Width effects of broad new resonances in loop observables and application to $(g-2)_\mu$

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(Received 25 November 2022; accepted 29 June 2023; published 17 July 2023)

In the phenomenology of strong interactions most physical states acquire a substantial width, and thus can only be defined in a model-independent way by pole positions and residues of the S-matrix. This information is incorporated in the Källén-Lehmann representation, whose spectral function characterizes the shape of the resonance and can be constrained by the dominant decay channels. Here, we argue that similar effects become important whenever beyond-the-Standard-Model particles possess a sizable decay width—as possible for instance in cases with a large branching fraction to a dark sector or strongly coupled scenarios—and show how their widths can be incorporated in the calculation of loop observables. As an application, we consider the anomalous magnetic moment of the muon, including both the direct effect of new physics and the possible indirect impact of a broad light Z' on $e^+e^- \to$ hadrons cross sections. Throughout, we provide results for a general spectral function and its reconstruction from the one-loop imaginary part, where the latter captures the leading two-loop effects.

DOI: 10.1103/PhysRevD.108.013005

I. INTRODUCTION

While the existence of physics beyond the Standard Model (BSM) is established by experimental observations, in particular of neutrino masses and dark matter at cosmological scales, the properties of the required new particles and interactions remain unclear. Therefore, these observations are insufficient to construct the fundamental theory superseding the SM. Anomalies in precision experiments, see, e.g., Refs. [1–3] for recent reviews, suggest certain patterns for the novel interactions, most notably the violation of lepton-flavor universality, but only the ratio of couplings over masses can be accessed and the widths of the new states remain elusive. Nonetheless, for a given fixed effect in low-energy observables, the couplings must increase with the mass, leading unavoidably to larger widths of the new states. In fact, bounds on the couplings of new particles from perturbativity and unitarity have been derived in this context [4–6]. Importantly, even below these bounds the width of the new particles is sizable and such large widths were used in several models to avoid or weaken collider bounds [7–9], in particular by decays to invisible final states [10–21]. Examples include $e^+e^- \to \mu^+\mu^- +$ invisible, $e^+e^- \to \gamma +$ invisible searches [22] as well as monophoton and monojet searches at the LHC [23–25]. Therefore, the question arises how to properly include broad width effects—in particular in strongly coupled regimes, as arise naturally in composite [26,27] or extradimensional [28–30] models—into the calculation of low-energy observables. In fact, this problem has, to the best of our knowledge, so far not been addressed in the context of BSM physics.

Here we argue that this outstanding problem can be solved via the application of the Källén-Lehmann (KL) spectral representation [31,32], which describes the general form of a time-ordered two-point function of an interacting quantum field theory. It can be derived on general grounds by inserting a complete set of states and using Lorentz invariance, with a result that makes the analytic structure of the two-point function manifest. The KL representation for a scalar particle is given by

$$\Delta_\phi(p^2) = \int_0^\infty ds\, \frac{\rho_\phi(s)}{p^2 - s + i\epsilon}, \tag{1}$$

where the propagator of the free theory with squared mass s is convolved with a general spectral density $\rho_\phi(s)$ [subject to the normalization condition $\int ds \rho_\phi(s) = 1$ and positivity constraint $\rho_\phi(s) \geq 0$]. A one-particle state corresponds to an isolated pole in $\rho_\phi(s)$, multiparticle states to a branch cut starting at the respective threshold s_{th}, and bound states to

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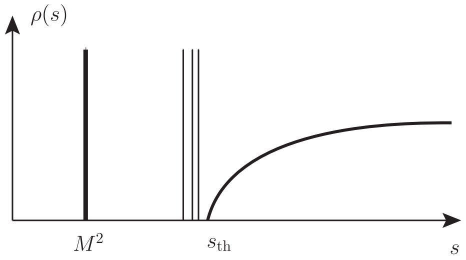

FIG. 1. Generic form of the KL spectral function $\rho(s)$. Single-particle poles show up as δ functions at the respective mass, $s = M^2$, while multiparticle states lead to a continuum that starts at some threshold $s_{\rm th}$, below which bound states are possible (indicated by the narrow lines).

additional poles below $s_{\rm th}$ (see Fig. 1). While the existence of such spectral representations is a general nonperturbative result, a practical application concerns the incorporation of width effects in the phenomenology of strong interactions, e.g., by improving Breit-Wigner (BW) parametrizations [33] through variants with good analytic properties [34–36], which can then be used in loop calculations without generating spurious imaginary parts. As an added benefit, the denominator of the KL representation can be efficiently integrated in higher-order computations as it resembles the standard Feynman propagator with $s = M^2$, so that merely the final result has to be convolved with the spectral function.

In general, once a resonance is located deep in the complex plane—the $f_0(500)$ [37–41] and the $K_0^*(700)$ [42,43] being typical examples—only a description in terms of the pole position and its residues is viable. However, for moderate widths—such as the $\rho(770)$, with $\Gamma_\rho/M_\rho \simeq 20\%$—a KL representation with spectral function derived from the main decay channels (potentially supplemental by centrifugal-barrier factors [44,45] to dampen the high-energy behavior) is often phenomenologically successful, with resonance properties close to the actual pole parameters [39,46,47]. In particular, representations of the form (1) can be used to implement resonance effects in loop observables without introducing unphysical imaginary parts below the respective thresholds [36,48].

The main point of this article is that a similar strategy can also be applied to physics beyond the SM, whenever the new particles acquire a sizable width. In general, the results are formulated in terms of a spectral function that needs to be determined from experiment—a prime example in the SM being hadronic vacuum polarization (HVP). In the absence of data, the spectral function can still be constrained from the perturbative imaginary part once a given decay channel is assumed, especially, the threshold behavior and the functional form in the vicinity of the resonance, and systematic improvements are possible by going to subleading orders in perturbation theory [49–51]. In the following, we will first present the general formalism, including the explicit form of the one-loop spectral functions for different quantum numbers of all particles involved, before turning to the application to the exemplary case of the anomalous magnetic moment of the muon.

II. SPECTRAL FUNCTIONS AND KÄLLÉN-LEHMANN REPRESENTATION

While the form of the KL representation in Eq. (1) holds in general for a scalar particle, for practical applications the spectral function is required. In principle, in the case of a broad resonance its form can be extracted experimentally, e.g., from scattering processes involving the decay products that give rise to the continuum in Fig. 1, but already for hadronic reactions a complete measurement of spectral functions is complicated. However, useful approximations to $\rho(s)$ can be obtained for instance by analytically improved versions of BW parametrizations [34–36], in which case the energy dependence can be constrained via the imaginary part of the self-energy, matching the imaginary part of Eq. (1) to the imaginary part of the resummed Dyson series. The result for the decay of a scalar ϕ with mass M, derived in Appendix A, can be written in the form

$$\rho_\phi(s) = \frac{Z}{\pi}\frac{\sqrt{s}\Gamma(s)}{(s-M^2)^2 + s[\Gamma(s)]^2},$$
$$\Gamma(s) = \sum_{K=\phi_1\phi_2, F_1F_2} \Gamma_{\phi\to K}\frac{\gamma_{\phi\to K}(s)}{\gamma_{\phi\to K}(M^2)}\theta(s-s_K), \quad (2)$$

with total width $\Gamma = \Gamma_{\phi\to\phi_1\phi_2} + \Gamma_{\phi\to F_1F_2}$. The energy dependence is described by

