Equality of the Hilbert Hamiltonian and the canonical Hamiltonian for gauge theories in a static spacetime

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(Received 17 February 2023; accepted 8 May 2023; published 21 June 2023)

The Hilbert energy-momentum tensor for gauge-fixed non-Abelian gauge theories, defined by the variational derivative of the action with respect to the spacetime metric, is a tensor under general coordinate transformations, symmetric in its indices, and BRST invariant. The canonical energy-momentum tensor has none of these properties but the canonical Hamiltonian does correctly generate the time dependence of the fields. It is shown that the Hilbert Hamiltonian $\int d^3x\sqrt{g}T^0_0$ is equal to the canonical Hamiltonian for a general gauge theory coupled to spin-1/2 and spin-0 matter fields (including an $R\phi^2$ term) in a static background metric ($\partial_0 g_{\mu\nu} = 0$ and $g_{0j} = 0$). The equality depends on on the Gauss's law constraint but not on the dynamical Euler-Lagrange equations.

DOI: 10.1103/PhysRevD.107.125021

I. INTRODUCTION

The Hilbert energy-momentum tensor [1] in an arbitrary background metric is determined from the Lagrangian density for matter and radiation by

$$\frac{\delta}{\delta g^{\mu\nu}(x)}\int d^4z\mathcal{L} = \frac{\sqrt{g}}{2}T_{\mu\nu}(x), \qquad (1.1)$$

where $g \equiv -\det(g_{\alpha\beta}) > 0$. $\mathcal{L}$ transforms as a scalar density under general coordinate transformations (i.e. $\mathcal{L}/\sqrt{g}$ is a coordinate scalar) and is gauge invariant except for a gauge-fixing term. $T_{\mu\nu}$ transforms as a tensor under general coordinate transformations, is symmetric in $\mu\nu$, and is BRST invariant [2,3]. It is the source of the gravitational field in the Einstein field equations. The covariant divergence of the mixed tensor is

$$(T^\mu_\nu)_{;\mu} = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}T^\mu_\nu) - \frac{1}{2}(\partial_\nu g_{\alpha\beta})T^{\alpha\beta}. \qquad (1.2)$$

If the fields that appear in $\mathcal{L}$ are required to satisfy the field equations then $(T^\mu_\nu)_{;\mu} = 0$ (see Sec. 94 of [4] or Sec. 12.3 of [5]); but it is not a true conservation law because of the second term in (1.2).

The so-called canonical energy-momentum tensor that results from Noether's first theorem [6] applied to the invariance of $\mathcal{L}$ under global spacetime translations is

$$\sqrt{g}\Theta^\mu_\nu = \sum_s \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_s)}\partial_\nu\chi_s - \delta^\mu_\nu\mathcal{L}, \qquad (1.3)$$

where χ_s runs over all the fields: gauge bosons, ghosts, spin 1/2 fermions, and scalar bosons. Though Θ^μ_ν is not a coordinate tensor it will be referred to as the canonical energy-momentum tensor because when $g_{\alpha\beta}(x)$ is replaced by the Minkowski metric $(\eta_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1)$ the resulting quantity is a Lorentz tensor.

For fields obeying Fermi statistics the ordering in (1.3) is not accurate. The ghost fields η_a and $\bar{\eta}_a$ are independent as are the spin-1/2 fields ψ and $\psi^\dagger$; the correct statement of the first term in (1.3) is

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\eta_a)}\partial_\nu\eta_a + (\partial_\nu\bar{\eta}_a)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\eta}_a)} \qquad (1.4)$$

for ghosts and

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi + (\partial_\nu\psi^\dagger)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)} \qquad (1.5)$$

for spin 1/2 fermions. This is rather cumbersome to repeatedly make explicit and so the simpler form (1.3) will often be used.

a. Θ^μ_ν improvements in Minkowski spacetime: The differences between T^μ_ν and Θ^μ_ν when $g_{\alpha\beta}(x)$ is replaced by the Minkowski metric was resolved for electrodynamics

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by Belinfante and Rosenfeld [7,8]. In modern approaches [9–15] an improved Θ^{μ}_{ν} is obtained by using both global translation invariance and global Lorentz invariance or with Noether's second theorem using local translation invariance. This plus the field equations lead to an improved Lorentz tensor that agrees with the Hilbert tensor in Minkowski spacetime,

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\alpha \Lambda^{\alpha\mu\nu}, \tag{1.6}$$

where the superpotential satisfies $\Lambda^{\alpha\mu\nu} = -\Lambda^{\mu\alpha\nu}$. Two results follow from the antisymmetry:

$$T^{0\nu} = \Theta^{0\nu} + \partial_j \Lambda^{j0\nu}, \tag{1.7}$$

$$\partial_\mu(T^{\mu\nu}) = \partial_\mu(\Theta^{\mu\nu}). \tag{1.8}$$

The first shows that

$$\int d^3x T^{0\nu} = \int d^3x \Theta^{0\nu}; \tag{1.9}$$

the second shows that the integrals (1.9) are time independent. Both require imposing the field equations.

An interesting generalization is presented in Ref. [14]. If $\mathcal{L}$ contains second derivatives, or higher, of the fields then $\Lambda^{\alpha\mu\nu}$ is not antisymmetric in the first two indices; nevertheless, $\Lambda^{00\nu}$ is a spatial divergence which adds another term to (1.7) so that (1.9) is still valid and $\partial_\mu\partial_\alpha\Lambda^{\alpha\mu\nu} = 0$ so (1.8) holds which makes (1.9) time independent.

b. Derivations of $T_{\mu\nu}$ for an arbitrary metric: The Hilbert energy-momentum tensor can be derived in a more geometrical manner using the spacetime diffeomorphism group [16] or fiber bundles [17]. Though it is natural to expect that a canonical energy-momentum tensor satisfying the requirements of gauge invariance, $\mu\nu$ symmetry, and covariant conservation would necessarily be equal to the Hilbert tensor, [18] treats an example of spin-2 fields, the linearized Gauss-Bonnet gravity model, in which this is not true.

c. Static metric with field equations: A static metric satisfies both $\partial_0 g_{\alpha\beta} = 0$ and $g_{0j} = 0$. The Schwarzschild and Reissner-Nordström metrics are of this type and the geometry shares many features of Minkowski spacetime [19–21]. The vanishing of g_{0j} means that Lagrangian terms like $\sqrt{g}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$, and analogous terms for gauge bosons and spin-1/2 fermions, do not violate time-reversal invariance; the field equations allow separation of variables (time vs three-space) and the propagators are invariant under global time translation. If the field equations are satisfied and the metric is static then $\partial_\mu(\sqrt{g}\Theta^{\mu}_{0}) = 0$; thus the canonical Hamiltonian is time independent,

$$\frac{d}{dt}\int d^3x \sqrt{g}\Theta^0{}_0 = 0. \tag{1.10}$$

Under the same conditions (static metric plus field equations) the Hilbert tensor satisfies $\partial_\mu(\sqrt{g}T^\mu{}_0) = 0$ [4,5] and the Hilbert Hamiltonian is time independent,

$$\frac{d}{dt}\int d^3x \sqrt{g}T^0{}_0 = 0. \tag{1.11}$$

It is plausible, but not guaranteed, that the two Hamiltonians are equal; more importantly, the argument gives no information about what happens when the field equations are not satisfied, as is the case in the functional integral formulation of field theory.

d. Outline: This paper investigates what happens when the metric is static and only the non-Abelian Gauss's law is imposed but none of the other field equations. It is assumed throughout that the field decrease at spatial infinity is sufficiently rapid as to allow spatial integration by parts with no boundary terms.

Section II introduces the Lagrangian density $\mathcal{L}$ for a general non-Abelian gauge theory containing five parts: gauge bosons, gauge fixing, ghosts, spin-1/2 fermions, and scalar bosons,

$$\mathcal{L} = \sum_{n=1}^{5} \mathcal{L}^n. \tag{1.12}$$

The spin-1/2 fermions and the scalar bosons are in arbitrary representations of the gauge group. The scalar bosons have Yukawa couplings to fermions and a coupling $\xi R\phi_i^2$ to the Ricci scalar curvature. The variational derivative of each action $\int d^4x\mathcal{L}^n$ gives the Hilbert energy-momentum tensor $T^n_{\mu\nu}$, which is then evaluated for a static metric. The value of $\sqrt{g}g^{00}T^n_{00} + \mathcal{L}^n$ is computed for a static metric.

