Induced energy-momentum tensor in de Sitter scalar QED and its implication for induced currents

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The aim of this research is to investigate the vacuum energy-momentum tensor of a quantized, massive, nonminimally coupled scalar field induced by a uniform electric field background in a four-dimensional de Sitter spacetime (dS_4). We compute the expectation value of the energy-momentum tensor in the in-vacuum state and then regularize it using the adiabatic subtraction procedure. The correct trace anomaly of the induced energy-momentum tensor that confirmed our results is significant. The nonconservation equation for the induced energy-momentum tensor imposes the renormalization condition for the induced electric current of the scalar field. The findings of this research indicate that there are significant differences between the two induced currents which are regularized by this renormalization condition and the minimal subtraction condition.

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I. INTRODUCTION

In the past several decades, quantum field theory in curved spacetime has played a major role in the study of the effects of gravity on quantum fields; for a pedagogical introduction, see [1–3]. A precise understanding of the quantum effects in curved spacetime was acquired by the late 1960s by Parker [4–6] and followed by investigations of others. Indeed, these early investigations focused primarily on the physical consequences of particle creation in the cosmological spacetimes. It has been illustrated that a time-varying gravitational field creates elementary particles from the vacuum. Parker discovered that this particle creation process can be analyzed by using the Bogoliubov transformations method [5]. Formulating a general framework of quantum field theory in curved spacetime involves nontrivial questions. The problem of particle concept is a deep one, and it is associated with one of the most fundamental difficulties of quantum field theory in curved spacetime, that there is no an unambiguous or unique vacuum state; a detailed discussion can be found in [1,2]. This ambiguity is reflected in the theory by the absence of an unambiguous or unique preferred mode solutions of the field equation, which is in turn a consequence of symmetry group of the spacetime. Since in a general nonstatic curved spacetime there is generically no any timelike Killing vector field, it is not possible to classify modes as positive frequency or negative frequency. Indeed, it is possible to construct a compleat set of modes. However, the problem is that there are many of such sets, and we will not have any criterion available to select a unique privileged choice of the modes. One of the key lessons learned from the development of this subject is that the notion of particle number does not generally have universal significant. Hence, the expectation value of the energy-momentum tensor in a suitable vacuum state is perhaps a more physically relevant object to probe the structure of quantum fields in curved spacetime rather than particle number quantity [1]. Part of reason is that the expectation value of the energy-momentum tensor in a fixed vacuum state transforms according to the usual tensor transformation law. In particular, if the vacuum expectation value of the energy-momentum tensor vanishes for an observer, it will vanish for all observers. On the contrary, the expectation of particle number is an essential observer-dependent quantity. The energy-momentum tensor is also important, because it is directly relevant to explore the consequences of the quantum field dynamics for the geometry of the spacetime through the Einstein equation. Hence, it is interesting to investigate the expectation value of the energy-momentum tensor in a suitable vacuum state.

Since the mid-seventies, significant advances have been made in the computation of the energy-momentum tensor of the quantum fields as well as its applications in the cosmological context. An essential feature of these computations is that they all involve the ultraviolet divergencies. In fact, such divergencies occur naturally in quantum field theory calculations. Various prescriptions have been developed for rendering the energy-momentum tensor

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finite. Among these prescriptions, Pauli-Villars regularization [7,8], dimensional regularization [9–11], and zeta-function regularization [11–14] were originally developed for use in Minkowski spacetime. Several sets of fictitious fields may be necessary to remove all of the divergences, making Pauli-Villars regularization rather complicated. In [13], it was pointed out that the dimensional regularization procedure is ambiguous in curved spacetime due to the fact that in some classes of spacetimes, there is no natural way to generalize the dimensionality of the spacetime, and the answer would be different in different extensions to D dimensions. This method also suffers from limitation that one needs, in principle, to solve the field equations exactly. Another often used regularization procedure is point-splitting technique [15–18] which is intrinsically designed to be used in the position-space representation of the composite operators. This regularization is implemented by placing the two quantum fields at distant points separated by an infinitesimal distance in a non-null direction, and then, the divergencies that arise in the coincidence limit are absorbed into the renormalized parameters. Although the point-splitting technique is the most applicable regularization scheme, it involves considerable technical complication. An alternative regularization prescription is adiabatic subtraction [19–23] which was originally invented by Parker [5] to obtain the average density of the scalar particles created in a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. This approach is only applicable in spacetimes with slowly varying curvature, and in fact, an adiabatic expansion is an expansion in number derivatives of the spacetime metric. Hence, it is especially useful for studies that involve numerical techniques and problems of cosmological interest [22]. In order to arrive a sensible semiclassical backreaction theory of quantum fields in curved spacetime, Wald has propounded [24] a minimal set of properties that the renormalized energy-momentum tensor should satisfy. These properties which are often called the Wald axioms, in the weaker set of axioms, can be expressed as follows [1,3]: (1) The expectation value of the energy-momentum tensor in any state at a point p in spacetime is covariant under general coordinate transformations (diffeomorphisms) and is independent of spacetime geometry at any point $q \neq p$. (2) The off-diagonal matrix element of the energy-momentum tensor between any orthogonal pairs of states is finite and unambiguous. (3) For all states, the expectation value of the energy-momentum tensor is covariantly conserved. (4) The expectation value of the energy-momentum tensor vanishes in the relevant vacuum state in the Minkowski spacetime. It was shown in [18] that the fifth axiom of Ref. [24] cannot be satisfied. Wald then proved a remarkable result, i.e., the uniqueness theorem, which states that if the expectation value of the energy-momentum tensor satisfies the Wald axioms (1)–(3), then it is unique up to the addition of a local conserved tensor [24]. Hence, the final results for the renormalized energy-momentum tensor do not depend on the employed regularization method [1,3]. Using these regularization techniques, the regularized and renormalized energy-momentum tensor of a neutral scalar field has been widely investigated in FLRW universes [19–22,25–33] and its physical implications for cosmological issues have been explored [34–36].

It is thought that the very early universe can be approximately described by de Sitter spacetime (dS) [2], which motivates us to study the quantum field theory in this spacetime. Gibbons and Hawking [37] originally discovered that an inertial observer with a particle detector at rest perceives the de Sitter invariant vacuum state as a bath of thermal radiation which apparently comes from the cosmological event horizon. In this context, a relatively simple model problem is that of a neutral scalar field with no self-interactions whose regularized and renormalized energy-momentum tensor has been analyzed [11,38–45]. In [39–41], it was subsequently shown that the vacuum state of the quantum field in dS is unstable to particle creation. Furthermore, the study of a semiclassical backreaction effect of the quantum corrections onto the Hubble rate in [42] indicated that these contributions can potentially result in a superacceleration phase, i.e., a phase when the Hubble rate increases as the dS expands. The spontaneous creation of pairs of charged particles from the vacuum by a strong electric field background in Minkowski spacetime is a well-known nonperturbative feature of quantum field theory [46–48]. Since the clearest version of this effect was worked out by Schwinger, it is named the Schwinger effect; reviews of this subject can be found in, e.g., [49,50]. The phenomenon of particle creation has revealed the close analogy between quantum field theory in dS and a constant, uniform electric field in Minkowski spacetime [40,41,51,52]. This analogy motivates us to explore the combined implications of the dS Gibbons-Hawking effect and the Schwinger effect. Aside from this analogy, there are arguments and evidences that require the existence of strong electromagnetic fields in the early universe [53,54]. This logical possibility provides a further reason for studying quantum field theory in the presence of an electric field in dS.

There has been numerous studies to investigate the Schwinger pair creation by a uniform electric field background with the constant energy density in a dS. Seminal contributions have been made by [55,56]. The Bogoliubov transformation method has been used to analyze the Schwinger effect for the charged scalar field in two- [55–60], four- [61], and arbitrary-dimensional [62] de Sitter spacetimes. The Bogoliubov transformation method has been used to investigate the Schwinger effect for the Dirac field in dS [63–67], in a manner similar to that used for the corresponding charged scalar field. In this method of analysis, which bring its own insights, the expectation value for the number of particles is related to the Bogoliubov coefficients which, in turn, can be determined by specifying the in-vacuum and out-vacuum states. A reasonable definition of the out-vacuum state requires that the parameters mass of the quantum field and the electric field strength satisfy the semiclassical condition. For a created semiclassical particle, this condition implies that either or both the mass and the magnitude of its electric potential energy acroses one Hubble radius must be much greater than the energy scale determined by the spacetime curvature [60,61]. An immediate consequence of the semiclassical condition is that the entire physical ranges of the parameters mass and electric field strength cannot be probed by this method. A useful alternative approach for studying the Schwinger effect in dS was introduced in Ref. [60]. In this approach the expectation values of the objects such as the electric current and the energy-momentum tensor of the quantum field in the in-vacuum state are investigated. The choice of the in-state as the vacuum is justified from various viewpoints; see [55]. Regarding the regularity properties, it is a Hadamard and adiabatic state [56,60]. The regularized expectation value of the electric current (also called induced current) in the in-vacuum state of the charged scalar field coupled to a constant, uniform electric field background has been evaluated in two- [60], three- [62], and four-dimensional [61,68] de Sitter spacetimes. In those analyses, the behavior of the induced current can be probed in the infrared regime for which the quantum field mass is smaller than the magnitude of the electric potential energy acroses one Hubble radius which in turn is smaller than the energy scale determined by the spacetime curvature. The authors found that, although the induced current has been computed using different regularization method, in the infrared regime the induced current increases with deceasing electric field strength [60–62,68]. This peculiar behavior was first observed in [60] and is called the infrared hyperconductivity. In [69], the authors gave an alternative derivation of the infrared hyperconductivity phenomenon in dS_4 using the uniform asymptotic approximation method. For a charged scalar field coupled to a uniform electromagnetic field background with a constant energy density electric field parallel to a conserved flux magnetic field in dS_4, the Schwinger effect using the Bogoliubov transformation method [70–72] and the induced current [70,71] in the in-vacuum state have been investigated, and a period of infrared hyperconductivity was found. The in-vacuum induced current of the Dirac field coupled to a constant, uniform electric field background in two- [65] and four-dimensional [73] de Sitter spacetimes has been analyzed in a manner similar to that used for the corresponding scalar field. It was subsequently found that, in contrast to the case of corresponding scalar field, the infrared hyperconductivity phenomenon does not occur in the induced current of the Dirac field. Another peculiar behavior of the induced current occurs when the direction of the induced current is opposite the direction of applied electric field background. This phenomenon which is called the negative current, has been reported for both the scalar field [61,68] with essentially small mass and the Dirac field [73] with any mass in the four-dimensional de Sitter spacetime. For the scalar field case, the negative current occurs in a finite interval of the electric field strength, and for the Dirac field case, it occurs below a certain value of the electric field strength which depends on the mass. In Refs. [74,75], some notable attempts have been made to explain and remedy these peculiarities of the induced current. Beside the induced current, the regularized expectation value of the energy-momentum tensor (also called the induced energy-momentum tensor) in the in-vacuum state of the charged scalar [76–78] and Dirac [79] quantum fields coupled to a constant, uniform electric field background have been analyzed in different dimensions of dS. In two-dimensional dS, the induced energy-momentum tensor of the charged scalar field has been analyzed in [76], and subsequently, the nonperturbative, regularized, one-loop effective Lagrangian of scalar quantum electrodynamics (QED) has been constructed. The induced energy-momentum tensor of the corresponding Dirac field has been derived in [79] and then applied to study of the gravitational backreaction effect. The trace of the induced energy-momentum tensor of the charged, massive scalar field conformally coupled to the Ricci scalar curvature of three- [77] and four-dimensional [78] de Sitter spacetimes has been computed, and to examine the evolution of the Hubble constant, the induced energy-momentum tensor has been obtained from the trace, along with the assumption that the created pairs act like a perfect fluid with a vacuum equation of state. By applying the Bogoliubov transformation method, the energy-momentum tensor of the Schwinger scalar pairs created by a constant, uniform electric field background in an arbitrary dimensional dS has been computed under the two limiting conditions, i.e., the heavy scalar field [62] and the strong electric field [80]. In both these cases, it was found that the Hubble constant decays as a consequence of the Schwinger pair creation. Before closing the present section, it is worthwhile to mention that the Schwinger effect has been studied in FLRW spacetimes [81] and in the framework of cosmological models [69,82–92]. The role of strong electromagnetic fields in astrophysics and cosmology was reviewed in [93]. The objectives of this article are to investigate the induced energy-momentum tensor of the charged scalar field coupled to a uniform electric field background in dS_4 and to analyze its nonconservation equation. The importance and originality of this study are that it calculates the induced energy-momentum tensor and explores a relation between the induced energy-momentum tensor and the induced current which leads to new insights into the regularization and behavior of the induced current.

