Letter

Electric strings in non-Abelian theories

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We find electric string solutions in Yang-Mills-Higgs theory.

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Cosmic strings have attracted a lot of attention for nearly half a century since their discovery [1,2]. Such strings carry magnetic flux and are analogous to vortices in superconductors. In contrast, strings that carry electric flux are expected to lead to confinement in QCD and classical solutions corresponding to electric strings are not known. Even if electric strings were to exist as classical solutions in a non-Abelian gauge theory, because gauge excitations ("gluons") are massless and carry a non-Abelian charge one might expect that rapid production of gluons by the Schwinger process [3,4] would dissipate such strings. In view of these expectations it is surprising that electric string solutions do exist in certain non-Abelian gauge theories and are protected against the Schwinger process.

An essential element in constructing an electric string solution[1] is that a non-Abelian electric field does not uniquely specify a gauge equivalent class of gauge fields. As Brown and Weisberger (BW) showed [7], there is a one-parameter family of gauge inequivalent gauge fields that all result in the same electric field. Unlike a uniform non-Abelian electric field produced in analogy with Maxwell theory, BW gauge fields have been shown to be immune from decay due to Schwinger pair production [8]. This suggests the question: can there be string solutions containing BW gauge fields? In pure non-Abelian gauge theory, BW gauge fields do not solve the classical equations of motion. Instead they require external current and charge densities. Such external sources may arise due to quantum effects—after all the classical equations are expected to get modified due to the backreaction of quantum fluctuations—or they may be due to other fields in the system. Here we consider $SU(2)$ gauge theory with a scalar field in the fundamental representation. The same solution can be embedded in models with larger gauge group that have an $SU(2)$ subgroup [9–11]. The Lagrangian under consideration is

$$L = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} + |D_\mu \Phi|^2 - V(\Phi), \tag{1}$$

where

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu, \tag{2}$$

$$D_\mu \Phi = \partial_\mu \Phi - i\frac{g}{2} W^a_\mu \sigma^a \Phi, \tag{3}$$

$$V(\Phi) = m^2 |\Phi|^2 + \lambda |\Phi|^4, \tag{4}$$

where σ^a are the Pauli spin matrices. The Lagrangian actually has an $SU(2) \times U(1)$ symmetry but only the $SU(2)$ is gauged, while the $U(1)$ is global. (This corresponds to the $\sin^2\theta_w = 0$ limit of the electroweak model.)

The equations of motion are

$$(\partial_\nu + g\epsilon^{abc} W^b_\nu) W^{\mu\nu c} = i\frac{g}{2}(\Phi^\dagger \sigma^a D^\mu \Phi - \text{H.c.}) \tag{5}$$

$$D^\mu D_\mu \Phi + m^2 \Phi + 2\lambda |\Phi|^2 \Phi = 0. \tag{6}$$

To find an electric-field solution to the equations of motion, we first write the BW gauge field in temporal gauge [8],

$$W^a_0 = 0, \qquad W^\pm_\mu = -\frac{\epsilon}{g} e^{\pm i\Omega t} f(r) \partial_\mu z, \qquad W^3_\mu = 0, \tag{7}$$

where $W^\pm_\mu = W^1_\mu \pm i W^2_\mu$, and we have introduced a cylindrical profile function, $f(r)$, with $f(0) = 1$. This gauge field needs sources as given by the right-hand side of (5). Requiring that Φ provides the currents that will produce the gauge field in (7) allows us to construct Φ, which we have to make sure also satisfies (6). Some algebra shows that the required form of Φ is

$$\Phi = \frac{\epsilon}{g}\sqrt{\frac{\Omega}{\omega}} f(r) \begin{pmatrix} z_1 e^{+i\omega t} \\ z_2 e^{-i\omega t} \end{pmatrix}, \tag{8}$$

[1]Some early work on classical solutions in non-Abelian gauge theories can be found in Refs. [5,6].

where $W_\mu^\pm = W_\mu^1 \pm iW_\mu^2$ and $|z_1|^2 + |z_2|^2 = 1$. Without loss of generality we take $\Omega > 0$ while ω can be positive or negative. The Φ field can then provide suitable sources for the gauge field as well as satisfy its classical equation of motion provided we take

$$\Omega = m[4(1 - 4\lambda/g^2)^2 - 1]^{-1/2}, \tag{9}$$

$$\omega = 2(1 - 4\lambda/g^2)\Omega, \tag{10}$$

with restrictions that are necessary for Ω to be real; namely, for $m^2 > 0$, we need $0 < \lambda < g^2/8$ ($\omega > 0$), or $\lambda > 3g^2/8$ ($\omega < 0$), and for $m^2 < 0$, $g^2/8 < \lambda < g^2/4$ ($\omega > 0$) and $g^2/4 < \lambda < 3g^2/8$ ($\omega < 0$). An explicit check (see Supplemental Material [12]) shows that the solution is only valid for $\omega > 0$, i.e., for $\lambda < g^2/4$. In what follows we will assume $\omega > 0$. These constraints are summarized in Fig. 1.

