Mass and force relations for Einstein-Maxwell-dilaton black holes

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We investigate various properties of extremal dyonic static black holes in Einstein-Maxwell-dilaton theory. Using the fact that the long-range force between two identical extremal black holes always vanishes, we obtain a simple first-order ordinary differential equation for the black hole mass in terms of its electric and magnetic charges. Although this equation appears not to be solvable explicitly for general values of the strength a of the dilatonic coupling to the Maxwell field, it nevertheless provides a powerful way of characterizing the black hole mass and the scalar charge. We make use of these expressions to derive general results about the long-range force between two nonidentical extremal black holes. In particular, we argue that the force is repulsive whenever $a > 1$ and attractive whenever $a < 1$ (it vanishes in the intermediate BPS case $a = 1$). The sign of the force is also correlated with the sign of the binding energy between extremal black holes, as well as with the convexity or concavity of the surface characterizing the extremal mass as a function of the charges. Our work is motivated in part by the repulsive force conjecture and the question of whether long range forces between nonidentical states can shed new light on the swampland.

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I. INTRODUCTION

Recent years have seen growing efforts to sharpen the constraints that theories of quantum gravity place on low energy effective field theories (EFTs). Within these efforts, one of the challenges has been to quantify the notion that gravity is the weakest force and to understand what it tells us about the structure of long range interactions, in theories with a quantum gravity UV completion. In particular, the attempts to understand this question have led to various generalizations of the weak gravity conjecture (WGC) [1] and extensive studies of its phenomenological consequences (see e.g. [2] for a comprehensive review). In its simplest form, the WGC requires the existence of superextremal charged particles, states whose mass is smaller than or equal to their charge (in Planck units). In flat space, a closely related—but not equivalent[1]—way to quantify the weakness of gravity has led to the repulsive force conjecture (RFC) [1,5,6], which roughly states that theories compatible with quantum gravity should contain *self-repulsive states*, i.e. states which would feel either a repulsive or vanishing force when placed asymptotically far from an identical copy of themselves. Both the WGC and the RFC place restrictions on low energy EFTs and have implications for the spectrum of states in the theory.

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[1]In theories of quantum gravity with massless scalars, the WGC and the RFC are distinct. The differences become even more apparent when one takes into account the effects of higher derivative corrections [3,4].

Moreover, the RFC constrains all the interactions that lead to long range self-forces, including those coming from massless scalar fields. A question that arises naturally, then, is what these conjectures teach us about binding energies and the existence of bound states.

Thus far most of the discussions of the RFC have centered around studies of long range interactions between two copies of the *same* state, i.e. self-forces. However, it may be useful to explore what happens to the force and binding energy when the states in consideration are *not* the same, using black holes as probes. Indeed, while asymptotically the force between two identical extremal black holes is known [7] to be zero,[2] even in the presence of scalar matter, very little is known about its structure when the black holes carry different charges and represent distinct states. Using nonidentical states as probes of long range interactions may uncover novel features and potentially new insights on the string theory landscape. As a concrete example, in [8] we saw that the long-range force between distinct, extremal KK dyonic black holes is *always repulsive*. While this is naively surprising, it might have a natural explanation, at the microscopic level, in terms of the interactions between the constituent D-branes (in this case D0-D6 branes) and properties of bound states in the theory. If this is indeed the case, long range forces might provide an easier way to access some of the information encoded in the microscopic description of the theory.

Motivated by the questions above and by our previous work [8], in this paper our goal is to understand whether long range forces between different extremal black holes display any *generic* features, and, if so, how the latter are correlated with specific properties of the theory they arise in. As we will see, the scalar couplings in the theory we examine will leave clear imprints on certain characteristics of the black hole solutions (such as their extremality relations and propensity to bind), which will then be imprinted on the behavior of the long range interactions between them.

We are going to work with extremal static black holes solutions to four-dimensional Einstein-Maxwell-dilaton theory, described by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{a\phi}F^{\mu\nu}F_{\mu\nu}\right), \qquad (1.1)$$

where the constant a characterises the strength of the exponential coupling of the dilaton to the Maxwell field. For certain values of a, namely $a = 0$, $a = 1$ and $a = \sqrt{3}$, the solutions for dyonic black holes, carrying both electric and magnetic charge, are known explicitly. Our focus will be on the properties of the extremal static dyonic black hole solutions for *arbitrary* values of a.[3] The mass M of such an extremal black hole will be a function of the electric and magnetic charges Q and P, with $M = \mathcal{F}(Q,P)$, but except for the exactly-solvable cases when $a = 0$, 1 or $\sqrt{3}$, the explicit form of the function $\mathcal{F}(Q,P)$ is unknown. One of the main results in our work is a simple nonlinear first-order ordinary differential equation for (a rescaled version of) $\mathcal{F}(Q,P)$. Although this equation is, as far as we know, exactly solvable only at the special values $a = 1$ and $\sqrt{3}$, the fact that we can express the mass in this relatively simple way allows us to probe a number of properties of the extremal black holes.

The long-range force F_{12} between two black holes takes the form

$$F_{12} = \frac{1}{r^2}\left[Q_1Q_2 + P_1P_2 - \frac{1}{4}M_1M_2 - \Sigma_1\Sigma_2\right], \qquad (1.2)$$

where Q, P, M, and Σ denote, respectively, the electric and magnetic charges,[4] the mass and the scalar charge, while the subscripts 1, 2 are used to distinguish the first from the second black hole. In our earlier work [8] we initiated a study of (1.2), focusing on a specific class of Toda theories which support extremal black holes that are not BPS. Here we extend our analysis to a much broader class of dyonic solutions, and identify a number of new features. Most notably, the range of the parameter a controlling the gauge kinetic coupling dictates certain geometric properties of the energy surface $M = \mathcal{F}(Q,P)$ of each extremal solution, as well as the sign of the long range interactions between distinct ones.

In particular, using a combination of approximations, and also numerical analysis, we conclude that the force between nonidentical extremal dyonic black holes is always repulsive if the dilaton coupling a appearing in (1.1) satisfies $a > 1$, and it is always attractive if $a < 1$. In the intermediate case $a = 1$, for which the extremal dyonic black holes are BPS, the force between them is always zero. Moreover, using geometrical arguments we show that when

[2] In this context, "identical" can mean that the set of electric and magnetic charges carried by the two extremal black holes are either exactly the same, or, more generally, that the set of charges carried by one of the black holes is an overall positive constant multiple of the set of charges carried by the other.

[3] Some properties of these extremal black holes were investigated numerically in [9] and analytically in [10]. The solutions exist for all values of a, but there is some degree of nonanalyticity on the horizon unless a is such that $a^2 = \frac{1}{2}k(k+1)$ where k is an integer. Scalar curvature invariants, such as $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $\nabla^\lambda R^{\mu\nu\rho\sigma}\nabla_\lambda R_{\mu\nu\rho\sigma}$, etc., are finite on the horizon, however, for all values of a.

[4] For all the nonidentical extremal black holes that we shall consider, we shall always take the *signs* of the charges of the first and the second black hole to be the same. The case corresponding to opposite signs for the charges would be of little interest, since the electrostatic forces would just reinforce the gravitational attraction. Thus, without loss of generality, we assume that all the electric and magnetic charges are positive.

$a > 1$ the energy surface $M = \mathcal{F}(Q, P)$ describing the mass of each solution is *convex*, while for $a < 1$ it is *concave*.

We also investigate the related question of what is the sign of the binding energy between extremal black holes. Thus, if the mass of an extremal black hole with charges Q and P is $M = \mathcal{F}(Q, P)$, we define the binding energy between two such black holes to be

$$\Delta M = \mathcal{F}(Q_1 + Q_2, P_1 + P_2) - \mathcal{F}(Q_1, P_1) - \mathcal{F}(Q_2, P_2). \tag{1.3}$$

Intuitively, one may expect that if ΔM is positive then the two constituent black holes with charges (Q_1, P_1) and (Q_2, P_2) should tend to repel one another, while if ΔM is negative they should attract. Indeed, we find that the sign of ΔM does correlate with the sign of the long-range force, for all the EMD extremal black holes. Perhaps not surprisingly, the sign of ΔM is also governed by the convexity or concavity of the surface $M = \mathcal{F}(Q, P)$.

The paper is organized as follows. In Sec. II we describe properties of the extremal EMD black hole solutions we will be working with. In Sec. III we derive a simple differential equation that controls the mass of the extremal black hole in terms of its electric and magnetic charges, and in Sec. IV we present some solutions valid in specific perturbative regimes. Section V is devoted to the computation of the long distance force between nonidentical black holes, while Sec. VI discusses the binding energies. Geometrical properties of the energy surface describing how the mass is related to the charges are discussed in Sec. VII. Finally, in Appendix A we include some examples of the numerical computations that support our results, while in Appendix B we prove certain properties of the binding energy near the special value of the coupling $a = 1$.

II. STATIC EXTREMAL BLACK HOLES IN EINSTEIN-MAXWELL-DILATON THEORY

Purely electric or magnetic static black holes in the EMD theory (1.1) for arbitrary values of the dilaton coupling were constructed in [11]. The system of equations for the most general dyonic static solutions was obtained in [12], where it was shown that they could be reduced to a Toda-like system. It was noted there that the equations became exactly those of the $SU(3)$ Toda system when $a = \sqrt{3}$, and of the $SU(2) \times SU(2)$ Toda (or $(\text{Liouville})^2$) system when $a = 1$, but that no explicit dyonic solutions could be obtained for generic values of the dilaton coupling. The dyonic solution for $a = \sqrt{3}$ had been obtained by [13,14].