$$\gamma_{\phi\to\phi_1\phi_2}(s) = \frac{\lambda^{1/2}(s, M_1^2, M_2^2)}{s^{3/2}},$$
$$\gamma_{\phi\to F_1F_2}(s) = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{s^{3/2}} \times (s - m_1^2 - m_2^2 - 2\xi_\phi m_1 m_2), \quad (3)$$

with the Källén function $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$, masses M_i (m_i) for the scalar (fermionic) decay products ϕ_i (F_i), and a parameter $\xi_\phi = [(C_S^\phi)^2 - (C_P^\phi)^2]/[(C_S^\phi)^2 + (C_P^\phi)^2] \in [-1, 1]$ that describes the chirality of the couplings, see Appendix B. The thresholds are $s_{\phi_1\phi_2} = (M_1 + M_2)^2$, $s_{F_1F_2} = (m_1 + m_2)^2$. Finally, the parameter Z is to be determined from the normalization condition $\int ds \rho_\phi(s) = 1$, and Eq. (2) generalizes accordingly when further decay channels contribute. In the form of Eq. (2), mass and width of the resonance (together with branching fractions and masses of the decay products) need to be provided as input. If the width is sizable but not too large, it can also be calculated perturbatively, see Eq. (B8), in which case inserting the KL representation into a one-loop diagram captures the leading two-loop effect, and similarly at subleading orders [49–51].

For nonchiral fermions the KL representation generalizes to [52]

$$\Delta_F(p) = \int_0^\infty ds \frac{\not{p}\rho_1(s) + \rho_2(s)}{p^2 - s + i\epsilon}, \tag{4}$$

involving two spectral functions with positivity conditions for $\rho_1(s)$ and $\sqrt{s}\rho_1(s) - \rho_2(s)$ as well as the normalization $\int ds\, \rho_1(s) = 1$. The determination of the spectral function in terms of the imaginary part of the self-energy is accurate up to terms of second order $\mathcal{O}((s-m^2)^2)$ in the expansion around the resonance mass m. At the same level of precision we find that $\rho_1(s) = \rho_2(s)/\sqrt{s} \equiv \rho_F(s)$, while the generalization to chiral fermions, given in Appendix A, can potentially introduce a first-order correction. The result for a broad fermion F with decays to fermion (F') and scalar (ϕ') or vector (X') becomes

$$\rho_F(s) = \frac{Z}{\pi} \frac{m\Gamma(s)}{(s-m^2)^2 + m\sqrt{s}[\Gamma(s)]^2},$$
$$\Gamma(s) = \sum_{K=F'\phi', F'X'} \Gamma_{F\to K} \frac{\gamma_{F\to K}(s)}{\gamma_{F\to K}(m^2)} \theta(s - s_K), \tag{5}$$

with total width $\Gamma = \Gamma_{F\to F'\phi'} + \Gamma_{F\to F'X'}$, thresholds $s_{F'\phi'} = (m_{F'} + M_{\phi'})^2$, $s_{F'X'} = (m_{F'} + M_{X'})^2$, and energy dependence

$$\gamma_{F\to F'\phi'}(s) = \frac{\lambda^{1/2}(s, m_{F'}^2, M_{\phi'}^2)}{s^{3/2}} \times (s + m_{F'}^2 - M_{\phi'}^2 + 2\xi_{\phi'} m_{F'}\sqrt{s}),$$
$$\gamma_{F\to F'X'}(s) = \frac{\lambda^{1/2}(s, m_{F'}^2, M_{X'}^2)}{s^{3/2}} [\lambda(s, m_{F'}^2, M_{X'}^2) + 3M_{X'}^2(s + m_{F'}^2 - M_{X'}^2 - 2\xi_{X'} m_{F'}\sqrt{s})], \tag{6}$$

where $\xi_{\phi'} = (C_S'^2 - C_P'^2)/(C_S'^2 + C_P'^2)$, $\xi_{X'} = (C_V'^2 - C_A'^2)/(C_V'^2 + C_A'^2)$ again determine the chirality of the couplings (see Appendix B) and Z follows from the normalization $\int ds\, \rho_F(s) = 1$.

Finally, the generalization of the KL representation to the spin-1 case becomes complicated by the presence of unphysical degrees of freedom in the covariant formulation, and thus the need to specify the choice of gauge [52]. In Feynman gauge, the Goldstone part can be evaluated using Eq. (2), while the spectral function for the transverse component

$$\Delta_X^{\mu\nu}(p) = g^{\mu\nu} \int_0^\infty ds \frac{\rho_X(s)}{p^2 - s + i\epsilon} \tag{7}$$

takes the form

$$\rho_X(s) = \frac{Z}{\pi} \frac{\sqrt{s}\Gamma(s)}{(s - M_X^2)^2 + s[\Gamma(s)]^2},$$
$$\Gamma(s) = \sum_{K=\phi_1\phi_2, F_1F_2} \Gamma_{X\to K} \frac{\gamma_{X\to K}(s)}{\gamma_{X\to K}(M_X^2)} \theta(s - s_K), \tag{8}$$

with

$$\gamma_{X\to\phi_1\phi_2}(s) = \frac{\lambda^{3/2}(s, M_1^2, M_2^2)}{s^{5/2}},$$
$$\gamma_{X\to F_1F_2}(s) = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{s^{5/2}} [-\lambda(s, m_1^2, m_2^2) + 3s(s - m_1^2 - m_2^2 + 2m_1 m_2 \xi_X)], \tag{9}$$

and $\xi_X = [(C_V^X)^2 - (C_A^X)^2]/[(C_V^X)^2 + (C_A^X)^2]$.

In the application of the preceding expressions in loop integrals, one additional subtlety concerns the high-energy behavior. To ensure the required decoupling limit for large momenta, Eq. (1) needs to behave as $\Delta_\phi(p^2) \sim 1/p^2$, which reproduces the normalization condition $\int ds\, \rho_\phi(s) = 1$ as a superconvergence relation [53]. However, the exchange of limits here is delicate, and in general subtractions may be necessary [54]. Given that our derivation of the spectral functions ρ_ϕ, ρ_F, ρ_X is only accurate up to terms $\mathcal{O}((s - M^2)^2)$ in the first place, a convenient way to ensure convergence is based on the observation that multiplication by the factor

$$[\xi(s)]^n = \left(\frac{2M\sqrt{s}}{s + M^2}\right)^n = 1 + \mathcal{O}((s - M^2)^2), \tag{10}$$

changes the high-energy behavior, without affecting the resonance physics at the claimed accuracy. Similar relations have already been used in the derivation of Eq. (5), see Appendix A, so that in this case the superconvergence relation is already well defined, while for the bosonic cases $n = 1$ is sufficient to ensure convergence. Physically, the necessity to introduce modifications of the high-energy behavior as in Eq. (10) originates from the momentum factors associated with higher spins. In hadronic physics, the same effect can be achieved by adding centrifugal-barrier factors [44,45], but, in either case, ultimately the behavior of the spectral function off the resonance needs to be extracted from experiment.

The expressions presented here determine the amount of information that can be gleaned from resumming the 1-loop self-energy diagrams, most importantly, the correct threshold behavior of the spectral function. We illustrate the spectral functions for scalars and spin-1 particles for the minimal integer n required to achieve convergence in Fig. 2, but emphasize that the formalism is much more general, and becomes most powerful in cases in which the spectral function can be constrained from experiment.