The profile function satisfies the equation,

$$f'' + \frac{f'}{r} + \Omega^2\left(1 - \frac{\epsilon^2}{2\omega\Omega}f^2\right)f = 0 \tag{11}$$

with boundary conditions $f(0) = 1$, $f'(0) = 0$.

For $\epsilon^2 \ll 2\omega\Omega$ the solution is closely approximated by the zeroth order Bessel function, $J_0(\Omega r)$. For $\epsilon^2/2\omega\Omega > 1$, there is no well-behaved solution. We define a rescaled profile $F(r) \equiv \epsilon f(r)/\sqrt{2\omega\Omega}$ and a rescaled coordinate $R = \Omega r$. A numerical solution for $F(R)$ vs R is shown in Fig. 2. The asymptotic form of the solution is approximately described by the asymptotic form of the Bessel function,

$$F(R) \sim F(0)\sqrt{\frac{2}{\pi R}}\cos\left(R - \frac{\pi}{4}\right), \qquad R \to \infty. \tag{12}$$

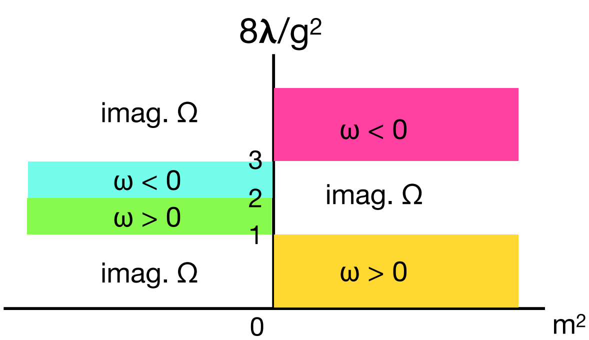

FIG. 1. Constraints on the parameters in the $m^2 - 8\lambda/g^2$ plane. The unshaded regions give imaginary Ω and are not allowed. The solution is only valid in the regions of parameter space where $\omega > 0$.

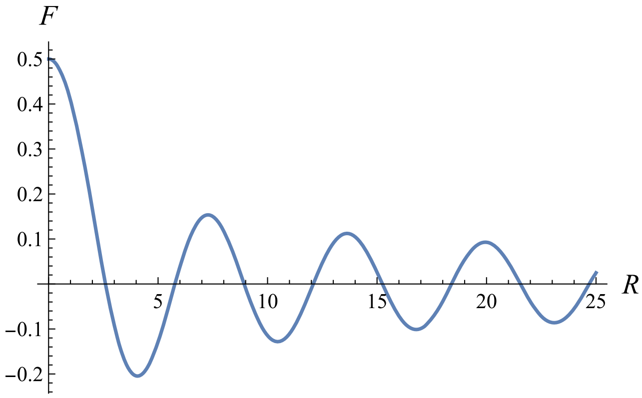

FIG. 2. $F(R)$ vs R for $F(0) = 0.5$.

The field strength for the solution is

$$W_{\mu\nu}^\pm = -\frac{\epsilon}{g}e^{\pm i\Omega t}[i\Omega f(r)(\partial_\mu t\partial_\nu z - \partial_\nu t\partial_\mu z) + f'(r)(\partial_\mu r\partial_\nu z - \partial_\nu r\partial_\mu z)] \tag{13}$$

and $W_{\mu\nu}^3 = 0$. An $SU(2)$ gauge rotation by

$$U = e^{i\sigma^1\Omega t/2}e^{-i\sigma^1\pi/4}e^{-i\sigma^2\pi/4} \tag{14}$$

brings the field strength to the form,

$$W'_{\mu\nu} = UW_{\mu\nu}U^\dagger, \tag{15}$$

where $W_{\mu\nu} = W_{\mu\nu}^a\sigma^a$. This gives static field strengths of the form in [7] (if we set $f = 1$),

$$W_{\mu\nu}^{1\prime} = 0, \tag{16}$$

$$W_{\mu\nu}^{2\prime} = -\frac{\epsilon}{g}f'(r)(\partial_\mu r\partial_\nu z - \partial_\nu r\partial_\mu z), \tag{17}$$

$$W_{\mu\nu}^{3\prime} = -\frac{\epsilon}{g}\Omega f(r)(\partial_\mu t\partial_\nu z - \partial_\nu t\partial_\mu z), \tag{18}$$