Section III employs Gauss's law to obtain

$$\int d^3x \sum_{n=1}^{5} \sqrt{g}\, g^{00} T^n_{00} = \int d^3x \left[\sum_s \Pi_s \partial_0 \chi_s - \sum_{n=1}^{5} \mathcal{L}^n\right], \tag{1.13}$$

which shows the equality of the Hilbert and the canonical Hamiltonians,

$$\int d^3x \sqrt{g}g^{00}T_{00} = \int d^3x \sqrt{g}g^{00}\Theta_{00}. \tag{1.14}$$

The dynamical Euler-Lagrange equations are not used. The integrals (1.14) could be called proto-Hamiltonians since they are time dependent.

At this point the equality of the two Hamiltonians could be a special feature of non-Abelian gauge theories, particularly since the curvature appeared only in the term $\xi R\phi_i^2$. To see if the Hamiltonian equality is more general a term of the form

$$\sqrt{g}R^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) \tag{1.15}$$

is investigated. Explicit calculation shows that the Hilbert energy density and the canonical energy density are very different but the Hamiltonians are equal. This rather tedious calculation is contained in Appendix D.

Section IV derives the Hamiltonian equality (1.14) in a general manner in which the details of the Lagrangian density are not specified except that it has only first derivatives of the fields. The result is

$$\partial_\mu(\sqrt{g}T^\mu_0) = \partial_\mu(\sqrt{g}\Theta^\mu_0) - \partial_j N^j, \tag{1.16}$$

using Gauss's law but not the dynamical field equations. The spatial integral is

$$\frac{d}{dt}\int d^3x\sqrt{g}T^0{}_0 = \frac{d}{dt}\int d^3x\sqrt{g}\Theta^0{}_0. \tag{1.17}$$

Since the fields have arbitrary time dependence the integrals must be equal, which confirms (1.14) in the more general case.

II. EXPLICIT RESULTS FOR THE HILBERT ENERGY-MOMENTUM TENSOR

This section will compute $T^n_{\mu\nu}$ for $n = 1, \ldots 5$ for a general time-dependent metric and then catalog, for a static metric, the combination $\sqrt{g}g^{00}T^n_{00} + \mathcal{L}^n$ for gauge bosons, ghosts, and spin-1/2 fermions ($n = 1,\ 3,\ 4$) and the integrated form $\int d^3x[\sqrt{g}g^{00}T^n_{00} + \mathcal{L}^n]$ for the gauge-fixing term and scalar bosons ($n = 2,\ 5$).

A. Gauge bosons

a. Gauge bosons in a general metric: The covariant field-strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ef^{abc}A^b_\mu A^c_\nu \tag{2.1}$$

is independent of the metric tensor. (e will be used for the gauge coupling since g is reserved for the absolute value of the determinant of the metric.) To make the metric dependence explicit $\mathcal{L}^1$ is written in terms of the covariant field strengths

$$\mathcal{L}^1 = -\frac{\sqrt{g}}{4}F^a_{\mu\alpha}F^a_{\nu\beta}g^{\mu\nu}g^{\alpha\beta}. \tag{2.2}$$

The variational derivative of the action gives

$$T^1_{\mu\nu} = -F^a_{\mu\alpha}F^a_{\nu\beta}g^{\alpha\beta} + \frac{g_{\mu\nu}}{4}F^a_{\kappa\alpha}F^a_{\lambda\beta}g^{\kappa\lambda}g^{\alpha\beta}. \tag{2.3}$$

b. Gauge bosons with a static metric: In this case

$$g^{00}T^1_{00} = -\frac{1}{2}F^{0j}_aF^a_{0j} + \frac{1}{4}F^a_{jk}F^{jk}_a, \tag{2.4}$$

and therefore

$$\sqrt{g}g^{00}T^1_{00} + \mathcal{L}^1 = -\sqrt{g}F^{0j}_aF^a_{0j}. \tag{2.5}$$

B. Gauge fixing

a. Gauge fixing with general metric: The Lagrange density

$$\mathcal{L}^2 = \frac{\lambda}{2}\sqrt{g}[(A^\mu_a)_{;\mu}]^2, \tag{2.6}$$

$$(A^\mu_a)_{;\mu} = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}A^a_\nu), \tag{2.7}$$

results in BRST invariance [2,3]. Variation of the action with respect to the metric gives for the energy-momentum tensor

$$T^2_{\mu\nu} = \lambda\left[-A^a_\mu\partial_\nu W_a - A^a_\nu\partial_\mu W_a + g_{\mu\nu}\left[\frac{1}{2}(W^a)^2 + A^\beta_a\partial_\beta W_a\right]\right]. \tag{2.8}$$

where for conciseness

$$W_a \equiv (A^\alpha_a)_{;\alpha} = \frac{1}{\sqrt{g}}\partial_\alpha[\sqrt{g}g^{\alpha\beta}A^a_\beta]. \tag{2.9}$$

b. Gauge-fixing with a static metric: Equation (2.8) with $\mu = \nu = 0$ leads to

$$\sqrt{g}g^{00}T^2_{00} + \mathcal{L}^2 = \lambda\sqrt{g}[-2A^0_a\partial_0 W_a + (W_a)^2 + A^\beta_a\partial_\beta W_a]. \tag{2.10}$$

In the term $(W_a)^2$ if one factor of W_a is expressed in terms of derivatives then a spatial integration by parts produces

$$\int d^3x[\sqrt{g}g^{00}T^2_{00} + \mathcal{L}^2] = \lambda\int d^3x\sqrt{g}[(\partial_0 A^0_a)W_a - A^0_a\partial_0 W_a]. \tag{2.11}$$

C. Ghost fields

a. Ghosts with general time-dependent metric: The Lagrangian density for the ghost fields is

$$\mathcal{L}^3 = \sqrt{g}g^{\mu\nu}(\partial_\mu\bar{\eta}_a)(D_\nu\eta)_a \tag{2.12}$$

where $\bar{\eta}_a$ and η_a obey Fermi statistics, transform in the adjoint representation, are not conjugates of each other, and

$$(D_\nu\eta)_a = \partial_\nu\eta_a - ef_{abc}A^b_\nu\eta_c. \tag{2.13}$$

Varying the action with respect to the metric gives

$$T^3_{\mu\nu} = (\partial_\mu\bar{\eta}_a)(D_\nu\eta)_a + (\partial_\nu\bar{\eta}_a)(D_\mu\eta)_a - g_{\mu\nu}g^{\alpha\beta}(\partial_\alpha\bar{\eta}_a)(D_\beta\eta)_a. \tag{2.14}$$

b. Ghosts with static metric: Equation (2.14) becomes

$$g^{00}T^3_{00} = g^{00}(\partial_0\bar{\eta}_a)(D_0\eta)_a - g^{jk}(\partial_j\bar{\eta}_a)(D_k\eta)_a \tag{2.15}$$

which leads to

$$\sqrt{g}g^{00}T^3_{00} + \mathcal{L}^3 = 2\sqrt{g}g^{00}(\partial_0\bar{\eta}_a)(D_0\eta)_a. \tag{2.16}$$

D. Spin-1/2 fermions

a. Fermions with general metric: The Lagrangian density for fermions

$$\mathcal{L}^4 = \sqrt{g}\left\{\frac{i}{2}\psi^\dagger h\gamma^\mu\nabla_\mu\psi - \frac{i}{2}(\nabla_\mu\psi)^\dagger h\gamma^\mu\psi - \psi^\dagger h(m_f + Y_i\phi_i)\psi\right\} \tag{2.17}$$

requires some explanation. First, m_f is a Hermitian mass matrix and Y_i are Hermitian Yukawa couplings to the real scalar fields that are discussed in Sec. II E. The spacetime-dependent Dirac matrices satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}I \qquad (\gamma^\mu)^\dagger = h\gamma^\mu h^{-1} \tag{2.18}$$

where $h^\dagger = h$ is the spin metric [22]. The spacetime-independent Dirac matrices satisfy

$$\{\gamma^{(\alpha)},\gamma^{(\beta)}\} = 2\eta^{\alpha\beta}I. \tag{2.19}$$

The spacetime dependence of the γ^μ is carried by vierbeins, $\gamma^\mu = e^\mu_{(\alpha)}\gamma^{(\alpha)}$ where $\eta^{\alpha\beta}e^\mu_{(\alpha)}e^\nu_{(\beta)} = g^{\mu\nu}$ and $g_{\mu\nu}e^\mu_{(\alpha)}e^\nu_{(\beta)} = \eta_{\alpha\beta}$. The spin metric $h = \gamma^{(0)}$ is not a function of spacetime nor is the matrix γ^5,

$$\gamma^5 = -i\sqrt{g}\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu/4!. \tag{2.20}$$

Consequently $\nabla_\mu\gamma^5 = 0$ and no additional effort is required if ψ_L and ψ_R are in different representations of the gauge group.