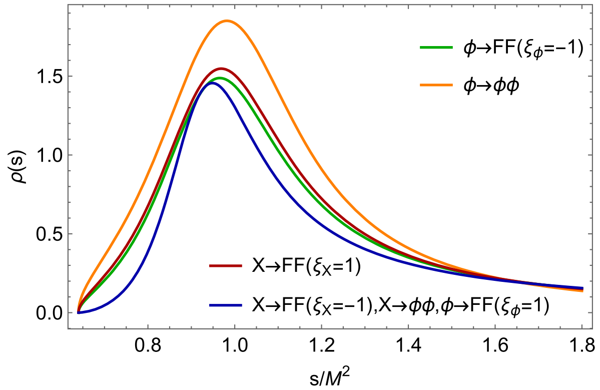

FIG. 2. Spectral functions for the decay of scalars and spin-1 particles to a pair of fermions or scalars with mass $m/M = 0.4$ and a width $\Gamma/M = 0.2$ for $\xi_{X,\phi} = \pm 1$.

III. ANOMALOUS MAGNETIC MOMENT OF THE MUON

Broad new states will also leave their imprint in low-energy precision observables. As a specific and very relevant example we consider here the anomalous magnetic moment of the muon a_μ (see, e.g., Refs. [55–67] for the analysis of generic BSM scenarios). In Appendix C we give the expressions for BSM effects in a_μ [56], generalized to the case of sizable widths using the KL representation. Depending on the size of the width such effects could significantly alter the parameter space required to explain the current 4.2σ discrepancy between experiment [68–72] and the prediction in the SM [73–97] when HVP is derived from $e^+e^- \to$ hadrons cross section data. In fact, in this data-driven evaluation a KL representation is used to implement, in a model-independent way, the effect of hadronic resonances.

The leading-order HVP contribution to a_μ can be represented by the master formula [98,99]

$$a_\mu^{\text{HVP}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{th}}}^\infty ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s),$$
$$R_{\text{had}}(s) = \frac{3s}{4\pi\alpha^2}\sigma(e^+e^- \to \text{hadrons}(+\gamma)), \tag{11}$$

where $\hat{K}(s)$ is an analytically known kernel function and the integration threshold $s_{\text{th}} = M_{\pi^0}^2$ is determined by the $\pi^0\gamma$ channel. Its derivation starts from a dispersion relation for the subtracted vacuum polarization function

$$\bar{\Pi}(k^2) = \frac{k^2}{\pi}\int_{s_{\text{th}}}^\infty ds \frac{\operatorname{Im}\Pi(s)}{s(s-k^2)}, \tag{12}$$

which amounts to a KL representation for the two-point function of two electromagnetic currents with a spectral function determined by the imaginary part

$$\operatorname{Im}\Pi(s) = -\frac{\alpha}{3}R_{\text{had}}(s). \tag{13}$$

The kernel function $\hat{K}(s)$ is obtained by performing the Feynman-parameter integral in a_μ, but the derivation via Eq. (12) also admits a spacelike master formula

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi}\int_0^1 dx(1-x)\bar{\Pi}(s_x), \qquad s_x = -\frac{x^2 m_\mu^2}{1-x}. \tag{14}$$

In this form, Eq. (14) can be formally evaluated for any polarization function $\bar{\Pi}(s)$ regardless of its analytic properties without running into obvious inconsistencies, but the numerical result can be altered dramatically. As an example, the calculation of the $\pi^0\gamma$ contribution from Ref. [100] using an asymptotic expansion misses the correct value by a factor 10, which can be traced back to the assumed form of $\bar{\Pi}(s)$ that does not fulfill the dispersion relation (12). This illustrates the importance of working with a representation of the two-point function with good analytic properties, see Appendix D for a more detailed analysis. In particular, the $\pi^0\gamma$ example shows that the mistake incurred when violating analyticity properties is not always suppressed by the width of the states, since in this case the correct imaginary part actually scales with the inverse of the small width of the ω, which ultimately produces the sizable enhancement of the $\pi^0\gamma$ contribution.

A simple example for a broad new state in a_μ is a neutral gauge boson (Z'). In particular, an $L_\mu - L_\tau$ symmetry [101–104] constitutes an anomaly-free extension of the SM and is known to be capable of explaining the tension in a_μ [105–108] while avoiding the bounds related to electrons (e.g., from $(g_2)_e$ and $e^+e^- \to e^+e^-$). Furthermore, it can serve as a portal to dark matter [109–115], which at the same time weakens collider bounds by introducing decays into invisible final states. In fact, a Z' with such a sizable invisible width was also studied in the literature as a possible solution to the $b \to s\ell^+\ell^-$ anomalies [10–21]. We therefore consider

$$\mathcal{L}_{Z'} = C_X g' \bar{\chi}\gamma^\mu \chi Z'_\mu + \text{H.c.}, \tag{15}$$

with a generic mass $M_{Z'}$, as an example to illustrate the impact of a large width.[1] As can be seen from Fig. 3, the contribution to a_μ decreases compared to the narrow-width limit, with the amount of the suppression depending on the mass of the decay products and the assumption for the high-energy behavior of the spectral function.

Another possible application concerns the determination of the HVP contribution itself, given that the global 2.1σ tension between e^+e^- data and the lattice-QCD calculation

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[1] While we are assuming that the related $U(1)'$ symmetry is broken spontaneously, the details of the scalar sector are not relevant for our purpose.

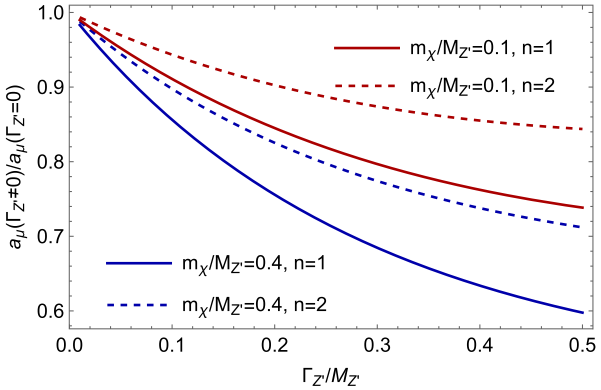

FIG. 3. Relative effect of taking into account a nonzero width, compared to the case in which it is neglected, of a Z' boson in a_μ for $M_{Z'} = 1$ GeV as a function of $\Gamma_{Z'}/M_{Z'}$ for different powers ($n = 1, 2$) in Eq. (10).