A formulation of the Toda-like equations for general values of a appeared also in a recent paper [15]. With some adaption of their notation to suit our conventions, the static black hole solutions to the equation of motion following from the Lagrangian (1.1) are given by

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (H_e H_m)^h d\Omega_2^2, \\ f(r) &= (H_e H_m)^{-h}\left(1 - \frac{\mu}{r}\right), \\ e^{a\phi} &= \left(\frac{H_e}{H_m}\right)^{2-h}, \\ F &= \frac{Q}{r^2} H_e^{-2} H_m^{2-2h} dt \wedge dr + P \sin\theta d\theta \wedge d\varphi, \end{aligned} \tag{2.1}$$

where Q and P denote the electric and magnetic charges, the parameter μ is positive for nonextremal black holes and equal to zero for extremal black holes, and

$$h = \frac{2}{1 + a^2}. \tag{2.2}$$

Defining the inverse radial coordinate

$$\rho = \frac{1}{r}, \tag{2.3}$$

the EMD equations of motion imply that the functions H_e and H_m obey the equations

$$\begin{aligned} \left(\frac{(1-\mu\rho)H_e'}{H_e}\right)' + \frac{8Q^2 H_m^{2-2h}}{hH_e^2} &= 0, \\ \left(\frac{(1-\mu\rho)H_m'}{H_m}\right)' + \frac{8P^2 H_e^{2-2h}}{hH_m^2} &= 0, \end{aligned} \tag{2.4}$$

where a prime denotes a derivative with respect to ρ. Since our focus in this paper is on extremal black holes we shall set $\mu = 0$ from now on. The EMD equations of motion also imply a constraint, which for $\mu = 0$ reads

$$\frac{H_e'^2}{H_e^2} + \frac{H_m'^2}{H_m^2} + \frac{2(h-1)H_e' H_m'}{H_e H_m} - \frac{8Q^2 H_m^{2-2h}}{hH_e^2} - \frac{8P^2 H_e^{2-2h}}{hH_m^2} = 0. \tag{2.5}$$

As already mentioned, except for the cases $a = 1$ and $a = \sqrt{3}$ (and, more trivially, $a = 0$), for which the equations are exactly solvable, no explicit solutions for dyonic black holes are known. One approach to studying the solutions in general is to look first for solutions as series expansions in the asymptotic region where r goes to infinity, which corresponds to $\rho \to 0$. Thus one may seek solutions of the form

$$\begin{aligned} H_e(\rho) &= 1 + e_1\rho + e_2\rho^2 + e_3\rho^3 + \cdots, \\ H_m(\rho) &= 1 + m_1\rho + m_2\rho^2 + m_3\rho^3 + \cdots, \end{aligned} \tag{2.6}$$

where the e_i and m_i are constants. Substituting these expansions into (2.4) with $\mu = 0$ (i.e., working at extremality), one finds that all the e_i and m_i for $i \geq 3$ can be solved for in terms of (e_1, e_2, m_1, m_2), with the electric and magnetic charges Q and P being given by

$$Q^2 = 2h(e_1^2 - 2e_2), \qquad P^2 = 2h(m_1^2 - 2m_2). \tag{2.7}$$

The constraint equation (2.5) implies one condition on the four free parameters (e_1, e_2, m_1, m_2), namely

$$e_2 + m_2 + (h-1)e_1 m_1 = 0. \tag{2.8}$$

We shall view the constraint as determining m_2 in terms of the three remaining parameters (e_1, e_2, m_1).

We know that the extremal dyonic black hole solutions should be characterized by just two parameters and not three. The three-parameter solutions will in fact generically describe *singular* spacetimes. This can be seen by performing a numerical integration of the equations, using the small-ρ (i.e. large-r) expansions characterized by (e_1, e_2, m_1) to set initial data for an integration to large ρ (i.e. approaching the horizon at $r = 0$, which is $\rho = \infty$). For generic choices of (e_1, e_2, m_1) the solution develops singularities at some finite value of ρ. By choosing values for two of the parameters (say e_1 and m_1), and then fine-tuning e_2, one can home in on a single specific value for e_2 that gives a solution that behaves properly for a black hole spacetime. In fact, the proper behavior is such that $H_e(\rho)$ and $H_m(\rho)$ go like

$$H_e(\rho) \sim c_e \rho^{1/h}, \qquad H_m \sim c_m \rho^{1/h} \tag{2.9}$$

at large ρ, so that the horizon has a finite and nonzero area (see the expression for the metric in Eqs. (2.1)]. By means of a "shooting method" approach, one can find the value of e_2, for given choices of e_1 and m_1 (and of course, also of the dilaton coupling a) that gives the well-behaved black hole solution. Since the mass of the extremal black hole is given by

$$M = 2h(e_1 + m_1), \tag{2.10}$$

this means that one can, very laboriously, collect numerical data that would allow one to plot the mass as a function of the electric and magnetic charges. (Recall that Q and P are specified in Eqs. (2.7) in terms of the e_i and m_i parameters).

In order to take into account the contribution of the massless scalar in (1.1) to the long range force, we need to extract the scalar charge Σ, which can be defined as the coefficient of r^{-1} in the large-r expansion of the dilaton ϕ,[5]

$$\phi = \phi_\infty + \frac{\Sigma}{r} + \mathcal{O}(r^{-2}). \tag{2.11}$$

Thus one has

$$\Sigma = \sqrt{h(2-h)}(m_1 - e_1). \tag{2.12}$$

In principle, therefore, having accumulated sufficient numerical data from computations that give the charges Q and P, the mass M and the scalar charge Σ for pairs of extremal black holes for a given choice of the dilaton coupling a, one could then calculate the force between the distantly-separated black holes using Eq. (1.2). It is evident that this would be a very time-consuming way of trying to learn about the nature of the force as a function of the charges Q and P for different choices of the dilaton coupling.

Note that by substituting the near-horizon behavior of the functions H_e and H_m, as given in Eq. (2.9), into the equations of motion (2.4), we find that the constants c_e and c_m are given by

$$c_e = 2^{\frac{3}{2h}} Q^{-\frac{1}{h(h-2)}} P^{\frac{h-1}{h(h-2)}}, \qquad c_m = 2^{\frac{3}{2h}} P^{-\frac{1}{h(h-2)}} Q^{\frac{h-1}{h(h-2)}}. \tag{2.13}$$

From this, it follows that the area of the horizon is given by

$$\mathcal{A}_H = 32\pi QP, \tag{2.14}$$

and so the entropy is $S = 8\pi QP$. From these expressions it follows that purely electric or purely magnetic black holes have vanishing horizon area, i.e. they are "small black holes."

One observation that is worth noting at this point is that if one uses (1.2) to calculate the force between a pair of *identical* objects characterized by the four parameters (e_1, m_1, e_2, m_2), taking the charges, mass and scalar charge to be given by (2.7), (2.10) and (2.12), then one finds that the force vanishes purely as a consequence of the constraint (2.8). In other words, regardless of whether one imposes the much more stringent condition that the solution should describe a genuine black hole, one already finds just from the large-r behavior of the fields that the force between two such identical objects will vanish. The vanishing of the force between two identical extremal black holes then follows as a consequence. This latter result is a particular manifestation of a general argument given in [7].

We shall make use of the fact that the force between identical extremal black holes vanishes in the next section, when we derive a simple equation that governs the mass of an extremal EMD black hole in terms of its electric and magnetic charges.

III. A DIFFERENTIAL EQUATION FOR THE MASS

In the EMD theory, for general values of a, we know that the mass of an extremal black hole carrying electric charge

[5]The constant ϕ_∞, the asymptotic value of ϕ at infinity, is zero in the present discussion. However, it is sometimes useful to allow ϕ_∞ to be nonzero, which can be accomplished using a global scaling symmetry of the EMD theory; see Sec. III where we discuss this and employ it in order to rewrite Σ in terms of derivatives of the mass with respect to the charges.

Q and magnetic charge P must be given by a formula of the form

$$M = \mathcal{F}(Q, P). \tag{3.1}$$

The function $\mathcal{F}(Q, P)$ will be different for different values of the dilaton coupling a, and furthermore it is only known explicitly for the three special cases $a = 0$, $a = 1$ and $a = \sqrt{3}$:

$$\begin{aligned} a = 0&: \ M = 2\sqrt{Q^2 + P^2}, \\ a = 1&: \ M = \sqrt{2}(Q + P), \\ a = \sqrt{3}&: \ M = (Q^{\frac{2}{3}} + P^{\frac{2}{3}})^{\frac{3}{2}}. \end{aligned} \tag{3.2}$$

We do know however, on dimensional grounds, that for any value of a the mass function $\mathcal{F}(Q, P)$ must be a homogeneous function of degree 1. That is, we know that

$$\mathcal{F}(\lambda Q, \lambda P) = \lambda \mathcal{F}(Q, P). \tag{3.3}$$

By differentiating this equation with respect to λ, and then setting $\lambda = 1$, it follows that

$$Q \frac{\partial \mathcal{F}(Q, P)}{\partial Q} + P \frac{\partial \mathcal{F}(Q, P)}{\partial P} = \mathcal{F}(Q, P). \tag{3.4}$$

Moreover, the scalar charge Σ can be expressed in terms of derivatives of the mass with respect to Q and P[6]:

$$\Sigma = \frac{1}{2} a \left(Q \frac{\partial M}{\partial Q} - P \frac{\partial M}{\partial P} \right). \tag{3.5}$$