The covariant derivative of the fermion field is

$$\nabla_\mu\psi = \partial_\mu\psi + \Gamma_\mu\psi - ieA^a_\mu T^a\psi, \tag{2.21}$$

where the spin connection is

$$\Gamma_\mu = \frac{1}{8}e^{(\alpha)}_\rho(\partial_\mu e^\rho_{(\beta)} + \Gamma^\rho_{\mu\lambda}e^\lambda_{(\beta)})[\gamma_{(\alpha)},\gamma^{(\beta)}]. \tag{2.22}$$

The detailed calculation of $T^4_{\mu\nu}$ is presented in Appendix B with the result

$$T^4_{\mu\nu} = \frac{i}{4}(\psi^\dagger h\gamma_\mu\nabla_\nu\psi + \psi^\dagger h\gamma_\nu\nabla_\mu\psi) - \frac{i}{4}((\nabla_\nu\psi)^\dagger h\gamma_\mu\psi + (\nabla_\mu\psi)^\dagger h\gamma_\nu\psi) - g_{\mu\nu}\mathcal{L}^4/\sqrt{g}. \tag{2.23}$$

b. Fermions with static metric: For a static metric (2.23) immediately gives

$$\sqrt{g}\,g^{00}T^4_{00} + \mathcal{L}^4 = \sqrt{g}\frac{i}{2}[\psi^\dagger h\gamma^0\nabla_0\psi - (\nabla_0\psi)^\dagger h\gamma^0\psi]. \tag{2.24}$$

For the static metric $e^{(0)}_0 = \sqrt{g_{00}}$ and $e^{(j)}_0 = 0$; the spin connection Γ_0 simplifies to

$$\Gamma_0 = \frac{1}{8}(\partial_j g_{00})[\gamma^0,\gamma^j]. \tag{2.25}$$

This satisfies $(\Gamma_0)^\dagger = -h\Gamma_0 h^{-1}$. The combination that appears in (2.24) is

$$h\gamma^0\Gamma_0 - \Gamma_0^\dagger h\gamma^0 = h\{\gamma^0,\Gamma_0\} = 0. \tag{2.26}$$

Thus, $\nabla_0\psi$ in (2.24) can be replaced by

$$D_0\psi = \partial_0\psi - ieA^a_0 T^a\psi \tag{2.27}$$

so that

$$\sqrt{g}\,g^{00}T^4_{00} + \mathcal{L}^4 = \sqrt{g}\frac{i}{2}[\psi^\dagger h\gamma^0 D_0\psi - (D_0\psi)^\dagger h\gamma^0\psi]. \tag{2.28}$$

E. Scalar bosons

a. Scalars with general metric: For a set of real scalar fields ϕ_i the Lagrangian density is

$$\mathcal{L}^5 = \sqrt{g}\left[\frac{g^{\mu\nu}}{2}(D_\mu\phi)_i(D_\nu\phi)_i - U(\phi) - \frac{1}{2}\xi R\phi_i^2\right], \tag{2.29}$$

where the gauge covariant derivative is

$$(D_\mu\phi)_i = \partial_\mu\phi_i - ieA^a_\mu(t^a)_{ij}\phi_j \tag{2.30}$$

with t^a imaginary and antisymmetric, $U(\phi)$ is a polynomial in the fields invariant under local gauge transformations, R is the Ricci scalar, and ξ is an arbitrary parameter.

The presence of the Ricci scalar R causes some complications in varying the action with respect to the metric tensor. It is convenient to separate the calculation into two parts:

$$\delta\int d^4x\mathcal{L}^5=\frac{1}{2}\int d^4x\sqrt{g}[T^{5A}_{\mu\nu}\delta g^{\mu\nu}-\xi(\delta R)\phi_i^2], \quad (2.31)$$

where $T^{5A}_{\mu\nu}$ results from varying everything except R,

$$T^{5A}_{\mu\nu}=(D_\mu\phi)_i(D_\nu\phi)_i - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}(D_\alpha\phi)_i(D_\beta\phi)_i-U(\phi)-\frac{1}{2}\xi R\phi_i^2\right]. \quad (2.32)$$

The variation of R required in (2.31) is available in Sec. 10.9 of [5],

$$\delta R=R_{\mu\nu}\delta g^{\mu\nu}-(\delta g^{\mu\nu})_{;\mu;\nu}+g_{\mu\nu}(\delta g^{\mu\nu})^{;\rho}{}_{;\rho}. \quad (2.33)$$

The result of the variation is

$$-\frac{1}{2}\xi\int d^4x\sqrt{g}(\delta R)\phi_i^2=\frac{1}{2}\int d^4x\sqrt{g}T^{5B}_{\mu\nu}\delta g^{\mu\nu}, \quad (2.34)$$

where

$$T^{5B}_{\mu\nu}=\xi[-R_{\mu\nu}\phi_i^2+(\phi_i^2)_{;\mu;\nu}-g_{\mu\nu}(\phi_i^2)^{;\rho}{}_{;\rho}]. \quad (2.35)$$

In summary

$$\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\mathcal{L}^5=\frac{1}{2}\sqrt{g}[T^{5A}_{\mu\nu}+T^{5B}_{\mu\nu}]. \quad (2.36)$$

Comment: $T^{5B}_{\mu\nu}$ has the property that its covariant divergence is

$$(g^{\mu\lambda}T^{5B}_{\lambda\nu})_{;\mu}=-\frac{1}{2}\xi(\partial_\nu R)\phi_i^2 \quad (2.37)$$

with no derivatives of the fields.

b. Scalars with static metric: Eq. (2.32) with $\mu=\nu=0$ and a static metric gives

$$\sqrt{g}\,g^{00}T^{5A}_{00}+\mathcal{L}^5=\sqrt{g}\,g^{00}(D_0\phi)_i(D_0\phi)_i. \quad (2.38)$$

For a static metric (2.37) is a true conservation law, $\partial_\mu[\sqrt{g}g^{00}T^{5B}_{00}]=0$ and so

$$\frac{d}{dt}\int d^3x\sqrt{g}\,g^{00}T^{5B}_{00}=0. \quad (2.39)$$

Since the spatial integral is constant and the fields have arbitrary spacetime dependence, it is natural to expect that the integral is zero. Appendix C confirms this,

$$\int d^3x\sqrt{g}\,g^{00}T^{5B}_{00}=0. \quad (2.40)$$

This and (2.38) may be summarized as

$$\int d^3x[\sqrt{g}\,g^{00}(T^{5A}_{00}+T^{5B}_{00})+\mathcal{L}^5] =\int d^3x\sqrt{g}\,g^{00}(D_0\phi)_i(D_0\phi)_i. \quad (2.41)$$

Comment: $T^1_{\mu\nu}, T^3_{\mu\nu}, T^4_{\mu\nu}, T^5_{\mu\nu}$ all agree with Table 1 of [17], which does not include gauge fixing.

III. EQUALITY OF $\int d^3x\sqrt{g}\,g^{00}T_{00}$ AND THE CANONICAL HAMILTONIAN

The results of the previous five subsections will now be combined. All equations assume a static metric but allow arbitrary spacetime dependence of the fields.