The remaining part of the article proceeds as follows: In Sec. II, we define and construct the basic elements of the formalism that are necessary in our subsequent discussions. We then carry out explicit computation of the induced energy-momentum tensor in Sec. III. The properties of the induced energy-momentum tensor are explored in detail in Sec. IV. In Sec. V, we analyze the nonconservation equation of the induced energy-momentum tensor; this investigation yields a relation between the induced energy-momentum tensor and the induced current. Then, the properties of the resulting induced current are discussed in detail. In Sec. VI, we present the findings of the research. In the Appendix we present further supplementary data associated with the calculation of the expectation values of the components of the energy-momentum tensor.

II. BASIC DEFINITIONS AND CONSTRUCTIONS

In this section, we define and construct the basic elements of the formalism that are necessary in our subsequent discussions.

A. Specification of the model

We have already mentioned that we consider a massive complex scalar field $\varphi(x)$, coupled to the electromagnetic vector potential $A_\mu(x)$, which describes a uniform electric field background with a constant energy density in the conformal Poincaré patch of dS_4. To represent this region of dS_4 which is conformally related to a region of Minkowski spacetime, we choose the coordinates $x^\mu = (\tau, \mathbf{x})$ that the ranges of the conformal time τ, and the comoving spatial coordinates $\mathbf{x}$ are given by

$$\tau \in (-\infty, 0), \qquad \mathbf{x} \in \mathbb{R}^3. \tag{1}$$

In terms of these coordinates, the metric of the spacetime takes the form

$$g_{\mu\nu} dx^\mu dx^\nu = \Omega^2(\tau)(d\tau^2 - d\mathbf{x}\cdot d\mathbf{x}), \tag{2}$$

with the conformal scale factor

$$\Omega(\tau) = -\frac{1}{H\tau}, \tag{3}$$

where H is the Hubble constant. The nonzero Christoffel symbols for the metric (2) are given by

$$\Gamma^0_{00} = \frac{\dot{\Omega}}{\Omega}, \qquad \Gamma^0_{ij} = \frac{\dot{\Omega}}{\Omega}\delta_{ij}, \qquad \Gamma^i_{0j} = \frac{\dot{\Omega}}{\Omega}\delta^i_j, \tag{4}$$

where the roman indices i, j denote only three spatial components and run from 1 to 3. We use the overdot to denote differentiation with respect to the conformal time τ. From Eq. (4), the Ricci tensor and hence the Ricci scalar can be calculated,

$$R_{\mu\nu} = 3H^2 g_{\mu\nu}, \qquad R = 12H^2. \tag{5}$$

We put a uniform electric field background with a constant energy density on the Poincaré patch represented in Eq. (2). Without loss of generality, we choose our coordinates so that this electric field to point in the x^1 direction. Thus, the nonzero components of the electromagnetic field tensor are

$$F_{01} = -F_{10} = \Omega^2(\tau)E, \tag{6}$$

where E is a constant. It is convenient to express this electromagnetic field tensor in terms of a vector potential in the Coulomb gauge [60–62] as

$$A_\mu(\tau) = -\frac{E}{H^2\tau}\delta^1_\mu. \tag{7}$$

The complete action S_{tot} of this theory can be represented as sum of a pure gravitational piece, an electromagnetic piece, and a scalar field piece,

$$S_{\mathrm{tot}} = S_{\mathrm{gr}} + S_{\mathrm{em}} + S. \tag{8}$$

Here, S_{gr} is the Einstein-Hilbert action that only includes the gravitation piece of the complete action, and is given by

$$S_{\mathrm{gr}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda_{\mathrm{c}}), \tag{9}$$

where G is Newton's gravitational constant, g is the determinant of the metric, R is the Ricci scalar of the spacetime, and Λ_{c} is the cosmological constant. The electromagnetic piece of the complete action is expressed in terms of the electromagnetic field tensor as

$$S_{\mathrm{em}} = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{10}$$

The dynamics of the complex scalar field $\varphi(x)$ of mass m coupled to the electromagnetic vector potential (7) with coupling e is governed by the last piece of the complete action which can be written as

$$S = \int d^4x\sqrt{-g}[g^{\mu\nu}(\partial_\mu + ieA_\mu)\varphi(\partial_\nu - ieA_\nu)\varphi^* - (m^2 + \xi R)\varphi\varphi^*], \tag{11}$$

where ξ is a dimensionless nonminimal coupling constant which describes the strength of the coupling φ to the Ricci scalar (5). The Euler-Lagrange equations of motion give the Klein-Gordon equation for the scalar field,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) + 2ieg^{\mu\nu}A_\mu\partial_\nu\varphi - e^2 g^{\mu\nu}A_\mu A_\nu\varphi + (m^2 + \xi R)\varphi = 0, \tag{12}$$

and the Maxwell equation for the electromagnetic field,

$$\nabla_\nu F^{\nu\mu} = j^\mu, \tag{13}$$

where ∇ denotes the covariant derivative operator, and j^μ is the electric current of the scalar field caused by the electric field background (6) which is defined by

$$j^\mu(x) = ieg^{\mu\nu}[(\partial_\nu\varphi + ieA_\nu\varphi)\varphi^* - \varphi(\partial_\nu\varphi^* - ieA_\nu\varphi^*)]. \tag{14}$$

It is straightforward to verify that $\nabla_\mu j^\mu = 0$; i.e., the electric current is conserved.

B. Preliminary definition of the energy-momentum tensor

The classical Einstein equation can be derived from the complete action (8) by demanding the invariance of $S_{\rm tot}$ under infinitesimal variation of the metric $\delta g_{\mu\nu}$, or equivalently, infinitesimal variation of the inverse metric $\delta g^{\mu\nu}$. This condition requires that

$$\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm tot}}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}}\left(\frac{\delta S_{\rm gr}}{\delta g^{\mu\nu}} + \frac{\delta S_{\rm em}}{\delta g^{\mu\nu}} + \frac{\delta S}{\delta g^{\mu\nu}}\right) = 0. \tag{15}$$

The variation of expression (9) with respect to $\delta g^{\mu\nu}$ leads to

$$\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm gr}}{\delta g^{\mu\nu}} = \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_{\rm c}g_{\mu\nu}\right). \tag{16}$$

The variations of the electromagnetic action $S_{\rm em}$, and the scalar field action S, with respect to $\delta g^{\mu\nu}$ define the energy-momentum tensor of the electromagnetic field $T^{(\rm em)}_{\mu\nu}$, and the energy-momentum tensor of the scalar field $T_{\mu\nu}$, respectively, as

$$\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm em}}{\delta g^{\mu\nu}} = T^{(\rm em)}_{\mu\nu}, \tag{17}$$

$$\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu}. \tag{18}$$

Plugging expressions (10) and (11) into Eqs. (17) and (18), respectively, and following the standard calculus of variations procedure yields the energy-momentum tensor of the electromagnetic field,

$$T^{(\rm em)}_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + g^{\rho\sigma}F_{\mu\rho}F_{\sigma\nu}, \tag{19}$$

and a preliminary expression for the energy-momentum tensor of the scalar field,

$$\begin{aligned} T_{\mu\nu} = {} & \left[(4\xi - 1)g^{\rho\sigma}(\partial_\rho + ieA_\rho)\varphi(\partial_\sigma - ieA_\sigma)\varphi^* + (1-4\xi)m^2\varphi\varphi^* + \left(\frac{1}{2} - 4\xi\right)\xi R\varphi\varphi^*\right]g_{\mu\nu} \\ & + (1-2\xi)(\partial_\mu\varphi\partial_\nu\varphi^* + \partial_\nu\varphi\partial_\mu\varphi^*) + ieA_\mu(\varphi\partial_\nu\varphi^* - \partial_\nu\varphi\varphi^*) + ieA_\nu(\varphi\partial_\mu\varphi^* - \partial_\mu\varphi\varphi^*) \\ & + 2e^2A_\mu A_\nu\varphi\varphi^* + 2\xi\Gamma^\rho_{\mu\nu}(\varphi\partial_\rho\varphi^* + \partial_\rho\varphi\varphi^*) - 2\xi(\partial_\mu\partial_\nu\varphi\varphi^* + \varphi\partial_\mu\partial_\nu\varphi^*). \end{aligned} \tag{20}$$