by BMWc [116] has now been confirmed at the level of about 4σ for the intermediate window in Euclidean time [117] by several lattice collaborations [118–122]. While independent lattice-QCD calculations of the entire HVP integral will require more time, some first conclusions can be drawn about the energy range in which changes to the $e^+e^- \to$ hadrons cross section would need to occur; first, the changes cannot occur too high in center-of-mass energy, otherwise serious tensions in the global electroweak fit would arise via the hadronic running of the fine-structure constant [123–126], and this condition is confirmed by direct lattice-QCD calculations of the hadronic running itself [116,127]. On the other hand, the deviation in the intermediate window shows that not all modifications can occur below 1 GeV [128], suggesting a more complicated pattern. In particular, already for the leading 2π channel large changes beyond the quoted experimental uncertainties would be required, and much larger relative changes for the subleading hadronic channels. In this situation, one could be inclined to entertain a BSM solution [129,130]. However, arguably the most promising candidate, a broad Z' interfering destructively with the SM signal, was shown to be excluded by other observables in Ref. [129], assuming a modification of the $e^+e^- \to \pi^+\pi^-$ cross section according to

$$\frac{\sigma_{\pi\pi}^{\mathrm{SM+NP}}}{\sigma_{\pi\pi}^{\mathrm{SM}}} = \left|1 + \frac{\epsilon_{Z'} s}{s - M_{Z'}^2 + iM_{Z'}\Gamma_{Z'}}\right|^2, \tag{16}$$

where $\epsilon_{Z'} = g_V'^e(g_V'^u - g_V'^d)/e^2$ collects the Z' couplings [129]. Since the interference before and after the resonance is opposite in sign (and cancels in the limit of a narrow resonance), the required net destructive interference mostly relies on the energy dependence of $\sigma_{\pi\pi}^{\mathrm{SM}}$ and the kernel function in Eq. (11). This effect can be enhanced by replacing the standard Z' propagator via a KL representation. An example is shown in Fig. 4, which illustrates how the asymmetry induced by increasing the threshold or changing the functional form above can substantially increase the change Δa_μ for the same set of couplings. We emphasize that the constraints derived in Ref. [129] remain severe (see Appendix E for the pion mass difference and Appendix F for LEP bounds), so that one would likely have to push the Z' mass beyond 1 GeV (including the effect on a variety of hadronic channels), introduce a large width, and tune the energy dependence of its spectral function to try and find a viable model.

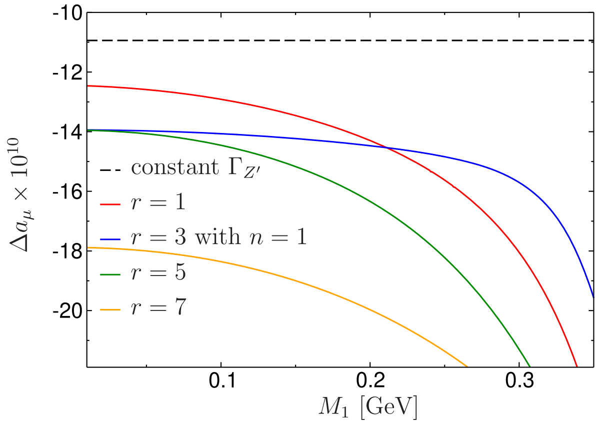

FIG. 4. Shift Δa_μ due to a BSM effect in HVP for $M_{Z'} = 0.8$ GeV, $\Gamma_{Z'} = 0.2$ GeV, and $\epsilon_{Z'} = 0.02$, as a function of M_1, for spectral functions with $\gamma_{Z'}(s) = (s - 4M_1^2)^{r/2}/s$. The case $r = 3$, see Eq. (9), requires $n = 1$ in Eq. (10), while the other variants, illustrating the impact of modifying the threshold behavior, converge for $n = 0$. The reference point for a constant width as in Eq. (16) (dashed line) should be treated with care due to the spurious imaginary parts below threshold.

IV. CONCLUSIONS

In this work we presented a general framework how sizable width effects of new resonances can be consistently and efficiently incorporated into loop calculations. In particular, we showed how the underlying Källén-Lehmann representation can be matched to the Dyson series to constrain the properties of the spectral function from the perturbatively calculated one-loop self-energy, capturing the leading two-loop effect in the case of a broad resonance. We calculated these spectral functions for several cases and showed how the calculation needs to be modified if new particles in one-loop diagrams acquire a large width, giving the general results for a_μ as an example. As a concrete application, we discussed how the effect of a broad Z' could be included both directly in the calculation of a_μ and indirectly via its impact on the $e^+e^- \to$ hadrons cross section. Our results are applicable in quite general circumstances, reducing the calculation to the narrow-width limit for general masses together with a subsequent convolution with the spectral function, and can therefore be applied to a wide class of processes.

ACKNOWLEDGMENTS

We thank Bastian Kubis, Luca Di Luzio, and Massimo Passera for helpful discussions. Financial support from the SNSF (Projects No. PP00P21_76884 and No. PCEFP2_181117) is gratefully acknowledged. This work was performed in part at the Aspen Center for Physics, which is supported by National Science Foundation Grant No. PHY-1607611.

APPENDIX A: DYSON SERIES AND KL REPRESENTATION

Writing the one-particle-irreducible self-energy of a scalar particle with bare mass M_0 as $\Sigma(p^2)$, the resummation of all self-energy insertions into a geometric series gives

$$i\Delta_D(p^2) = \frac{i}{p^2 - M_0^2 + \Sigma(p^2)}. \tag{A1}$$

To recast this resummation into a KL representation (1), one can proceed as follows: absorbing the real part of the self energy into the mass renormalization (up to quadratic corrections) by means of a Z factor defined by

$$p^2 - M_0^2 + \mathrm{Re}\,\Sigma(p^2) = Z^{-1}(p^2 - M^2) + \mathcal{O}[(p^2 - M^2)^2], \tag{A2}$$

one finds the well-known expression

$$i\Delta_D(p^2) = \frac{iZ}{p^2 - M^2 + iZ\,\mathrm{Im}\,\Sigma(p^2)} \tag{A3}$$

for the resummed propagator in terms of the on shell renormalized mass M, which is valid up to higher orders in the expansion around the pole.[2] Equation (A3) reproduces a BW parametrization [33] once the energy-dependent width $\Gamma(s)$ is identified via $Z\,\mathrm{Im}\,\Sigma(s) = \sqrt{s}\Gamma(s)$. The key idea in deriving the spectral function from such BW parametrizations is that, while the analytic properties of the resummed result (A3) are at odds with the KL representation (1), the imaginary part does provide a useful approximation as long as the resonance does not become too broad. Matching the imaginary parts then gives

$$\rho_\phi(s) = \frac{Z}{\pi}\frac{\sqrt{s}\Gamma(s)}{(s - M^2)^2 + s[\Gamma(s)]^2}, \tag{A4}$$

where Z can be determined from the normalization condition $\int ds\rho_\phi(s) = 1$. In particular, $\rho_\phi(s)$ vanishes below the first threshold s_{th} starting at which a nonvanishing imaginary part is generated in the self energy, as required by analyticity, and the correct threshold behavior is inherited from $\mathrm{Im}\,\Sigma(s)$ as well. In the limit of a narrow resonance, $\Gamma(s) = \Gamma \to 0$, Eq. (A4) collapses to a δ function,

$$\rho_\phi(s) \to \delta(s - M^2), \tag{A5}$$

where we used that the normalization determines $Z = 1$ in this case, and Eq. (1) reduces to the free propagator with mass M.

