma}_{ab \rightarrow g^\dagger g^\dagger}}{d\hat{t}} \frac{|\mathbf{R}_C|^2}{M_C^2} \sin^2 \theta_C H_1^{\leq g}(z_{hhC}, \cos \theta_C, M_C^2) \frac{|\mathbf{R}_D|^2}{M_D^2} \sin^2 \theta_D H_1^{\leq g}(z_{hhD}, \cos \theta_D, M_D^2) \right\}, \quad (\text{A8}) \end{aligned}$$

where the momenta and the angles of the second hadron pair are defined in complete analogy with the first pair by replacing the labels c, C with d, D . The additional delta functions are due to momentum conservation in the elementary $ab \rightarrow cd$ process both in the longitudinal direction of the $\hat{z}$ axis, identified with $\mathbf{P}_A$, and in the transverse plane.

In collinear kinematics, the conservation in the transverse plane is trivially $\mathbf{P}_{CT}/z_{hhC} = -\mathbf{P}_{DT}/z_{hhD}$. This implies that the above cross section is integrated in the azimuthal angles of $\mathbf{P}_{CT}$ and $\mathbf{P}_{DT}$ with the condition $\phi_C = \phi_D + \pi$ and that the moduli are constrained by [8]

$$\begin{aligned} \delta\left(\frac{|\mathbf{P}_{CT}|}{z_{hhC}} - \frac{|\mathbf{P}_{DT}|}{z_{hhD}}\right) &= \frac{z_{hhD}^2 z_{hhC}}{|\mathbf{P}_{CT}||\mathbf{P}_{DT}|} \delta(z_{hhD} - \bar{z}_{hhD}), \\ \bar{z}_{hhD} &= \frac{|\mathbf{P}_{DT}|}{\sqrt{s}} \frac{e^{\eta_C} + e^{\eta_D}}{x_A}. \end{aligned} \quad (\text{A9})$$

The conservation along the $\hat{z}$ axis in the c.m. frame of the annihilation reads $x_A P_{Az} - x_B P_{Bz} = P_{Cz}/z_{hhC} + P_{Dz}/z_{hhD}$. Using the previous delta function, after some manipulation it can be rewritten as

$$\begin{aligned} \delta\left(\eta_C + \eta_D + \log \frac{x_B}{x_A}\right) &= x_B \delta(x_B - \bar{x}_B), \\ \bar{x}_B &= x_A e^{-\eta_C} e^{-\eta_D}. \end{aligned} \quad (\text{A10})$$

Finally, the third delta function is the analog of Eq. (10). Because of Eq. (A10), it can be rewritten as

$$\begin{aligned} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) &= z_{hhC} \delta(z_{hhC} - \bar{z}_{hhC}), \\ \bar{z}_{hhC} &= \frac{|\mathbf{P}_{CT}|}{\sqrt{s}} \frac{x_A e^{-\eta_C} + x_B e^{\eta_C}}{x_A x_B} = \frac{|\mathbf{P}_{CT}|}{\sqrt{s}} \frac{e^{\eta_C} + e^{\eta_D}}{x_A}. \end{aligned} \quad (\text{A11})$$

In Eq. (A8), the elementary cross sections $d\Delta\hat{\sigma}_{ab \rightarrow c^\dagger d^\dagger}$ involve only quarks for the final partons c, d , while $d\Delta\hat{\sigma}_{ab \rightarrow g^\dagger g^\dagger}$ contain only final gluons linearly polarized in the transverse plane. Hence, the $H_1^{\leq g}$ function describes the fragmentation of such linearly polarized gluons into pairs of unpolarized hadrons. For both cases of final polarized quarks and gluons, all nonvanishing combinations are listed in the Appendix of Ref. [38].

By introducing the same transformation of variables used for the inclusive production of a single hadron pair, the cross section of Eq. (A8) can be rewritten as