A. Assembly of the results

The spatial integral of (2.5), (2.16), and (2.28) when added to the integrated results (2.11) and (2.41) give

$$\int d^3x[\sqrt{g}\,g^{00}T_{00}+\mathcal{L}]=\int d^3x\sqrt{g}[\Omega_1+\Omega_2], \quad (3.1)$$

where Ω_1 comes from the gauge boson $\mathcal{L}^1$ and Ω_2 from the gauge fixing, ghosts, spin-1/2 fermions, and scalars,

$$\Omega_1=-F^{0j}_aF^a_{0j},$$
$$\Omega_2=g^{00}[\lambda(\partial_0A^a_0)W_a-\lambda A^a_0(\partial_0W_a)+2(\partial_0\bar{\eta}_a)(D_0\eta)_a +\frac{i}{2}(\psi^\dagger h\gamma_0D_0\psi-(D_0\psi)^\dagger h\gamma_0\psi)+(D_0\phi)_i(D_0\phi)_i]. \quad (3.2)$$

A more explicit expression of Ω_1 is

$$\sqrt{g}\Omega_1=\sqrt{g}[-F^{0j}_a\partial_0A^a_j+F^{0j}_a(D_jA_0)^a]. \quad (3.3)$$

The spatial integral, after an integration by parts, is

$$\int d^3x\sqrt{g}\Omega_1=-\int d^3x[\sqrt{g}F^{0j}_a\partial_0A^a_j+[D_j(\sqrt{g}F^{0j})]_aA^a_0]. \quad (3.4)$$

Gauss's law requires $[D_j(\sqrt{g}F^{0j})]_a=\sqrt{g}J^0_a$, where

$$\sqrt{g}J^0_a=-\lambda\sqrt{g}\,g^{00}\partial_0W_a+\sum_{n=3}^{5}\frac{\partial\mathcal{L}^n}{\partial A^a_0} \quad (3.5)$$

and therefore

$$\int d^3x\sqrt{g}\Omega_1 = -\int d^3x\sqrt{g}[F_a^{0j}\partial_0 A_j^a + J_a^0 A_0^a]. \quad (3.6)$$

The charge density is

$$J_a^0 = g^{00}[-\lambda\partial_0 W_a + e(\partial_0\bar{\eta}_b)f_{abc}\eta_c + e\psi^\dagger h\gamma_0 T^a\psi - ie(D_0\phi)_i t_{ij}^a\phi_j]. \quad (3.7)$$

Using J_a^0 in (3.6) and adding this to the spatial integral of $\sqrt{g}\Omega_2$ produces cancellations of some of the A_0^a dependence and results in

$$\int d^3x[\sqrt{g}T_0^0 + \mathcal{L}] = \int d^3x\sqrt{g}\{-F_a^{0j}\partial_0 A_j^a + g^{00}[\lambda(\partial_0 A_0^a)W^a + (\partial_0\bar{\eta}_a)(\partial_0\eta_a) + (\partial_0\bar{\eta}_a)(D_0\eta)_a + \frac{i}{2}(\psi^\dagger h\gamma_0\partial_0\psi - (\partial_0\psi)^\dagger h\gamma_0\psi) + (D_0\phi)_i(\partial_0\phi_i)\}. \quad (3.8)$$

B. Canonical momenta

There are seven canonical momenta:

$$\pi_a^j = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_j^a)} = \sqrt{g}F^{j0},$$
$$\pi_a^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0^a)} = \lambda\sqrt{g}g^{00}W_a,$$
$$p_a = \frac{\partial\mathcal{L}}{\partial(\partial_0\eta_a)} = \sqrt{g}g^{00}\partial_0\bar{\eta}_a,$$
$$\bar{p}_a = \frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\eta}_a)} = \sqrt{g}g^{00}(D_0\eta)_a,$$
$$\pi_\psi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\frac{\sqrt{g}}{2}\psi^\dagger h\gamma^0,$$
$$\pi_{\psi^\dagger} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi^\dagger)} = -i\frac{\sqrt{g}}{2}h\gamma^0\psi,$$
$$\pi_i^\phi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)} = \sqrt{g}g^{00}(D_0\phi)_i. \quad (3.9)$$

The rather cumbersome equation (3.8) is more recognizable when expressed in terms of the canonical momenta,

$$\int d^3x[\sqrt{g}T_0^0 + \mathcal{L}] = \int d^3x\{\pi_a^j\partial_0 A_j^a + \pi_a^0\partial_0 A_0^a + p_a\partial_0\eta_a + (\partial_0\bar{\eta}_a)\bar{p}_a + \pi_\psi\partial_0\psi + (\partial_0\psi^\dagger)\pi_{\psi^\dagger} + \pi_i^\phi\partial_0\phi_i\}. \quad (3.10)$$

Of the seven terms involving the canonical momenta the fourth and the sixth have the canonical momenta on the right, as they should be. Eq. (3.10) may be summarized as

$$\int d^3x\sqrt{g}T^0{}_0 = \int d^3x\left[\sum_s \Pi_s\partial_0\chi_s - \mathcal{L}\right]. \quad (3.11)$$

The right-hand side is the Legendre transform of the $\partial_0\chi_s$ dependence of the Lagrangian to the Π_s dependence of the canonical Hamiltonian. This proves the equality

$$\int d^3x\sqrt{g}T^0{}_0 = \int d^3x\sqrt{g}\Theta^0{}_0 \quad (3.12)$$

using Gauss's law but not the dynamical Euler-Lagrange equations.

It is perhaps worth noting that the constraint $[D_j(\sqrt{g}F^{0j}]_a = \sqrt{g}J_a^0$ may be written in terms of the canonical momenta

$$[D_j\pi^j]_a = \pi_a^0\partial_0 W_a + ef_{abc}p_b\eta_c - ie\pi_\psi T^a\psi - ie\pi_i^\phi t_{ij}^a\phi_j, \quad (3.13)$$

and this relation is independent of the metric.

As mentioned in the Introduction, Appendix D contains another test of the equality with a Lagrangian density that contains the Ricci tensor $R^{\mu\nu}$.

IV. GENERAL ARGUMENT

This section investigates a more general context in which the form of the Lagrangian density is not specified, except that it has only first derivatives of the fields. The equality of the two Hamiltonians for a static background metric is again demonstrated.

a. Divergence of the canonical EMT: The divergence of the canonical tensor (1.3) is

$$\partial_\mu(\sqrt{g}\Theta^\mu{}_\nu) = \sum_s\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_s)}\partial_\nu\chi_s\right] - \partial_\nu\mathcal{L}. \quad (4.1)$$

The spacetime dependence of $\mathcal{L}$ occurs in both the fields and the metric,

$$\partial_\nu\mathcal{L} = \sum_s\left[\frac{\partial\mathcal{L}}{\partial\chi_s}(\partial_\nu\chi_s) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_s)}\partial_\mu\partial_\nu\chi_s\right] - \partial_\nu\mathcal{L}\Big|_\chi. \quad (4.2)$$

The last term requires differentiating the spacetime dependence of the metric while keeping all the fields χ_s fixed. Substitution above gives

$$\partial_\mu(\sqrt{g}\Theta^\mu{}_\nu) = \sum_s\mathcal{M}_s(\partial_\nu\chi_s) - \partial_\nu\mathcal{L}\Big|_\chi, \quad (4.3)$$

$$\mathcal{M}_s \equiv \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_s)}\right) - \frac{\partial\mathcal{L}}{\partial\chi_s}. \tag{4.4}$$

$\mathcal{M}_s$ vanishes if the Euler-Lagrange equations are imposed but even then $\partial_\mu(\sqrt{g}\Theta^\mu_\nu) \neq 0$ for a general metric.

b. Divergence of the Hilbert EMT: The covariant divergence of $T^\mu{}_\nu$ is computed from the fact that the action is invariant under a coordinate transformation $x^\nu \to x'^\nu$; see Sec. 94 of [4] or Sec. 12.3 of [5]. The metric and the fields (scalar, ghost, spin-1/2 fermion, and gauge) transform as follows:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x), \tag{4.5}$$

$$\begin{aligned}\phi_i{}'(x') &= \phi_i(x),\\ \eta'_a(x') &= \eta_a(x),\\ \psi'(x') &= \psi(x),\\ A'^a_\alpha(x') &= \frac{\partial x^\lambda}{\partial x'^\alpha} A^a_\lambda(x).\end{aligned} \tag{4.6}$$

(The covariant vierbein transforms the same as the covariant gauge field but it will not be needed.) Invariance of the action means that

$$\int d^4x'\mathcal{L}(g'_{\mu\nu}(x'),\chi'(x')) = \int d^4x\mathcal{L}(g_{\mu\nu}(x),\chi(x)).$$