In the spin-1/2 case, the KL representation generalizes to Eq. (4). To derive expressions for the spectral functions $\rho_{1/2}(s)$ from the imaginary part of the self energy we again start from the Dyson series

$$i\Delta_D(p) = \frac{i}{\not{p} - m_0 + \not{p}\Sigma_1(p^2) + \Sigma_2(p^2)}, \tag{A6}$$

where we separated the self-energy, $\Sigma(p) = \not{p}\Sigma_1(p^2) + \Sigma_2(p^2)$ in analogy to the spectral function. Up to higher orders in the expansion around the renormalized mass m this gives

$$i\Delta_D(p) = \frac{iZ}{\not{p} - m + iZ(\not{p}\,\mathrm{Im}\,\Sigma_1(p^2) + \mathrm{Im}\,\Sigma_2(p^2))}, \tag{A7}$$

where the narrow-width limit suggests the identification of an energy-dependent width $\Gamma(p) = \not{p}\Gamma_1(p^2) + \Gamma_2(p^2)$ with $\Gamma_1(s) = 2Z\,\mathrm{Im}\,\Sigma_1(s)$, $\Gamma_2(s) = 2Z\,\mathrm{Im}\,\Sigma_2(s)$. Neglecting higher orders in the $\Gamma_i(s)$, this gives the matching relations

$$\begin{aligned}\rho_1(s) &= \frac{Z}{\pi}\frac{\frac{s+m^2}{2}\Gamma_1(s) + m\Gamma_2(s)}{(s - m^2)^2 + s[\Gamma(s)]^2},\\ \rho_2(s) &= \frac{Z}{\pi}\frac{ms\Gamma_1(s) + \frac{s+m^2}{2}\Gamma_2(s)}{(s - m^2)^2 + s[\Gamma(s)]^2},\\ s[\Gamma(s)]^2 &= \frac{s + m^2}{2}(s[\Gamma_1(s)]^2 + [\Gamma_2(s)]^2)\\ &\quad + 2ms\Gamma_1(s)\Gamma_2(s).\end{aligned} \tag{A8}$$

In particular, in the limit $s = m^2$ one can read off $\Gamma = m\Gamma_1(m^2) + \Gamma_2(m^2)$ to make the identification with the constant width of the resonance. The positivity constraints translate to

$$\begin{aligned}\frac{s + m^2}{2}\Gamma_1(s) + m\Gamma_2(s) &\geq 0,\\ \sqrt{s}\Gamma_1(s) - \Gamma_2(s) &\geq 0,\end{aligned} \tag{A9}$$

and Z can again be determined from the normalization condition for $\rho_1(s)$. For $\Gamma(s) = \Gamma \to 0$, Eq. (A8) reduces to

$$\rho_1(s) \to \delta(s - m^2), \qquad \rho_2(s) \to m\delta(s - m^2), \tag{A10}$$

where $Z = 1$ follows from the normalization of $\rho_1(s)$. In the narrow-width limit one thus has $\sqrt{s}\rho_1(s) - \rho_2(s) = 0$ and Eq. (4) reduces to the free propagator with mass m.

---

[2] At higher orders in perturbation theory additional subtleties arise in the definition of the resonance parameters, see Refs. [49–51] for the W and Z propagators in the SM.

For the case of chiral fermions, Eq. (4) generalizes to

$$\Delta_F(p) = \int_0^\infty ds \frac{\not{p}[P_R\rho_1^R(s) + P_L\rho_1^L(s)] + \rho_2(s)}{p^2 - s + i\epsilon}, \quad \text{(A11)}$$

where the spectral functions follow from the matching relations

$$\rho_1^{R/L}(s) = \frac{Z}{\pi}\frac{\frac{s+m^2}{2}\bar{\Gamma}_1(s) \mp \frac{s-m^2}{2}\Delta\Gamma_1(s) + m\Gamma_2(s)}{(s-m^2)^2 + s[\Gamma(s)]^2},$$
$$\rho_2(s) = \frac{Z}{\pi}\frac{ms\bar{\Gamma}_1(s) + \frac{s+m^2}{2}\Gamma_2(s)}{(s-m^2)^2 + s[\Gamma(s)]^2},$$
$$s[\Gamma(s)]^2 = \frac{s+m^2}{2}(s[\bar{\Gamma}_1(s)]^2 + [\Gamma_2(s)]^2) + 2ms\bar{\Gamma}_1(s)\Gamma_2(s) + \frac{s(s-m^2)}{2}[\Delta\Gamma_1(s)]^2, \quad \text{(A12)}$$

with

$$\bar{\Gamma}_1 = \frac{1}{2}(\Gamma_1^R(s) + \Gamma_1^L(s)),$$
$$\Delta\Gamma_1 = \frac{1}{2}(\Gamma_1^R(s) - \Gamma_1^L(s)), \quad \text{(A13)}$$

decomposing $\Sigma_1(s)$ and $\Gamma_1(s)$ into their left- and right-handed components accordingly. This results shows that, as long as Γ_1 is replaced by the mean of left- and right-handed contributions, the chirality difference enters always suppressed by a kinematical factor $s - m^2$, so that its effect vanishes on the resonance. For most situations with moderate widths in which a KL representation with spectral functions reconstructed from the self energy applies, it should therefore be sufficient to work with Eq. (4).

By the same argument, we can reexamine the relations (A8). If corrections that scale with $(s - m^2)^2$ are ignored, e.g.,

$$\frac{s+m^2}{2}(s[\Gamma_1(s)]^2 + [\Gamma_2(s)]^2) + 2ms\Gamma_1(s)\Gamma_2(s)$$
$$= m\sqrt{s}[\sqrt{s}\Gamma_1(s) + \Gamma_2(s)]^2 + \frac{s[\Gamma_1(s)]^2 + [\Gamma_2(s)]^2}{2(\sqrt{s}+m)^2}(s-m^2)^2, \quad \text{(A14)}$$

the spectral functions simplify to

$$\sqrt{s}\rho_1(s) = \rho_2(s) = \frac{Z}{\pi}\frac{m\sqrt{s}\Gamma(s)}{(s-m^2)^2 + m\sqrt{s}[\Gamma(s)]^2},$$
$$\Gamma(s) = \sqrt{s}\Gamma_1(s) + \Gamma_2(s), \quad \text{(A15)}$$

where again the constant width is identified as $\Gamma = \Gamma(m^2) = m\Gamma_1(m^2) + \Gamma_2(m^2)$, and both positivity constraints are automatically fulfilled as long as $\Gamma(s) \geq 0$.

APPENDIX B: SELF-ENERGIES

For the decay of a scalar particle ϕ according to

$$\mathcal{L}_\phi = A_\phi\phi\phi_1\phi_2^\dagger + \bar{F}_1(C_S^\phi + C_P^\phi\gamma_5)F_2\phi + \text{H.c.}, \quad \text{(B1)}$$

the imaginary part of the self-energy is given by

$$\text{Im}\,\Sigma^\phi = \frac{q}{8\pi\sqrt{s}}(4(C_S^\phi)^2[s - (m_1+m_2)^2] + 4(C_P^\phi)^2[s - (m_1-m_2)^2] + 2A_\phi^2), \quad \text{(B2)}$$

with center-of-mass momentum

$$q = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} \quad \text{(B3)}$$

for the decay into particles with masses m_1, m_2, and step functions $\theta(s - s_{\text{th}})$, $s_{\text{th}} = (m_1+m_2)^2$ implied in Eq. (B2) and the following. Since we are mainly interested in the energy dependence, as input for Eq. (2), we only give a minimal variant of Eq. (B2) for distinguishable complex scalars ϕ_1, ϕ_2 and Dirac fermions F_1, F_2; symmetry factors may need to be added for other cases.

