Letter

New form of the Kerr-Newman solution

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A new form of the Kerr-Newman solution is presented. The solution involves a time coordinate which represents the local proper time for a charged massive particle released from rest at spatial infinity. The chosen coordinates ensure that the solution is well-behaved at horizons and enable an intuitive description of many physical phenomena. If the charge of the particle $e = 0$, the coordinates reduce to Doran coordinates for the Kerr solution with the replacement $M \to M - Q^2/(2r)$, where M and Q are the mass and charge of the black hole, respectively. Such coordinates are valid only for $r \geq Q^2/(2M)$, however, which corresponds to the region that a neutral particle released from rest at infinity can penetrate. By contrast, for $e \neq 0$ and of opposite sign to Q, the new coordinates have a progressively extended range of validity as $|e|$ increases and tend to advanced Eddington-Finkelstein (EF) null coordinates as $|e| \to \infty$, hence becoming global in this limit. The Kerr solution (i.e. with $Q = 0$) may also be written in terms of the new coordinates by setting $eQ = -\alpha$, where α is a real parameter unrelated to charge; in this case the coordinate system is global for all non-negative values of α and the limits $\alpha = 0$ and $\alpha \to \infty$ correspond to Doran coordinates and advanced EF null coordinates, respectively, without any need to transform between them.

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The reexpression in different coordinate systems of exact solutions in general relativity may at first seem a pointless exercise, since the theory is covariant under such transformations by construction. Nonetheless, a judicious choice of coordinates can assist both in the mathematical calculation and physical intuition associated with general-relativistic phenomena. Since one is often interested in processes related to particle motion in the background spacetime, it is thus natural to construct coordinate systems that are based thereon.

In asymptotically flat black hole backgrounds, on which we will focus our attention, consideration of ingoing or outgoing principal null geodesics (which reduce to ingoing or outgoing radial photon trajectories in the limit of a nonrotating black hole) lead to advanced and retarded Eddington-Finkelstein (EF) coordinates, respectively, which are well behaved at horizons and cover the entire spacetime [1,2]. Equally, one may instead construct coordinate systems by considering the trajectories of massive particles. The most natural of these are based on the motion of such particles released from rest at spatial infinity (which are often referred to as "raindrops") and, in particular, take the time coordinate T (say) to represent the local proper time of the raindrop [3–8].

For the Schwarzschild solution [9], demanding that $\dot{T} = 1$, $\dot{\theta} = 0 = \dot{\phi}$ along a raindrop trajectory (where a dot denotes the derivative with respect to the raindrop proper time τ) yields Painlevé-Gullstrand (PG) coordinates [3,4], in which the metric takes the form[1]

$$ds^2 = dT^2 - \left(dr + \sqrt{\frac{2M}{r}}dT\right)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where r lies in the range $0 < r < \infty$, and θ and ϕ take their usual meanings. Such coordinates are clearly global and well-behaved at the horizon $r = 2M$, unlike their standard Schwarzschild counterparts, and so are useful in clarifying many physical phenomena [6,10,11], especially black-hole thermodynamics [12,13]. In particular, the 4-velocity of a raindrop in PG coordinates $x^\mu = (T, r, \theta, \phi)$ has the simple form $\dot{x}^\mu = (1, -\sqrt{2M/r}, 0, 0)$.

Following an analogous procedure for the Reissner-Nordström (RN) solution [14,15], one obtains a line-element in PG coordinates again of the form (1), but with $M \to M - Q^2/(2r) \equiv \mathcal{M}$, where Q is the charge of the black hole [10,11]. Although the PG coordinates are again regular at horizons and useful for investigating many physical phenomena, it is clear that the coordinates are not global, since they are valid only for $r > Q^2/(2M)$ [16].

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[1]We adopt the "mostly minus" metric signature $(+,-,-,-)$ and employ geometric units $c = G = 1$.

This occurs because, in general, coordinates based on the motion of some test particle are valid only in the region of the spacetime that such a particle can penetrate, as is well known (e.g. [17,18]); it is easily shown that $r = Q^2/2M$ marks the innermost radius that can be reached by a raindrop in the RN solution. As one might expect intuitively, for static, spherically symmetric spacetimes such as the RN solution, one can address this shortcoming by instead constructing coordinates systems based on the radial motion of massive particles with nonzero ingoing coordinate speed at infinity v_∞. This yields the so-called Martel-Poisson class of coordinates [19], which clearly coincide with PG coordinates when $v_\infty = 0$, but have a progressively larger range of validity as v_∞ increases and tend to advanced EF null coordinates as $v_\infty \to 1$, hence becoming global in this limit.