Plugging three expressions (16)–(18) into Eq. (15) gives the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_{\rm c}g_{\mu\nu} = -8\pi G(T^{(\rm em)}_{\mu\nu} + T_{\mu\nu}). \tag{21}$$

An important property of the Einstein equation is that both sides of Eq. (21) have identically vanishing covariant divergences, which implies

$$\nabla_\mu T^{\mu\nu} = -\nabla_\mu T^{(\rm em)\mu\nu}. \tag{22}$$

Using the Klein-Gordon equation (12), it is straightforward to show that the covariant divergence of expression (20) is given by

$$\nabla_\mu T^{\mu\nu} = -j_\mu F^{\mu\nu}. \tag{23}$$

Apparently, the energy-momentum tensor of the scalar field is not covariantly conserved in the presence of the electromagnetic field, as the consequence of the electromagnetic interactions. We show that the nonconservation of $T_{\mu\nu}$ is compatible with the nonconservation of $T^{(\rm em)}_{\mu\nu}$ so that the relation (22) is satisfied, and hence, the total energy momentum tensor is covariantly conserved. By taking the covariant divergence of the expression (19) and using the Maxwell equation (13), it is seen that

$$\nabla_\mu T^{(\rm em)\mu\nu} = j_\mu F^{\mu\nu}. \tag{24}$$

As a result of Eqs. (23) and (24), it is evident that the relation (22) is satisfied. Hence, the total energy-momentum tensor $T_{\mu\nu} + T^{(\rm em)}_{\mu\nu}$ is manifestly conserved.

It is important to state that we treat the classical gravitational field (2) and the classical electromagnetic field (7) as fixed field configurations which are unaffected by the dynamics of the quantum complex scalar field $\varphi(x)$ in response to these backgrounds. In fact, the Einstein equation (21) describes the backreaction effects on the gravitational field, and the Maxwell equation (13) and Eq. (24) describe the backreaction effects on the electromagnetic field. Indeed, due to the conceptual importance of Eq. (23) in our subsequent discussions, we have presented Eqs. (13), (21), and (24) to provide a fairly detailed discussion of its derivation. It is then clear that we do not discuss these equations further in this article.

C. Quantizing the complex scalar field

Quantizing the complex scalar field $\varphi(x)$, in the classical gravitational (2) and electromagnetic (7) field backgrounds, is completely straightforward and follows exactly the same route as that of a complex scalar field in Minkowski spacetime. We solve the Kline-Gordon equation (12), and then adopting the canonical quantization method, we can define the in-vacuum state to obtain the expectation value of the energy-momentum tensor operator.

Taking account of the fact that the gravitational (2) and electromagnetic (7) backgrounds are invariant under spatial translations, it is convenient to write the mode solution of Eq. (12) as

$$U_{\mathbf{k}}(x) = \Omega^{-1}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}f(\tau), \tag{25}$$

where $\mathbf{k}$ is the comoving momentum. Plugging Eqs. (2), (7), and (25) into Eq. (12), we can put the differential equation in a standard form by the change of variable $z = 2ik\tau$ so that $k = |\mathbf{k}|$. Then, it becomes

$$\frac{d^2 f}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \gamma^2}{z^2}\right)f(z) = 0, \tag{26}$$

where the dimensionless parameters are defined through

$$\mu = \frac{m}{H}, \qquad \lambda = -\frac{eE}{H^2}, \qquad \bar{\xi} = \xi - \frac{1}{6},$$
$$r = \frac{k_x}{k}, \qquad \kappa = -i\lambda r, \qquad \gamma = \sqrt{\frac{1}{4} - \lambda^2 - \mu^2 - 12\bar{\xi}}. \tag{27}$$

Two values of ξ are of particular interest and are the minimally coupled case $\xi = 0$ and conformally coupled case $\xi = 1/6$ which imply $\bar{\xi} = 0$. The variable k_x which appears in the definition of the parameter r, denotes the component of the comoving momentum $\mathbf{k}$ along the electric field background. Equation (26) is the Whittaker equation, and the solutions are called Whittaker functions; see, e.g., [94]. We define the in-vacuum state $|\text{in}\rangle$ so that in the remote past ($\tau \to -\infty$) where the spacetime is asymptotically Minkowskian, an inertial observer there would identify this state with a physical vacuum. This vacuum state may be represented by a mode solution of the Whittaker equation (26) which behaves like a mode function in Minkowski spacetime in the limit of $\tau \to -\infty$. Hence, the solution of Eq. (26) with the desired asymptotic form in the limit of $|z| \to \infty$ which can be represented in terms of a Mellin-Barnes integral [94] is

$$W_{\kappa,\gamma}(z) = \frac{e^{-\frac{z}{2}}}{\Gamma(\frac{1}{2}+\gamma-\kappa)\Gamma(\frac{1}{2}-\gamma-\kappa)} \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i}\Gamma\left(\frac{1}{2}+\gamma+s\right) \times \Gamma\left(\frac{1}{2}-\gamma+s\right)\Gamma(-\kappa-s)z^{-s}, \tag{28}$$

with the condition that the phase of the variable z and the values of the parameters γ and κ must satisfy the following inequalities

$$|\text{ph}(z)| < \frac{3\pi}{2}, \qquad \frac{1}{2} \pm \gamma - \kappa \neq 0, -1, -2, \cdots. \tag{29}$$

In expression (28), the factors denoted by Γ are the gamma functions. The contour of the Mellin-Barnes integral (28) consists of a straight vertical line from minus infinity to infinity, parallel to imaginary axis in the complex plane, and of a semicircle at infinity with indentations if necessary to avoid poles of the integrand in a way that separates the poles of $\Gamma(\frac{1}{2}+\gamma+s)$ and $\Gamma(\frac{1}{2}-\gamma+s)$ from the poles of $\Gamma(-\kappa-s)$. The normalized mode functions which behave like the positive frequency Minkowski mode functions in the remote past are given by [61,62],

$$U_{\mathbf{k}}(x) = \frac{1}{\sqrt{2k}} e^{\frac{i\pi\kappa}{2}}\Omega^{-1}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}W_{\kappa,\gamma}(2ik\tau). \tag{30}$$

Besides, the normalized mode functions which behave like the negative frequency Minkowski mode functions in the remote past are found to be [61,62]

$$V_{\mathbf{k}}(x) = \frac{1}{\sqrt{2k}} e^{-\frac{i\pi\kappa}{2}}\Omega^{-1}(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}W_{\kappa,\gamma}(-2ik\tau). \tag{31}$$

We have conventionally normalized the mode functions (30) and (31) such that their Wronskian is

$$U_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} - U^*_{\mathbf{k}}\dot{U}_{\mathbf{k}} = V^*_{\mathbf{k}}\dot{V}_{\mathbf{k}} - V_{\mathbf{k}}\dot{V}^*_{\mathbf{k}} = i\Omega^{-2}(\tau). \tag{32}$$

These mode functions will be orthonormal with respect to the conserved scalar product integrated over the constant τ hypersurface [62]. The usual canonical quantization will proceed by introducing the creation $a^\dagger_{\mathbf{k}}$, and annihilation $a_{\mathbf{k}}$ operators for each of mode functions $U_{\mathbf{k}}$, and similarly the creation $b^\dagger_{\mathbf{k}}$, and annihilation $b_{\mathbf{k}}$ operators for each of mode functions $V_{\mathbf{k}}$. The creation and annihilation operators satisfy the commutation rules

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = [b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{33}$$

with all other commutators vanishing. Then, the complex scalar field operator can be expanded in terms of the creation and annihilation operators in the standard manner as

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3}[a_{\mathbf{k}}U_{\mathbf{k}}(x) + b^\dagger_{\mathbf{k}}V_{\mathbf{k}}(x)]. \quad (34)$$

The in-vacuum state $|\text{in}\rangle$ is characterized by the fact that it is annihilated by each of $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ operators

$$a_{\mathbf{k}}|\text{in}\rangle = b_{\mathbf{k}}|\text{in}\rangle = 0, \quad \forall\, \mathbf{k}. \quad (35)$$

III. CONSTRUCTION OF THE INDUCED ENERGY-MOMENTUM TENSOR

Having built a foundation for our discussion and presented the definition for the energy-momentum tensor of the scalar field in Eq. (20), we proceed to present the regularized in-vacuum expectation value of the energy-momentum tensor of the scalar field.

A. Unregularized expectation values in the in-vacuum state

Integral representations for the in-vacuum expectation values of the components of the energy-momentum tensor can be obtained by substituting the mode expansion (34) for the quantum scalar field $\varphi(x)$ into the definition (20), and then calculating the expectation values in the in-vacuum state using relations (33) and (35). By using Eq. (12) and some algebra, we have the following integral expressions in terms of the positive frequency mode function (30) for the expectation values of the components. The integral expression of the timelike component is given by

$$\langle\text{in}|T_{00}|\text{in}\rangle = \int \frac{d^3k}{(2\pi)^3}[\dot{U}_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} - 6\xi\tau^{-1}(U_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} + \dot{U}_{\mathbf{k}}U^*_{\mathbf{k}}) + \tau^{-2}(k^2\tau^2 + 2\lambda rk\tau + \lambda^2 + \mu^2 + 6\xi)U_{\mathbf{k}}U^*_{\mathbf{k}}]. \quad (36)$$

For the diagonal spacelike components, we get

$$\langle\text{in}|T_{11}|\text{in}\rangle = \int \frac{d^3k}{(2\pi)^3}\{(1-4\xi)\dot{U}_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} - 2\xi\tau^{-1}(U_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} + \dot{U}_{\mathbf{k}}U^*_{\mathbf{k}}) + \tau^{-2}[(4\xi - 1 + 2r^2)k^2\tau^2 + 2(4\xi+1)\lambda rk\tau + (4\xi+1)\lambda^2 + (4\xi-1)\mu^2 + 6\xi(8\xi-1)]U_{\mathbf{k}}U^*_{\mathbf{k}}\}, \quad (37)$$

and

$$\langle\text{in}|T_{22}|\text{in}\rangle = \langle\text{in}|T_{33}|\text{in}\rangle = \int \frac{d^3k}{(2\pi)^3}\{(1-4\xi)\dot{U}_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} - 2\xi\tau^{-1}(U_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} + \dot{U}_{\mathbf{k}}U^*_{\mathbf{k}}) + \tau^{-2}[(4\xi-1)k^2\tau^2 + 2k_z^2\tau^2 + 2(4\xi-1)\lambda rk\tau + (4\xi-1)\lambda^2 + (4\xi-1)\mu^2 + 6\xi(8\xi-1)]U_{\mathbf{k}}U^*_{\mathbf{k}}\}. \quad (38)$$

