Letter

Supersymmetric black holes and $T\bar{T}$ deformation

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The entropy of supersymmetric black holes in string theory compactifications can be related to that of a D- or M-brane system, which in many cases can be further reduced to a two-dimensional conformal field theory (2D CFT). For black holes in M theory, this relation involves a decoupling limit where the black hole mass diverges. We suggest that moving away from this limit corresponds to a specific irrelevant perturbation of the 2D CFT, namely the supersymmetric completion of the $T\bar{T}$ deformation. We demonstrate that the black hole mass matches precisely with the $T\bar{T}$ deformed energy levels, upon identifying the $T\bar{T}$ deformation parameter with the inverse of the leading term of the black hole mass. We discuss various implications of this novel realization of the $T\bar{T}$ deformation, including a Hagedorn temperature for wrapped M5-branes, and potential change of degeneracies in the deformed theory.

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I. INTRODUCTION

Black holes are known to host an enormous amount of entropy, known as Bekenstein-Hawking entropy $S_{\mathrm{BH}} = \frac{A}{4\ell_P^2}$, where A is the area of the event horizon and ℓ_P is the Planck length [1–5]. This formula is particularly striking since the black hole entropy scales as area rather than volume. This intriguing observation has given rise to the idea of holography [6,7]. Another remarkable feature is the appearance of the Planck length, the characteristic length scale of quantum gravity, although the entropy formula was derived in the realm of classical gravity. This suggests that understanding of black hole entropy is key to the understanding of quantum gravity. The immediate puzzle is that black hole microstates cannot be distinguished and enumerated within classical gravity.

For supersymmetric black holes, one can overcome this stumbling block, by utilizing the fact that the degeneracy, or index to be precise [8–10], is locally constant as a function of continuous parameters of the theory. For black holes in string theory, a decoupling limit in which Newton's constant becomes small leads to a complementary description in terms of D-branes, often described by a two-dimensional conformal field theory (CFT). Microstates have a clear meaning in this picture and their counting reproduces S_{BH} correctly [11,12]. It is an interesting question to extend the microscopic description away from the limit.

In this article, we consider four-dimensional compactifications of string theory with $\mathcal{N} = 2$ supersymmetry. This theory gives rise supersymmetric black holes, that is to say half-Bogomol'nyi-Prasad-Sommerfield (BPS) states, since they preserve four out of the eight supercharges. There is a great variety of these states, with intricate dependence on asymptotic values of the vector multiplet scalars, which parametrize the Kähler moduli space [13–23]. We are interested in a family of black holes whose degrees of freedom reduce to a 2D CFT, more precisely the Maldacena-Strominger-Witten (MSW) CFT [12,24], in a decoupling limit of the Kähler moduli space. This limit corresponds to the "infinite volume limit" in string units, $|J| \to \infty$ with J being the real Kähler modulus. Newton's constant G_4 vanishes in this limit.

For this family of black holes, moving to finite mass coincides with moving away from the infinite volume limit. In this article, we argue that moving to the "finite volume regime" of a family of half-BPS black holes in $\mathcal{N} = 2$ theory is captured by the $T\bar{T}$ deformation of the MSW CFT. This deformation is an irrelevant deformation constructed in terms of the energy-momentum tensor [25]. Even though irrelevant, the theory remains solvable, and the degeneracies do not lift under this deformation. Our main argument, that moving to finite volume corresponds to the $T\bar{T}$ deformation, is that the expression for the black hole masses matches with the formula for $T\bar{T}$ deformed energy levels.

II. D4-BRANE BLACK HOLES AND CFT

Extremal black holes in $\mathcal{N}=2$ supergravity are characterized by their magnetic and electric charges $\gamma = (P^0, P^a, Q_a, Q_0)$, $a = 1, \ldots, n_v$, where n_v is the number of vector multiplets. Such black holes preserve four out of total eight supercharges. By the attractor mechanism for extremal black holes, irrespective of their values at spatial infinity, the scalars $t^a \coloneqq \frac{Y^a}{Y^0}$ flow to their "attractor values" t^a_γ at the black hole horizon [26]. For the family of black holes with $P^0 = 0$, the Bekenstein-Hawking entropy of the black hole is $S_{\mathrm{BH}} = \pi\sqrt{\frac{2}{3}P^3\hat{Q}_{\bar{0}}}$ [27], where $P^3 = d_{abc}P^aP^bP^c$, with d_{abc} the 3-tensor of the tree-level prepotential $F(Y) = \frac{d_{abc}Y^aY^bY^c}{6Y^0}$, with Y^a's being the vector multiplet scalars. Moreover, $\hat{Q}_{\bar{0}} = -Q_0 + \frac{1}{2}d^{ab}Q_aQ_b$, with d^{ab} being the inverse of the quadratic form $d_{ab} = d_{abc}P^c$. The signature of d_{ab} is $(1, b_2 - 1)$.