Relabeling of the integration variable gives

$$\int d^4x\mathcal{L}(g'_{\mu\nu}(x),\chi'(x)) = \int d^4x\mathcal{L}(g_{\mu\nu}(x),\chi(x)).$$

The action is invariant under a change in the functional form of the metric and the fields at the same position x,

$$\Delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x), \tag{4.7}$$

$$\Delta\chi_s(x) \equiv \chi'_s(x) - \chi_s(x). \tag{4.8}$$

Let

$$x'^\nu = x^\nu + \xi^\nu(x), \tag{4.9}$$

where $\xi^\nu(x)$ is an arbitrary, infinitesimal function. The change in the metric is

$$\Delta g_{\mu\nu} = -(\xi_\mu)_{;\nu} - (\xi_\nu)_{;\mu}. \tag{4.10}$$

For the scalars, ghosts, and spin-1/2 fermions

$$\Delta\chi_s = -\xi^\nu\partial_\nu\chi_s; \tag{4.11}$$

and for the vector potential there is an additional term

$$\Delta A^a_\alpha = -\xi^\nu\partial_\nu A^a_\alpha - (\partial_\alpha\xi^\nu)A^a_\nu. \tag{4.12}$$

Under these variations the action is invariant. The variation with respect to covariant metric gives the contravariant energy-momentum tensor but with the opposite sign from (1.1),

$$0 = -\int d^4x\frac{\sqrt{g}}{2}T^{\mu\nu}\Delta g_{\mu\nu} + \sum_s\int d^4x\mathcal{M}_s\xi^\nu\partial_\nu\chi_s + \int d^4x\mathcal{M}^\mu_a(\partial_\mu\xi^\nu)A^a_\nu. \tag{4.13}$$

In the second term the sum on fields s includes a term for the gauge potential; in the third term the explicit form is

$$\mathcal{M}^\mu_a = \partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda A^a_\mu)}\right) - \frac{\partial\mathcal{L}}{\partial A^a_\mu}. \tag{4.14}$$

An integration by parts in the first and third term yields

$$\int d^4x\sqrt{g}(T^{\mu\nu})_{:\mu}\xi_\nu = \sum_s\int d^4x\mathcal{M}_s\xi^\nu\partial_\nu\chi_s - \int d^4x\partial_\mu(\mathcal{M}^\mu_a A^a_\nu)\xi^\nu. \tag{4.15}$$

Since $\xi^\nu(x)$ is an arbitrary function, the integrands must be equal,

$$\sqrt{g}(T^\mu{}_\nu)_{;\mu} = \sum_s\mathcal{M}_s(\partial_\nu\chi_s) - \partial_\mu(\mathcal{M}^\mu_a A^a_\nu). \tag{4.16}$$

The first term in (4.14) is $-\partial_\mu(\sqrt{g}F^{\mu\alpha})_a$ which combines with $\partial\mathcal{L}^1/\partial A^a_\alpha$ to give the gauge-covariant derivative $-[D_\mu(\sqrt{g}F^{\mu\alpha})]_a$. The remaining terms from the gauge fixing, ghosts, spin-1/2 fermions, and scalars define the current density

$$\sqrt{g}J^\alpha_a = -\lambda\partial_\nu(\sqrt{g}g^{\nu\alpha}W_a) + \sum_{n=3}^{5}\frac{\partial\mathcal{L}^n}{\partial A^a_\alpha} \tag{4.17}$$

and so

$$\mathcal{M}^\alpha_a = [D_\mu(\sqrt{g}F^{\mu\alpha})]_a + \sqrt{g}J^\alpha_a. \tag{4.18}$$

The constraint of Gauss's law requires

$$\mathcal{M}^0_a = 0. \tag{4.19}$$

Therefore, the last term in (4.16) is really only a spatial divergence $\partial_j(\mathcal{M}^j_a A^a_\nu)$ though the form $\partial_\mu(M^\mu_a A^a_\nu)$ will at times be used below.

c. Comparison of divergences of the Hilbert EMT and the canonical EMT: The terms of the form $\mathcal{M}_s(\partial_\nu\chi_s)$ in (4.16) also appear in the canonical divergence (4.3) and so (4.16) can be expressed as

$$\partial_\mu(\sqrt{g}T^\mu{}_\nu) - \frac{\sqrt{g}}{2}(\partial_\nu g_{\alpha\beta})T^{\alpha\beta} = \partial_\mu(\sqrt{g}\Theta^\mu{}_\nu) + \partial_\nu \mathcal{L}|_\chi - \partial_\mu(\mathcal{M}^\mu_a A^a_\nu). \quad (4.20)$$

This holds for a general metric and arbitrary fields using Gauss's law but not the dynamical Euler-Lagrange equations.

For simplicity suppose $\mathcal{L}$ has only first and second derivatives of the metric, as is the case for the general gauge theory in Secs. II and III. The second and fourth terms of Eq. (4.20) require the derivatives

$$-\frac{\sqrt{g}}{2}T^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu}}\right] + \partial_\mu\partial_\rho\left[\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu\rho}}\right]$$

$$\partial_\nu\mathcal{L}|_\chi = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}}g_{\alpha\beta,\nu} + \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu}}g_{\alpha\beta,\nu\mu} + \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu\rho}}g_{\alpha\beta,\nu\mu\rho}.$$

The difference between these two is a total derivative; Eq. (4.20) becomes

$$\partial_\mu(\sqrt{g}T^\mu{}_\nu) = \partial_\mu(\sqrt{g}\Theta^\mu{}_\nu) + \partial_\mu\Sigma^\mu{}_\nu - \partial_\mu(A^a_\nu\mathcal{M}^\mu_a), \quad (4.21)$$

where

$$\Sigma^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu}}g_{\alpha\beta,\nu} + \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu\rho}}g_{\alpha\beta,\nu\rho} - \partial_\rho\left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu\rho}}\right)g_{\alpha\beta,\nu}.$$

The derivatives of the metric are not tensors and so $\Sigma^\mu{}_\nu$ is not a tensor. (If $\mathcal{L}$ has any number of metric derivatives of the metric the form (4.21) still holds but $\Sigma^\mu{}_\nu$ is more complicated.) Because of Gauss's law (4.19) the spatial integral of (4.21) is

$$\frac{d}{dt}\int d^3x\sqrt{g}T^0{}_\nu = \frac{d}{dt}\int d^3x[\sqrt{g}\Theta^0{}_\nu + \Sigma^0{}_\nu]. \quad (4.22)$$

Since the time dependence of the fields and the metric is arbitrary the integrals must be equal,

$$\int d^3x\sqrt{g}T^0{}_\nu = \int d^3x[\sqrt{g}\Theta^0{}_\nu + \Sigma^0{}_\nu]. \quad (4.23)$$

Case 1: Minkowski metric. If the metric is chosen to be $(\eta_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1)$, then $\Sigma^\mu{}_\nu = 0$ in (4.23). The resulting equality holds for arbitrary fields and thus is more general than the usual result (1.9).

Case 2: Arbitrary time-dependent metric but $\mathcal{L}$ has no derivatives of the metric. These conditions are satisfied by the gauge boson $\mathcal{L}^1$ and by the scalar boson $\mathcal{L}^5$ if the $\xi R\phi_i^2$ term is omitted. Obviously $\Sigma^\mu{}_\nu = 0$.

Case 3: Static metric. This includes the case of principle interest since $\mathcal{L}$ for a general gauge theory contains first derivatives of the metric in the gauge-fixing term and the spin connection Γ_μ for fermions, and second derivatives of the metric in the $\xi R\phi_i^2$ term. For a static metric $\Sigma^0{}_0 = 0$ and so

$$\int d^3x\sqrt{g}T^0{}_0 = \int d^3x\sqrt{g}\Theta^0{}_0. \quad (4.24)$$

The dynamical field equations have not been used. This agrees with the explicit calculations leading to (3.12) and explains the miraculous cancellations for the example considered in Appendix D.