$$\begin{aligned}
\frac{d\sigma_{UU}}{d\eta_C d|\mathbf{P}_{CT}| d\zeta_C d\mathbf{R}_{C\perp} d\eta_D d|\mathbf{P}_{DT}| d\zeta_D d\mathbf{R}_{D\perp}} &= \frac{|\mathbf{P}_{CT}||\mathbf{P}_{DT}|}{8\pi^2} \sum_{a,b} \int \frac{dx_A dx_B dz_{hhC} dz_{hhD}}{z_{hhC}^2 z_{hhD}^2} f_1^a(x_A) f_1^b(x_B) \\
&\times \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) \delta\left(\frac{|\mathbf{P}_{CT}|}{z_{hhC}} - \frac{|\mathbf{P}_{DT}|}{z_{hhD}}\right) \delta\left(x_A P_{Az} - x_B P_{Bz} - \frac{P_{Cz}}{z_{hhC}} - \frac{P_{Dz}}{z_{hhD}}\right) \\
&\times \left\{ \sum_{c,d} \left[\frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} D_1^c(z_{hhC}, \zeta_C, \mathbf{R}_{C\perp}^2) D_1^d(z_{hhD}, \zeta_D, \mathbf{R}_{D\perp}^2) \right. \right. \\
&+ \cos(\phi_{R_C} - \phi_{R_D}) \frac{d\Delta\hat{\sigma}_{ab \rightarrow c^\dagger d^\dagger}}{d\hat{t}} \frac{|\mathbf{R}_{C\perp}|}{M_C} H_1^{\triangleleft c}(z_{hhC}, \zeta_C, \mathbf{R}_{C\perp}^2) \frac{|\mathbf{R}_{D\perp}|}{M_D} H_1^{\triangleleft d}(z_{hhD}, \zeta_D, \mathbf{R}_{D\perp}^2) \\
&\left. + \cos(2\phi_{R_C} - 2\phi_{R_D}) \frac{d\Delta\hat{\sigma}_{ab \rightarrow g^\dagger g^\dagger}}{d\hat{t}} \frac{|\mathbf{R}_{C\perp}|^2}{M_C^2} H_1^{\triangleleft g}(z_{hhC}, \zeta_C, \mathbf{R}_{C\perp}^2) \frac{|\mathbf{R}_{D\perp}|^2}{M_D^2} H_1^{\triangleleft g}(z_{hhD}, \zeta_D, \mathbf{R}_{D\perp}^2) \right\}. \quad (\text{A12})
\end{aligned}$$

APPENDIX B: ELEMENTARY HARD CROSS SECTION FOR $a+b \rightarrow c+d$

We compare the elementary cross section used in Refs. [29,38] and in the relevant literature for inclusive production of a hadronic final state in hadron-hadron collisions.

In Refs. [88,89], the cross section for the process $A + B \rightarrow C + X$ reads

$$\begin{aligned}
\frac{E_C d\sigma}{d\mathbf{P}_C} &= \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_C}{\pi z_C^2} f_1^a(x_A) f_1^b(x_B) \\
&\times \frac{d\hat{\sigma}_{ab \rightarrow cd}}{d\hat{t}} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) D_1(z_C). \quad (\text{B1})
\end{aligned}$$

The same structure of cross section can be obtained in Ref. [90] after making the transformation of variables $\hat{v} = 1 + \hat{t}/\hat{s}$ and $\hat{w} = -\hat{u}/(\hat{s} + \hat{t})$, but obtaining the above elementary cross section multiplied by $\hat{s}$. By labeling the $d\hat{\sigma}$ of Refs. [88,89] as $d\hat{\sigma}^{\text{ToCa}}$ and the one of Ref. [90] as $d\hat{\sigma}^{\text{ACGG}}$, we get

$$\frac{d\hat{\sigma}^{\text{ACGG}}_{ab \rightarrow cd}}{d\hat{t}} = \hat{s} \frac{d\hat{\sigma}^{\text{ToCa}}_{ab \rightarrow cd}}{d\hat{t}}. \quad (\text{B2})$$

Equation (B1) can be further integrated in the angle of $\mathbf{P}_T$ and made differential in the pseudorapidity η of the final hadron through the transformation $2\pi|\mathbf{P}_T|d|\mathbf{P}_T|dP_z/E \rightarrow 2\pi|\mathbf{P}_T|d|\mathbf{P}_T|d\eta$:

$$\begin{aligned}
\frac{d\sigma}{d\eta d|\mathbf{P}_T|} &= 2|\mathbf{P}_T| \sum_{a,b,c,d} \int \frac{dx_A dx_B dz_C}{z_C^2} f_1^a(x_A) f_1^b(x_B) \\
&\times \frac{d\hat{\sigma}^{\text{ToCa}}_{ab \rightarrow cd}}{d\hat{t}} \hat{s} \delta(\hat{s} + \hat{t} + \hat{u}) D_1(z_C). \quad (\text{B3})
\end{aligned}$$