Then the static electric field is in the third $SU(2)$ direction and the spatial z-direction, while the static magnetic field is in the second $SU(2)$ direction and in the spatial azimuthal direction. The structure of the solution consists of a tube of electric field along z, wrapped by a magnetic field along the azimuthal direction $\hat\phi$, which is then within a sheath of electric field in the $-z$ direction, wrapped in magnetic field in the $-\hat\phi$ direction, and so on. The distinction between electric and magnetic fields is frame dependent and so we calculate the Lorentz invariant $-W_{\mu\nu}^a W^{\mu\nu a} \propto F^2 - F'^2$ in Fig. 3, confirming the alternating sequence of electric and magnetic fields.

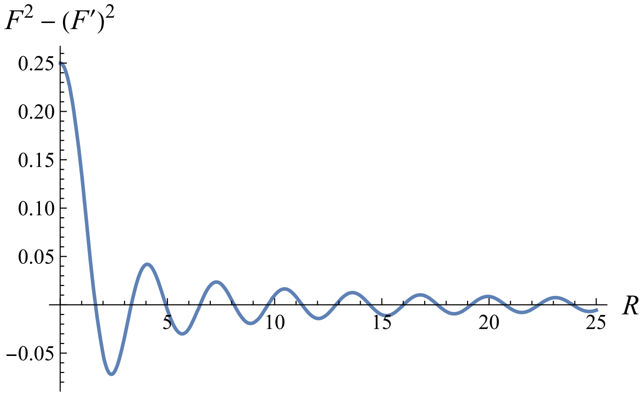

FIG. 3. $F(R)^2 - F'(R)^2$ vs R for $F_0 = 0.5$. The field strength is electriclike where $F(R)^2 - F'(R)^2$ is positive and magneticlike where $F(R)^2 - F'(R)^2$ is negative.

The gauge transformation U when applied to Φ gives

$$\Phi' = U\Phi = \frac{\epsilon}{g}\sqrt{\frac{\Omega}{2\omega}} f(r) \begin{pmatrix} z_1' e^{i\omega' t} - z_2' e^{-i\omega' t} \\ z_1' e^{i\omega' t} + z_2' e^{-i\omega' t} \end{pmatrix}, \qquad (19)$$

where $\omega' = \omega + \Omega/2$ and $z_1' = z_1 e^{-i\pi/4}$, $z_2' = z_2 e^{i\pi/4}$.

The energy density, $\mathcal{E}$ of the solution can be calculated from the expression,

$$\mathcal{E} = \frac{1}{2}(W_{0i}^a)^2 + \frac{1}{4}(W_{ij}^a)^2 + |D_t\Phi|^2 + |D_i\Phi|^2 + V(\Phi). \qquad (20)$$

In terms of rescaled variables, and restricting to $\omega > 0$,

$$\mathcal{E}' \equiv \frac{g^2}{2\omega\Omega^3}\mathcal{E} = \left(\frac{1}{2} + \frac{1}{\kappa}\right)F'^2 + \left(\frac{1}{2} + 2\kappa - \frac{1}{\kappa}\right)F^2 + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\kappa}\right)F^4, \qquad (21)$$

where [see (10)] $0 < \kappa \equiv \omega/\Omega < 2$. In Fig. 4 we show an example of $\mathcal{E}'$ vs R.

The slow falloff $\propto 1/\sqrt{R}$ of the gauge fields in (12) implies that $\mathcal{E} \propto 1/R$ and that the energy per unit length, μ, diverges linearly with radial distance. Hence, the string is not localized as in a Nielsen-Olesen string but is more like a global string that has a logarithmically divergent energy per unit length, or like a global monopole with linearly divergent energy [2].[2]

We evaluate μ numerically by integrating over $R \in [0, R_c]$ where R_c is a radial cutoff to get

[2] It is appropriate to call the solution an "electric string" because it contains electric flux, has cylindrical symmetry, the energy density peaks at the origin and falls off at large distances.