APPENDIX A: CHRISTOFFEL SYMBOL AND CURVATURE TENSOR FOR STATIC METRIC

A static metric is time independent and $g_{0j} = 0$. Christoffel symbols with an odd number of time components vanish,

$$\Gamma^0_{00} = \Gamma^j_{k0} = \Gamma^0_{k\ell} = 0. \quad (A1)$$

The Christoffel symbols with two 0's are

$$\Gamma^j_{00} = -\frac{1}{2}g^{jk}\partial_k g_{00}, \quad (A2)$$

$$\Gamma^0_{k0} = \frac{1}{2}g^{00}\partial_k g_{00}. \quad (A3)$$

$\Gamma^j_{k\ell}$ is nonvanishing and independent of g_{00}. Two useful contractions are

$$\Gamma^j_{jk} = \frac{1}{\sqrt{\gamma}}\partial_k\sqrt{\gamma}, \quad (A4)$$

$$g^{jk}\Gamma^\ell_{jk} = -\frac{1}{\sqrt{\gamma}}\partial_j(\sqrt{\gamma}g^{j\ell}), \quad (A5)$$

where $|\det(g_{\alpha\beta})| = g_{00}\gamma$. The Riemann-Christoffel tensor with four spatial components is independent of g_{00}: ${}^{(4)}R_{ijk\ell} = {}^{(3)}R_{ijk\ell}$. If there is one time component $R_{0jk\ell} = 0$. If there are two time components

$$R_{0j0k} = -\frac{1}{2}\partial_j\partial_k g_{00} + \frac{1}{2}\Gamma^\ell_{jk}\partial_\ell g_{00} + \frac{1}{4}\frac{(\partial_j g_{00})(\partial_k g_{00})}{g_{00}}. \quad (A6)$$

The Ricci tensor with two spatial indices is

$${}^{(4)}R_{jk} = g^{00}R_{0j0k} + {}^{(3)}R_{jk}, \quad (A7)$$

where $g^{ab}R_{ajbk} = {}^{(3)}R_{jk}$ is independent of g_{00} if there are two time indices

$${}^{(4)}R_{00} = g^{jk}R_{0j0k} = -\frac{g_{00}}{2\sqrt{g}}\partial_j\left[\sqrt{g}g^{jk}\frac{\partial_k g_{00}}{g_{00}}\right], \quad (A8)$$

and if one time index ${}^{(4)}R_{0k} = 0$. The Ricci scalar is

$$ {}^{(4)}R = 2g^{00}g^{jk}R_{0j0k} + {}^{(3)}R. \quad (A9) $$

APPENDIX B: CALCULATION OF $T^4_{\mu\nu}$ FOR FERMIONS WITH ARBITRARY TIME-DEPENDENT METRIC

This appendix contains the detailed calculation of $T^4_{\mu\nu}$ displayed in Eq. (2.23). The fermion field ψ is not required to satisfy the Dirac equation.

The Lagrangian density (2.17) may be written

$$ \mathcal{L}^4 = \sqrt{g}\frac{1}{2}[\psi^\dagger(K\psi) + (K\psi)^\dagger\psi], \quad K = ih\gamma^\mu\nabla_\mu - h(m_f + Y_i\phi_i). \quad (B1) $$

The variation of $\mathcal{L}^4$ with respect to the metric is

$$ \delta\mathcal{L}^4 = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\mathcal{L}^4 + \frac{1}{2}\sqrt{g}[\psi^\dagger(\delta K\psi) + (\delta K\psi)^\dagger\psi]. \quad (B2) $$

In [22] the variations are shown to be

$$ \begin{aligned} \delta K &= \frac{i}{2}(\delta g^{\mu\nu})h\gamma_\mu\nabla_\nu + (\delta G)K + K(\delta G) + \delta X, \\ \delta G &= \frac{1}{8}\eta^{\alpha\beta}e^\lambda_{(\alpha)}[\gamma_\lambda, \gamma_\nu]\delta e^\nu_{(\beta)}, \\ \delta X &= \frac{i}{8}(\delta\Gamma^\mu_{\lambda\nu})h\gamma^\lambda[\gamma_\mu, \gamma^\nu]. \end{aligned} \quad (B3) $$

Because $(\delta X)^\dagger = -\delta X$ it disappears from (B2) and so

$$ \begin{aligned} \delta\mathcal{L}^4 &= -\frac{1}{2}g_{\mu\nu}\mathcal{L}^4\delta g^{\mu\nu} + \frac{i}{4}\sqrt{g}[\psi^\dagger h\gamma_\mu\nabla_\nu\psi - (\nabla_\nu\psi)^\dagger h\gamma_\mu\psi]\delta g^{\mu\nu} \\ &\quad + \frac{1}{2}[\Delta_1 + \Delta_2], \\ \Delta_1 &\equiv \sqrt{g}[\psi^\dagger(\delta GK\psi) + (\delta GK\psi)^\dagger\psi], \\ \Delta_2 &\equiv \sqrt{g}[\psi^\dagger(K\delta G\psi) + (K\delta G\psi)^\dagger\psi]. \end{aligned} \quad (B4) $$

Δ_2 contains derivatives of the vierbein because of $K\delta G$. To simplify Δ_2 the following identity valid for arbitrary spinor fields ψ_1 and ψ_2, is useful

$$ \sqrt{g}\psi_1^\dagger(K\psi_2) - \sqrt{g}(K\psi_1)^\dagger\psi_2 = -\partial_\lambda(\sqrt{g}\psi_1^\dagger h\gamma^\lambda\psi_2). \quad (B5) $$

For the first term in Δ_2 take $\psi_1 = \psi$ and $\psi_2 = \delta G\psi$ and for the second term in Δ_2 take $\psi_1 = K\delta G\psi$ and $\psi_2 = \psi$. The result is

$$ \Delta_2 = \Delta_1 + i\partial_\lambda[\sqrt{g}\psi^\dagger h\gamma^\lambda\delta G\psi - \sqrt{g}(\delta G\psi)^\dagger h\gamma^\lambda\psi]. \quad (B6) $$

Since $(\delta G)^\dagger = -h(\delta G)h^{-1}$ this is

$$ \Delta_2 = \Delta_1 + i\partial_\lambda[\sqrt{g}\psi^\dagger h\{\gamma^\lambda, \delta G\}\psi]. \quad (B7) $$

The variation of $\mathcal{L}^4$ becomes

$$ \begin{aligned} \delta\mathcal{L}^4 &= -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\mathcal{L}^4 + i\frac{\sqrt{g}}{4}[\psi^\dagger h\gamma_\mu\nabla_\nu\psi - (\nabla_\nu\psi)^\dagger h\gamma_\mu\psi]\delta g^{\mu\nu} \\ &\quad + \sqrt{g}[\psi^\dagger(\delta G)K\psi + (K\psi)^\dagger(\delta G)^\dagger\psi] \\ &\quad + i\partial_\lambda[\sqrt{g}\psi^\dagger h\{\gamma^\lambda, \delta G\}\psi]. \end{aligned} \quad (B8) $$

The total derivative in the fourth line does not contribute to the variation of the action. From (B3) the derivative of G with respect to the vierbein is

$$ e_{\mu(\beta)}\frac{\partial G}{\partial e^\nu_{(\beta)}} = \frac{1}{8}[\gamma_\mu, \gamma_\nu], \quad (B9) $$

and the derivative with respect to the metric tensor is

$$ \frac{\partial G}{\partial g^{\mu\nu}} = \frac{1}{2}\left[e_{\mu(\beta)}\frac{\partial G}{\partial e^\nu_{(\beta)}} + e_{\nu(\beta)}\frac{\partial G}{\partial e^\mu_{(\beta)}}\right] = 0. \quad (B10) $$

Consequently the energy-momentum tensor is determined by just the first two lines of (B8),

$$ \begin{aligned} T^4_{\mu\nu} &= \frac{i}{4}[\psi^\dagger h\gamma_\mu\nabla_\nu\psi + \psi^\dagger h\gamma_\nu\nabla_\mu\psi] \\ &\quad - \frac{i}{4}[(\nabla_\nu\psi)^\dagger h\gamma_\mu\psi + (\nabla_\mu\psi)^\dagger h\gamma_\nu\psi] - g_{\mu\nu}\mathcal{L}^4/\sqrt{g}. \end{aligned} \quad (B11) $$

APPENDIX C: PROOF THAT $\int d^3x\sqrt{g}g^{00}T^{5B}_{00} = 0$

The "extra" piece (2.35) in the energy-momentum tensor for scalar bosons is

$$ T^{5B}_{\mu\nu} = -R_{\mu\nu}\Psi + \Psi_{;\mu;\nu} - g_{\mu\nu}\Psi^{;\rho}_{;\rho}, \quad (C1) $$

where $\Psi = \xi\phi_i^2$. This came from the term $-\frac{1}{2}\xi\sqrt{g}(\delta R)\phi_i^2$ in (2.31). The following applies to any action of the form

$$ -\frac{1}{2}\int d^4x\sqrt{g}\Psi R, \quad (C2) $$

where Ψ is a coordinate scalar. A Lagrangian density of the form $\sqrt{g}X^\mu\partial_\mu R$ or $\sqrt{g}Y^{\mu\nu}R_{;\mu;\nu}$ has an action of the form (C2) after an integration by parts.