The only nonvanishing in-vacuum expectation values of the off-diagonal components can be expressed as

$$\langle\text{in}|T_{01}|\text{in}\rangle = \langle\text{in}|T_{10}|\text{in}\rangle = i\tau^{-1}\int \frac{d^3k}{(2\pi)^3}(rk\tau + \lambda)(U_{\mathbf{k}}\dot{U}^*_{\mathbf{k}} - \dot{U}_{\mathbf{k}}U^*_{\mathbf{k}}). \quad (39)$$

Using the asymptotic expansion of the Whittaker function (28) for large values of its argument [94], it can be shown that the mode function (30) is proportional to $k^{-\frac{1}{2}-i\lambda r}$ in the limit of $k \to \infty$. Then, inspection of integrals in Eqs. (36)–(39) shows that these expressions are ultraviolet divergent. Thus, we first regulate them by cutting them off at a large momentum K. A further description of the calculation of the expressions (36)–(38) is available in the Appendix. We find the final expression for the unregularized in-vacuum expectation value of the timelike component

$$\langle\text{in}|T_{00}|\text{in}\rangle = \Omega^2(\tau)\frac{H^4}{8\pi^2}\bigg\{\Lambda^4 + \left(\mu^2 - 6\bar{\xi} + \frac{2\lambda^2}{3}\right)\Lambda^2 - \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 - \frac{\lambda^2}{6}\right)\log(2\Lambda) + 54\bar{\xi}^2 - \frac{2\lambda^4}{15} + \frac{\mu^2}{4} + 6\bar{\xi}\mu^2 + \frac{\mu^4}{8} - \frac{19\lambda^2}{72} - 2\lambda^2\bar{\xi} - \frac{\lambda^2\mu^2}{2} + \frac{\gamma}{24\pi}\left(\frac{45}{\pi^2} - 6 - 96\bar{\xi} + 4\lambda^2 - 26\mu^2\right)\frac{\cosh(2\pi\lambda)}{\sin(2\pi\gamma)} - \frac{\gamma}{48\pi^2\lambda}\left(\frac{45}{\pi^2} - 6 - 96\bar{\xi} + 64\lambda^2 - 26\mu^2\right) \times\frac{\sinh(2\pi\lambda)}{\sin(2\pi\gamma)} + i\csc(2\pi\gamma)\int_{-1}^{+1}C_{0r}\left[(e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right) - (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right)\right]dr + \frac{i\pi}{12}(3\mu^4 + 36\bar{\xi}\mu^2 - \lambda^2)\bigg\}, \quad (40)$$

where the coefficient C_{0r} is given by

$$C_{0r} = -\frac{5}{8}\lambda^4 r^4 + \frac{1}{8}(6\mu^2 + 6\lambda^2 + 36\bar{\xi} + 1)\lambda^2 r^2 - \frac{1}{8}(\mu^2 + \lambda^2)(\mu^2 + \lambda^2 + 12\bar{\xi}). \tag{41}$$

The final results of our evaluation of the unregularized in-vacuum expectation values of the diagonal spacelike components are

$$\begin{aligned}\langle \mathrm{in}|T_{11}|\mathrm{in}\rangle = \Omega^2(\tau)\frac{H^4}{8\pi^2}\Big\{&\frac{\Lambda^4}{3} + \left(2\bar{\xi} - \frac{\mu^2}{3} + \frac{14\lambda^2}{15}\right)\Lambda^2 + \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 - \frac{\lambda^2}{6}\right)\log(2\Lambda) - 54\bar{\xi}^2 - \frac{26\lambda^4}{105} - \frac{7\mu^4}{24} - 8\bar{\xi}\mu^2 \\ &- \frac{18\bar{\xi}}{5}\lambda^2 - \frac{\mu^2}{4} + \frac{19\lambda^2}{72} + \frac{\lambda^2\mu^2}{30} - \frac{\gamma}{24\pi\lambda^2}\left[\frac{15}{\pi^2}(105\pi^{-2} - 15 - 132\bar{\xi} + 35\lambda^2 - 11\mu^2) - 66\lambda^2 - 336\bar{\xi}\lambda^2 - 4\lambda^4\right. \\ &\left. - 46\lambda^2\mu^2\right]\frac{\cosh(2\pi\lambda)}{\sin(2\pi\gamma)} + \frac{\gamma}{48\pi^2\lambda^3}\left[\frac{15}{\pi^2}(105\pi^{-2} - 15 - 132\bar{\xi} + 175\lambda^2 - 11\mu^2) - 366\lambda^2 - 2976\bar{\xi}\lambda^2 + 136\lambda^4\right. \\ &\left. - 266\lambda^2\mu^2\right]\frac{\sinh(2\pi\lambda)}{\sin(2\pi\gamma)} + i\csc(2\pi\gamma)\int_{-1}^{+1} C_{1r}\left[(e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right)\right. \\ &\left. - (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right)\right]dr - \frac{i\pi}{12}(3\mu^4 + 36\bar{\xi}\mu^2 - \lambda^2)\Big\},\end{aligned} \tag{42}$$

where the coefficient C_{1r} is given by

$$\begin{aligned}C_{1r} = \frac{35}{8}\lambda^4 r^6 &- \frac{1}{8}(70\lambda^4 + 30\lambda^2\mu^2 + 360\bar{\xi}\lambda^2 + 25\lambda^2)r^4 + \frac{1}{8}(39\lambda^4 + 3\mu^4 + 30\lambda^2\mu^2 + 396\bar{\xi}\lambda^2 + 72\bar{\xi}\mu^2 + 20\lambda^2 \\ &+ 6\mu^2 + 432\bar{\xi}^2 + 72\bar{\xi})r^2 - \frac{1}{8}(4\lambda^4 + 4\lambda^2\mu^2 + 60\bar{\xi}\lambda^2 + 12\bar{\xi}\mu^2 + 2\lambda^2 + 2\mu^2 + 144\bar{\xi}^2 + 24\bar{\xi}),\end{aligned} \tag{43}$$

and we also have

$$\begin{aligned}\langle \mathrm{in}|T_{22}|\mathrm{in}\rangle &= \langle \mathrm{in}|T_{33}|\mathrm{in}\rangle \\ &= \Omega^2(\tau)\frac{H^4}{8\pi^2}\Big\{\frac{\Lambda^4}{3} + \left(2\bar{\xi} - \frac{\mu^2}{3} - \frac{2\lambda^2}{15}\right)\Lambda^2 + \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 + \frac{\lambda^2}{6}\right)\log(2\Lambda) - 54\bar{\xi}^2 + \frac{2\lambda^4}{35} \\ &- \frac{7\mu^4}{24} - 8\bar{\xi}\mu^2 + \frac{4\bar{\xi}}{5}\lambda^2 - \frac{\mu^2}{4} - \frac{19\lambda^2}{72} + \frac{2}{5}\lambda^2\mu^2 + \frac{\gamma}{16\pi\lambda^2}\left[\frac{5}{\pi^2}(105\pi^{-2} - 15 - 132\bar{\xi} + 38\lambda^2 - 11\mu^2) - 24\lambda^2\right. \\ &\left. - 144\bar{\xi}\lambda^2\right]\frac{\cosh(2\pi\lambda)}{\sin(2\pi\gamma)} - \frac{\gamma}{32\pi^2\lambda^3}\left[\frac{5}{\pi^2}(105\pi^{-2} - 15 - 132\bar{\xi} + 178\lambda^2 - 11\mu^2) - 124\lambda^2 - 1024\bar{\xi}\lambda^2 + \frac{200\lambda^4}{3}\right. \\ &\left. - \frac{220}{3}\lambda^2\mu^2\right]\frac{\sinh(2\pi\lambda)}{\sin(2\pi\gamma)} + i\csc(2\pi\gamma)\int_{-1}^{+1} C_{2r}\left[(e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right)\right. \\ &\left. - (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right)\right]dr - \frac{i\pi}{12}(3\mu^4 + 36\bar{\xi}\mu^2 + \lambda^2)\Big\},\end{aligned} \tag{44}$$

where the coefficient C_{2r} is given by

$$C_{2r} = -\frac{C_{1r}}{2} - \frac{5}{16}\lambda^4 r^4 + \frac{1}{16}(6\lambda^2 - 6\mu^2 + 36\bar{\xi} + 1)\lambda^2 r^2 + \frac{1}{16}(3\mu^4 + 2\lambda^2\mu^2 + 12\bar{\xi}\lambda^2 + 36\bar{\xi}\mu^2 - \lambda^2). \tag{45}$$

Plugging the resulting expression given in Eq. (32) for the Wronskian of the mode functions $U_{\mathbf{k}}$ into Eq. (39) and integrating over momentum phase space, we obtain the unregularized expression for the only nonvanishing off-diagonal components,

$$\langle \mathrm{in}|T_{01}|\mathrm{in}\rangle = \langle \mathrm{in}|T_{10}|\mathrm{in}\rangle = \Omega^2(\tau)\frac{H^4\lambda}{6\pi^2}\Lambda^3. \tag{46}$$

The expressions (40), (42), (44), and (46) clearly have ultraviolet divergences when $\Lambda \to \infty$. This was expected, because the expectation values in the Hadamard in-vacuum state suffer from the same ultraviolet divergence properties as Minkowski spacetime.