The long-range force F between two identical extremal black holes is given by

$$r^2 F = Q^2 + P^2 - \frac{1}{4} M^2 - \Sigma^2, \tag{3.6}$$

and, as we discussed in the previous section, this must vanish. Plugging the expression (3.5) for the scalar charge into the equation $F = 0$ gives a differential equation for $\mathcal{F}(Q, P)$. Using the homogeneity relation (3.4) we can replace the $P\partial\mathcal{F}/\partial P$ term in the differential equation by $\mathcal{F} - Q\partial\mathcal{F}/\partial Q$, thus giving an ordinary differential equation in which Q is viewed as the independent variable and P is just viewed as a fixed parameter. If we furthermore define $\tilde{f}(Q)$ by writing

$$\mathcal{F}(Q, P) = \sqrt{8PQ}\tilde{f}(Q), \tag{3.7}$$

we see that the equation following from requiring the vanishing of the force becomes

$$Q^2 + P^2 - 2PQ\tilde{f}(Q)^2 - 8a^2 PQ^3 (\partial_Q \tilde{f}(Q))^2 = 0. \tag{3.8}$$

This equation can be simplified even more by defining a new independent variable x in place of Q, where

$$Q = Pe^{2ax}, \qquad -\infty \le x \le \infty. \tag{3.9}$$

Finally, introducing $f(x)$, which is just $\tilde{f}(Q)$ with Q replaced using the definition (3.9), we obtain the simple equation

$$\boxed{f^2 + f'^2 = \cosh 2ax,} \tag{3.10}$$

where f' means $df(x)/dx$. In view of the duality symmetry under the exchange of electric and magnetic charges, it is evident that the required solution for $f(x)$ must obey

$$f(-x) = f(x). \tag{3.11}$$

When $x = 0$, meaning $Q = P$, the dilaton becomes constant and the black hole will just be the extremal Reissner-Nordström $Q = P$ dyon for all values of a, with mass $M = \sqrt{8}Q$. It follows therefore that

$$f(0) = 1, \qquad f'(0) = 0. \tag{3.12}$$

As will be seen in Appendix A, our findings from studying the solutions of Eq. (3.10) numerically are that, for all values of $a > 0$ the function $f(x)$ increases monotonically from $f(0) = 1$, as x increases from 0. The gradient $f'(x)$ also increases monotonically from $f'(0) = 0$ as x increases from 0. Of course because of Eq. (3.11), $f(x)$ also increases monotonically as x becomes increasingly negative, and $f'(x)$ becomes monotonically more negative as x becomes increasingly negative.

If we could solve Eq. (3.10), it would give us $\mathcal{F}(Q, P)$, and hence we would have an explicit mass formula for extremal black holes for all values of a, namely

$$M = \mathcal{F}(Q, P) = \sqrt{8QP} f(x) = \sqrt{8} P e^{ax} f(x),$$
$$\text{where } x = \frac{1}{2a} \log \frac{Q}{P}. \tag{3.13}$$

From Eq. (3.5), we can also express the scalar charge in terms of $f(x)$, finding

$$\Sigma = \frac{1}{2} \sqrt{8QP} f'(x) = \sqrt{2} P e^{ax} f'(x). \tag{3.14}$$

[6]This was shown in [3], based on the result in [16] that the scalar charge Σ, defined as in Eq. (2.11), can also be written as $\frac{\partial M}{\partial \phi_\infty}$, where generically the scalar ϕ is taken to be equal to ϕ_∞ at infinity. As observed in [3] (and adapted to our notation here), there is a global shift symmetry of the EMD theory under which $\phi \to \phi + \phi_\infty, A_\mu \to e^{-\frac{1}{2}a\phi_\infty} A_\mu$, implying for the charges that $Q \to e^{\frac{1}{2}a\phi_\infty} Q$ and $P \to e^{-\frac{1}{2}a\phi_\infty} P$. Calculating $\frac{\partial}{\partial \phi_\infty} M(e^{\frac{1}{2}a\phi_\infty} Q, e^{-\frac{1}{2}a\phi_\infty} P)$ in the shifted system, and then restoring $\phi_\infty = 0$, gives (3.5).

As a check, we can calculate what $f(x)$ is explicitly for the known special cases. Actually $a = 0$ is a degenerate case in this parametrization, because of the redefinition (3.9). For the others, we see from Eqs. (3.2) that we have

$$a = 1 \colon\ f(x) = \cosh x,$$
$$a = \sqrt{3} \colon\ f(x) = \left(\cosh \frac{2x}{\sqrt{3}}\right)^{3/2}. \tag{3.15}$$

As can be verified, these expressions indeed satisfy Eq. (3.10).[7]

Although the ODE in Eq. (3.10) looks very simple, it is not clear how to solve it explicitly. We can, however, use the results obtained above in order to make some observations about the force between two nonidentical extremal black holes. Suppose we have two such black holes, with charges (Q_1, P_1) and (Q_2, P_2). The force F_{12} between them is given by Eq. (1.2), and so, using the definitions given above,

$$r^2 F_{12} = 2P_1 P_2 e^{a(x_1+x_2)}[\cosh a(x_1 + x_2) - f_1 f_2 - f_1' f_2'], \tag{3.16}$$

where f_1 means $f(x_1)$, f_1' means $f'(x_1)$, and so on. Thus once $f(x)$ is known (for a given value of a), we can calculate the force between any pair of extremal black holes.

Note that Eq. (3.10) provides a major computational simplification, compared to the methods previously available to us for determining M. Previously, we would have to do a completely new numerical integration for each choice of Q and P. Each such calculation required the use of the shooting method to find the right choice for the expansion coefficient e_2 that gave a well-behaved solution with a regular black hole horizon. Now, by contrast, we simply have to carry out one numerical calculation to obtain the result for $f(x)$. With this result, we can then immediately find the numerical result for $\mathcal{F}(Q, P)$, and hence for the mass, for any choice of Q and P that we like. (Of course in both approaches, we first pick a value for the dilaton coupling a.)

[7] In [17] Rasheed conjectured the mass formula $M = 2(1 + a^2)^{-1/2}(Q^b + P^b)^{1/b}$, with the constant b given by $b = (2\log 2)/(\log(2 + 2a^2))$, for arbitrary values of a. This would imply that our function $f(x)$ would become $f(x) = (1 + a^2)^{-1/2} 2^{(2-b)/(2b)} \times (\cosh abx)^{1/b}$. However, one can easily verify that except for $a = 1$ and $a = \sqrt{3}$, for which the Rasheed conjecture does indeed (by construction) yield the known solutions, for all other generic values of a the $f(x)$ that results from the Rasheed conjecture fails to satisfy Eq. (3.10). This supports a result in [18], which also found that Rasheed's conjecture could not be correct for general values of a.

IV. PERTURBATIVE SOLUTIONS OF $f(x)$

In the absence of an explicit closed-form solution for $f(x)$ for general values of the dilaton coupling a, we can look at the solution in various regimes where perturbative techniques can be applied. Before doing so, we should point out that the special choice of coupling $a = 1$ will play a crucial role in our discussion. Indeed, when $a = 1$ the extremal solutions are BPS, and the long distance force between any two of them is always zero. As we will see further below, the long distance nature of the force (i.e., whether it is repulsive or attractive) will be correlated with whether a is above or below this special value.

A. Perturbations around $a = 1$

We know that when $a = 1$, the solution to the Eq. (3.10) is $f(x) = \cosh x$. We may now consider perturbing around $a = 1$, by writing

$$a = 1 + \epsilon, \qquad f(x) = \cosh x + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots. \tag{4.1}$$

At order ϵ, we therefore find that $u_1(x)$ must satisfy

$$2 \sinh x u_1' + 2 \cosh x u_1 - 2x \sinh 2x = 0. \tag{4.2}$$

The solution that is regular at $x = 0$ is given by

$$u_1 = \frac{2x \cosh 2x - \sinh 2x}{4 \sinh x}. \tag{4.3}$$

Thus for $a = 1 + \epsilon$ we have the solution

$$f(x) = \cosh x + \epsilon \frac{2x \cosh 2x - \sinh 2x}{4 \sinh x} + \mathcal{O}(\epsilon^2). \tag{4.4}$$

One could also consider an expansion around the other exactly-solvable case, $a = \sqrt{3}$, but this would be of less interest than the expansion around $a = 1$. Indeed, as we mentioned above, the case $a = 1$ has special significance because it supports BPS extremal solutions, whose long range interactions vanish independently of the values of the charges of each black hole solution. Thus, by probing the regime close to $a = 1$ we shall be able to see the transition from having attractive forces when $a < 1$ and repulsive forces when $a > 1$.

B. Perturbative expansion for small x (Q close to P)

First, we consider a perturbative expansion for $f(x)$ in the regime where x is small. In view of the relation $Q = Pe^{2ax}$ in Eq. (3.9), this corresponds to the situation where Q is close to P.

We can look for the perturbative solution for $f(x)$ in Eq. (3.10), by expanding $f(x)$ in the form

$$f(x) = \sum_{n\geq 0} a_{2n} x^{2n}. \tag{4.5}$$

(The required solutions should be symmetric under $x \to -x$, since the black hole metrics must be invariant under exchanging the electric and magnetic charges.) Writing a in terms of a parameter k, where

$$a^2 = \frac{1}{2} k(k+1), \tag{4.6}$$

and expanding Eq. (3.10) in powers of x, one can solve for the coefficients in Eq. (4.5), finding

$$a_0 = 1, \qquad a_2 = \frac{k}{2}, \qquad a_4 = \frac{k^2(2k^2+4k-1)}{24(4k+1)},$$
$$a_6 = \frac{k^3(24k^5+64k^4+52k^3-18k^2+34k+19)}{720(4k+1)^2(6k+1)} \tag{4.7}$$

and so on. In fact, this small-x expansion can be seen to be in agreement with an expansion for the mass of extremal EMD black holes that was obtained in Eq. (4.9) of [18] for the case where Q and P are nearly equal (the ϵ parameter in [18] is equivalent to $2x$ in our expansion here). We have verified the agreement up to order x^{10} in the expansion (4.5).