Returning to PG coordinates, it is of particular note in (1) that the metric coefficient g_{00} (or lapse function) is unity and the spacelike 3-surfaces $T =$ constant are flat. Indeed, these criteria comprise the original definition [3,4] of (strong) PG coordinates.[2] For the Schwarzschild and RN solutions (and a number of other static, spherically symmetric spacetimes), the notions of a raindrop-based time coordinate and a PG form for the metric coincide [6,10,11,18]. Indeed, in such cases, raindrops are often also called PG observers.

For rotating black holes, the situation is rather different. Although the Lense-Thirring solution [20,21] (the slowly rotating limit of the Kerr solution) can be expressed in (strong) PG form [22–25], the exact Kerr(-Newman) metric cannot be put in strong, weak or even conformal PG form [26]. Nonetheless, one can still construct useful coordinate systems for the Kerr(-Newman) solution by again considering the motion of raindrops, although the link between such coordinates and the PG form for the metric is broken in this case.

The Kerr metric [27] in standard Boyer-Lindquist (BL) coordinates [28] may be written as[3]

$$ds^2 = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[a dt - (r^2+a^2)d\phi]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \quad (2)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$. As will become clear from our further analysis below, by considering the usual geodesic equations in BL coordinates for a raindrop and introducing new coordinates T, Θ and Φ, for which one demands $\dot{T} = 1$, $\dot{\Theta} = 0 = \dot{\Phi}$ (or, equivalently, $dT = d\tau$, $d\Theta = 0 = d\Phi$) along a raindrop trajectory, one immediately obtains $d\Theta = d\theta$ and

$$dT = dt + \frac{\sqrt{2Mr(r^2+a^2)}}{\Delta}dr, \quad (3a)$$

$$d\Phi = d\phi + \frac{a}{\Delta}\sqrt{\frac{2Mr}{r^2+a^2}}dr. \quad (3b)$$

In terms of these new coordinates (retaining the original θ-coordinate, which is clearly unchanged) the line element (2) may be written as

$$ds^2 = dT^2 - \left[\frac{\rho}{\sqrt{r^2+a^2}}dr + \frac{\sqrt{2Mr}}{\rho}(dT - a\sin^2\theta d\Phi)\right]^2 - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\Phi^2, \quad (4)$$

which corresponds to Doran coordinates [29], although they were not originally derived in the above manner. It is clear that this coordinate system, like PG coordinates for the Schwarzschild solution, is both regular at horizons and global, and also useful for describing many physical phenomena. In particular, the 4-velocity of a raindrop in Doran coordinates $x^\mu = (T, r, \theta, \Phi)$ has the simple form $\dot{x}^\mu = (1, -\sqrt{2Mr(r^2+a^2)}/\rho^2, 0, 0)$. Various other coordinate systems for the Kerr solution, based on raindrop trajectories or otherwise, and their relationship to Doran coordinates are discussed in [8,30–32]. It is also worth noting that all these coordinate systems are related to advanced EF null coordinates by transformations of similar complexity to those in (3).

The Kerr-Newman (KN) solution [33,34] is the most general asymptotically flat, stationary solution of the Einstein-Maxwell equations in general relativity. By analogy with the transition from the Schwarzschild to RN solution, the standard KN metric in BL coordinates may be obtained from (2) by again making the substitution $M \to M - Q^2/(2r) \equiv \mathcal{M}$ throughout. Similarly, the transition to the KN solution in Doran coordinates is also immediately obtained by setting $M \to \mathcal{M}$ in (4) [7,35]. For the KN solution, however, it is clear that one encounters an analogous difficulty to that discussed earlier for the RN solution, namely that the Doran coordinates are not global since they are valid only for $r > Q^2/(2M)$ [16]; this is again easily shown to correspond to the innermost radius to which a raindrop trajectory can penetrate.

By analogy with the RN solution, one might hope to address this shortcoming by constructing coordinate systems based on the motion of massive particles with nonzero ingoing coordinate speed at infinity v_∞, which should penetrate to smaller values of r, thereby extending the notion of the Martel-Poisson class of coordinate systems

[2]For weak PG coordinates, the lapse function is allowed to be non-trivial, while spacelike 3-surfaces are still required to be flat. One may also define conformal strong or weak PG coordinates, for which the spacetime line element is conformal to either the strong or weak versions of the PG form.