$\mathcal{N}=2$ supergravity can in turn be realized as a low-energy effective description of string theory, compactified on a Calabi-Yau threefold X. In type IIA string theory description, the charge γ is carried by D6-D4-D2-D0-branes, wrapped around appropriate cycles of X. Number of vector multiplets n_v equals the second Betti number b_2 of X. In type IIB string theory, these states manifest themselves in hypermultiplet geometry [28–30].

In the large volume limit, the Arnowitt-Deser-Misner mass $M_\gamma(t)$ of a D4-brane black hole carrying charge $\gamma = (0, P, Q, Q_0)$ equals $|Z(\gamma, t)|$ with $Z(\gamma, t)$ the tree-level holomorphic central charge, $Z(\gamma, t) = -\frac{1}{2\ell_s^5}Pt^2 + \frac{1}{\ell_s^3}Qt - \frac{Q_0}{\ell_s}$, where ℓ_s is the string length and $t^a = \frac{Y^a}{Y^0} = B^a + iJ^a$ are the complexified Kähler moduli fields. We will also use $t^a = B^a + i\lambda\underline{J}^a$, with $\underline{J}$ the normalized Kähler modulus, $P\underline{J}^2 = \ell_s^4$. Thus $\lambda \sim (V_X/\ell_s^6)^{1/3}$ is a dimensionless measure for the volume of 2-cycles in string units. The infinite volume limit then corresponds to $\lambda \to \infty$. We will set $\ell_s = 1$, unless otherwise mentioned. The (renormalized) rest mass $M_\gamma(t)$ simplifies in the infinite volume limit $\lambda \to \infty$ [19,31–34], rendering various supergravity partition functions amenable to analytics. Note the black hole mass $M_\gamma(t)$ diverges in this limit.

For vanishing D6-brane charge, type IIA string theory can be further uplifted to M theory, by introducing the M-theory circle S^1_M with radius $R = g_s\ell_s/2\pi$, where g_s is the string coupling. In M theory the D4-brane is elevated to a M5-brane wrapping the 4-cycle P times the M-theory circle S^1_M. D2-brane charges are realized as fluxes in the world volume of the M5-brane, whereas the D0-charge becomes the momentum along S^1_M. The M5-brane dynamics is captured by MSW CFT in the decoupling limit [12,24]. In this limit, the 11-dimensional Planck length vanishes, $\ell_{11} = g_s^{1/3}\ell_s \to 0$, such that

$$R/\ell_{11} \to \infty, \qquad V_X/\ell_{11}^6 \ \text{fixed but large.} \tag{1}$$

The second quantity is fixed since it corresponds to a hypermultiplet scalar.

The MSW CFT has (4,0) supersymmetry, that is to say four chiral supercharges, which matches the number of preserved supersymmetries of the black hole. The bosons of the theory comprise three nonchiral scalars corresponding to movement of the brane system along transverse $\mathbb{R}^3$. In addition, there are nonchiral real scalars describing movement of the D4-brane inside X, as well as left-moving and right-moving scalars descending from the M5-brane world volume self-dual 3-form field strength. The bosons and their fermionic partners can be arranged in (4,0) supermultiplets, whose numbers depend on Calabi-Yau data as well as the divisor. The four supersymmetries broken by the brane configuration lead to a universal Goldstino supermultiplet [24]. The left and right central charges arising from field content are $c_L = P^3 + \frac{1}{2}c_2.P$ and $c_R = P^3 + c_2.P$, with c_2 the second Chern class of X. The combination $\hat{Q}_{\bar{0}}$ is bounded below by $-c_L/24$ with c_L the central charge of the left-moving degrees of freedom. The Cardy formula for the left movers reproduces the one-loop corrected Bekenstein-Hawking entropy [12].