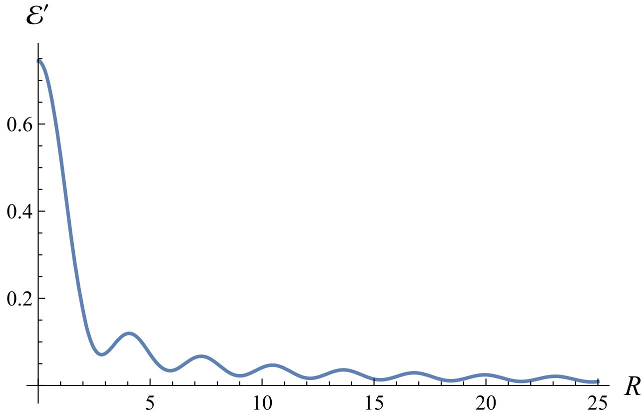

FIG. 4. $\mathcal{E}'$ vs R for $F_0 = 0.5$ and $\kappa = 1.5$.

$$\mu(R_c) = 2\pi \int_0^{R_c} dRR\mathcal{E}' \approx \left[\left(\frac{1}{2} + \frac{1}{\kappa}\right)0.54 + \left(\frac{1}{2} + 2\kappa - \frac{1}{\kappa}\right)0.54\right]R_c + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\kappa}\right)0.09\ln(R_c), \qquad (\omega > 0) \qquad (22)$$

where, from (10), κ may also be written as

$$\kappa = 2(1 - 4\lambda/g^2). \qquad (23)$$

In the $m^2 < 0$ case, we should add a constant piece to the potential so that $V = 0$ at its minimum. In that case, however, the solution still has $|\Phi| \to 0$ in the asymptotic region. Since the true vacuum has $|\Phi| \neq 0$, the solution has divergent energy per unit length and the divergence will go as R_c^2 instead of R_c.

Our electric string solution is with a scalar field in the fundamental representation. This leads to the question whether scalar fields in other representations can also provide suitable sources for the gauge fields in (7). We have examined the case of a scalar field in the adjoint representation, ϕ^a, and can show that a solution does not exist. Briefly, the gauge field equation is $D_\nu W^{\mu\nu a} = j^{\mu a}$ where the current $j^{\mu a}$ is now due to the adjoint scalar and has a form such that $\phi^a j_\mu^a = 0$ for every value of the index μ. This constraint then requires $\phi^a D_\nu W^{\mu\nu a} = 0$ for every μ, where the gauge field is given in (7). The constraint is strong enough that it fixes the form of ϕ^a. Then we can calculate j_μ^a using ϕ^a and we see that it does not satisfy the gauge field equation of motion, $j^{\mu a} = D_\nu W^{\mu\nu a}$.

Another interesting question is if electric string solutions exist in the electroweak model. The difference between the model in (1) and the bosonic sector of the electroweak model is that the global $U(1)$ symmetry of (1) is now gauged with coupling g' and Φ has nonvanishing $U(1)$ gauge charge (known as "hypercharge"). Then the $U(1)$ hypercharge current is

$$j_\mu^Y = i\frac{g'}{2}(\Phi^\dagger D_\mu\Phi - \text{H.c.}) \quad (24)$$

Using (7) and (8) we find,

$$j_\mu^Y = -\frac{g'\epsilon^2}{g^2}\Omega f^2(|z_1|^2 - |z_2|^2)\partial_\mu t - \frac{g'\epsilon^3}{2g^2}\frac{\Omega}{\omega}f^3[z_1 z_2^* e^{i(\Omega+2\omega)t} + \text{c.c.}]\partial_\mu z, \quad (25)$$

which shows that in general there is a hypercharge charge density as well as a three current. If j_μ^Y is nonvanishing, then necessarily the hypercharge gauge field, Y^μ, is nonvanishing because it satisfies

$$\partial_\nu Y^{\mu\nu} = j^{Y\mu}. \quad (26)$$

Our solution does not include a hypercharge component and so the question is if we can choose parameters such that $j_\mu^Y = 0$. The $\mu = 0$ component can be made to vanish by choosing $|z_1|^2 = |z_2|^2$. The $\mu = 3$ component is time dependent except if $\Omega + 2\omega = 0$, which corresponds to $\omega = -\Omega/2 < 0$. Since the solution is only valid for $\omega > 0$, it does not hold in the electroweak model with $Y_\mu = 0$. An alternate possibility is that there exist solutions with nonvanishing hypercharge gauge field Y_μ and then we do not need to require that j_μ^Y vanish. We have not been able to construct such solutions.