For our purposes later in the paper, it will suffice to keep the terms just up to order x^2 in the expansion, and so we have the small-x expansion

$$f(x) = 1 + \frac{1}{2} k x^2 + \mathcal{O}(x^4). \tag{4.8}$$

C. Large-x expansion (large hierarchy between charges)

When x is very large and positive, so that there is a large hierarchy between Q and P, Eq. (3.10) becomes approximately

$$f^2 + f'^2 = \frac{1}{2} e^{2ax}. \tag{4.9}$$

Thus $f(x)$ for large positive x takes the form

$$f(x) \approx \frac{e^{ax}}{\sqrt{2(1+a^2)}}. \tag{4.10}$$

Since we must have $f(-x) = f(x)$, the solution when x is large and negative takes the form

$$f(x) \approx \frac{e^{-ax}}{\sqrt{2(1+a^2)}}. \tag{4.11}$$

V. FORCE BETWEEN NONIDENTICAL EXTREMAL BLACK HOLES

We are now ready to examine the main issue we want to address in this paper, which is whether long range forces between nonidentical black holes display any generic features that are tied to the structure of the theory and of the scalar couplings. As we are about to see, the behavior of long range interactions in the EMD model is controlled in a simple way by the dilatonic coupling a. In particular, the sign of the force between distinct extremal black holes—whether it is attractive or repulsive—is correlated with whether $a > 1$ or $a < 1$.

In Eq. (3.16) we gave a general expression for the long-range force between two nonidentical extremal static black holes in the EMD theory, for an arbitrary value of the dilaton coupling a, which we repeat here for convenience:

$$r^2 F_{12} = 2P_1 P_2 e^{a(x_1+x_2)} [\cosh a(x_1+x_2) - f_1 f_2 - f_1' f_2']. \tag{5.1}$$

The expression is written in terms of the electric and magnetic charges (Q_1, P_1) and (Q_2, P_2) for the two black holes, with Q_1 and Q_2 written as

$$Q_1 = e^{2ax_1} P_1, \qquad Q_2 = e^{2ax_2} P_2. \tag{5.2}$$

Note that the force $F(Q_1, P_1; Q_2, P_2)$ between extremal black holes with charges (Q_1, P_1) and (Q_2, P_2) necessarily has the homogeneity property that,[8]

$$F(\lambda_1 Q_1, \lambda_1 P_1; \lambda_2 Q_2, \lambda_2 P_2) = \lambda_1 \lambda_2 F(Q_1, P_1; Q_2, P_2). \tag{5.3}$$

Thus when looking at the force as a function of the charges (Q_1, P_1, Q_2, P_2), there is really only a two-dimensional parameter space of nontrivially inequivalent configurations to explore, rather than the three-dimensional parameter space one might naively have expected. That is to say, there is not only the obvious overall scaling, under which $F(Q_1, P_1; Q_2, P_2)$ would scale by k^2 if all four charges were scaled by k, but there are the two separate, independent scalings of the two sets of charges (Q_1, P_1) and (Q_2, P_2), as seen in Eq. (5.3).

This double scaling homogeneity is seen in the expression (5.1) for the force between the two black holes, with the product of the two magnetic charges appearing in the prefactor. The nontrivial charge-dependence of the force (i.e. dependence that is not merely taking the form of an overall scaling of the force) is then encapsulated by the two parameters x_1 and x_2, which characterise the ratio of the

[8] This homogeneity accounts, in particular, for the fact, mentioned in footnote 2, that the zero-force property for two extremal black holes that have identical sets of charges also holds if the set of charges of one of the black holes is an overall multiple of the set of charges of the other black hole.

electric to magnetic charge for each of the black holes through the relations (5.2).

Since the function $f(x)$ is not known explicitly for general values of a, we would have to determine it numerically, by solving Eq. (3.10), in order to explore the full (x_1, x_2) space of nontrivial parameter values. We can however obtain some analytical results in certain regimes, as we shall now discuss.

A. Force between extremal black holes with *a* near to 1

We saw in Sec. IV A that if we consider values of a close to $a = 1$, by writing $a = 1 + \epsilon$, the solution to Eq. (3.10) up to order ϵ is given by Eq. (4.4). Substituting this expression for $f(x)$ into Eq. (5.1) gives the force

$$r^2 F_{12} = \epsilon P_1 P_2 e^{a(x_1+x_2)} \sinh(x_1 - x_2)[H(x_1) - H(x_2)] + \mathcal{O}(\epsilon^2), \tag{5.4}$$

where we have defined

$$H(x) = \frac{\sinh 2x - 2x}{2\sinh^2 x}. \tag{5.5}$$

The function $H(x)$ is monotonically increasing from $H(-\infty) = -1$ to $H(\infty) = +1$, with $H(-x) = -H(x)$. Therefore we see that the coefficient of ϵ in Eq. (5.4) is always non-negative, for all x_1 and x_2. It follows that F_{12} is positive (repulsive) when ϵ is small and positive (i.e. $a > 1$) and the force is negative (attractive) when ϵ is small and negative (i.e. $a < 1$).

This calculation for the case where a is close to 1 suggests that it may more generally be true that the force between nonidentical extremal static EMD black holes is always positive (repulsive) when $a > 1$, and always negative (attractive) when $a < 1$. We shall now establish further evidence to support this proposition, by considering other regimes where we can perform analytical calculations.

B. Force between extremal black holes with *x* small

Since $Q = e^{2ax} P$, the black holes with Q close to P correspond to the case where x is small. Using the small-x expansion for $f(x)$ that we obtained in Sec. IV B, we may substitute $f(x)$ given by Eqs. (4.5) and (4.7) into the expression (5.1) for the force between two such black holes, finding

$$r^2 F_{12} = \frac{1}{2} P_1 P_2 [k(k-1)(x_1 - x_2)^2 + \cdots], \tag{5.6}$$

where the ellipses denote terms of higher than quadratic order in the small quantities x_1 and x_2. Since $a^2 = \frac{1}{2} k(k+1)$, we see that in this regime (where the x parameters are small) the force between two extremal black holes is again positive whenever $a > 1$ and negative whenever $a < 1$.

C. Force between extremal black holes with *x* large

Recall that large x corresponds to a large hierarchy between the associated charges. There are several cases of interest that we may consider here:

(i) x_1 large and positive, $x_2 = 0$:

Using the expression (4.10) for $f(x_1)$, and Eqs. (3.12) for $f(x_2)$, we find that the force (5.1) between the two black holes becomes

$$r^2 F_{12} \sim P_1 P_1 e^{2ax_1} \left[1 - \sqrt{\frac{2}{1+a^2}}\right] \tag{5.7}$$

as x_1 becomes large. As in the previous specializations we considered, here too the force will be positive if $a > 1$ and negative if $a < 1$.

(ii) x_1 large and positive, x_2 large and negative:

Let us take

$$x_1 = k + \lambda, \qquad x_2 = -k, \tag{5.8}$$

where k is large and positive, and λ is held fixed. Using Eq. (4.10) for $f(x_1)$ and Eq. (4.11) for $f(x_2)$, we find that the force in Eq. (5.1) becomes

$$r^2 F_{12} \sim P_1 P_2 e^{2a(k+\lambda)} \frac{a^2 - 1}{a^2 + 1}. \tag{5.9}$$

In this regime also, the force is positive when $a > 1$ and negative when $a < 1$.

(iii) x_1 and x_2 large and positive:

Taking $x_1 = k + \lambda$ and $x_2 = k$, and sending k to infinity while holding λ fixed, just gives the result that F_{12} goes to zero, if we work at the level of approximation of the expression (4.10) for $f(x)$ at large x. This is not too surprising, since when x_1 and x_2 are both becoming very large, it means that the electric charges of the two black holes are completely overwhelming their magnetic charges. Thus with their charges becoming effectively (after rescaling) of the form $(\tilde{Q}_1, 0)$ and $(\tilde{Q}_2, 0)$, it means that the charges of one black hole are just a multiple of the charges of the other, and so, just as for identical black holes, the force will be zero.

D. Force between nearly identical extremal black holes

Next, we take two extremal black holes with charges (Q_1, P_1) and (Q_2, P_2), for the case

$$Q_1 = P_1 e^{2ax}, \qquad Q_2 = P_2 e^{2a(x+\epsilon)}, \tag{5.10}$$

with ϵ a perturbatively small parameter. Substituting into the expression (5.1) for the force between the black holes and expanding to order ϵ^2 we find, after making use of Eq. (3.10) and its first two derivatives, that

$$r^2F_{12} = \epsilon^2 e^{2ax}P_1P_2(a^2\cosh 2ax - f(x)f''(x) - f'(x)f'''(x)) + \mathcal{O}(\epsilon^3). \quad (5.11)$$

Thus, the sign of the force is governed by the factor

$$G = a^2\cosh 2ax - f(x)f''(x) - f'(x)f'''(x). \quad (5.12)$$

By making use of the Eq. (3.10) for $f(x)$, and its derivatives, one can recast the expression for G into the form

$$G = -a^2\cosh 2ax + f'(x)^2 + f''(x)^2. \quad (5.13)$$

One can also rewrite the expression for G in such as way that all derivatives of f are eliminated, by making use of (3.10) and its derivatives. This gives

$$G = (1-a^2)\cosh 2ax + \frac{a^2\sinh^2 2ax}{\cosh 2ax - f(x)^2} - \frac{2af\sinh 2ax}{\sqrt{\cosh 2ax - f(x)^2}}. \quad (5.14)$$

If one solves Eq. (3.10) numerically for some specified value of a, and then substitutes into (5.11) or one of the expressions for G above, one finds that the force is positive when $a > 1$ and negative when $a < 1$, in accordance with previous expectations. From the standpoint of numerical accuracy, it is advantageous to use the expression for G in Eq. (5.14), where derivatives of the numerically determined function $f(x)$ are not needed.