This expression is formally identical to the cross section for the $A + B \rightarrow (C_1 C_2) + X$ process of Eq. (A1) after integrating over $d\cos\theta_C, d\phi_R$ and using Eq. (10) (apart from the dependence on the dihadron invariant mass M_{hh} through the unpolarized DiFF D_1 , which does not affect the argument about the elementary cross section). By labeling the $d\hat{\sigma}$ of Eq. (A1) as $d\hat{\sigma}^{\text{DiFF}}$, we get

$$\frac{d\hat{\sigma}^{\text{ACGG}}_{ab \rightarrow cd}}{d\hat{t}} = \hat{s} \frac{d\hat{\sigma}^{\text{ToCa}}_{ab \rightarrow cd}}{d\hat{t}} = \hat{s} \frac{d\hat{\sigma}^{\text{DiFF}}_{ab \rightarrow cd}}{d\hat{t}}. \quad (\text{B4})$$

The above relation can be cross-checked by inspecting the elementary cross sections of various partonic channels in the appendix of Refs. [90,38], respectively.

By recalling the relations (9) and (10), Eq. (17) has the same structure as Eq. (B3) (apart for the dependence on the variables $(z_h, \mathbf{j}_\perp)$ describing the jet substructure through the unpolarized jTMDFF $\mathcal{D}_1$, which does not affect the argument about the elementary cross section) provided that

$$\frac{\pi z_{JC}}{\hat{s}} H_{ab \rightarrow cd}^U = \frac{d\hat{\sigma}^{\text{ToCa}}_{ab \rightarrow cd}}{d\hat{t}}. \quad (\text{B5})$$

Because of Eq. (B4), the above relation leads to Eq. (18).

- [1] T. C. Rogers and P. J. Mulders, *Phys. Rev. D* **81**, 094006 (2010).
- [2] J. Collins and J.-W. Qiu, *Phys. Rev. D* **75**, 114014 (2007).
- [3] J. C. Collins and A. Metz, *Phys. Rev. Lett.* **93**, 252001 (2004).

- [4] X. Ji, J.-P. Ma, and F. Yuan, *Phys. Rev. D* **71**, 034005 (2005).
- [5] J. C. Collins and D. E. Soper, *Nucl. Phys. B* **193**, 381 (1981).
- [6] J. C. Collins, D. E. Soper, and G. Sterman, in *Perturbative Quantum Chromodynamics*, edited by A. H. Mueller (World Scientific, Singapore, 1989).