For the fermion Lagrangian

$$\mathcal{L}_F = \bar{F}[C'_V\gamma^\mu + C'_A\gamma^\mu\gamma_5]F'X'_\mu + \bar{F}[C'_S + C'_P\gamma_5]F'\phi' + \text{H.c.}, \quad \text{(B4)}$$

we get

$$\text{Im}\,\bar{\Sigma}_1^F = \frac{q}{8\pi\sqrt{s}}\left(C_V'^2\left[1 + \frac{(\sqrt{s} - m_{F'})^2}{2M_{X'}^2}\right]\frac{s + m_{F'}^2 - M_{X'}^2}{s} + C_S'^2\frac{s + m_{F'}^2 - M_{\phi'}^2}{2s}\right) + \{C_V'^2 \to C_A'^2, C_S'^2 \to C_P'^2, m_{F'} \to -m_{F'}\},$$
$$\text{Im}\,\Sigma_2^F = \frac{q}{8\pi\sqrt{s}}\left(C_V'^2\left[\frac{(\sqrt{s} - m_{F'})^2}{M_{X'}^2} - 4\right] + C_S'^2\right)m_{F'} + \{C_V'^2 \to C_A'^2, C_S'^2 \to C_P'^2, m_{F'} \to -m_{F'}\}, \quad \text{(B5)}$$

where the spin-1 contribution has been evaluated in Feynman gauge together with the appropriate Goldstone-boson contributions. We checked explicitly in general R_ξ gauge that the linear combination $\sqrt{s}\,\text{Im}\,\bar{\Sigma}_1^F(s) + \text{Im}\,\Sigma_2^F(s)$, which enters in Eq. (A15), is indeed gauge invariant.

Finally, for the decay of a spin-1 particle we use the Lagrangian

$$\mathcal{L}_X = iC_X(\phi_1\partial_\mu\phi_2^\dagger - \phi_2^\dagger\partial_\mu\phi_1)X^\mu + \bar{F}_1(C_V^X\gamma^\mu + C_A^X\gamma^\mu\gamma_5)F_2X_\mu + \text{H.c.}, \quad \text{(B6)}$$

leading to

$$\operatorname{Im}\Sigma^X = \frac{q}{8\pi\sqrt{s}}\left(\frac{8}{3}q^2C_X^2 + 8m_1m_2[(C_V^X)^2-(C_A^X)^2] + 4[(C_V^X)^2+(C_A^X)^2]\left(s-m_1^2-m_2^2-\frac{4}{3}q^2\right)\right), \tag{B7}$$

where Σ^X is defined by the coefficient of the $g^{\mu\nu}$ term, as required for the calculation in Feynman gauge. The spectral function for the Goldstone-boson contribution can be derived as for the scalar case above.

For completeness, we also provide the explicit perturbative expressions for the partial decay widths as they arise in the narrow-width approximation

$$\begin{aligned}
\Gamma_{\phi\to\phi_1\phi_2} &= \frac{\lambda^{1/2}(M_\phi^2,M_1^2,M_2^2)}{8\pi M_\phi^3}A_\phi^2,\\
\Gamma_{\phi\to F_1F_2} &= \frac{\lambda^{1/2}(M_\phi^2,m_1^2,m_2^2)}{4\pi M_\phi^3}((C_S^\phi)^2[M_\phi^2-(m_1+m_2)^2]+(C_P^\phi)^2[M_\phi^2-(m_1-m_2)^2]),\\
\Gamma_{F\to F'\phi'} &= \frac{\lambda^{1/2}(m_F^2,m_{F'}^2,M_{\phi'}^2)}{16\pi m_F^3}(C_S'^2[(m_F+m_{F'})^2-M_{\phi'}^2]+C_P'^2[(m_F-m_{F'})^2-M_{\phi'}^2]),\\
\Gamma_{F\to F'X'} &= \frac{\lambda^{1/2}(m_F^2,m_{F'}^2,M_{X'}^2)}{16\pi m_F^3M_{X'}^2}(C_V'^2[\lambda(m_F^2,m_{F'}^2,M_{X'}^2)+3M_{X'}^2((m_F-m_{F'})^2-M_{\phi'}^2)]\\
&\quad + C_A'^2[\lambda(m_F^2,m_{F'}^2,M_{X'}^2)+3M_{X'}^2((m_F+m_{F'})^2-M_{\phi'}^2)]),\\
\Gamma_{X\to\phi_1\phi_2} &= \frac{\lambda^{3/2}(M_X^2,M_1^2,M_2^2)}{24\pi M_X^5}C_X^2,\\
\Gamma_{X\to F_1F_2} &= \frac{\lambda^{1/2}(M_X^2,m_1^2,m_2^2)}{12\pi M_X^5}((C_V^X)^2(2M_X^2+(m_1+m_2)^2)(M_X^2-(m_1-m_2)^2)\\
&\quad + (C_A^X)^2(2M_X^2+(m_1-m_2)^2)(M_X^2-(m_1+m_2)^2)),
\end{aligned} \tag{B8}$$

where again symmetry factors may need to be applied for cases other than distinguishable complex scalars ϕ_i and Dirac fermions F_i.

APPENDIX C: GENERAL EXPRESSIONS FOR $(g-2)_\mu$

General expressions for a_μ have been derived in Ref. [56] for new interactions of the form

$$\begin{aligned}
\mathcal{L} &= \bar{\mu}[C_V\gamma^\mu + C_A\gamma^\mu\gamma_5]FX_\mu\\
&\quad + \bar{\mu}[C_S+C_P\gamma_5]F\phi + \text{H.c.},
\end{aligned} \tag{C1}$$

including new fermions F, (pseudo)scalars ϕ, and (axial-)vectors X_μ, with masses m_F, M_ϕ, M_X, respectively. For completeness, we reproduce the result

$$\begin{aligned}
a_\mu &= \frac{-Q_Fm_\mu^2C_V^2}{4\pi^2}\int_0^1\frac{dx}{\Delta^{(1)}(x,m_F^2,M_X^2)}\left[x(1-x)\left(x+\frac{2\Delta}{m_\mu}\right)+\frac{x^2\Delta^2}{2M_X^2}\left(1-x+\frac{m_F}{m_\mu}\right)\right]\\
&\quad + \frac{Q_Xm_\mu^2C_V^2}{4\pi^2}\int_0^1\frac{dx}{\Delta^{(2)}(x,m_F^2,M_X^2)}\left[x^2\left(1-x+\frac{2\Delta}{m_\mu}\right)+\frac{x(1-x)\Delta^2}{2M_X^2}\left(x+\frac{m_F}{m_\mu}\right)\right]\\
&\quad - \frac{Q_Fm_\mu^2C_S^2}{8\pi^2}\int_0^1\frac{dx}{\Delta^{(1)}(x,m_F^2,M_\phi^2)}x^2\left(1-x+\frac{m_F}{m_\mu}\right)+\frac{Q_\phi m_\mu^2C_S^2}{8\pi^2}\int_0^1\frac{dx}{\Delta^{(2)}(x,m_F^2,M_\phi^2)}x(1-x)\left(x+\frac{m_F}{m_\mu}\right)\\
&\quad + \{C_V^2\to C_A^2, C_S^2\to C_P^2, m_F\to -m_F\},
\end{aligned} \tag{C2}$$

where

$$\begin{aligned}\Delta^{(1)}(x,m_F^2,M^2)&=m_\mu^2x^2+(m_F^2-m_\mu^2)x+M^2(1-x),\\ \Delta^{(2)}(x,m_F^2,M^2)&=m_\mu^2x^2+(M^2-m_\mu^2)x+m_F^2(1-x),\\ \Delta&=m_F-m_\mu,\end{aligned}\tag{C3}$$

and the charges Q_F, Q_X, Q_ϕ, are given in conventions in which $Q_\mu=Q_F+Q_X=Q_F+Q_\phi=-1$. In Ref. [56] these relations were derived in unitary gauge for the (axial-) vector contribution, but in the form given in Eq. (C2) it is straightforward to read off how the full result emerges from summing the spin-1 part in Feynman gauge and the additional (pseudo)scalar contribution proportional to Δ^2 (for $Q_X>0$ there are also mixed diagrams with both spin-1 and Goldstone-boson propagators). The matching of the couplings, $C_S^2=C_V^2\Delta^2/M_X^2$, can be derived independently by demanding that the tree-level μ–F scattering amplitude be gauge independent.