[3]This originally proposed form for the Kerr metric in BL coordinates [28] directly yields the form usually quoted (e.g. [17]) on multiplying out and collecting coefficients of differentials.

to rotating black hole backgrounds. While this approach is valid, in principle, we find that it leads to a rather cumbersome and unintuitive set of coordinate systems that moreover do not tend to advanced EF null coordinates in the limit $v_\infty \to 1$. We therefore instead consider the alternative approach of constructing a coordinate system based on the motion of a *charged* raindrop. The basic intuition in so doing is that a raindrop with charge of opposite sign to that of the black hole experiences an additional inwards electromagnetic force that should allow it to penetrate to smaller values of r than a neutral raindrop.

We begin our analysis by considering the orbit equations in BL coordinates for a particle of unit mass and charge e in the KN spacetime, which read [36]

$$\rho^2\dot{t} = -a(ak\sin^2\theta - h) + \frac{r^2+a^2}{\Delta}P(r), \tag{5a}$$

$$\rho^2\dot{r} = \pm\sqrt{P(r)^2 - \Delta[r^2 + (h-ak)^2 + C]}, \tag{5b}$$

$$\rho^2\dot{\theta} = \pm\sqrt{C - \cos^2\theta\left[a^2(1-k^2) + \frac{h^2}{\sin^2\theta}\right]} \tag{5c}$$

$$\rho^2\dot{\phi} = -\left(ak - \frac{h}{\sin^2\theta}\right) + \frac{a}{\Delta}P(r), \tag{5d}$$

where $P(r) \equiv k(r^2+a^2) - ah - eQr$ and the orbit is characterized by the particle specific energy k, specific angular momentum h and specific Carter's constant C, all of which are conserved quantities.

For a raindrop, the conserved quantities along the orbit have the values $k = 1$, $h = 0 = C$. In this special case, the orbit equations (5) are greatly simplified. In particular, one sees that this case is unique in yielding $\dot{\theta} = 0$. Moreover, one obtains

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\frac{(\rho^2 - eQr)(r^2+a^2) + 2\mathcal{M}ra^2\sin^2\theta}{\Delta\sqrt{R(r)}}, \tag{6a}$$

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = -\frac{(2\mathcal{M} - eQ)ra}{\Delta\sqrt{R(r)}}, \tag{6b}$$

where we have assumed that the raindrop is ingoing and have defined the cubic $R(r) \equiv 2(\mathcal{M} - eQ)r(r^2+a^2) + e^2Q^2r^2$. If one now introduces new coordinates T, Θ and Φ, for which one demands $\dot{T} = 1$, $\dot{\Theta} = 0 = \dot{\Phi}$ (or, equivalently, $dT = d\tau$, $d\Theta = 0 = d\Phi$) along a raindrop trajectory, one immediately obtains $d\Theta = d\theta$ and

$$dT = dt + \frac{r^2+a^2}{\Delta}F(r)dr \tag{7a}$$

$$d\Phi = d\phi + \frac{a}{\Delta}F(r)dr. \tag{7b}$$

where we have defined $F(r) = (2\mathcal{M} - eQ)r/\sqrt{R(r)}$.

Before rewriting the KN metric in terms of these coordinates, it is useful to consider some limits of the above transformations. As expected, if $e = 0$ then $R(r) = 2\mathcal{M}r(r^2+a^2)$ and $F(r) = \sqrt{2\mathcal{M}r/(r^2+a^2)}$, such that the transformations (7) reduce directly to those in (3) with the replacement $M \to \mathcal{M}$, which define Doran coordinates for the Kerr-Newman solution that are valid for $r > Q^2/(2M)$; clearly these coordinates further reduce to the original globally valid Doran coordinates for the Kerr solution if $Q = 0$. If $e \neq 0$ and is of opposite sign to Q, the coordinates defined by the transformations (7) enjoy a progressively extended range of validity as $|e|$ increases. The innermost value of r for which the coordinates are valid is clearly given by the single real root of the cubic $R(r)$; the corresponding expression for this root is rather cumbersome and unenlightening, but always lies in the range $0 < r < Q^2/(2M - eQ)$. In the limit $|e| \to \infty$ (again with $eQ < 0$), $F(r) \to 1$ and so the transformations (7) tend to those defining advanced EF null coordinates (where dT is usually denoted by du) [36]. Indeed, from (6) in the same limit, one finds that the particle trajectory coincides will the principal null geodesics, along which $dT = 0$, as expected. In this limit, the coordinate system thus becomes global.