On the gravity side, D4-brane black holes develop AdS_3 throats [34] dual to the MSW CFT, after uplifting from four to five dimensions [35,36] and then taking the decoupling limit (1). Since $\lambda \sim V_X^{1/3}/\ell_s^2 = (V_X/\ell_{11}^6)^{1/3}(R/\ell_{11})$, the limit (1) is equivalent to the infinite volume limit in string units, $\lambda \to \infty$.

An important aspect which complicates the state counting of single-center black holes are multicenter black holes [15,19,37,38]. Their low-energy dynamics is captured by $\mathcal{N}=4$ quiver quantum mechanics [37]. Five-dimensional multicenter solutions with centers carrying nonvanishing D6-brane charge have a distance scale $\sim\ell_5^3/R^2 \sim R/\lambda^3$, whereas those with vanishing D6-brane charge have a distance scale $\sim\ell_5$, with $\ell_5 = \ell_{11}^3/(4\pi V_X^{1/3})$ the five-dimensional Planck length. Upon appropriate coordinate redefinition, configurations with distance scale $\sim\ell_5^3/R^2$ go over to a single AdS_3 throat in this limit. On the other hand, multicenter solutions with distance scale $\sim\ell_5$ form multiple throats [34]. Scaling black holes [19,39–41] with centers carrying vanishing D6-brane charges present an intriguing case, since these can approach each other arbitrarily close. In particular taking their mutual distance to scale as ℓ_5^3, they are seen to merge in a single AdS_3 throat [34]. Upon dimensional reduction, this AdS_3 throat reduces to an AdS_2, which can also be seen to arise in the near coincident regime of scaling black holes in four dimensions [42,43]. The black holes merging in a single AdS_3 throat have been argued to be captured by the

CFT [34]. In terms of the Kähler moduli, the 4D supergravity states which correspond to the CFT are those which exist at the infinite volume attractor point [30,34], $t_\gamma^\infty = \lim_{\lambda\to\infty} t_\gamma^\lambda$, with

$$(t_\gamma^\lambda)^a = d^{ab}Q_b\ell_s^2 + i\lambda\ell_s^2\frac{P^a}{\sqrt{P^3}}. \tag{2}$$

Then $\lambda = (p^3)^{1/6}(6^{1/3}/2)R/\ell_5$. Note that the attractor value t_γ^λ (2) differs for different D2-brane charge, even for states within the same CFT.

The ground state of the CFT corresponds to the "bare" D4-brane, whereas excited states carry additional D0- and D2-brane charge. The energies of excitations of the CFT correspond to the infinite volume limit of the D4-brane mass, renormalized by subtracting the leading term $PJ^2/2$. For $B = 0$, this gives for the CFT energy E_γ and momentum Π_γ

$$\begin{aligned} RE_\gamma &= \lim_{|J|\to\infty}\ell_s\left(M_\gamma(t) - \frac{1}{2}PJ^2/\ell_s^5\right) \\ &= -Q_0 + \frac{(Q.J)^2}{PJ^2}, \\ R\Pi_\gamma &= Q_0. \end{aligned} \tag{3}$$

The expression for $B \neq 0$ is invariant under translations in the electric-magnetic duality group $\mathrm{Sp}(2+2b_2,\mathbb{Z})$. Note Q_0 is the momentum along the M-theory circle S_M^1. Equation (3) in turn implies for the Virasoro operators $L_0 = \frac{(Q.J)^2}{2PJ^2} + \frac{c_L}{24}$ and $\bar{L}_0 = -Q_0 + \frac{(Q.J)^2}{2PJ^2} + \frac{c_R}{24}$. L_0 saturates the BPS bound for half-BPS states, which preserve four fermionic symmetries.

Altogether, one has the CFT partition function [19,44,45]

$$\begin{aligned} \mathcal{Z}_{\mathrm{CFT}}(\tau,\bar{\tau}) = &\sum_{Q_0,Q}\Omega(\gamma;t_\gamma^\infty)q^{(E_\gamma+\Pi_\gamma)/2}\bar{q}^{(E_\gamma-\Pi_\gamma)/2} \\ &\times\int d^3\vec{p}\,e^{-\beta\frac{\vec{p}^2}{2m_5}}, \end{aligned} \tag{4}$$