B. Construction of the counterterms and adiabatic subtractions

To render the in-vacuum expectation values given by Eqs. (40), (42), (44), and (46) finite, we provide a set of needed adiabatic counterterms. The adiabatic regularization method consists of subtracting the appropriate adiabatic counterterms from the corresponding unregularized expressions. To adjust the set of appropriate counterterms, we treat the conformal scale factor $\Omega(\tau)$, and the electromagnetic vector potential $A_\mu(\tau)$ as quantities of zero adiabatic order. As pointed out by Wald [18], in four dimensions, to obtain a renormalized energy-momentum tensor which is consistent with the Wald axioms, the subtraction counterterms should be expanded up to fourth adiabatic order; see also [1,2]. Thus, to construct the expectation value of the energy-momentum tensor with the desired physical properties, we expand the subtraction counterterms up to fourth adiabatic order. To construct the appropriate counterterms, we need an expansion for the mode functions up to fourth adiabatic order. We use the definition $z = 2ik\tau$ to turn the Klein-Gordon equation (26) into the convenient form

$$\frac{d^2\mathcal{F}_A}{d\tau^2} + (\omega_0^2(\tau) + \Delta(\tau))\mathcal{F}_A = 0, \tag{47}$$

where $\mathcal{F}_A$ is the positive frequency adiabatic solution, and the conformal time dependent frequencies read

$$\omega_0(\tau) = (k^2 + 2eA_1kr + e^2A_1^2 + m^2\Omega^2)^{\frac{1}{2}}, \tag{48}$$

$$\Delta(\tau) = 12\bar{\xi}\left(\frac{\dot{\Omega}}{\Omega}\right)^2. \tag{49}$$

It is then obvious that ω_0 is of zero adiabatic order, and Δ is of second adiabatic order. The adiabatic form of $\mathcal{F}_A$ is the Wentzel-Kramers-Brillouin (WKB) solution of Eq. (47). This WKB solution can be written as

$$\mathcal{F}_A(\tau) = \frac{1}{\sqrt{2\mathcal{W}(\tau)}}\exp\left[-i\int^\tau \mathcal{W}(\tau')d\tau'\right], \tag{50}$$

where $\mathcal{W}$ satisfies the exact equation

$$\mathcal{W}^2 = \omega_0^2 + \Delta - \frac{\ddot{\mathcal{W}}}{2\mathcal{W}} + \frac{3\dot{\mathcal{W}}^2}{4\mathcal{W}^2}. \tag{51}$$

To put the solution into the desired form, it is convenient to write $\mathcal{W}$ as

$$\mathcal{W} = \mathcal{W}^{(0)} + \mathcal{W}^{(2)} + \mathcal{W}^{(4)}, \tag{52}$$

where the superscript numbers in parentheses denote the adiabatic order approximation to $\mathcal{W}$. Substituting the expansion (52) into Eq. (51), we find the zero adiabatic order approximation

$$\mathcal{W}^{(0)} = \omega_0. \tag{53}$$

The next iteration gives the second adiabatic order approximation

$$\mathcal{W}^{(2)} = \frac{\Delta}{2\omega_0} - \frac{\ddot{\omega}_0}{4\omega_0^2} + \frac{3\dot{\omega}_0^2}{8\omega_0^3}. \tag{54}$$

Repeated iteration yields the fourth adiabatic order approximation

$$\mathcal{W}^{(4)} = -\frac{\mathcal{W}^{(2)2}}{2\omega_0} - \frac{\ddot{\mathcal{W}}^{(2)}}{4\omega_0^2} + \frac{\ddot{\omega}_0\mathcal{W}^{(2)}}{4\omega_0^3} + \frac{3\dot{\omega}_0\dot{\mathcal{W}}^{(2)}}{4\omega_0^3} - \frac{3\dot{\omega}_0^2\mathcal{W}^{(2)}}{4\omega_0^4}. \tag{55}$$

It should be remarked that all terms in the adiabatic expansion of $\mathcal{W}$ of odd adiabatic order vanish. Assembling the pieces given in Eqs. (25), (50), and (52)–(55), we find the positive frequency adiabatic solution to fourth adiabatic order approximation,

$$U_{\mathbf{k}}^{[4]}(x) = \Omega^{-1}(\tau)\frac{1}{\sqrt{2\omega_0}}\left(1 - \frac{\mathcal{W}^{(2)}}{2\omega_0} - \frac{\mathcal{W}^{(4)}}{2\omega_0} + \frac{\mathcal{W}^{(2)2}}{2\omega_0^2}\right) \times \exp\left[i\mathbf{k}\cdot\mathbf{x} - i\int^\tau \mathcal{W}(\tau')d\tau'\right]. \tag{56}$$

We use the superscript symbol [4] to indicate that the cross terms from the field expansion products that are of adiabatic order greater than 4 are to be discarded. To obtain expansions of the required counterterms to adiabatic order four, it is only necessary to calculate (36)–(39) with $U_{\mathbf{k}}$ replaced by $U_{\mathbf{k}}^{[4]}$. Doing this, we find the counterterm to adiabatic order four for the timelike component,

$$\begin{aligned}\mathcal{T}_{00}^{[4]} = \Omega^2(\tau)\frac{H^4}{8\pi^2}\Bigg[&\Lambda^4 + \left(\mu^2 - 6\bar{\xi} + \frac{2\lambda^2}{3}\right)\Lambda^2 \\ &- \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 - \frac{\lambda^2}{6}\right)\log\left(\frac{2\Lambda}{\mu}\right) + 72\bar{\xi}^2 - \frac{\lambda^4}{15} + \frac{\mu^2}{6} \\ &+ 9\bar{\xi}\mu^2 + \frac{\mu^4}{8} - \frac{2\lambda^2}{9} - 2\lambda^2\bar{\xi} - \frac{\lambda^2\mu^2}{3} - \frac{1}{60} \\ &+ \frac{7\lambda^4}{240\mu^4} + \frac{\lambda^2}{30\mu^2} - \frac{3\bar{\xi}\lambda^2}{2\mu^2}\Bigg].\end{aligned} \tag{57}$$

We find the counterterms to adiabatic order four for the diagonal spacelike components,

$$\mathcal{T}_{11}^{[4]} = \Omega^2(\tau)\frac{H^4}{8\pi^2}\left[\frac{\Lambda^4}{3} + \left(2\bar{\xi} - \frac{\mu^2}{3} + \frac{14\lambda^2}{15}\right)\Lambda^2 + \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 - \frac{\lambda^2}{6}\right)\log\left(\frac{2\Lambda}{\mu}\right) - 72\bar{\xi}^2 - \frac{13\lambda^4}{105} - \frac{7\mu^4}{24} - 11\bar{\xi}\mu^2 - \frac{14\bar{\xi}}{5}\lambda^2 - \frac{\mu^2}{6} + \frac{7\lambda^2}{18} - \frac{\lambda^2\mu^2}{15} + \frac{1}{60} - \frac{7\lambda^4}{240\mu^4} + \frac{\lambda^2}{18\mu^2} + \frac{3\bar{\xi}\lambda^2}{2\mu^2}\right], \tag{58}$$

and

$$\mathcal{T}_{22}^{[4]} = \mathcal{T}_{33}^{[4]} = \Omega^2(\tau)\frac{H^4}{8\pi^2}\left[\frac{\Lambda^4}{3} + \left(2\bar{\xi} - \frac{\mu^2}{3} - \frac{2\lambda^2}{15}\right)\Lambda^2 + \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 + \frac{\lambda^2}{6}\right)\log\left(\frac{2\Lambda}{\mu}\right) - 72\bar{\xi}^2 + \frac{\lambda^4}{35} - \frac{7\mu^4}{24} - 11\bar{\xi}\mu^2 + \frac{2\bar{\xi}}{5}\lambda^2 - \frac{\mu^2}{6} - \frac{7\lambda^2}{18} + \frac{\lambda^2\mu^2}{5} + \frac{1}{60} + \frac{7\lambda^4}{720\mu^4} - \frac{\lambda^2}{45\mu^2} - \frac{\bar{\xi}\lambda^2}{2\mu^2}\right]. \tag{59}$$

We obtain the counterterms to adiabatic order four for the only nonvanishing off-diagonal components,

$$\mathcal{T}_{01}^{[4]} = \mathcal{T}_{10}^{[4]} = \Omega^2(\tau)\frac{H^4\lambda}{6\pi^2}\Lambda^3. \tag{60}$$

Then, the adiabatic regularization procedure is carried out by subtracting the counterterms (57)–(60) from the corresponding unregularized in-vacuum expectation values (40), (42), (44), and (46). Thus, we obtain our final expression for the timelike component of the regularized energy-momentum tensor,

$$\begin{aligned} T_{00} &= \langle \mathrm{in}|T_{00}|\mathrm{in}\rangle - \mathcal{T}_{00}^{[4]} \\ &= \Omega^2(\tau)\frac{H^4}{8\pi^2}\left\{\frac{1}{60} - \frac{7\lambda^4}{240\mu^4} - \frac{\lambda^2}{30\mu^2} + \frac{3\bar{\xi}\lambda^2}{2\mu^2} - 18\bar{\xi}^2 - \frac{\lambda^4}{15} + \frac{\mu^2}{12} - 3\bar{\xi}\mu^2 - \frac{\lambda^2}{24} - \frac{\lambda^2\mu^2}{6}\right. \\ &\quad + \frac{\gamma}{24\pi}\left(\frac{45}{\pi^2} - 6 - 96\bar{\xi} + 4\lambda^2 - 26\mu^2\right)\frac{\cosh(2\pi\lambda)}{\sin(2\pi\gamma)} - \frac{\gamma}{48\pi^2\lambda}\left(\frac{45}{\pi^2} - 6 - 96\bar{\xi} + 64\lambda^2 - 26\mu^2\right)\frac{\sinh(2\pi\lambda)}{\sin(2\pi\gamma)} \\ &\quad + i\csc(2\pi\gamma)\int_{-1}^{+1} C_{0r}\left[(e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right) - (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right)\right]dr \\ &\quad \left. - \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 - \frac{\lambda^2}{6}\right)\log(\mu) + \frac{i\pi}{12}(3\mu^4 + 36\bar{\xi}\mu^2 - \lambda^2)\right\}. \end{aligned} \tag{61}$$