In terms of the new coordinates (7), the KN solution, which is given in BL coordinates by (2) with $M \to \mathcal{M}$, becomes

$$ds^2 = \frac{\Delta}{\rho^2}\left[\frac{\rho^2}{\Delta}F(r)dr - (dT - a\sin^2\theta d\Phi)\right]^2 - \frac{\sin^2\theta}{\rho^2}\left[adT - (r^2+a^2)d\Phi\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \tag{8}$$

which in fact holds for arbitrary $F(r)$ in (7) (other forms for $F(r)$ are considered in [37]). An equivalent form of the KN solution in the new coordinates is given by

$$ds^2 = dT^2 - \frac{2(\mathcal{M} - eQ)r\rho^2}{R(r)}\left[dr + \frac{\sqrt{R(r)}}{\rho^2}(dT - a\sin^2\theta d\Phi)\right]\left[dr + \frac{\mathcal{M}}{\mathcal{M} - eQ}\frac{\sqrt{R(r)}}{\rho^2}(dT - a\sin^2\theta d\Phi)\right] - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\Phi^2, \tag{9}$$

which may be more useful in some physical applications. In particular, one sees immediately that for a charged raindrop, for which $\rho^2 dr = -\sqrt{R(r)}dT$ and $d\theta = 0 = d\Phi$, the line element reduces to $ds^2 = dT^2$, as required, so the raindrop 4-velocity is $\dot{x}^\mu = (1, -\sqrt{R(r)}/\rho^2, 0, 0)$. It is further worth noting that $ds^2 = dT^2$ *also* holds for particles for which $\rho^2 dr = -\mathcal{M}\sqrt{R(r)}dT/(\mathcal{M} - eQ)$ and $d\theta = 0 = d\Phi$.

For completeness, we note that the electromagnetic vector potential of the KN solution in the new coordinate system is trivially obtained from its standard form in BL coordinates, most succinctly expressed as the 1-form $A = A_\mu dx^\mu = (Qr/\rho^2)(dt - a\sin^2\theta d\phi)$, by using the transformations (7) to obtain

$$A = \frac{Qr}{\rho^2}\left[dT - \frac{\rho^2}{\Delta}F(r)dr - a\sin^2\theta d\Phi\right]. \tag{10}$$

For $e = 0$, the line element (9) is easily shown to reduce to the Doran form (4) with $M \to \mathcal{M}$, as expected. As also anticipated, in the limit $|e| \to \infty$ (with $eQ < 0$), for which $F(r) \to 1$, the line element (8) is quickly found to reduce to the KN solution in advanced EF null coordinates (where dT is usually denoted by du), namely

$$\begin{aligned} ds^2 = &\left(1 - \frac{2\mathcal{M}r}{\rho^2}\right)dT^2 - 2dTdr + \frac{4\mathcal{M}ra\sin^2\theta}{\rho^2}dTd\Phi \\ &+ 2a\sin^2\theta drd\Phi - \rho^2 d\theta^2 \\ &- \left(r^2 + a^2 + \frac{2\mathcal{M}ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\Phi^2. \end{aligned} \tag{11}$$

It is worth pointing out that the Kerr solution (for which $Q = 0$) may also be written in terms of the new coordinate system by setting $\mathcal{M} = M$ and $eQ = -\alpha$ throughout (9), where α should now be considered simply as a real parameter that is unrelated to charge. In this case, the coordinate system is regular at horizons and global for any non-negative value of α, and the limits $\alpha = 0$ and $\alpha \to \infty$ correspond to Doran coordinates and advanced EF null coordinates, respectively, without any need to transform between them.

Finally, one should note that the trajectories of charged or neutral massive test particles, as usually considered, are somewhat hypothetical compared to the motion of physical massive particles falling into a black hole. In particular, for a charged particle one should take into account the back reaction on the particle resulting from the emission of electromagnetic radiation along its trajectory (e.g. [38,39]). Even for a neutral particle, one should in principle consider the back reaction due to the emission of gravitational radiation, although this will be a much smaller effect. It is worth noting, however, that tidal forces on a infalling neutral physical massive particle will in any case eventually tear it apart into its charged fundamental constituents (unless the original neutral massive particle is a Higgs boson, which is the only such fundamental particle in the Standard Model). Thus, the 4-velocity of a physical massive particle in a black-hole background will not, in general, take a simple form in any coordinate system based on the idealized motion of test particles.

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