E. Force inequalities

We conclude this section by mentioning that the force is constrained to obey certain bounds. Consider the identities

$$\begin{aligned}(f_1 \pm f_2)^2 + (f_1' \pm f_2')^2 &= f_1^2 + f_1'^2 + f_2^2 + f_2'^2 \pm 2(f_1f_2 + f_1'f_2') \\ &= \cosh 2ax_1 + \cosh 2ax_2 \pm 2(f_1f_2 + f_1'f_2'),\end{aligned} \quad (5.15)$$

where we have used Eq. (3.10) in order to obtain the second line. Thus we see that

$$-\cosh 2ax_1 - \cosh 2ax_2 \le 2(f_1f_2 + f_1'f_2') \le \cosh 2ax_1 + \cosh 2ax_2. \quad (5.16)$$

In consequence, it follows from Eq. (3.16) that the force F_{12} between two extremal black holes obeys the bounds

$$-\sinh^2\frac{a(x_1-x_2)}{2} \le \hat{F}_{12} \le \cosh^2\frac{a(x_1-x_2)}{2}, \quad (5.17)$$

where we have defined $\hat{F}_{12}$ to be the positive multiple of the force F_{12} given by

$$r^2F_{12} = 4P_1P_2e^{a(x_1+x_2)}\cosh a(x_1+x_2)\hat{F}_{12}. \quad (5.18)$$

The inequalities are not powerful enough to provide useful information about the sign of the force, but they do place constraints on the magnitude of the force as a function of a and the charges.

VI. BINDING ENERGY

Now that we have seen evidence of a direct connection between the range of the coupling a and the long distance nature of the force, we want to examine what happens to the binding energies in the theory. The naive expectation is that they should exhibit the same kind of behavior, and indeed they do. As we will show below, depending on whether $a > 1$ or $a < 1$ the binding energies will be positive or negative. Moreover, their sign will be correlated with the convexity or concavity of the surface $M = \mathcal{F}(Q,P)$ which describes the dependence of the mass on the charges.

Consider two extremal EMD black holes, with charges (Q_1,P_1) and (Q_2,P_2). We may also consider a “composite” extremal black hole, with charges (Q_1+Q_2, P_1+P_2), and then define a notion of binding energy ΔM as the difference between the mass $\hat{M}$ of the composite black hole and the sum of the masses M_1 and M_2 of the two constituents[9]:

$$\Delta M = \hat{M} - M_1 - M_2. \quad (6.1)$$

Since we have introduced the function $\mathcal{F}(Q,P)$ as giving the mass of an extremal black hole with charges (Q,P) [see Eq. (3.1)], we therefore have

$$\Delta M = \mathcal{F}(Q_1+Q_2, P_1+P_2) - \mathcal{F}(Q_1,P_1) - \mathcal{F}(Q_2,P_2). \quad (6.2)$$

It would seem intuitively reasonable to expect that if the binding energy ΔM is positive, then it would be energetically favorable for the composite extremal black hole to separate into its two component black holes. In other words, we might expect that if ΔM is positive, then there should be a repulsive force between the two constituent black holes with charges (Q_1,P_1) and (Q_2,P_2). On the other hand, if ΔM is negative, we might expect that there would be an attractive force between the two constituent black holes.

To examine the structure of ΔM it turns out to be convenient to define the two-dimensional energy surface in $\mathbb{R}^3$ with (x,y,z) coordinates $(Q,P,\mathcal{F}(Q,P))$. The sign of ΔM defined in Eq. (6.2) is then related to the convexity or

[9]We are assuming for simplicity that the composite state is extremal. However, it does not have to be.

concavity of the surface. First, note that the homogeneity property $\mathcal{F}(\lambda Q, \lambda P) = \lambda \mathcal{F}(Q, P)$ means that we can rewrite Eq. (6.2) as

$$\Delta M = 2\mathcal{F}\left(\frac{Q_1+Q_2}{2}, \frac{P_1+P_2}{2}\right) - \mathcal{F}(Q_1, P_1) - \mathcal{F}(Q_2, P_2). \tag{6.3}$$

The two extremal black holes with charges (Q_1, P_1) and (Q_2, P_2) define two points on the energy surface, namely

$$(Q, P, \mathcal{F}(Q, P)) \quad \text{and} \quad (Q', P', \mathcal{F}(Q', P')). \tag{6.4}$$

If we draw a straight line in $\mathbb{R}^3$ joining these two points, its midpoint will lie at

$$\left(\frac{Q+Q'}{2}, \frac{P+P'}{2}, \frac{\mathcal{F}(Q,P)+\mathcal{F}(Q',P')}{2}\right) \tag{6.5}$$

in $\mathbb{R}^3$, and in general it will *not* lie on the energy surface. Now consider the point in $\mathbb{R}^3$, which *does* lie on the two-dimensional surface, whose coordinates are

$$\left(\frac{Q+Q'}{2}, \frac{P+P'}{2}, \mathcal{F}\left(\frac{Q+Q'}{2}, \frac{P+P'}{2}\right)\right). \tag{6.6}$$

Since the x and y coordinates of the two points (6.5) and (6.6) are the same, the two points sit vertically one above the other.

We can now see that if the energy surface defined by the equation $M = \mathcal{F}(Q, P)$ is *convex*, then the point (6.6) will lie *above* the point (6.5). On the other hand, if the energy surface is *concave*, then the point (6.6) will lie *below* the point (6.5). In other words, we have

$$\begin{aligned} \text{Convex:}\ \mathcal{F}\left(\frac{Q+Q'}{2}, \frac{P+P'}{2}\right) &> \frac{\mathcal{F}(Q,P)+\mathcal{F}(Q',P,)}{2}, \\ \text{Concave:}\ \mathcal{F}\left(\frac{Q+Q'}{2}, \frac{P+P'}{2}\right) &< \frac{\mathcal{F}(Q,P)+\mathcal{F}(Q',P,)}{2}. \end{aligned} \tag{6.7}$$

Thus, from Eq. (6.3), it follows that we have:

$$\begin{aligned} \text{Convex:}\ \Delta M &> 0, \\ \text{Concave:}\ \Delta M &< 0, \end{aligned} \tag{6.8}$$

tying the sign of the binding energy to the shape of the energy surface.

A. Binding energy for nearly identical extremal black holes

One can also look at the relation between the sign of the binding energy and the concavity or convexity of the energy surface at the infinitesimal level. Consider two extremal black holes with charges (Q, P) and $((1+\alpha)Q, (1+\beta)P)$, where α and β are infinitesimal. Calculating the binding energy ΔM given in Eq. (6.3), we find, up to quadratic order in α and β,

$$\Delta M = -\frac{1}{4}(\alpha^2 Q^2 \partial_Q^2 + \beta^2 P^2 + 2\alpha\beta QP)\mathcal{F}(Q, P). \tag{6.9}$$

Thus again we see that if the energy surface is locally convex then ΔM is positive, while if the energy surface is locally concave then ΔM is negative.

If we use Eq. (3.13) to write $\mathcal{F}(Q, P)$ in terms of $f(x)$, as $\mathcal{F}(Q, P) = \sqrt{8} P e^{ax} f(x)$ (recalling that $Q = e^{2ax} P$), then the expression (6.9) for the binding energy becomes

$$\Delta M = \frac{P e^{ax}}{4\sqrt{2} a^2}(\alpha - \beta)^2 [a^2 f(x) - f''(x)]. \tag{6.10}$$

Thus, the sign of the binding energy is governed by the sign of $a^2 f(x) - f''(x)$.

Using our expansion in Eq. (4.4) for the case when $a = 1 + \epsilon$, we see that up to linear order in ϵ

$$a^2 f(x) - f''(x) = \frac{\epsilon}{2\sinh^3 x}(\sinh 2x - 2x). \tag{6.11}$$

Since $(\sinh 2x - 2x)$ is positive (negative) when x is positive (negative), it follows that $a^2 f(x) - f''(x)$ is positive when $a = 1 + \epsilon$ with ϵ positive, and negative when ϵ is negative. In other words, at least in the regime where a is close to 1, the sign of the binding energy is indeed correlated with the sign of the force between extremal black holes that we saw previously.

As another check, we can look at the leading-order term in $a^2 f(x) - f''(x)$ when x is small, using the expansion for $f(x)$ in Eqs. (4.6) and (4.7). We find

$$a^2 f(x) - f''(x) = \frac{1}{2} k(k-1) + \mathcal{O}(x^2), \tag{6.12}$$

which, since $a^2 = \frac{1}{2} k(k+1)$, shows that in this small-x regime ΔM is again positive when $a > 1$ and negative when $a < 1$.

To probe the entire parameter space we have to resort to numerical methods in order to solve for $f(x)$ for some chosen value of a. Our numerical results indicate that our observations above are robust, i.e. they show that $a^2 f(x) - f''(x)$, and hence the binding energy, is always positive when $a > 1$ and negative when $a < 1$.