Static metric: To analyze T^{5B}_{00} for a static metric, organize the $\mu = \nu = 0$ component as

$$\sqrt{g}g^{00}T^{5B}_{00} = A^0_0 + B^0_0,$$
$$A^0_0 = -\frac{\sqrt{g}}{g_{00}}R_{00}\Psi,$$
$$B^0_0 = \sqrt{g}[\Psi^{;0}{}_{;0} - \Psi^{;\lambda}{}_{;\lambda}]. \tag{C3}$$

From (A8)

$$A^0{}_0 = \frac{1}{2}\partial_j\left[\sqrt{g}g^{jk}\frac{\partial_k g_{00}}{g_{00}}\right]\Psi. \tag{C4}$$

In $B^0{}_0$ the time derivatives of the fields all cancel and leave

$$B^0{}_0 = -\frac{\sqrt{g}}{g_{00}}\Gamma^k_{00}\partial_k\Psi - \partial_k[\sqrt{g}g^{kj}\partial_j\Psi)]. \tag{C5}$$

Using (A2) gives a more complicated looking result,

$$B^0{}_0 = \frac{1}{2}\sqrt{g}g^{jk}\frac{\partial_j g_{00}}{g_{00}}\partial_k\Psi - \partial_k[\sqrt{g}g^{jk}\partial_j\Psi)], \tag{C6}$$

but the sum with $A^0{}_0$ gives another total derivative,

$$A^0{}_0 + B^0{}_0 = \frac{1}{2}\partial_j\left[\sqrt{g}g^{jk}\frac{\partial_k g_{00}}{g_{00}}\Psi\right] - \partial_k[\sqrt{g}g^{jk}\partial_j\Psi)]. \tag{C7}$$

Consequently

$$\int d^3x\sqrt{g}g^{00}T^{5B}_{00} = 0. \tag{C8}$$

APPENDIX D: A COMPLICATED EXAMPLE

This appendix presents the details of the example mentioned in Sec. I in which $\mathcal{L}$ depends on the Ricci curvature tensor $R^{\mu\nu}$. Despite the complications the result is again that the Hilbert Hamiltonian is equal to the canonical Hamiltonian for a static metric even without using the field equation. The Lagrangian density is

$$\mathcal{L} = \sqrt{g}R^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) \tag{D1}$$

and could be added to the conventional Lagrangian density for a scalar field if multiplied by a coefficient of mass dimension M^{-2}. The question is not whether this $\mathcal{L}$ is a physically acceptable addition but whether the resulting Hamiltonians, Hilbert and canonical, are equal. The canonical energy-momentum pseudotensor (1.3) is

$$\sqrt{g}\Theta^\mu{}_\nu = 2\sqrt{g}R^{\mu\alpha}(\partial_\alpha\phi)(\partial_\nu\phi) - \delta^\mu{}_\nu\mathcal{L}. \tag{D2}$$

For a static background metric

$$\Theta^0{}_0 = R^{00}\partial_0\phi\partial_0\phi - R^{jk}\partial_j\phi\partial_k\phi. \tag{D3}$$

The field equations have not been used.

a. Hilbert $T_{\mu\nu}$ for a general metric: The Hilbert energy-momentum tensor for a general metric is more complicated to compute. It is convenient to organize $\mathcal{L}$ as

$$\mathcal{L} = \sqrt{g}R_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}\Phi_{\mu\nu}, \tag{D4}$$

where $\Phi_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi)$ is a tensor independent of the metric. One needs the variation of the covariant Ricci tensor with respect to the contravariant $\delta g^{\mu\nu}$ given in Eq. (10.9.3) of [5],

$$\delta R_{\alpha\beta} = \frac{1}{2}g_{\mu\nu}(\delta g^{\mu\nu})_{;\alpha;\beta} + \frac{1}{2}g_{\alpha\mu}g_{\beta\nu}(\delta g^{\mu\nu})^{;\lambda}{}_{;\lambda}$$
$$-\frac{1}{2}g_{\alpha\mu}(\delta g^{\mu\nu})_{;\beta;\nu} - \frac{1}{2}g_{\beta\nu}(\delta g^{\mu\nu})_{;\alpha;\mu}. \tag{D5}$$

The variational derivative of the action allows the covariant derivatives of $\delta g^{\mu\nu}$ to be shifted to the fields and leads to

$$T_{\mu\nu} = 2\Phi_{\mu\alpha}R^\alpha{}_\nu + 2\Phi_{\nu\alpha}R^\alpha{}_\mu - g_{\mu\nu}\Phi_{\alpha\beta}R^{\alpha\beta}$$
$$+ g_{\mu\nu}(\Phi_{\alpha\beta})^{;\alpha;\beta} + (\Phi_{\mu\nu})^{;\lambda}{}_{;\lambda}$$
$$- (\Phi_{\mu\alpha})_{;\nu}{}^{;\alpha} - (\Phi_{\nu\alpha})_{;\mu}{}^{;\alpha}. \tag{D6}$$

This holds for a general metric.

b. Hilbert T_{00} for a static metric: Specialize to a static metric, set $\mu = \nu = 0$, and use $R^0{}_j = R^j{}_0 = 0$ to obtain

$$T_{00} = P_{00} + Q_{00},$$
$$P_{00} = g_{00}[3R^{00}\Phi_{00} - R^{jk}\Phi_{jk}],$$
$$Q_{00} = g_{00}(\Phi_{\alpha\beta})^{;\alpha;\beta} + (\Phi_{00})^{;\lambda}{}_{;\lambda} - 2(\Phi_{0\alpha})_{;0}{}^{;\alpha}. \tag{D7}$$

Note that P_{00} contains a factor of 3 not present in Θ_{00}. In Q_{00} all the double time derivatives cancel and leave

$$Q_{00} = g_{00}[(\Phi_{0\ell})^{;\ell;0} - (\Phi_{0\ell})^{;0;\ell}] + g_{jk}(\Phi_{00})^{;j;k} + g_{00}(\Phi_{jk})^{;j;k}.$$

It is convenient to change contravariant derivatives to covariant derivatives

$$Q_{00} = g_{00}[(\Phi^{0\ell})_{;\ell;0} - (\Phi^{0\ell})_{;0;\ell}] + g^{jk}(\Phi_{00})_{;j;k} + g_{00}(\Phi^{jk})_{;j;k}.$$

The term in square brackets may be evaluated in terms of the curvature tensor

$$(\Phi^{0\ell})_{;\ell;0} - (\Phi^{0\ell})_{;0;\ell} = -R^\ell{}_{0\ell 0}\Phi^{00} + R^0{}_{j0k}\Phi^{jk}$$
$$= -R_{00}\Phi^{00} + [R_{jk} - R^\ell{}_{j\ell k}]\Phi^{jk}$$

and therefore

$$P_{00} + Q_{00} = g_{00} 2R^{00}\Phi_{00} - g_{00} R^{\ell}{}_{j\ell k}\Phi^{jk} + g^{jk}(\Phi_{00})_{;j;k} + g_{00}(\Phi^{jk})_{;j;k}. \quad \text{(D8)}$$