We obtain our final expressions for the diagonal spacelike components of the regularized energy-momentum tensor,

$$\begin{aligned} T_{11} &= \langle \mathrm{in}|T_{11}|\mathrm{in}\rangle - \mathcal{T}_{11}^{[4]} \\ &= \Omega^2(\tau)\frac{H^4}{8\pi^2}\left\{-\frac{1}{60} + \frac{7\lambda^4}{240\mu^4} - \frac{\lambda^2}{18\mu^2} - \frac{3\bar{\xi}\lambda^2}{2\mu^2} + 18\bar{\xi}^2 - \frac{13\lambda^4}{105} + 3\bar{\xi}\mu^2 - \frac{4\bar{\xi}\lambda^2}{5} - \frac{\mu^2}{12} - \frac{\lambda^2}{8} + \frac{\lambda^2\mu^2}{10}\right. \\ &\quad - \frac{\gamma}{24\pi\lambda^2}\left[\frac{15}{\pi^2}(105\pi^{-2} - 15 - 132\bar{\xi} + 35\lambda^2 - 11\mu^2) - 66\lambda^2 - 336\bar{\xi}\lambda^2 - 4\lambda^4 - 46\lambda^2\mu^2\right]\frac{\cosh(2\pi\lambda)}{\sin(2\pi\gamma)} \\ &\quad + \frac{\gamma}{48\pi^2\lambda^3}\left[\frac{15}{\pi^2}(105\pi^{-2} - 15 - 132\bar{\xi} + 175\lambda^2 - 11\mu^2) - 366\lambda^2 - 2976\bar{\xi}\lambda^2 + 136\lambda^4 - 266\lambda^2\mu^2\right]\frac{\sinh(2\pi\lambda)}{\sin(2\pi\gamma)} \\ &\quad + i\csc(2\pi\gamma)\int_{-1}^{+1} C_{1r}\left[(e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right) - (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right)\right]dr \\ &\quad \left. + \left(\frac{\mu^4}{2} + 6\bar{\xi}\mu^2 - \frac{\lambda^2}{6}\right)\log(\mu) - \frac{i\pi}{12}(3\mu^4 + 36\bar{\xi}\mu^2 - \lambda^2)\right\}, \end{aligned} \tag{62}$$

and

$$
\begin{aligned}
T_{22} = T_{33} &= \langle \mathrm{in}|T_{33}|\mathrm{in}\rangle - \mathcal{T}^{[4]}_{33} \\
&= \Omega^2(\tau)\frac{H^4}{8\pi^2}\left\{-\frac{1}{60}-\frac{7\lambda^4}{720\mu^4}+\frac{\lambda^2}{45\mu^2}+\frac{\bar{\xi}\lambda^2}{2\mu^2}+18\bar{\xi}^2+\frac{\lambda^4}{35}+3\bar{\xi}\mu^2+\frac{2\bar{\xi}\lambda^2}{5}-\frac{\mu^2}{12}+\frac{\lambda^2}{8}+\frac{\lambda^2\mu^2}{5}\right. \\
&\quad+\frac{\gamma}{16\pi\lambda^2}\left[\frac{5}{\pi^2}(105\pi^{-2}-15-132\bar{\xi}+38\lambda^2-11\mu^2)-24\lambda^2-144\bar{\xi}\lambda^2\right]\frac{\cosh(2\pi\lambda)}{\sin(2\pi\gamma)} \\
&\quad-\frac{\gamma}{32\pi^2\lambda^3}\left[\frac{5}{\pi^2}(105\pi^{-2}-15-132\bar{\xi}+178\lambda^2-11\mu^2)-124\lambda^2-1024\bar{\xi}\lambda^2+\frac{200\lambda^4}{3}-\frac{220}{3}\lambda^2\mu^2\right]\frac{\sinh(2\pi\lambda)}{\sin(2\pi\gamma)} \\
&\quad+i\csc(2\pi\gamma)\int_{-1}^{+1}C_{2r}\left[(e^{2\pi\lambda r}+e^{2i\pi\gamma})\psi\left(\frac{1}{2}+\gamma+i\lambda r\right)-(e^{2\pi\lambda r}+e^{-2i\pi\gamma})\psi\left(\frac{1}{2}-\gamma+i\lambda r\right)\right]dr \\
&\quad\left.+\left(\frac{\mu^4}{2}+6\bar{\xi}\mu^2+\frac{\lambda^2}{6}\right)\log(\mu)-\frac{i\pi}{12}(3\mu^4+36\bar{\xi}\mu^2+\lambda^2)\right\}. \qquad (63)
\end{aligned}
$$

We see that the nonvanishing unregularized expectation values of the off-diagonal components, given by Eq. (46), are exactly canceled by their counterterms (60),

$$T_{01} = T_{10} = \langle \mathrm{in}|T_{10}|\mathrm{in}\rangle - \mathcal{T}^{[4]}_{10} = 0. \qquad (64)$$

Thus, we have arrived at the desired expressions for the regularized in-vacuum expectation values of all the components of the energy-momentum tensor, also called the induced energy-momentum tensor.

An alternative and very fruitful approach to the issue of scalar QED in de Sitter spacetime has been established in the series of remarkable papers [95–97]. In this series, the Starobinsky stochastic formulation [98] of inflationary quantum field theory has been extended to the setup of scalar QED in de Sitter spacetime. In the framework of the Starobinsky stochastic formalism, the long and short wavelength modes of the light scalar field are split via a window function. Or equivalently, in the momentum-space, the low- and high-momentum modes are split via an appropriate cutoff. Then, the short-wavelength part of the field inside the Hubble horizon is treated as a source term in the classical equation of motion for the long-wavelength part of the field outside the Hubble horizon; for a recent review of this subject, see [99]. Hence, the Starobinsky stochastic formalism treatment is in contrast to the canonical quantization procedure adopted in this paper where the scalar field operator is built from an appropriate linear superposition of all the mode functions with an equal weight; see Eq. (34). In [95], the Starobinsky stochastic formalism applied to massless, minimally coupled scalar QED in de Sitter spacetime, where the electromagnetic vector field was integrated out, and then, the renormalized effective potential and scalar QED energy-momentum tensor were derived. Furthermore, the Starobinsky stochastic equation was generalized to the effective scalar theory and solved at late times. In particular, their result for the effective energy-momentum tensor implies a decay of the effective cosmological constant due to the inflationary pair creation. In [96,97], the two-loop vacuum expectation values of the gauge invariant bilinears and the energy-momentum tensor of massless, minimally coupled scalar QED in de Sitter spacetime have been evaluated using the dimensional regularization method. It was found that the results are in agreement with the predictions of the stochastic formalism which was derived in [95]. Our result for the induced energy-momentum tensor (61)–(63) differs considerably from that of Refs. [95–97], because our viewpoint in this investigation has been quite different from that of Refs. [95–97]. First, instead of the Starobinsky stochastic formalism, we have adopted the canonical quantization procedure, as discussed in Sec. II C. Second, in Refs. [95–97], the electromagnetic field was considered as a dynamical quantum field which was integrated out in Ref. [95], and was treated perturbatively using the formalism of Feynman diagrams at two-loop order in Refs. [96,97]. However, as we have noted already in Sec. II B, in our analysis, the electromagnetic field (7) is treated as a fixed classical background field.

IV. PROBING THE INDUCED ENERGY-MOMENTUM TENSOR

The nonvanishing components of the induced energy-momentum tensor are given by Eqs. (61)–(63). Observe that all the off-diagonal components of the induced energy-momentum tensor are zero, and the components T_{22} and T_{33} are equal as consequences of the underlying symmetries of the backgrounds (2) and (6). Furthermore, since the electric field background (6) is not invariant under full symmetries of de Sitter spacetime, this indeed violates the time reversal symmetry [51], and causes an electric current along its direction; we observe that T_{00}, T_{11}, and T_{22} are not equal to one another. Thus, we expect that in the limit of vanishing electric field background that the in-vacuum state possesses the full set of de Sitter invariances, and the induced energy-momentum tensor takes the maximally invariant form under the transformations of de Sitter group, i.e., must be proportional to the de Sitter metric. By setting $\lambda = 0$ in Eqs. (61)–(63), which corresponds to vanishing electric field background, the induced energy-momentum tensor reduced to the form

$$T_{\mu\nu} = \frac{H^4}{32\pi^2}\left\{\frac{1}{15} - 72\bar{\xi}^2 - 12\bar{\xi}\mu^2 - \frac{2\mu^2}{3} + \mu^2(12\bar{\xi} + \mu^2) \times \left[\psi\left(\frac{3}{2}+\gamma_0\right) + \psi\left(\frac{3}{2}-\gamma_0\right) - \log(\mu^2)\right]\right\} g_{\mu\nu}, \quad (65)$$

where $\gamma_0 = \sqrt{(1/4) - \mu^2 - 12\bar{\xi}}$ is obtained by setting $\lambda = 0$ in the definition of γ. The expression (65) accords with the result of computing the renormalized vacuum energy-momentum tensor of a real scalar field in dS_4 obtained in Refs. [11,38], except for the overall factor of 2 in Eq. (65). This factor of 2 is consistent with the complex scalar field as being made of two real scalar fields with the number of degrees of freedom doubling up.

A. Behavior of the induced energy-momentum tensor

Figures 1–3 provide some useful insight into the general behavior of the induced energy-momentum tensor. The absolute values of the expressions (61)–(63) as functions of the electric field parameter λ, for various values of the scalar field mass parameter μ, and two values of the coupling constant ξ are shown on the graphs in Figs. 1–3, respectively. Note especially that the scales are logarithmic on both axes; hence, on the graphs, zero values of the expressions, where the signs of the plots change, are displayed as singularities. Therefore, these figures signal that the induced energy-momentum tensor is analytic and varies continuously with the parameters λ, μ, and ξ; this statement is consistent with the requirements discussed in [100]. Figures 1–3 also illustrate the outstanding qualitative features of the induced energy-momentum tensor. For fixed values of μ and ξ, the absolute values of the nonvanishing components of induced energy-momentum tensor are increasing functions of λ, but by excluding a neighborhood of the zero value points, this behavior is assured. For fixed values of λ and ξ, the absolute values of the nonvanishing components are decreasing functions of μ. For fixed values of λ and μ, the nonvanishing components do not vary significantly with the parameter ξ in the range $0 \leq \xi \ll \lambda, \mu$. The qualitative behaviors shown in Figs. 1–3 can be given quantitative treatments by inspection of expressions (61)–(63) in the limiting regimes. We concentrate our attention on three regimes of interest: (1) The strong electric field regime with the criterion

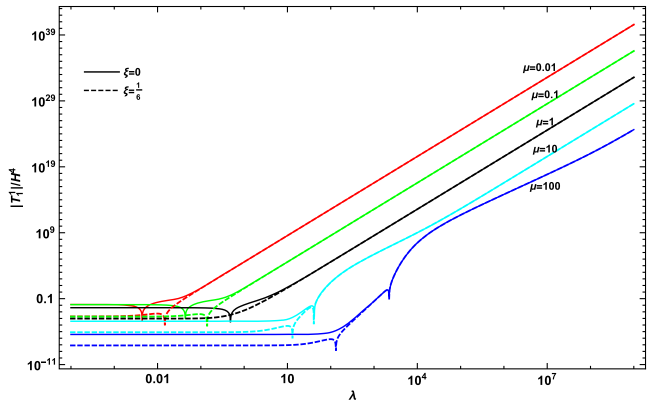

FIG. 2. The absolute value of T^1_1 component of the induced energy-momentum tensor is plotted in unit of H^4 as a function of the electric field parameter $\lambda = -eE/H^2$. The lines correspond to different values of the mass parameter $\mu = m/H$, and the coupling constant ξ as indicated. Both axes have logarithmic scales.