B. Binding energy between extremal black holes with a near to 1

The general expression in Eq. (6.2) for the binding energy between extremal black holes with charges (Q_1, P_1)

and (Q_2, P_2) becomes, upon using the expression (3.13) for $\mathcal{F}(Q,P)$ in terms of $f(x)$,

$$\frac{\Delta M}{\sqrt{8}} = (P_1+P_2)e^{a\hat{x}}f(\hat{x}) - P_1 e^{ax_1}f(x_1) - P_2 e^{ax_2}f(x_2), \tag{6.13}$$

where

$$Q_1 = e^{2ax_1}P_1, \qquad Q_2 = e^{2ax_2}P_2, \\ (Q_1+Q_2) = e^{2a\hat{x}}(P_1+P_2). \tag{6.14}$$

The parameter $\hat{x}$ for the composite extremal black holes is thus determined in terms of P_1, P_2, x_1 and x_2 by the equation

$$(e^{2a\hat{x}} - e^{2ax_1})P_1 + (e^{2a\hat{x}} - e^{2ax_2})P_2 = 0. \tag{6.15}$$

We may assume without loss of generality that $x_2 < x_1$. It then follows from Eq. (6.15) that $\hat{x}$ must satisfy

$$x_2 < \hat{x} < x_1. \tag{6.16}$$

Rather than viewing $\hat{x}$ as a derived quantity after P_1, P_2, x_1 and x_2 are specified, we may instead view P_1, x_1, $\hat{x}$ and x_2, subject to Eq. (6.16), as the four parameters characterizing the parameter space of the two extremal black holes. Since P_1 will then be merely an overall multiplicative factor in the expression for the binding energy, we have just three nontrivial parameters to consider, namely x_1, $\hat{x}$ and x_2, subject again to (6.16).

Now, let us consider the situation where a is close to 1, so as before we write $a = 1+\epsilon$. To linear order in ϵ, the function $f(x)$ is given by Eq. (4.4). We then find that to linear order in ϵ, the binding energy given in Eq. (6.13) becomes

$$\Delta M = \frac{\epsilon P_1 e^{ax_1} W}{\sqrt{8}\sinh x_1 \sinh x_2 \sinh\hat{x}\sinh(\hat{x}-x_2)}, \tag{6.17}$$

where

$$W(x_1,\hat{x},x_2) = (\hat{x}-x_2)\sinh 2x_1 + (x_1-\hat{x})\sinh 2x_2 \\ + x_2\sinh 2(x_1-\hat{x}) + x_1\sinh 2(\hat{x}-x_2) \\ - \hat{x}\sinh 2(x_1-x_2) - (x_1-x_2)\sinh 2\hat{x}. \tag{6.18}$$

It can easily be verified numerically, by testing numerous random choices for triples $(x_1,\hat{x},x_2)$ obeying the inequality (6.16), that the coefficient of ϵ in Eq. (6.17) is indeed always positive. This indicates that the binding energy is positive when $\epsilon > 0$ and negative when $\epsilon < 0$.

We have also constructed an analytical proof of the positivity of the coefficient of ϵ in Eq. (6.17). Since the proof is a little intricate, we have relegated it to Appendix B.

VII. A TWO-DIMENSIONAL PICTURE

Next, we would like to examine in some more detail the connection between the shape of the energy surface and the sign of the binding energy, which we mentioned in (6.8). Owing to the homogeneity $\mathcal{F}(\lambda Q,\lambda P) = \lambda\mathcal{F}(Q,P)$ of the mass function, another way to picture the properties of the energy surface is divide out the equation $M = \mathcal{F}(Q,P)$ by M and hence obtain

$$\mathcal{F}(u,v) = 1, \quad \text{where } u = \frac{Q}{M}, \qquad v = \frac{P}{M}. \tag{7.1}$$

The shape of the curve $\mathcal{F}(u,v) = 1$ in the (u,v) plane captures the characteristics of the energy surface. From the expressions $M = \sqrt{8QP}f(x)$ and $Q = Pe^{2ax}$ in Sec. III we see that the curve $\mathcal{F}(u,v) = 1$ can be written parametrically as

$$u(x) = \frac{e^{ax}}{\sqrt{8}f(x)}, \qquad v(x) = \frac{e^{-ax}}{\sqrt{8}f(x)}. \tag{7.2}$$

In the positive quadrant that we are considering, the $\mathcal{F}(u,v) = 1$ curves run from a point on the positive v axis (corresponding to $x = -\infty$) to a point on the u axis (corresponding to $x = +\infty$).

As can be seen in the examples plotted in Appendix A, the $\mathcal{F}(u,v) = 1$ curves are concave if $a > 1$ and convex if $a < 1$. Note that when the curve $\mathcal{F}(u,v) = 1$ is concave, the energy surface $M = \mathcal{F}(Q,P)$ is convex, and vice versa. This is just an inherent feature of the two different but equivalent ways of characterizing the same information contained in the mass function $\mathcal{F}(Q,P)$. This may be seen explicitly as follows:

Consider first the case where the energy surface is *convex*, which means that $\Delta M > 0$ [see Eq. (6.8)]. By Eq. (6.2), this means that

$$\Delta M = \mathcal{F}(Q_1+Q_2, P_1+P_2) - \mathcal{F}(Q_1,P_1) - \mathcal{F}(Q_2,P_2) > 0. \tag{7.3}$$

Since $\mathcal{F}(Q_1,P_1) = M_1$ and $\mathcal{F}(Q_2,P_2) = M_2$, it follows that

$$\Delta M = (M_1+M_2)\left[\mathcal{F}\left(\frac{Q_1+Q_2}{M_1+M_2}, \frac{P_1+P_2}{M_1+M_2}\right) - 1\right] > 0, \tag{7.4}$$

where we have used the homogeneity property $\mathcal{F}(\lambda Q, \lambda P) = \lambda\mathcal{F}(Q, P)$. The points (u_1, v_1) and (u_2, v_2) in the (u, v) plane lie on the curve $\mathcal{F}(u, v) = 1$, where

$$u_1 = \frac{Q_1}{M_1}, \quad v_1 = \frac{P_1}{M_1}, \quad u_2 = \frac{Q_2}{M_2}, \quad v_2 = \frac{P_2}{M_2}. \tag{7.5}$$

Now define the point $(\bar{u}, \bar{v})$, where

$$\bar{u} = \frac{Q_1 + Q_2}{M_1 + M_2}, \qquad \bar{v} = \frac{P_1 + P_2}{M_1 + M_2}. \tag{7.6}$$

This point lies on the straight line joining (u_1, v_1) to (u_2, v_2), since it is of the form

$$(\bar{u}, \bar{v}) = (u_1, v_1) + \lambda(u_2 - u_1, v_2 - v_1) \tag{7.7}$$

with

$$\lambda = \frac{M_2}{M_1 + M_2}. \tag{7.8}$$

Since Eq. (7.8) implies that λ lies somewhere between 0 and 1, it follows that the point $(\bar{u}, \bar{v})$ must lie *between* (u_1, v_1) and (u_2, v_2). It follows from Eq. (7.4) and the definitions (7.6) that when $\Delta M > 0$ we must have $\mathcal{F}(\bar{u}, \bar{v}) > 1$. In other words, the point $(\bar{u}, \bar{v})$ in the (u, v) plane lies *outside* the curve $\mathcal{F}(u, v) = 1$.[10] That is to say, the curve $\mathcal{F}(u, v) = 1$ must be *concave*.

In summary, we have shown that if the energy surface is *convex* then the curve $\mathcal{F}(u, v) = 1$ is *concave*. Of course the converse is true also.

Finally, using the standard formula for the radius of curvature of a parametric curve $(u(x), v(x))$, namely

$$K = \frac{u'v'' - v'u''}{(u'^2 + v'^2)^{3/2}}, \tag{7.9}$$

we see from (7.2) that here

$$u'v'' - v'u'' = \frac{2a}{f^3(x)}(a^2 f(x) - f''(x)). \tag{7.10}$$

The radius of curvature K is therefore a positive quantity multiplied by Eq. (7.10). Thus the sign of $a^2 f - f''$ provides the criterion for determining the convexity or concavity at each point along the curve. As is to be expected from our previous discussions of convexity and the sign of the binding energy, the quantity $a^2 f - f''$ is the same one that governed the sign of the binding energy in Eq. (6.10). Thus, we see a direct correlation between the range of the coupling a (larger or smaller than one) and the curvature of the energy surface.

[10]Note that because of the homogeneity of the function $\mathcal{F}(u, v)$, the curve $\mathcal{F}(u, v) = k$ is a scaled version of the curve $\mathcal{F}(u, v) = 1$, with the points (u, v) on the curve scaled to (ku, kv).

VIII. CONCLUSIONS

One way in which gravity being weak affects low energy EFTs is through the behavior of long range interactions in the theory. The latter can also be mediated by moduli—massless scalars fields with vanishing potentials—which are ubiquitous in string theories. Thus, despite their simplicity, long range forces can potentially carry useful information about quantum gravity signatures on low energy physics. This observation was one of the motivations behind the RFC. However, thus far essentially all of the work on probing long range interactions in the context of the Swampland program has been restricted to self-forces. It is natural to wonder if interactions between nonidentical states can teach us new lessons, beyond what can be accessed by inspecting self-forces.