where $m_5 = \frac{\pi PJ^2}{g_s\ell_s^5}$ is the mass of the wrapped MSW string and $\vec{p}$ is the momenta in $\mathbb{R}^3$. The coefficient $\Omega(\gamma;t_\gamma^\infty)$ is the appropriate (rational) BPS index [21,30]; it is a specialization of $\Omega(\gamma;t)$ which is independent of hypermultiplet scalars while only locally constant as a function of the vector multiplet scalars through its dependence on t. Moreover, $q = e^{2\pi i\tau}$ with $\tau = C_0 + i\frac{\beta}{\ell_s g_s}$ the modular parameter of the torus $S_M^1\times S_\beta^1$, where S_β^1 is the thermal circle and $\tau_2 \coloneqq \mathrm{Im}(\tau)$ is the ratio of circumferences of S_β^1 and S_M^1. Similarly, $\tau_1 \coloneqq \mathrm{Re}(\tau)$ is related to the IIA Ramond-Ramond 1-form C_1 as $C_1 = C_0\frac{dt}{\beta}$ and describes the tilt of S_β^1 with respect to S_M^1. We suppress further nonholomorphic contributions related to mock modular forms [30,32,46], which are not relevant for the present discussion.

III. MODULARITY

The MSW CFT is invariant under large reparametrizations of the torus, that is to say modular transformations: $\tau \to \frac{a\tau+b}{c\tau+d}$, $a,b,c,d\in\mathbb{Z}$, $ad-bc=1$. The modular weight of $\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar{\tau})$ is (2,0), since $\Omega(\gamma;t)$ is the second helicity supertrace. The integral on the second line in Eq. (4) is the integral over the three momenta, which evaluates to a factor $(\frac{2\pi^2PJ^2}{g_s^2\tau_2})^{3/2} = (\frac{2\pi^2PJ^2}{\beta g_s})^{3/2}$. This factor has modular weight $(\frac{3}{2},\frac{3}{2})$, since J has weight $(\frac{1}{2},\frac{1}{2})$ and the area of the torus βg_s is modular invariant. The spectrum allows for a theta function decomposition of $\mathcal{Z}_{\mathrm{CFT}}$, such that the degeneracies are enumerated by a weakly holomorphic vector-valued modular form of weight $-1-b_2/2$ [19,44,45]. These functions can be explicitly determined in special cases [45,47–50].

Compactifying M theory on S_M^1 and further applying T duality along the S_β^1, one arrives at type IIB string theory. This theory exhibits an $SL(2,\mathbb{Z})$ S-duality group. In this context, the modularity of the CFT partition function and S duality reinforce each other in a nontrivial way [28,30,32].

IV. D4-BRANE BLACK HOLES AT FINITE VOLUME

Let us now turn to black holes with finite λ. The mass $M_\gamma(t)$ can be expressed in terms of E_γ and Π_γ (3) as

$$M_\gamma(t) = \frac{1}{\ell_s}\sqrt{\left(\frac{1}{2}PJ^2/\ell_s^4\right)^2 + (PJ^2/\ell_s^4)RE_\gamma + R^2\Pi_\gamma^2}. \tag{5}$$

Next, we introduce the energy $\mathcal{E}_\gamma(t)$ through

$$R\mathcal{E}_\gamma(t) = \ell_s M_\gamma(t) - \frac{1}{2}PJ^2/\ell_s^4 \tag{6}$$

for finite PJ^2. In the infinite volume limit, $\mathcal{E}_\gamma$ simply reduces to E_γ (3), the energy levels of the MSW CFT. Thus for finite J, we expect $\mathcal{E}_\gamma$ to correspond to the energy spectra of the microscopic theory describing attractor black holes in the finite volume regime.

Remarkably with (5) substituted, $\mathcal{E}_\gamma(t)$ (6) is precisely of the form of the energy levels of the $T\bar{T}$ deformation of a two-dimensional CFT. This motivates us to propose that at finite volume, the microscopic description of D4-brane black holes is furnished by $T\bar{T}$ deformation of MSW CFT. We provide further justifications for the proposal in remainder of this article.

Until now we have worked in the rest frame of the black hole. To include the space-time momenta in the deformed theory, recall that the rest mass $\frac{2\pi}{g_s}M_\gamma$ (as in the conventions of Ref. [19]) is replaced by $\sqrt{(\frac{2\pi}{g_s}|Z|)^2+\vec{p}^2}$ in a moving frame. Equation (6) is then generalized to

$$\mathcal{H}_\gamma(t) = -\frac{1}{2}PJ^2 + \sqrt{\left(\frac{1}{2}PJ^2\right)^2 + PJ^2E_\gamma + \Pi_\gamma^2 + \left(\frac{g_s}{2\pi}\right)^2\vec{p}^2}. \tag{7}$$

This reduces in the large volume limit to $E_\gamma + \vec{p}^2/2m_5$ (4).