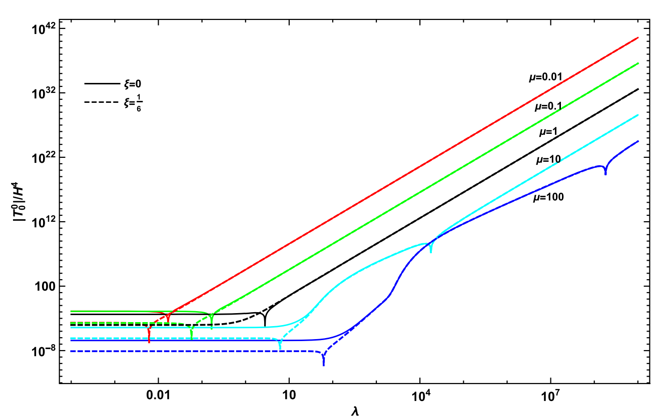

FIG. 1. The absolute value of T^0_0 component of the induced energy-momentum tensor is plotted in unit of H^4 as a function of the electric field parameter $\lambda = -eE/H^2$. The lines correspond to different values of the mass parameter $\mu = m/H$, and the coupling constant ξ as indicated. Both axes have logarithmic scales.

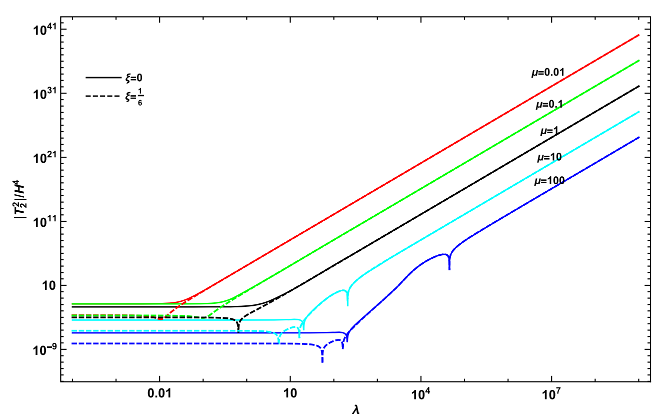

FIG. 3. The absolute value of T^2_2 component of the induced energy-momentum tensor is plotted in unit of H^4 as a function of the electric field parameter $\lambda = -eE/H^2$. The lines correspond to different values of the mass parameter $\mu = m/H$, and the coupling constant ξ as indicated. Both axes have logarithmic scales. Recall that $T^2_2 = T^3_3$.

$\lambda \gg \max(1, \mu, \xi)$. (2) The heavy scalar field regime with the criterion $\mu \gg \max(1, \lambda, \xi)$. (3) The infrared regime with the criteria $\mu \ll 1$, $\lambda \ll 1$, and $\xi = 0$.

1. Strong electric field regime

In the strong electric field regime $\lambda \gg \max(1, \mu, \xi)$, it is appropriate to find the approximate behavior of the induced energy-momentum tensor in the limit $\lambda \to \infty$. By expanding expressions (61)–(63) around $\lambda = \infty$ with μ and ξ fixed, we find the dominant terms in the components of the induced energy-momentum tensor,

$$T_{00} = -T_{11} = 3T_{22} = 3T_{33} = -\Omega^2 \frac{H^4}{8\pi^2}\left(\frac{7\lambda^4}{240\mu^4}\right). \tag{66}$$

Thus, in this regime, the absolute value of the nonvanishing components of induced energy-momentum tensor increase monotonically with increasing λ but decrease monotonically with increasing μ, as we see on the right end of any graphs in Figs. 1–3.

2. Heavy scalar field regime

In the heavy scalar field regime $\mu \gg \max(1, \lambda, \xi)$, it is appropriate to find approximate behavior of the induced energy-momentum tensor in the limit $\mu \to \infty$. By expanding expressions (61)–(63) around $\mu = \infty$ with λ and ξ fixed, we find the dominant terms in the components of the induced energy-momentum tensor,

$$\begin{aligned} T_{00} &= \Omega^2 \frac{H^4}{8\pi^2}\left(\frac{a}{\mu^2} + \frac{b}{\mu^4} + \frac{c_0\lambda^2}{\mu^4} + \mathcal{O}(\mu^{-6})\right), \\ T_{11} &= -\Omega^2 \frac{H^4}{8\pi^2}\left(\frac{a}{\mu^2} + \frac{b}{\mu^4} + \frac{c_1\lambda^2}{\mu^4} + \mathcal{O}(\mu^{-6})\right), \\ T_{22} = T_{33} &= -\Omega^2 \frac{H^4}{8\pi^2}\left(\frac{a}{\mu^2} + \frac{b}{\mu^4} + \frac{c_2\lambda^2}{\mu^4} + \mathcal{O}(\mu^{-6})\right), \end{aligned} \tag{67}$$

where the coefficients a, b, c_0, c_1 and c_2 are given by

$$\begin{aligned} a &= -\frac{2}{315} + \frac{\bar{\xi}}{5} - 72\bar{\xi}^3, \\ b &= -\frac{1}{210}(1 - 32\bar{\xi} + 504\bar{\xi}^2 - 90720\bar{\xi}^4), \\ c_0 &= -\frac{1}{315} - \frac{7\bar{\xi}}{15} + 12\bar{\xi}^2, \qquad c_1 = -\frac{22}{315} + \frac{3\bar{\xi}}{5} + 12\bar{\xi}^2, \\ c_2 &= \frac{2}{105} - \frac{\bar{\xi}}{3}. \end{aligned} \tag{68}$$

The asymptotic forms in Eq. (67) reveal that the induced energy-momentum tensor falls off as H^2/m^2 in the limit $(m/H) \to \infty$. Thus, the behavior of the induced energy-momentum tensor is inconsistent with the behavior of the semiclassical energy-momentum tensor [39,62] that falls of as $\exp(-2\pi m/H)$ in the heavy scalar field regime where $m \gg H$. A similar feature arises for the induced electric current of both scalar [61] and Dirac [73] fields in dS_4. Studies [74,75] have proposed explanations in physical terms for this feature of the induced electric current.

3. Infrared regime

To find approximate behavior of the induced energy-momentum tensor in the infrared regime, where $\mu \ll 1$, $\lambda \ll 1$ and $\xi = 0$, it is appropriate to make Taylor series expansions of the expressions (61)–(63) around $\mu = 0$ and $\lambda = 0$, and set $\xi = 0$. We find that the dominant terms in the expansions of (61) and (63) are given by

$$T_{00} = \Omega^2 \frac{H^4}{8\pi^2}\left(\frac{61}{60} - \frac{17\lambda^2}{60\mu^2} - \frac{7\lambda^4}{240\mu^4} + \mathcal{O}(\lambda^2, \mu^2)\right), \tag{69}$$

$$T_{22} = T_{33} = -\Omega^2 \frac{H^4}{8\pi^2}\left(\frac{61}{60} + \frac{11\lambda^2}{180\mu^2} + \frac{7\lambda^4}{720\mu^4} + \mathcal{O}(\lambda^2, \mu^2)\right), \tag{70}$$

which are valid for $\mu \ll \lambda \ll 1$ as well as $\lambda \ll \mu \ll 1$. The expansion of expression (62) for $\mu \ll \lambda \ll 1$ takes the form

$$T_{11} = \Omega^2 \frac{H^4}{8\pi^2}\left(\frac{299}{60} + \frac{7\lambda^2}{36\mu^2} + \frac{7\lambda^4}{240\mu^4} + \mathcal{O}(\lambda^2, \mu^2)\right), \tag{71}$$

while for $\lambda \ll \mu \ll 1$, it is approximated by

$$T_{11} = -\Omega^2 \frac{H^4}{8\pi^2}\left(\frac{61}{60} - \frac{223\lambda^2}{36\mu^2} + \frac{1433\lambda^4}{240\mu^4} + \mathcal{O}(\lambda^2, \mu^2)\right). \tag{72}$$

Observe that all the asymptotic expansions (69)–(71) diverge as m^{-4} in the exactly massless case, i.e., $m = 0$. We can understand the origin of these infrared-divergent terms by looking at the counterterms (57)–(59). These terms arise from the contribution of the zero modes in the massless case to the counterterms. Therefore, the signal for the massless case of the method of adiabatic regularization cannot be used because it leads to singularities in the counterterms, as pointed out in [19,20,61]. We emphasize that the result (72) is an approximation valid only for $\lambda \ll \mu \ll 1$, and hence, we cannot use it in the limit of zero mass for a fixed value of λ.