In this paper, motivated by the results of [8], we have examined long range forces and binding energies between nonidentical static extremal dyonic black hole solutions to the simple EMD model (1.1). These solutions are known explicitly only for three special values of the dilaton coupling constant, namely $a = 0$, $a = 1$ and $a = \sqrt{3}$. For generic values of a the black hole solutions can only be obtained numerically. Although in principle the extremal black hole mass and the scalar charge must be determined purely in terms of the electric and magnetic charges, it would be very laborious to explore the parameter space of the solutions, in order then to calculate the force between two black holes, by such numerical methods.

A key result in this paper is that we were able to find the simple first-order ordinary differential equation (3.10) for a function $f(x)$, from which the mass and scalar charge can then be calculated. Although we have not been able to find the explicit solution to this equation (except at $a = 1$ and $a = \sqrt{3}$), it is very easy to solve it numerically, performing just one numerical integration for any chosen value of a. Having obtained the numerical solution for the chosen value of a, all information about the mass and the scalar charge is then accessible. In various special cases, such as when the ratio of Q to P is very large or very small, or when Q is very close to P, or when the dilaton coupling a is very near to 1, one can solve for $f(x)$ by perturbation methods.

We were then able to identify a number of novel features in the behavior of the mass function, and the force between nonidentical extremal black holes. First of all, the range of a determines the shape of the surface relating the extremal mass of each black hole solution to its electric and magnetic charges, $M = \mathcal{F}(Q, P)$. In particular,

$$a > 1 \Rightarrow \text{M is convex} \tag{8.1}$$

$$a < 1 \Rightarrow \text{M is concave.} \tag{8.2}$$

Moreover, the sign of the long range force between distinct extremal black holes is also correlated with the range of the coupling, meaning that

$$a > 1 \Rightarrow \text{repulsive force} \tag{8.3}$$

$$a < 1 \Rightarrow \text{attractive force}, \tag{8.4}$$

with the borderline case $a = 1$, for which the corresponding dyonic black holes are BPS, always giving a vanishing force. Finally, as naively expected, the sign of the binding energy ΔM is correlated with the behavior of the long-range force.

A caveat of our discussion about the binding energy is that our analysis assumes—for simplicity—that both the initial and final states are extremal black holes. This, of course, does not have to be the case. Indeed, the naive correlation between the long distance force and the ability to form bound states might cease to exist if the final states are nonextremal. It would be interesting to better understand this case, and what the implications of an attractive or repulsive force would be in that case.

A natural question is whether our results can be connected with the RFC—especially its strongest formulation, in terms of strongly self-repulsive states. It is interesting that our results indicate that in the EMD theory with $a < 1$, all extremal dyonic black holes attract all other such solutions (assuming they carry different charges). This statement in itself is compatible with the RFC which, after all, does not require strongly self-repulsive multiparticle states to be black holes. However, it does raise the question of whether there is a more fundamental distinction between theories with $a > 1$ or $a < 1$, and if so, what is its origin. Independently of the RFC, it would be valuable to identify sharp criteria for the existence or absence of bound states. A more detailed understanding of binding energies and bound states is not only relevant to flat space but also to anti–de Sitter space, where the concept of repulsive forces needs to be expressed in an entirely different way (see e.g. [19–21]). We wonder if some of the features we have identified have an analog description in anti–de Sitter, and how they may be encoded in the dual CFT. We would like to return to these questions in the future.

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APPENDIX A: NUMERICAL RESULTS

In this appendix, we collect a few representative plots constructed by first obtaining numerically-integrated

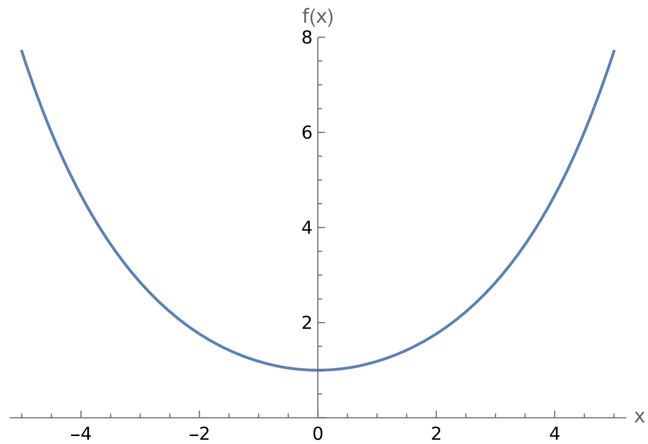

FIG. 1. The function $f(x)$, calculated numerically for $a = \frac{1}{2}$.

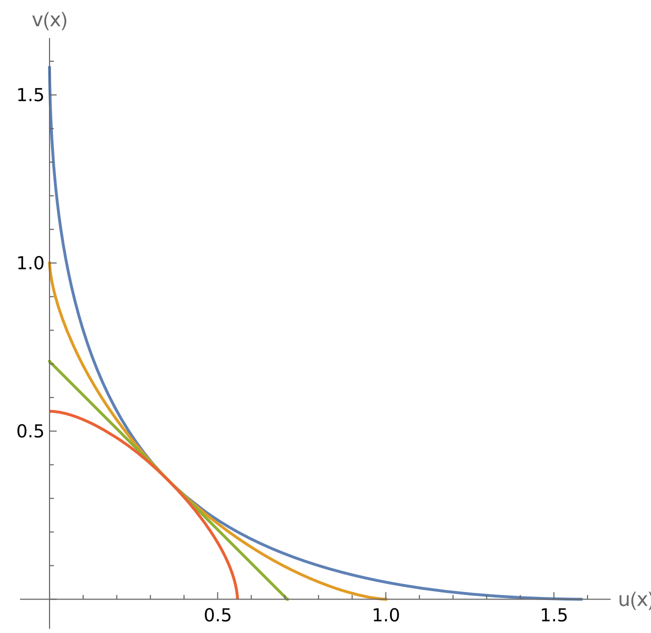

FIG. 2. Plots of $\mathcal{F}(u, v) = 1$ for $a = \frac{1}{2}$ (convex); $a = 1$ (flat); $a = \sqrt{3}$ (concave); and $a = 3$ (concave, outside the $a = \sqrt{3}$ curve). Note that the endpoints of the curves occur at $u = \frac{1}{2}\sqrt{1 + a^2}$, $v = 0$ when $x = \infty$ and $u = 0$, $v = \frac{1}{2}\sqrt{1 + a^2}$ when $x = -\infty$. All the curves pass through the point $u = v = \frac{1}{\sqrt{2}}$ when $x = 0$, which corresponds to the $Q = P$ dyonic Reissner-Nordström solution for all values of a.

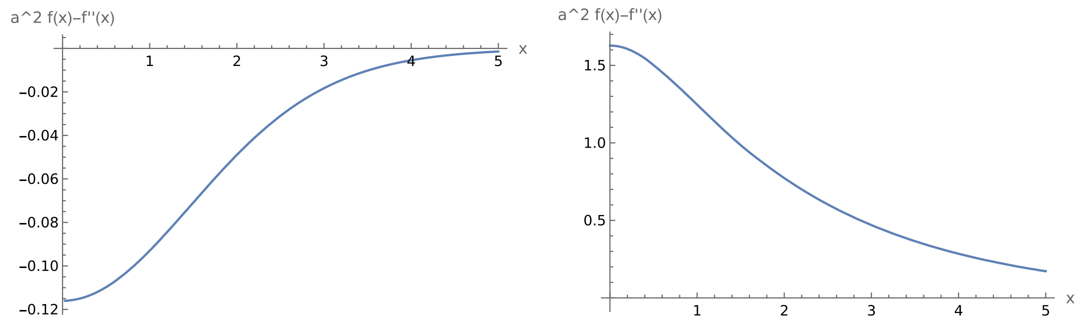

FIG. 3. The function $a^2 f(x) - f''(x)$ plotted for $a = \frac{1}{2}$ (left-hand figure) and $a = 2$ (right-hand figure). These illustrate the fact that this function is negative for a value of $a < 1$, and positive for a value of $a > 1$. From Eq. (7.10) this shows that the curve is convex for $a = \frac{1}{2}$ and concave for $a = 2$.

solutions of Eq. (3.10) for the function $f(x)$, for some representative values of the dilaton coupling a.

First, in Fig. 1, we give a plot of $f(x)$ for an example value of a. Since the function $f(x)$ looks broadly similar for all values of a it is not particularly instructive to display it for different values. We have chosen to plot it for $a = \frac{1}{2}$.

Figure 2 is a plot of the curves $\mathcal{F}(u, v) = 1$ for representative choices of the dilaton coupling a, where $u = Q/M$ and $v = P/M$. The curves were obtained by solving Eq. (3.10) numerically for the various choices of a and then plotting the parametric curve $(u(x), v(x))$ as given by Eq. (7.2).

In Fig. 3, we plot the function $a^2 f(x) - f''(x)$ for a couple of values of a. The sign of this governs the sign of the radius of curvature of the curve $\mathcal{F}(u, v) = 1$.

APPENDIX B: POSITIVITY PROOFS FOR BINDING ENERGY NEAR $a = 1$

Here we establish some results for the positivity of the coefficient of ϵ in Eq. (6.17). Recall that, as discussed in Sec. VI B, the parameter space of the charges (Q_1, P_1) and (Q_2, P_2) of the two extremal black holes in this discussion can, without loss of generality, be fully characterized by the three nontrivial parameters x_1, $\hat{x}$ and x_2 subject to the condition (6.16), i.e. $x_2 \le \hat{x} \le x_1$, together with the charge P_1, which just enters the expression for the binding energy as an overall scaling factor. (See Eqs. (6.14) for the definitions of x_1, $\hat{x}$ and x_2 in terms of the charges.)