V. $T\bar{T}$ DEFORMATION

In a seminal paper [25], Zamolodchikov showed that for a two-dimensional quantum field theory, the composite operator $T\bar{T}$ is free from short-distance divergences, even though this operator is an irrelevant operator modifying the ultraviolet behavior of the theory. Recent years have seen a flurry of activities in this topic [51–92].

The action of S^μ of the $T\bar{T}$ deformation of conformal field theory satisfies $dS^{(\mu)}/d\mu = \int dz d\bar{z} T^{(\mu)}\bar{T}^{(\mu)}$, with the stress energy tensors those of the deformed theory with action $S^{(\mu)}$. This universal deformation is also known as the double-trace $T\bar{T}$ deformation to distinguish the deformation from a similar but different single-trace deformation which can be introduced for a special class of CFTs [51–54,56,72,73]. Remarkably, the energy levels $E_n(R,\mu)$ of the deformed theory can be determined exactly [93,94] in terms of the momenta $P_n(R)$ and undeformed energy levels $E_n(R,0)$:

$$E_n(R,\mu) = -\frac{R}{2\mu} + \left[\frac{R^2}{4\mu^2} + \frac{R}{\mu}E_n(R,0) + P_n(R)^2\right]^{1/2}, \tag{8}$$

where R is the radius of the compact spatial dimension. The sign of μ gives rise to qualitatively different behavior. For $\mu < 0$, $E_n(R,\mu)$ becomes imaginary for large $E_n(R,0)$, whereas for $\mu > 0$ $E_n(R,\mu)$ becomes imaginary for low-level states. We will be concerned with positive μ, or the "right sign," which changes the UV dynamics of the theory. This is reflected in high-energy density of states, which exhibits Hagedorn growth with Hagedorn temperature $T_H \sim 1/\sqrt{\mu}$, as opposed to Cardy growth.

The deformed energy levels satisfy an inviscid Burgers equation, which exhibits shock singularities. This is related to the singularity which arises for $E_n(R,0) < 0$, that is to say the low-lying states in a CFT. For such states, $E_n(R,\mu)$ becomes imaginary for sufficiently large μ. At the crossover, the expression under the square root in Eq. (8) vanishes.

Now let us consider the $T\bar{T}$ deformation of the MSW CFT. Then Eq. (8) demonstrates that the deformed energy levels equal $\mathcal{E}_\gamma(t)$, with the identification

$$R^2/\mu \leftrightarrow PJ^2/\ell_s^4. \tag{9}$$

This suggests that at finite ℓ_{11}, the M5-brane degrees of freedom correspond to a $T\bar{T}$ deformed CFT. The identification demonstrates that μ scales as ℓ_5^2. Moving away from the infinite volume attractor point (2) is naturally an irrelevant deformation. This can be analyzed in detail for the D1–D5 system with (4,4) supersymmetry in two dimensions [95].

The shock waves mentioned above are in the black hole context related to wall-crossing phenomena. The crossover happens when $M_\gamma(t)$ vanishes; i.e. BPS states become massless. This situation can only arise for polar states, i.e. states with $\hat{Q}_{\bar{0}} < 0$. These states are realized as $D6-\overline{D6}$ bound states and they decay across a wall of marginal stability, before reaching the massless point [19].

For various free theories, even the action of the deformed theory can be determined exactly [94,96] and has the form of nonlocal Dirac-Born-Infeld (DBI) action, which arises as effective action for D-branes. Although D-branes spontaneously break some of the supersymmetry, the DBI action realizes the full Poincaré symmetry, albeit in a nonlinear fashion.

VI. SUPERSYMMETRY

Irrespective of the moduli, half-BPS black holes in $\mathcal{N}=2$ string theory preserve four supercharges, which must also be the case for their microscopic description. This works out nicely in the infinite volume limit, since the MSW CFT has (4,0) supersymmetry. As per our proposal, this requires for finite volume regime (4,0) supersymmetry of the $T\bar{T}$ deformed theory. We expect that such a supersymmetric completion of the $T\bar{T}$ deformation can be derived, as was obtained earlier for theories with (1,1) and (1,0) supersymmetries [70] and (2,2) and (0,2) supersymmetries [68,69]. The supersymmetric completion of the $T\bar{T}$ deformation for the MSW CFT will be left invariant by the SO(4) R symmetry.