As a first generalization we consider the case in which the fermion width is neglected. This gives

$$\begin{aligned}a_\mu=&\frac{-Q_Fm_\mu^2C_V^2}{4\pi^2}\int ds\int_0^1\frac{dx}{\Delta^{(1)}(x,m_F^2,s)}\left[x(1-x)\left(x+\frac{2\Delta}{m_\mu}\right)\rho_X(s)+\frac{x^2\Delta^2}{2s}\left(1-x+\frac{m_F}{m_\mu}\right)\rho_G(s)\right]\\ &+\frac{Q_Xm_\mu^2C_V^2}{4\pi^2}\int ds\,dt\int_0^1dx\,L_{st}^{(2)}(x,m_F^2)\left[x\left(1-x+\frac{3\Delta}{2m_\mu}\right)\rho_X(s)\rho_X(t)+\frac{x\Delta}{2m_\mu}\rho_X(s)\rho_G(t)\right.\\ &\left.+\frac{(1-x)\Delta^2}{2\sqrt{st}}\left(x+\frac{m_F}{m_\mu}\right)\rho_G(s)\rho_G(t)\right]-\frac{Q_Fm_\mu^2C_S^2}{8\pi^2}\int ds\,\rho_\phi(s)\int_0^1\frac{dx}{\Delta^{(1)}(x,m_F^2,s)}x^2\left(1-x+\frac{m_F}{m_\mu}\right)\\ &+\frac{Q_\phi m_\mu^2C_S^2}{8\pi^2}\int ds\,dt\rho_\phi(s)\rho_\phi(t)\int_0^1dx(1-x)\left(x+\frac{m_F}{m_\mu}\right)L_{st}^{(2)}(x,m_F^2)+\{C_V^2\to C_A^2,C_S^2\to C_P^2,m_F\to-m_F\},\end{aligned}\tag{C4}$$

where

$$L_{st}^{(2)}(x,m_F^2)=\frac{\log\frac{\Delta^{(2)}(x,m_F^2,s)}{\Delta^{(2)}(x,m_F^2,t)}}{s-t},\tag{C5}$$

and ρ_X, ρ_G, ρ_ϕ are the respective spectral functions (ρ_G denotes the Goldstone-boson part). As a final step, we provide the general result when also a finite width in the fermion propagators is admitted,

$$\begin{aligned}a_\mu=&\frac{-Q_Fm_\mu C_V^2}{4\pi^2}\int ds\,dt\,du\int_0^1dx\{(1-x)L_{tu}^{(1)}(x,s)[m_\mu(x-2)\rho_1^F(t)\rho_1^F(u)+2\bar\rho_{tu}]\rho_X(s)\\ &+\frac{x\Delta_t\Delta_u}{2s}[m_\mu(1-x)\rho_1^F(t)\rho_1^F(u)+\bar\rho_{tu}]L_{tu}^{(1)}(x,s)\rho_G(s)+\frac{\Delta_t\Delta_u}{2s}\Delta\rho_{tu}[\bar\Delta_{tu}^{(1)}(s)L_{tu}^{(1)}(x,s)-x]\rho_G(s)\}\\ &+\frac{Q_Xm_\mu C_V^2}{4\pi^2}\int ds\,dt\,du\int_0^1dx\,L_{st}^{(2)}(x,u)\left\{x\left[\frac{3}{2}\rho_2^F(u)-m_\mu\left(x+\frac{1}{2}\right)\rho_1^F(u)\right]\rho_X(s)\rho_X(t)+\frac{x\Delta_u}{2}\rho_1^F(u)\rho_X(s)\rho_G(t)\right.\\ &\left.+\frac{(1-x)\Delta_u^2}{2\sqrt{st}}(m_\mu x\rho_1^F(u)+\rho_2^F(u))\rho_G(s)\rho_G(t)\right\}\\ &-\frac{Q_Fm_\mu C_S^2}{8\pi^2}\int ds\,\rho_\phi(s)\int dt\,du\int_0^1dx\{x[m_\mu(1-x)\rho_1^F(t)\rho_1^F(u)+\bar\rho_{tu}]L_{tu}^{(1)}(x,s)+\Delta\rho_{tu}[\bar\Delta_{tu}^{(1)}(s)L_{tu}^{(1)}(x,s)-x]\}\\ &+\frac{Q_\phi m_\mu C_S^2}{8\pi^2}\int ds\,dt\rho_\phi(s)\rho_\phi(t)\int du\int_0^1dx(1-x)(m_\mu x\rho_1^F(u)+\rho_2^F(u))L_{st}^{(2)}(x,u)\\ &+\{C_V^2\to C_A^2,C_S^2\to C_P^2,\rho_2^F\to-\rho_2^F\},\end{aligned}\tag{C6}$$

with fermionic spectral functions ρ_1^F, ρ_2^F, $\Delta_s=\sqrt{s}-m_\mu$, as well as

$$
\begin{aligned}
L_{tu}^{(1)}(x,s) &= \frac{\log\frac{\Delta^{(1)}(x,t,s)}{\Delta^{(1)}(x,u,s)}}{t-u},\\
\bar{\Delta}_{tu}^{(1)}(s) &= \frac{1}{2}\left(\Delta^{(1)}(x,t,s)+\Delta^{(1)}(x,u,s)\right),\\
\Delta\rho_{tu} &= \frac{\rho_1^F(t)\rho_2^F(u)-\rho_1^F(u)\rho_2^F(t)}{t-u},\\
\bar{\rho}_{tu} &= \frac{1}{2}\left(\rho_1^F(t)\rho_2^F(u)+\rho_1^F(u)\rho_2^F(t)\right). \qquad \text{(C7)}
\end{aligned}
$$

APPENDIX D: $\pi^0\gamma$ CONTRIBUTION TO HVP

In a vector-meson-dominance (VMD) picture, the cross section for $e^+e^- \to \pi^0\gamma$ can be expressed as

$$
\begin{aligned}
\sigma(e^+e^-\to\pi^0\gamma) &= \frac{2\pi^2\alpha^3}{3}\left(1-\frac{M_{\pi^0}^2}{s}\right)^3 |F_{\pi^0\gamma^*\gamma}(s,0)|^2,\\
\frac{F_{\pi^0\gamma^*\gamma}(s,0)}{F_{\pi^0\gamma^*\gamma}(0,0)} &= 1+\frac{1}{2}\sum_{V=\rho,\omega}\frac{s}{M_V^2-s-iM_V\Gamma_V},\\
F_{\pi^0\gamma^*\gamma}(0,0) &= F_{\pi\gamma\gamma} = \frac{1}{4\pi^2F_\pi}, \qquad \text{(D1)}
\end{aligned}
$$

which upon insertion into Eq. (11) produces $a_\mu^{\pi^0\gamma} \simeq 4.23\times10^{-10}$, close to the full result $a_\mu^{\pi^0\gamma}\simeq 4.38(6)\times10^{-10}$ [84]. In contrast, the asymptotic expansion from Ref. [100], also based on a VMD model for $F_{\pi^0\gamma^*\gamma}(s,0)$, gives $a_\mu^{\pi^0\gamma}\simeq 0.37\times10^{-10}$, an order of magnitude smaller than the correct result. The reason for this mismatch can be traced back to the analytic structure; the VMD model employed in Ref. [100] does provide a realistic description in the spacelike region, but it does not properly include the imaginary part of the HVP function. Setting $M_\rho = M_\omega = M_V$, the VMD model produces

$$
\begin{aligned}
\bar{\Pi}(k^2) &= -\alpha^2F_{\pi\gamma\gamma}^2M_V^4\left[\left(\frac{M_V^2}{M_V^2-k^2}\right)^2 I(k^2)-I(0)\right],\\
I(k^2) &= \int_0^1 dx\int_0^{1-x}dy\,\frac{y}{xM_{\pi^0}^2+yM_V^2-k^2x(1-x)}. \qquad \text{(D2)}
\end{aligned}
$$