An important feature of the gauge field in the solution is that it is stationary. In other words, consider perturbations of only the gauge field,

$$W_\mu^\pm = A_\mu^\pm + e^{\pm i\Omega t}Q_\mu^a, \qquad W_\mu^3 = Q_\mu^3, \qquad \Phi = \Phi_0, \quad (27)$$

where A_μ^a, Φ_0 denote the electric string solution. Then the action for Q_μ^a does not contain any terms that are explicitly time dependent and hence the solution is protected from decay to Schwinger pair production of gauge field excitations as shown in Ref. [8]. The inclusion of the scalar field, Φ, does not make any difference to the analysis in Ref. [8] because the quadratic order interaction between Φ and Q_μ^a is simply $g^2|\Phi|^2(Q_\mu^a)^2/4$ and $|\Phi|^2$ is time independent. (The linear-order terms vanish because the solution obeys the classical equations of motion.) Fluctuations of the scalar field Φ can indeed get excited by the time dependence of the solution, and this means that Schwinger pair production of Φ excitations will occur. (This is similar to the Schwinger pair production of quarks on QCD strings.) In the limit of large mass parameter m, the Schwinger pair production of Φ will be suppressed.

A Maxwell electric field for example, with $W_z^3 = -Et$ and all other components zero, is a classical solution of the SU(2) pure gauge theory. However, the Schwinger process for gluons in this background is non-vanishing at all momentum scales [4] and the Maxwell electric field will decay and evolve into another configuration. Since a BW electric field is stable to the Schwinger process, it is likely that it is the final state. In other words, if initially we start with a Maxwell electric field, it may evolve into a BW electric field.

In Ref. [13] the *classical* stability of a homogeneous electric field of the BW type was analyzed. Several unstable modes were found for the homogeneous configuration. For the electric string, these unstable modes will be suppressed due to the $|\Phi|^2(Q_\mu^a)^2$ term, since this term provides an effective mass to the gauge excitations wherever $|\Phi|^2$ is nonzero. The solutions with $m^2 < 0$ will almost certainly be unstable since Φ approaches $|\Phi| = 0$ in the asymptotic region and this is at the top of the Mexican hat potential. We plan to carry out a detailed stability analysis in a future publication.

The electric string is a solution of the classical equations of motion and it is interesting to consider what might become of it in quantum theory, especially in the regime of strong coupling. Generally, to quantize a classical solution, quantum fields are split into the classical background and quantum operators that live on top of the background. For example, we may write $W_\mu^a = W_\mu^{(0)a} + Q_\mu^a$, where $W_\mu^{(0)a}$ represents the background and Q_μ^a are quantum operators. (Similarly $\Phi = \Phi^{(0)} + \psi$, where ψ is a quantum operator.) For weakly-coupled systems, the quantum fluctuations are simple harmonic oscillators whose eigenfrequencies can be determined by diagonalizing the fluctuation Hamiltonian. Provided the background is classically stable, the ground state of the field theory is simply given by the ground state of the simple harmonic oscillators. In the regime of strong coupling, the quantum operators Q_μ^a and ψ will be strongly coupled and the ground state is no longer given by a collection of simple harmonic oscillators. Furthermore, the quantum state will backreact on the background and modify it, just as quantum fluctuations modify the true minimum of an effective potential. These strongly coupled quantum effects are difficult to evaluate and lattice methods may be the only hope to quantify them. It is possible that the asymptotic structure of the electric string solution will get modified due to strong coupling at long wavelengths while the small scale structure at the core survives due to asymptotic freedom.

We close with a speculative remark. The solution we have found may be of interest in the context of pure non-Abelian gauge theories. In that case, the field Φ must arise as an effective degree of freedom due to quantum backreaction in the classical equations of motion. In this connection, it has been conjectured that magnetic monopoles at strong coupling transform as a scalar degree of freedom in the fundamental representation of a dual symmetry group [14]. While there are no classical monopole solutions in pure non-Abelian gauge theory, one can still write configurations that resemble the gauge fields of magnetic monopoles,

$$W_i^a = \frac{(1 - k(r))}{gr} \epsilon^{aij} \hat{x}^j, \tag{28}$$

where $k(r)$ is a suitable profile function. Can the back-reaction due to such monopole configurations behave like our scalar field Φ and provide the necessary sources for electric strings?

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