Along with four preserved supercharges, there are four broken supersymmetries leading to Goldstinos, which realize supersymmetry in a nonlinear fashion. This is evident in MSW CFT [12] and is expected to continue after $T\bar{T}$ deformation. It is encouraging that for free supersymmetric seed theory, the $T\bar{T}$ deformed theory is known to realize supersymmetry in a nonlinear fashion [97,98].

VII. MODULARITY REVISITED

It has been established that the $T\bar{T}$ deformation preserves modularity of the partition function [88,99], which has modular weight (0,0) if the dimensionless deformation parameter μ/R^2 has weight $(-1,-1)$:

$$\frac{\mu}{R^2} \to \frac{1}{|c\tau + d|^2} \frac{\mu}{R^2}. \tag{10}$$

Clearly, the identification of the deformation parameter (9) in our model agrees with this transformation, implying that the partition function of the modified MSW CFT is also modular invariant. On the other hand, since the weight of the elliptic genus is nontrivial, the relation between the deformed and undeformed elliptic genus is more complicated [100]. The type IIB perspective [28,29] can be relevant for this question, since it is valid for finite BPS mass.

VIII. HOLOGRAPHY

An important question is the holographic dual of the $T\bar{T}$ deformed CFT, which has been addressed in various papers [52,65,72,73,75,80,101]. As a first step toward holography for the model discussed above, we present the metric of the five-dimensional uplift of a single-centered black hole at the large volume attractor point (2) and including the dependence on ℓ_5:

$$\frac{1}{\ell_5^2} ds_{5d}^2 = \mathcal{N}^{-1}\left[-\frac{(\rho^2-\rho_*^2)^2}{\rho^2} dt^2 + \rho^2\left(d\alpha + \frac{\rho_*^2}{\rho^2} dt\right)^2\right] + \mathcal{N}^2\left[\frac{4U^2\rho^2 d\rho^2}{(\rho^2-\rho_*^2)^2} + U^2 d\Omega_2^2\right], \tag{11}$$

with $\mathcal{N} = 1 + \ell_5^2(\rho^2 - \rho_*^2)$, $\rho_*^2 \coloneqq -\frac{4\hat{Q}_0}{UR^2}$, and $U^3 = P^3/6$. The 4D radial coordinate r is in terms of these variables $r = \ell_5^3 U(\rho^2 - \rho_*^2)$. We have seen above that ℓ_5^2 scales as $T\bar{T}$ deformation μ in the decoupling limit (1). Indeed in this limit, the (t, α, ρ) coordinates of the metric (11) parametrize a Bañados-Teitelboim-Zanelli (BTZ) black hole [see Eq. (4.4) in Ref. [34]].

As per UV-IR correspondence in holography [102], the bulk asymptotic region and deep interior are related to the ultraviolet and infrared, respectively, of the boundary theory. Since the $T\bar{T}$ deformation is irrelevant, we expect the bulk asymptotics of (11) to change yet the metric near the BTZ singularity to remain unchanged. Indeed, the ℓ_5^2 deformation becomes negligible near the BTZ singularity for the "infrared" limit $\rho \to \rho_*$, whereas in the "ultraviolet" or asymptotic region $\rho \to \infty$ the geometry asymptotes to $\mathbb{R}^2 \times S^1$.

We naturally expect that the holographic description of the $T\bar{T}$ deformed model is in terms of metrics which have the same asymptotics as (11) for $\rho \to \infty$. An important complication for this analysis stems from the fact that even small ℓ_5 effects cannot be treated as perturbation. For example, if (11) is expanded to $\mathcal{O}(\ell_5^2)$, then the resultant metric has wrong signature in the asymptotic region $\rho^2 > \ell_5^{-2} + \rho_*^2$. A resolution may be to put an outward cutoff, reversely to the inward cutoff for the $\mu < 0$ deformation [75,80]. In fact, the closest analog of asymptotic AdS_3 is the crossover region $\rho_*^2 \ll \rho^2 \ll 1/\ell_5^2$, which exists whenever $\ell_5^2 \ll 1/\rho_*^2$ and might be important for holography for $\ell_5 \neq 0$. We leave further study for future work.