The difference between the Hilbert energy density and the canonical energy density is

$$\sqrt{g}[T^0{}_0 - \Theta^0{}_0] = \sqrt{g}[R^{00}\Phi_{00} + R^{jk}\Phi_{jk}] + X_1 + X_2 + X_3, \quad \text{(D9)}$$

$$X_1 \equiv \sqrt{g} g^{jk}(\Phi^0{}_0)_{;j;k},$$
$$X_2 \equiv \sqrt{g}(\Phi^{jk})_{;j;k},$$
$$X_3 \equiv -\sqrt{g} R^{\ell}{}_{j\ell k}\Phi^{jk}, \quad \text{(D10)}$$

c. Analysis of X_1: To simplify X_1 use (A1) and (A5) to obtain

$$g^{jk}(\Phi^0{}_0)_{;j;k} = g^{jk}[\partial_k(\Phi^0{}_0)_j - \Gamma^{\ell}_{jk}(\Phi^0{}_0)_{;\ell}] = g^{jk}\partial_k(\Phi^0{}_0)_{;j} + \frac{1}{\sqrt{\gamma}}\partial_k(\sqrt{\gamma} g^{k\ell})(\Phi^0{}_0)_{;\ell} = \frac{1}{\sqrt{\gamma}}\partial_k[\sqrt{\gamma} g^{jk}(\Phi^0{}_0)_{;j}]. \quad \text{(D11)}$$

Therefore,

$$X_1 = \sqrt{g_{00}}\partial_k[\sqrt{\gamma} g^{jk}(\Phi^0{}_0)_{;j}]. \quad \text{(D12)}$$

In the volume integral of X_1 a spatial integration by parts gives

$$\int d^3x X_1 = -\frac{1}{2}\int d^3x \sqrt{g} g^{jk}\frac{\partial_k g_{00}}{g_{00}}(\Phi^0{}_0)_{;j}. \quad \text{(D13)}$$

Since $(\Phi^0{}_0)_{;j} = \partial_j \Phi^0{}_0$ another integration by parts gives

$$\int d^3x X_1 = \frac{1}{2}\int d^3x \partial_j\left[\sqrt{g} g^{jk}\frac{\partial_k g_{00}}{g_{00}}\right]\Phi^0{}_0. \quad \text{(D14)}$$

From the identity (A8) this is

$$\int d^3x X_1 = -\int d^3x \sqrt{g} R^0{}_0 \Phi^0{}_0 = -\int d^3x \sqrt{g} R^{00}\Phi_{00}. \quad \text{(D15)}$$

d. Analysis of X_2*:* To simplify X_2 use (A4)

$$(\Phi^{jk})_{;j;k} = \partial_k[(\Phi^{jk})_{;j}] + \Gamma^k_{k\ell}(\Phi^{j\ell})_{;j} = \partial_k[(\Phi^{jk})_{;j}] + \frac{1}{\sqrt{\gamma}}(\partial_\ell \sqrt{\gamma})(\Phi^{j\ell})_{;j} = \frac{1}{\sqrt{\gamma}}\partial_k[\sqrt{\gamma}(\Phi^{jk})_{;j}]. \quad \text{(D16)}$$

Therefore,

$$X_2 = \sqrt{g_{00}}\partial_k[\sqrt{\gamma}(\Phi^{jk})_{;j}]. \quad \text{(D17)}$$

In the volume integral of X_2 a spatial integration by parts yields

$$\int d^3x X_2 = -\frac{1}{2}\int d^3x \sqrt{g}(\Phi^{jk})_{;j}\frac{\partial_k g_{00}}{g_{00}}. \quad \text{(D18)}$$

The necessary covariant derivative is

$$(\Phi^{jk})_{;j} = \partial_j \Phi^{jk} + \Gamma^j_{j\ell}\Phi^{\ell k} + \Gamma^k_{j\ell}\Phi^{j\ell} = \frac{1}{\sqrt{\gamma}}\partial_j[\sqrt{\gamma}\Phi^{jk}] + \Gamma^k_{j\ell}\Phi^{j\ell}. \quad \text{(D19)}$$

Substitution into (D18) and integration by parts gives

$$\int d^3x X_2 = \int d^3x \frac{\sqrt{g}}{g_{00}}\left[\frac{\partial_j \partial_k g_{00}}{2} - \frac{\partial_j g_{00}\partial_k g_{00}}{4 g_{00}}\right]\Phi^{jk} - \frac{1}{2}\int d^3x \frac{\sqrt{g}}{g_{00}}(\partial_\ell g_{00})\Gamma^\ell_{jk}\Phi^{jk}. \quad \text{(D20)}$$

Comparison with (A6) shows that

$$\int d^3x X_2 = -\int d^3x \sqrt{g} g^{00} R_{0j0k}\Phi^{jk}. \quad \text{(D21)}$$

Combining this with X_3 from (D10) gives

$$\int d^3x [X_2 + X_3] = -\int d^3x \sqrt{g}[R^0{}_{j0k} + R^\ell{}_{j\ell k}]\Phi^{jk} = -\int d^3x \sqrt{g} R_{jk}\Phi^{jk}. \quad \text{(D22)}$$

e. Result: From (D15) and (D22)

$$\int d^3x [\sqrt{g} R^{00}\Phi_{00} + X_1] = 0, \quad \text{(D23)}$$

$$\int d^3x [\sqrt{g} R^{jk}\Phi_{jk} + X_2 + X_3] = 0, \quad \text{(D24)}$$

and so, from (D10)

$$\int d^3x \sqrt{g}[T^0{}_0 - \Theta^0{}_0] = 0, \quad \text{(D25)}$$

which shows that the Hilbert and canonical Hamiltonians are equal.

[1] D. Hilbert, Die Grundlagen der Physik, Nachr. Ges. Wiss. Gott., Math-Phys., Kl. **27**, 395 (1915).
[2] C. Becchi, A. Rouet, and R. Stora, Renormalization of the Abelian Higgs-Kibble model, Commun. Math. Phys. **42**, 127 (1975).
[3] I. V. Tyutin, Gauge invariance in field theory and statistical physics in operator formalism, Lebedev Institute preprint N39, arXiv:0812.0580.
[4] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, New York, 1975).
[5] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972).
[6] E. Noether, Invariante variationsprobleme, Gott. Nachr. **1918**, 235 (1918), E. Noether and M. A. Tavel, Invariant variational problems, arXiv:physics/0503066.
[7] F. J. Belinfante, On the spin angular momentum of mesons, Physica **6**, 887 (1939); On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields, Physica **7**, 449 (1940).
[8] L. Rosenfeld, Sur le tenseur d'impulsion-énergie, Mém. Acad. R. Belg. Sci. **18**, 1 (1940).
[9] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, UK (1995), Vol. I, p. 316.
[10] G. Muñoz, Lagrangian field theories and energy-momentum tensors, Am. J. Phys. **64**, 1153 (1996).
[11] R. E. Gamboa Saravi, On the energy-momentum tensor, J. Phys. A **37**, 9573 (2004).
[12] J. M. Pons, Noether symmetries, energy-momentum tensors, and conformal invariance in classical field theory, J. Math. Phys. (N.Y.) **52**, 012904 (2011).
[13] D. N. Blaschke, F. Gieres, M. Reboud, and M. Schweda, The energy-momentum tensor(s) in classical gauge theories, Nucl. Phys. **B912**, 192 (2016).
[14] R. V. Ilin and S. A. Paston, Exact relation between canonical and metric energy-momentum tensors for higher derivative tensor field theories, Eur. Phys. J. Plus **134**, 21 (2019).
[15] A. Freese, Noether's theorems and the energy-momentum tensor in quantum gauge theories, Phys. Rev. D **106**, 125012 (2022).
[16] M. J. Gotay and J. E. Marsden, Stress-energy-momentum tensors and the Belinfante-Rosenfeld formula, Contemp. Math. **132**, 367 (1992).
[17] M. Forger and H. Römer, Currents and the energy-momentum tensor in classical field theory: A fresh look at an old problem, Ann. Phys. (Amsterdam) **309**, 306 (2004).
[18] M. R. Baker, N. Kiriushcheva, and S. Kuzmin, Noether and Hilbert (metric) energy-momentum tensors are not, in general, equivalent, Nucl. Phys. **B962**, 115240 (2021).
[19] M. Sánchez, On the geometry of static spacetimes, Nonlinear Anal. **63**, e455 (2005).
[20] S. A. Fulling, Nonuniqueness of canonical field quantization in Riemannian spacetime, Phys. Rev. D **7**, 2850 (1973).
[21] J. Dereziński and D. Siemssen, Feynman propagators on static spacetimes, Rev. Math. Phys. **30**, 1850006 (2018).
[22] H. A. Weldon, Fermions without vierbeins in curved spacetime, Phys. Rev. D **63**, 104010 (2001).