- [7] C. J. Bomhof, P. J. Mulders, and F. Pijlman, *Phys. Lett. B* **596**, 277 (2004).
- [8] A. Bacchetta, C. J. Bomhof, P. J. Mulders, and F. Pijlman, *Phys. Rev. D* **72**, 034030 (2005).
- [9] M. G. A. Buffing, A. Mukherjee, and P. J. Mulders, *Phys. Rev. D* **86**, 074030 (2012).
- [10] M. G. A. Buffing, A. Mukherjee, and P. J. Mulders, *Phys. Rev. D* **88**, 054027 (2013).
- [11] J. C. Collins, *Nucl. Phys.* **B396**, 161 (1993).
- [12] M. Anselmino, M. Boglione, U. D'Alesio, A. Kotzinian, F. Murgia, A. Prokudin, and C. Türk, *Phys. Rev. D* **75**, 054032 (2007).
- [13] M. Anselmino, M. Boglione, U. D'Alesio, S. Melis, F. Murgia, and A. Prokudin, *Phys. Rev. D* **87**, 094019 (2013).
- [14] M. Anselmino, M. Boglione, U. D'Alesio, J. O. Gonzalez Hernandez, S. Melis, F. Murgia, and A. Prokudin, *Phys. Rev. D* **92**, 114023 (2015).
- [15] J. C. Collins, S. F. Heppelmann, and G. A. Ladinsky, *Nucl. Phys.* **B420**, 565 (1994).
- [16] R. L. Jaffe, X. Jin, and J. Tang, *Phys. Rev. Lett.* **80**, 1166 (1998).
- [17] M. Radici, R. Jakob, and A. Bianconi, *Phys. Rev. D* **65**, 074031 (2002).
- [18] A. Bacchetta and M. Radici, *Phys. Rev. D* **67**, 094002 (2003).
- [19] D. Boer, R. Jakob, and M. Radici, *Phys. Rev. D* **67**, 094003 (2003).
- [20] H. H. Matevosyan, A. Bacchetta, D. Boer, A. Courtoy, A. Kotzinian, M. Radici, and A. W. Thomas, *Phys. Rev. D* **97**, 074019 (2018).
- [21] A. Courtoy, A. Bacchetta, M. Radici, and A. Bianconi, *Phys. Rev. D* **85**, 114023 (2012).
- [22] M. Radici, A. Courtoy, A. Bacchetta, and M. Guagnelli, *J. High Energy Phys.* **05** (2015) 123.
- [23] M. Radici, A. M. Ricci, A. Bacchetta, and A. Mukherjee, *Phys. Rev. D* **94**, 034012 (2016).
- [24] M. Radici and A. Bacchetta, *Phys. Rev. Lett.* **120**, 192001 (2018).
- [25] F. Yuan, *Phys. Rev. Lett.* **100**, 032003 (2008).
- [26] M. Procura and I. W. Stewart, *Phys. Rev. D* **81**, 074009 (2010); **83**, 039902(E) (2011).
- [27] A. Jain, M. Procura, and W. J. Waalewijn, *J. High Energy Phys.* **04** (2012) 132.
- [28] Z.-B. Kang, X. Liu, F. Ringer, and H. Xing, *J. High Energy Phys.* **11** (2017) 068.
- [29] Z.-B. Kang, A. Prokudin, F. Ringer, and F. Yuan, *Phys. Lett. B* **774**, 635 (2017).
- [30] X. Liu, F. Ringer, W. Vogelsang, and F. Yuan, *Phys. Rev. Lett.* **122**, 192003 (2019).
- [31] M. Arratia, Z.-B. Kang, A. Prokudin, and F. Ringer, *Phys. Rev. D* **102**, 074015 (2020).
- [32] Z.-B. Kang, K. Lee, D. Y. Shao, and F. Zhao, *J. High Energy Phys.* **11** (2021) 005.
- [33] D. Boer, M. Diehl, R. Milner, R. Venugopalan, W. Vogelsang *et al.*, [arXiv:1108.1713](https://arxiv.org/abs/1108.1713).
- [34] A. Bacchetta and M. Radici, *Phys. Rev. D* **69**, 074026 (2004).
- [35] S. Pisano and M. Radici, *Eur. Phys. J. A* **52**, 155 (2016).
- [36] A. Bianconi, S. Boffi, R. Jakob, and M. Radici, *Phys. Rev. D* **62**, 034008 (2000).
- [37] A. Bianconi, S. Boffi, R. Jakob, and M. Radici, *Phys. Rev. D* **62**, 034009 (2000).
- [38] A. Bacchetta and M. Radici, *Phys. Rev. D* **70**, 094032 (2004).
- [39] A. Bacchetta, U. D'Alesio, M. Diehl, and C. A. Miller, *Phys. Rev. D* **70**, 117504 (2004).
- [40] S. Gliske, A. Bacchetta, and M. Radici, *Phys. Rev. D* **90**, 114027 (2014); **91**, 019902(E) (2015).
- [41] A. Bacchetta, M. Diehl, K. Goeke, A. Metz, P. J. Mulders, and M. Schlegel, *J. High Energy Phys.* **02** (2007) 093.
- [42] A. Courtoy, S. Baessler, M. Gonzalez-Alonso, and S. Liuti, *Phys. Rev. Lett.* **115**, 162001 (2015).
- [43] D. Dubbers and M. G. Schmidt, *Rev. Mod. Phys.* **83**, 1111 (2011).
- [44] T. Bhattacharya, V. Cirigliano, S. D. Cohen, A. Filipuzzi, M. Gonzalez-Alonso *et al.*, *Phys. Rev. D* **85**, 054512 (2012).
- [45] M. Constantinou *et al.*, *Prog. Part. Nucl. Phys.* **121**, 103908 (2021).
- [46] S. J. Brodsky *et al.*, *Int. J. Mod. Phys. E* **29**, 2030006 (2020).
- [47] R. Abdul Khalek *et al.*, *Nucl. Phys.* **A1026**, 122447 (2022).
- [48] R. Abdul Khalek *et al.*, [arXiv:2203.13199](https://arxiv.org/abs/2203.13199).
- [49] A. Efremov and P. Schweitzer, *J. High Energy Phys.* **08** (2003) 006.
- [50] R. L. Jaffe and X. Ji, *Phys. Rev. Lett.* **67**, 552 (1991).
- [51] X.-D. Ji, *Phys. Rev. Lett.* **74**, 1071 (1995).
- [52] X. Ji, *Nucl. Phys.* **B960**, 115181 (2020).
- [53] C. Lorcé, *Eur. Phys. J. C* **78**, 120 (2018).
- [54] C. Lorcé, A. Metz, B. Pasquini, and S. Rodini, *J. High Energy Phys.* **11** (2021) 121.
- [55] J. R. Ellis, K. A. Olive, and C. Savage, *Phys. Rev. D* **77**, 065026 (2008).
- [56] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG)), *Eur. Phys. J. C* **82**, 869 (2022).
- [57] R. Jaffe and X.-D. Ji, *Nucl. Phys.* **B375**, 527 (1992).
- [58] M. Wakamatsu, *Phys. Lett. B* **509**, 59 (2001).
- [59] P. Schweitzer, *Phys. Rev. D* **67**, 114010 (2003).
- [60] C. Cebulla, J. Ossmann, P. Schweitzer, and D. Urbano, *Acta Phys. Pol. B* **39**, 609 (2008), <https://inspirehep.net/literature/764352>.
- [61] A. Mukherjee, *Phys. Lett. B* **687**, 180 (2010).
- [62] H. Avakian, A. V. Efremov, P. Schweitzer, and F. Yuan, *Phys. Rev. D* **81**, 074035 (2010).
- [63] C. Lorcé, B. Pasquini, and P. Schweitzer, *J. High Energy Phys.* **01** (2015) 103.
- [64] B. Pasquini and S. Rodini, *Phys. Lett. B* **788**, 414 (2019).
- [65] S. Bastami, A. V. Efremov, P. Schweitzer, O. V. Teryaev, and P. Zavada, *Phys. Rev. D* **103**, 014024 (2021).
- [66] P. J. Mulders and R. D. Tangerman, *Nucl. Phys.* **B461**, 197 (1996); **B484**, 538(E) (1997).
- [67] A. V. Efremov, K. Goeke, and P. Schweitzer, *Phys. Rev. D* **67**, 114014 (2003).
- [68] H. Avakian *et al.* (CLAS Collaboration), *Phys. Rev. D* **69**, 112004 (2004).
- [69] A. Bacchetta, P. J. Mulders, and F. Pijlman, *Phys. Lett. B* **595**, 309 (2004).
- [70] F. Yuan, *Phys. Lett. B* **589**, 28 (2004).
- [71] L. P. Gamberg, D. S. Hwang, and K. A. Oganessyan, *Phys. Lett. B* **584**, 276 (2004).
- [72] W. Yang, X. Wang, Y. Yang, and Z. Lu, *Phys. Rev. D* **99**, 054003 (2019).