Extracting the imaginary part from $I(k^2)$, this diagrammatic calculation reproduces Eq. (D1) by means of Eq. (13), but, crucially, with widths $\Gamma_V = 0$. The resulting function $\bar{\Pi}(k^2)$ as defined by the VMD model, Eq. (D2), therefore cannot be continued into the timelike region. However, good analytic properties are important for the correct evaluation of the loop integral, and it is therefore no surprise that the phenomenological value is severely underestimated. Indeed, plugging in $\bar{\Pi}(k^2)$ from Eq. (D2) into the spacelike master formula (14) gives a value of 0.41×10^{-10}, very close to the result quoted in Ref. [100]. As the derivation of Eq. (14) assumed the validity of the dispersion relation (12), this illustrates how the evaluation of loop integrals can fail if the correct analytic structure is not respected. In this case, it is ultimately the narrow width of the ω meson that leads to a sizable enhancement of the $\pi^0\gamma$ contribution.

APPENDIX E: PION MASS DIFFERENCE

A key constraint on the couplings of a Z' to the light quarks in Ref. [129] derives from the pion mass difference. In the SM, most of the difference [131–133]

$$
\Delta M_\pi^2 = M_{\pi^+}^2 - M_{\pi^0}^2 = 1.26116(13)\times10^{-3}\ \text{GeV}^2, \qquad \text{(E1)}
$$

comes from electromagnetic interactions, with strong isospin breaking suppressed by $(m_u-m_d)^2$. This QCD effect can be estimated as [134]

$$
\begin{aligned}
\Delta M_\pi^2|_{\text{QCD}} &= \frac{2l_7M_\pi^4}{F_\pi^2}\left(\frac{m_u-m_d}{m_u+m_d}\right)^2\\
&= 0.024(13)\times10^{-3}\ \text{GeV}^2, \qquad \text{(E2)}
\end{aligned}
$$

where we used $l_7 = 2.5(1.4)\times10^{-3}$ [135] (in line with Refs. [134,136]). With $M_{\pi^+}-M_{\pi^0}|_{\text{QED}} = \{4.622(95), 4.534(60\}$ MeV obtained in Refs. [137,138], respectively, we conclude that

$$
\begin{aligned}
\Delta M_\pi^2|_{Z'} &= \Delta M_\pi^2 - \Delta M_\pi^2|_{\text{QCD}} - \Delta M_\pi^2|_{\text{QED}}\\
&= \{-0.032(29), -0.008(21)\}\times10^{-3}\ \text{GeV}^2 \qquad \text{(E3)}
\end{aligned}
$$

could still originate from a Z' contribution. The biggest effect for $M_{Z'} \lesssim 1$ GeV comes from the elastic contribution, which can be evaluated within the Cottingham approach [139–147]. Following Ref. [147], one has

$$
\begin{aligned}
M_\pi^2|_{\text{QED}}^{\text{el}} &= \frac{\alpha}{8\pi}\int_0^\infty ds[F_\pi^V(-s)]^2\left(4W+\frac{s}{M_\pi^2}(W-1)\right)\\
&= 1.3(3)\times10^{-3}\ \text{GeV}^2, \qquad \text{(E4)}
\end{aligned}
$$

where $W=\sqrt{1+4M_\pi^2/s}$, i.e., the elastic part saturates the observed mass difference within uncertainties. The analogous formula for the Z' contribution reads

$$
\begin{aligned}
M_\pi^2|_{Z'}^{\text{el}} &= \frac{(g_V^{\prime u}-g_V^{\prime d})^2}{32\pi^2}\int_0^\infty ds\,\frac{s}{s+M_{Z'}^2}[F_\pi^V(-s)]^2\\
&\quad\times\left(4W+\frac{s}{M_\pi^2}(W-1)\right), \qquad \text{(E5)}
\end{aligned}
$$

notably of the opposite sign as Eq. (E3). Allowing for a Z' contamination at the level of the uncertainty in Eq. (E3) produces bounds

$$
|g_V^{\prime u}-g_V^{\prime d}| \lesssim 0.05\ldots0.08,\quad M_{Z'} = (0\ldots1)\ \text{GeV}, \qquad \text{(E6)}
$$

corroborating the estimate from Ref. [129]. The limit becomes weaker with increasing Z' mass according to the decoupling with $1/(s+M_{Z'}^2)$, but in the mass range interesting for $e^+e^- \to$ hadrons data it remains very stringent.

APPENDIX F: LEP BOUNDS FOR Z' COUPLINGS

Including, compared to Ref. [129], the contribution from the SM Z boson, we find for the modification of the $e^+e^- \to q\bar{q}$ cross section due to a light Z' at LEP [148]

$$\frac{\sigma_{qq}^{\text{SM+NP}}}{\sigma_{qq}^{\text{SM}}} = 1 + \frac{2g_V'^e g_V'^q}{e^2 Q_q} \frac{1 - \frac{g_V^e g_V^q}{4Q_q c_W^2 s_W^2}\frac{s}{s-M_Z^2}}{1 - \frac{g_V^e g_V^q}{2Q_q c_W^2 s_W^2}\frac{s}{s-M_Z^2} + \frac{[(g_V^e)^2+(g_A^e)^2][(g_V^q)^2+(g_A^q)^2]}{(4Q_q c_W^2 s_W^2)^2}\left(\frac{s}{s-M_Z^2}\right)^2}, \tag{F1}$$

where $s_W = \sin\theta_W$, $c_W = \cos\theta_W$, and the Z-boson couplings $g_{V,A}^e$, $g_{V,A}^q$ are given in the conventions of Ref. [131] (no summation over quark flavors is implied). Assuming again at most a 1% change in the cross section, the Z-boson contribution tends to weaken the limits, with $|\epsilon_{Z'}| \leq 3.3(1.7) \times 10^{-3}$ changing to $|\epsilon_{Z'}| \leq 5.1(5.9) \times 10^{-3}$ for $q = u$ $(q = d)$ at $\sqrt{s} \simeq 200$ GeV (for smaller center-of-mass energies probed at LEP the Z-boson contribution becomes more important and thus the limit weaker). Moreover, the measured cross sections are the sum of $q = u, d, s, c, b$, and therefore the limit is further diluted if the Z' does not couple in a flavor-universal way, by a factor of 5 if only the coupling to a single flavor is assumed.

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