Taking into account the signs of the sinh denominators in Eq. (6.17), it can be seen that the propositions that we wish to establish are as follows:

(1) If $0 \le x_2 \le x_1$ then $W(x_1, \hat{x}, x_2) \ge 0$.
(2) If $x_2 \le 0 \le x_1$ then $W(x_1, \hat{x}, x_2) \ge 0$ if $x_2 \le \hat{x} \le 0$ and $W(x_1, \hat{x}, x_2) \le 0$ if $0 \le \hat{x} \le x_1$.
(3) If $x_2 \le x_1 \le 0$ then $W(x_1, \hat{x}, x_2) \le 0$.

The strategy that we shall follow in order to establish these properties of the function $W(x_1, \hat{x}, x_2)$ is to think first of fixed endpoints x_1 and x_2, with $x_2 \le x_1$, and then allow $\hat{x}$ to range in the interval between the endpoints. An important feature of the function $W(x_1, \hat{x}, x_2)$, defined in Eq. (6.18), is that although it depends on x_1, $\hat{x}$ and x_2 both as arguments of hyperbolic functions and with linear dependence as prefactors, it has the property that after differentiating with respect to any of x_1, $\hat{x}$ or x_2, the resulting function has dependence on that coordinate only as an argument in hyperbolic functions. This will mean that we can easily and explicitly solve for the locations of stationary points of $W(x_1, \hat{x}, x_2)$.

To establish proposition 1, we first note that $W(x_1, \hat{x}, x_2)$, defined in Eq. (6.18), vanishes when $\hat{x} = x_2$ or $\hat{x} = x_1$. Next, viewing $W(x_1, \hat{x}, x_2)$ as a function of $\hat{x}$, we look for the values of $\hat{x}$ for which $\frac{d}{d\hat{x}} W(x_1, \hat{x}, x_2) = 0$. This gives a quadratic equation for $\hat{d} \equiv e^{2\hat{x}}$:

$$\begin{aligned} &\hat{d}^2(x_1 - x_2 + x_2 e^{-2x_1} - x_1 e^{-2x_2}) \\ &\quad - 4\hat{d} \sinh x_1 \sinh x_2 \sinh(x_1 - x_2) \\ &\quad + x_1 - x_2 + x_2 e^{2x_1} - x_1 e^{2x_2} = 0. \end{aligned} \tag{B1}$$

It can easily be verified that only one for the two roots for $\hat{d}$ corresponds to a value of $\hat{x}$ that lies inside the interval $x_2 \le \hat{x} \le x_1$. Thus we know that $W(x_1, \hat{x}, x_2)$, viewed as a function of $\hat{x}$, vanishes at $\hat{x} = x_2$ and $\hat{x} = x_1$, and it has just one stationary point in the interval $x_2 \le \hat{x} \le x_1$. It remains only to establish whether $W(x_1, \hat{x}, x_2)$ increases from 0 to a positive maximum and then decreases to 0 again as $\hat{x}$ ranges from x_2 to x_1, or whether instead it decreases from 0 to a negative minimum and then increases to 0 again. This question can be settled by looking at the sign of $H_2(x_1, x_2) \equiv \frac{d}{d\hat{x}} W(x_1, \hat{x}, x_2)|_{\hat{x}=x_2}$.

Viewing $H_2(x_1, x_2)$ as a function of $x_1 \ge x_2$ with x_2 held fixed, its can be seen that $H_2(x_1, x_2)$ vanishes when $x_1 = x_2$, and that $\frac{d}{dx_1} H_2(x_1, x_2)$ has two zeros, when

$$e^{2x_1} = e^{2x_2}, \quad \text{and} \quad e^{2x_1} = \left[1 - \frac{2(\sinh 2x_1 - 2x_1)}{e^{2x_2} - 1 - 2x_2}\right] e^{2x_2}. \tag{B2}$$

The first root lies at $x_1 = x_2$, while the second occurs for $x_1 < x_2$, which lies outside the range $x_2 \le x_1$ that we are considering. With $H_2(x_1, x_2)$ and $\frac{d}{dx_1}H_2(x_1, x_2)$ both vanishing at $x_1 = x_2$ we turn to the second derivative, finding

$$\frac{d^2}{dx_1^2}H_2(x_1, x_2)\Big|_{x_1=x_2} = 4(\sinh 2x_2 - 2x_2). \tag{B3}$$

Since x_2 is assumed to be non-negative this implies that the second derivative is non-negative, and therefore since $H_2(x_1, x_2)$ has no turning points for $x_1 > x_2$, it follows that we must have

$$H_2(x_1, x_2) \geq 0 \quad \text{for } x_1 \geq x_2. \tag{B4}$$

Consequently, we have shown that when $0 \leq x_2 \leq x_1$, the function $W(x_1, \hat{x}, x_2)$ obeys

$$W(x_1, \hat{x}, x_2) > 0 \quad \text{for } x_2 < \hat{x} < x_1, \tag{B5}$$

with $W(x_1, \hat{x}, x_2)$ being equal to zero for $\hat{x} = x_2$ and $\hat{x} = x_1$. This completes the proof of proposition 1.

To establish proposition 2, which is for the case where $x_2 \leq 0 \leq x_1$, we note that $W(x_1, \hat{x}, x_2)$ vanishes when $\hat{x} = x_2$, when $\hat{x} = x_1$, and when $\hat{x} = 0$. Since, as we showed previously, $\frac{d}{d\hat{x}}W(x_1, \hat{x}, x_2)$ vanishes at just two values of $\hat{h}$, it must therefore be that one of these roots lies in the range $x_2 < \hat{x} < 0$ and the other in the range $0 < \hat{x} < x_1$. This means that one possibility is that $W(x_1, \hat{x}, x_2)$ increases from 0 at $\hat{x} = x_2$, then falls to 0 again at $\hat{x} = 0$, and then decreases as $\hat{x}$ becomes positive, before rising to zero again as $\hat{x}$ reaches x_1. The other possibility is that $W(x_1, \hat{x}, x_2)$ decreases from 0 at $\hat{x} = x_2$, then rises to 0 again at $\hat{x} = 0$, and then increases as $\hat{x}$ becomes positive, before falling to zero again as $\hat{x}$ reaches x_1. To settle which of these occurs, we can examine the sign of $\frac{d}{d\hat{x}}W(x_1, \hat{x}, x_2)$ at $\hat{x} = 0$.

Defining $H_0(x_1, x_2) = \frac{d}{d\hat{x}}W(x_1, \hat{x}, x_2)|_{\hat{x}=0}$, it can be seen that $\frac{d}{dx_1}H_0(x_1, x_2) = 0$ at two values of x_1, namely where

$$e^{2x_1} = 1, \quad \text{and} \quad e^{2x_1} = \frac{e^{2x_2} - 1 - 2x_2}{e^{-2x_2} - 1 + 2x_2}. \tag{B6}$$

Writing x_2, which by assumption is negative here, as $x_2 = -\frac{1}{2}p$ where $p > 0$, we see that the second root is at

$$e^{2x_1} = \frac{e^{-p} - 1 + p}{e^p - 1 - p} = \frac{\frac{1}{2}p^2 - \frac{1}{6}p^3 + \frac{1}{24}p^4 + \cdots}{\frac{1}{2}p^2 + \frac{1}{6}p^3 + \frac{1}{24}p^4 + \cdots} < 1, \tag{B7}$$

and therefore this root occurs for $x_1 < 0$, which is outside the assume range $x_1 > 0$. Thus the function $H_0(x_1, x_2)$, viewed as a function of x_1, has no turning points in the range $x_1 > 0$.

With $H_0(x_1, x_2)$ and $\frac{d}{dx_1}H_0(x_1, x_2)$ both vanishing at $x_1 = 0$, we calculate the second derivative, finding

$$\frac{d^2}{dx_1^2}H_0(x_1, x_2)\Big|_{x_1=0} = 4(\sinh 2x_2 - 2x_2), \tag{B8}$$

which is negative since x_2 is assumed to be negative here. Therefore since $H_0(x_1, x_2)$ has no turning points when x_1 is positive, it follows that $H_0(x_1, x_2)$ is negative for all $x_1 > 0$. Thus we have shown that the sign of $\frac{d}{d\hat{x}}W(x_1, \hat{x}, x_2)$ is negative when $\hat{x} = 0$. By the earlier argument, we therefore have that in this $x_2 < 0 < x_1$ case under discussion,

$$\begin{aligned} W(x_1, \hat{x}, x_2) &> 0 \quad \text{when } x_2 < \hat{x} < 0, \\ W(x_1, \hat{x}, x_2) &< 0 \quad \text{when } 0 < \hat{x} < x_1. \end{aligned} \tag{B9}$$

This completes the proof of proposition 2.

Finally, to establish proposition 3 we note that the function $W(x_1, \hat{x}, x_2)$ has the symmetry $W(-x_2, -\hat{x}, -x_1) = -W(x_1, \hat{x}, x_2)$. Therefore having already established in proposition 1 that when $0 \leq x_2 < \hat{x} \leq x_1$ it must be that $W(x_1, \hat{x}, x_2) \geq 0$, it immediately follows that when $x_2 \leq \hat{x} \leq x_1 \leq 0$ it must be that $W(x_2, \hat{x}, x_1) \leq 0$. This proves proposition 3.

In summary, the results above provide a general proof that the binding energy ΔM defined by Eq. (6.1) is positive when the dilaton coupling is of the form $a = 1 + \epsilon$ and ϵ is small and positive, and ΔM is negative when ϵ is small and negative.

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