- [73] A. Courtoy, A. S. Miramontes, H. Avakian, M. Mirazita, and S. Pisano, *Phys. Rev. D* **106**, 014027 (2022).
- [74] A. Bacchetta and M. Radici, *Phys. Rev. D* **74**, 114007 (2006).
- [75] S. Albino *et al.*, [arXiv:0804.2021](#).
- [76] A. Bacchetta, A. Courtoy, and M. Radici, *Phys. Rev. Lett.* **107**, 012001 (2011).
- [77] A. Bacchetta, A. Courtoy, and M. Radici, *J. High Energy Phys.* **03** (2013) 119.
- [78] L. Adamczyk *et al.* (STAR Collaboration), *Phys. Rev. Lett.* **115**, 242501 (2015).
- [79] L. Adamczyk *et al.* (STAR Collaboration), *Phys. Lett. B* **780**, 332 (2018).
- [80] B. R. Pokhrel (STAR Collaboration), *SciPost Phys. Proc.* **8**, 047 (2022).
- [81] A. Vossen *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **107**, 072004 (2011).
- [82] A. Airapetian *et al.* (HERMES Collaboration), *J. High Energy Phys.* **06** (2008) 017.
- [83] C. Adolph *et al.* (COMPASS Collaboration), *Phys. Lett. B* **713**, 10 (2012).
- [84] C. Adolph *et al.* (COMPASS Collaboration), *Phys. Lett. B* **736**, 124 (2014).
- [85] C. Braun (COMPASS Collaboration), *EPJ Web Conf.* **85**, 02018 (2015).
- [86] L. Adamczyk *et al.* (STAR Collaboratio), *Phys. Rev. D* **97**, 032004 (2018).
- [87] M. Abdallah *et al.* (STAR Collaboration), *Phys. Rev. D* **106**, 072010 (2022).
- [88] U. D'Alesio and F. Murgia, *Phys. Rev. D* **70**, 074009 (2004).
- [89] M. Anselmino, M. Boglione, U. D'Alesio, E. Leader, and F. Murgia, *Phys. Rev. D* **71**, 014002 (2005).
- [90] F. Aversa, P. Chiappetta, M. Greco, and J. P. Guillet, *Nucl. Phys.* **B327**, 105 (1989).