IX. HAGEDORN TRANSITION

Keeping with the Cardy formula [103], the degeneracies $\Omega(\gamma; t_\gamma^\infty)$ grow as exponential of $\pi\sqrt{\frac{2}{3}P^3\hat{Q}_{\bar{0}}}$ for large $\hat{Q}_{\bar{0}}$. In general, E_γ is bounded below by $\hat{Q}_{\bar{0}}$, which is positive in the Cardy regime, while in the limit of large D2-brane charge with Q_0 fixed, $2\hat{Q}_{\bar{0}} \le E_\gamma$. In the latter regime, the energy $\mathcal{E}_\gamma$ behaves as $\sqrt{PJ^2E_\gamma}$. With the lower bound for E_γ, we then have

$$\Omega(\gamma; t_\gamma^\infty) e^{-\mathcal{E}_\gamma/T} \le e^{\pi\sqrt{\frac{2}{3}P^3\hat{Q}_{\bar{0}}} - \sqrt{2PJ^2\hat{Q}_{\bar{0}}}/T}. \tag{12}$$

Consequently, the system gives rise to a Hagedorn temperature T_H above which the sum over D2-charges diverges. Equation (12) shows that $\frac{1}{\pi}\sqrt{3PJ^2/P^3}$ is an upper bound for T_H.

The Hagedorn temperature [104] indicates the existence of a different high-temperature phase. Various systems in string theory, including little string theory [105], superstring theory [106–108], $\mathcal{N} = 4$ super Yang-Mills on compact spaces [109,110], Banks-Fischler-Shenker-Susskind matrix model in the presence of an IR cutoff [111] exhibit Hagedorn transition. In the present case, since the relevant two-dimensional theories descend from M5-brane world volume theory, this predicts a Hagedorn temperature for wrapped M5 branes.

X. CHANGE OF DEGENERACIES?

It has been argued that the degeneracies are not lifted under the $T\bar{T}$ deformation [25] and similarly that supersymmetric indices remain unchanged [67]. This suggests that the $\Omega(\gamma; t_\gamma^\infty)$ remain the same as functions of γ. On the other hand, moving to finite volume suggests that the natural BPS index is $\Omega(\gamma; t_\gamma^\lambda)$ for finite λ. While for simple systems of D4-branes, such as those with a irreducible magnetic charge, these indices are indeed equal, this may not be the case for more involved systems. For example, Ref. [40] described a family of scaling solutions, whose degeneracy is subleading to that of a single-center black hole. These solutions are present for finite λ but decouple in the limit $\lambda \to \infty$ [34]. Thus the supergravity picture gives some indication that degeneracies may change upon the $T\bar{T}$ deformation in sufficiently intricate CFTs. Of course for sufficiently large μ, the deformation may give rise to wall crossing, under which the degeneracies will also change.

XI. DISCUSSION

We have presented evidence that the microscopic description of D4-brane black holes with finite mass is furnished by a $T\bar{T}$ deformation of MSW CFT.

A first principle derivation of $T\bar{T}$ deformation from the microscopic side is clearly desirable. One possibility is that this arises from integrating out gravitational effects. This could also explain nonlocality of the $T\bar{T}$ deformed theory and is, in fact, the case for infinitesimal $T\bar{T}$ deformations [84]. More generally, $T\bar{T}$ deformation has been shown to arise as a result of coupling a CFT with flat space Jackiw-Teitelboim (JT) gravity [74,91]. It would be interesting to see if JT gravity emerges in the world volume of MSW string.

It may be worthwhile to explore other brane systems such as the D1–D5 system. Another relevant brane system is that of NS5-branes and fundamental strings [51–54,56,72,73]. As alluded to earlier, this system can be deformed by a $T\bar{T}$-like deformation, which is known as the single-trace deformation. This deformation is introduced on the string world sheet and expected to lead to a symmetric product of deformed CFTs, which is to be distinguished from the double-trace deformation, i.e. the $T\bar{T}$ deformation of the symmetric product CFT. While the single-trace deformation also vanishes in the limit $\ell_s \to 0$ [51], our setup is notably different, since the MSW CFT does not take the form of a symmetric product of a simpler CFT. Still it would be interesting to see if these holographic duals of irrelevantly deformed CFTs can give mutual insights.

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