B. Trace anomaly

It is reassuring to do a consistency check, to see that whether the induced energy momentum tensor yields the well-known predicted trace anomaly for a free, massless, conformally invariant scalar field in dS_4. Putting the metric (2) and components (61)–(63) together, we obtain the trace of the induced energy-momentum tensor,

$$T = g^{\mu\nu}T_{\mu\nu} = \frac{H^4}{8\pi^2}\left\{\frac{1}{15} - \frac{7\lambda^4}{180\mu^4} - \frac{\lambda^2}{45\mu^2} + \frac{2\bar{\xi}\lambda^2}{\mu^2} - 72\bar{\xi}^2 + \frac{\mu^2}{3} - 12\bar{\xi}\mu^2 - \frac{\lambda^2}{6} - \frac{2\lambda^2\mu^2}{3} - \frac{3\mu^2\gamma}{2\pi^2\lambda\sin(2\pi\gamma)}(2\pi\lambda\cosh(2\pi\lambda) - \sinh(2\pi\lambda)) + \frac{i\mu^2}{2\sin(2\pi\gamma)}\int_{-1}^{+1}(3\lambda^2r^2 - \lambda^2 - \mu^2 - 12\bar{\xi})\left[(e^{2\pi\lambda r} + e^{2i\pi\gamma}) \times\psi\left(\frac{1}{2}+\gamma+i\lambda r\right) - (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2}-\gamma+i\lambda r\right)\right]dr - \mu^2(\mu^2+12\bar{\xi})\log(\mu^2) + i\pi\mu^2(\mu^2+12\bar{\xi})\right\}. \quad (73)$$

We see that the trace anomaly of the free, massless, conformally coupled complex scalar field is given by

$$\lim_{\lambda\to 0}\lim_{\mu\to 0}\lim_{\xi\to\frac{1}{6}} T = \frac{H^4}{120\pi^2} = \frac{2}{2880\pi^2}\left(\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu}\right), \quad (74)$$

and in arriving at the second equality, we have expressed H^4 in terms of a combination of the Ricci tensor and the scalar curvature of dS_4 which are given by Eq. (5), to put the result into a familiar form. The trace anomaly (74) is in agreement with the result of computing the trace anomaly [101] of a free, massless, conformally coupled real scalar field in dS_4, except for the overall factor of 2 in Eq. (74) as explained below Eq. (65).

V. IMPLICATIONS FOR THE INDUCED CURRENT

The generalization of the nonconservation equation (23) for the classical energy-momentum tensor of the scalar field to the induced energy-momentum tensor $T_{\mu\nu}$, and the induced current j_μ of the scalar field has important implications for the induced current. Recall that the nonvanishing components of $T_{\mu\nu}$ are given by Eqs. (61)–(63), and the nonzero components of $F_{\mu\nu}$ are given by Eq. (6). The only nontrivial relation that arises from Eq. (23) is obtained by setting $\nu = 0$, which leads to

$$\partial_0 T^{00} + H\Omega(5T^{00} + T^{11} + T^{22} + T^{33}) = -\Omega^{-2}Ej_1, \quad (75)$$

where we have used Eqs. (4) and (6). The relation (75), along with the relations

$$\partial_0 T^{00} = -2H\Omega(\tau)T^{00}, \qquad T = g_{\mu\nu}T^{\mu\nu},$$
$$-\Omega^{-2}Ej_1 = H\Omega^{-1}A.j, \quad (76)$$

suffice to show that the timelike component of the induced energy-momentum tensor can be written in terms of the trace T, and the effective electromagnetic potential energy $A.j$ as

$$T_{00} = \frac{1}{4}\Omega^2(T + A.j). \quad (77)$$

The induced current j_μ is defined as the regularized expectation value of the scalar field electric current operator (14) in the in-vacuum state specified in Eq. (35). It can be verified that the induced current j_μ is conserved and whose only nonvanishing component is j_1. Thus, the induced current flows along the electric field background direction. For convenience, we write the induced current as

$$j_\mu = \Omega(\tau)J_{[n]}\delta^1_\mu. \quad (78)$$

Here, we use the subscript $[n]$ to indicate the adiabatic order n of the subtracted counterterms to obtain the induced current $J_{[n]}$. Comparison of Eqs. (61), (73), and (77) allows us to read off the effective electromagnetic potential energy $A.j$, and then, we use the last relation in Eq. (76) and definition (78) to extract $J_{[4]}$. This gives

$$J_{[4]} = \frac{eH^3}{8\pi^2}\left\{\frac{4\bar{\xi}\lambda}{\mu^2} - \frac{\lambda}{9\mu^2} - \frac{7\lambda^3}{90\mu^4} + \frac{\lambda}{3}\log(\mu^2) - \frac{i\pi\lambda}{3} - \frac{4\lambda^3}{15} + \frac{\gamma}{6\pi\lambda}\left(\frac{45}{\pi^2} - 6 - 96\bar{\xi} + 4\lambda^2 - 8\mu^2\right)\frac{\cosh(2\pi\lambda)}{\sin(2\pi\gamma)} - \frac{\gamma}{12\pi^2\lambda^2}\left(\frac{45}{\pi^2} - 6 - 96\bar{\xi} + 64\lambda^2 - 8\mu^2\right)\frac{\sinh(2\pi\lambda)}{\sin(2\pi\gamma)} - \frac{i\lambda}{2\sin(2\pi\gamma)}\int_{-1}^{+1}(5\lambda^2r^4 - (1 + 36\bar{\xi} + 6\lambda^2 + 3\mu^2)r^2 + \lambda^2 + \mu^2 + 12\bar{\xi})\left[(e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2}+\gamma+i\lambda r\right) - (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2}-\gamma+i\lambda r\right)\right]dr\right\}, \quad (79)$$

where the subscript [4] indicates $J_{[4]}$ has been derived from the expressions which have been regularized by the counterterms expanded up to fourth adiabatic order. This should be clear from Eqs. (61), (77), and (78). To verify our result (79), we have evaluated directly the induced current by calculating the expectation value of the current operator (14) in the in-vacuum state and then subtracting the corresponding counterterms expanded up to fourth adiabatic order. The expression for $J_{[4]}$ that follows from this direct analysis reproduces exactly the expression given by Eq. (79).

In Ref. [61], the induced current of the massive, minimally coupled $\xi = 0$, scalar field has been evaluated by calculating the expectation value of the current operator (14) in the in-vacuum state which is represented by the mode functions given in Eqs. (30) and (31) with $\xi = 0$ for this case. In order to regularize the expectation value, the method of adiabatic subtraction was employed. Those authors expanded the required counterterm up to second adiabatic order that it suffices to remove the divergences, and the regularized expression resulting from use of it reduces to the expected results in the Minkowski spacetime limit. We refer to this prescription as minimal subtraction. Furthermore, they argued that including the contribution of fourth adiabatic order in the counterterm spoils the expected behavior in the Minkowski spacetime limit. Following these restrictions and arguments, it was subsequently found in Ref. [61] that the induced current is given in terms of $J_{[2]}$ as in Eq. (78) by

$$J_{[2]} = \frac{eH^3}{8\pi^2}\Bigg\{\frac{\lambda}{3}\log(\mu^2) - \frac{i\pi\lambda}{3} - \frac{4\lambda^3}{15} + \frac{\bar{\gamma}}{6\pi\lambda}\left(\frac{45}{\pi^2} + 10 + 4\lambda^2 - 8\mu^2\right)\frac{\cosh(2\pi\lambda)}{\sin(2\pi\bar{\gamma})} - \frac{\bar{\gamma}}{12\pi^2\lambda^2}\left(\frac{45}{\pi^2} + 10 + 64\lambda^2 - 8\mu^2\right) \times \frac{\sinh(2\pi\lambda)}{\sin(2\pi\bar{\gamma})} - \frac{i\lambda}{2\sin(2\pi\bar{\gamma})}\int_{-1}^{+1}(5\lambda^2 r^4 + (5 - 6\lambda^2 - 3\mu^2)r^2 + \lambda^2 + \mu^2 - 2)\Big[(e^{2\pi\lambda r} + e^{2i\pi\bar{\gamma}}) \times \psi\left(\frac{1}{2} + \bar{\gamma} + i\lambda r\right) - (e^{2\pi\lambda r} + e^{-2i\pi\bar{\gamma}})\psi\left(\frac{1}{2} - \bar{\gamma} + i\lambda r\right)\Big]dr\Bigg\}, \tag{80}$$

where $\bar{\gamma} = \sqrt{9/4 - \lambda^2 - \mu^2}$ is obtained by setting $\xi = 0$ in the definition of γ, and the subscript [2] indicates $J_{[2]}$ has been regularized by the counterterms expanded up to second adiabatic order. It is important to note that there are two differences between expressions (79) and (80). First, the coupling constant ξ is treated as arbitrary real value parameter in the expression (79), whereas the expression (80) has been computed for fixed value $\xi = 0$. Second, expression (79) contains contributions from the fourth adiabatic order expansion of the counterterm. With these two differences, (79) is a generalization of (80). Therefore, the difference $J_{[4]}(\xi = 0) - J_{[2]}$ would give only the fourth adiabatic order contribution of the counterterm to $J_{[4]}$, that is

$$J_{[4]}(\xi = 0) - J_{[2]} = -\frac{eH^3}{8\pi^2}\left(\frac{7\lambda}{9\mu^2} + \frac{7\lambda^3}{90\mu^4}\right). \tag{81}$$

In our subsequent discussion, we demonstrate that the expected behavior of $J_{[4]}$ in Minkowski spacetime is not spoiled by these fourth adiabatic order contributions. Furthermore, we show that these contributions alter the behavior of the $J_{[4]}$ when compared to $J_{[2]}$, especially in the two regimes of the infrared hyperconductivity and the strong electric field.

A. Minkowski spacetime limit

Here, we explore the behavior of the result (79) in the Minkowski spacetime limit. Note that in the limit where the Hubble constant tends to zero, i.e., $H \to 0$, we recover Minkowski spacetime. As was mentioned in the discussion of Eq. (81), the difference between the expressions (79) and (80) comes from the contribution of fourth adiabatic order in the adiabatic expansion of the counterterm. Comparison of the expressions (79) and (80) shows that these fourth adiabatic order contributions are

$$\delta J = \frac{eH^3}{8\pi^2}\left(\frac{4\bar{\xi}\lambda}{\mu^2} - \frac{\lambda}{9\mu^2} - \frac{7\lambda^3}{90\mu^4}\right) = -\frac{e}{8\pi^2}\left(\frac{eE}{m^2}\right)\left(4\bar{\xi}H^3 - \frac{H^3}{9} - \frac{7(eE)^2}{90m^2}H\right), \tag{82}$$

where the second equality comes from using the definitions of λ and μ given in Eq. (27). It is clear from the second equality in Eq. (82) that all these terms vanish in the limit $H \to 0$ for fixed and finite values of E and m. This observation will automatically insure the validity of (79) in the Minkowski spacetime limit. If the electric field background E and the scalar field mass m are regarded as fixed and finite, then by taking the limit $H \to 0$ of the expressions (79), we find

$$\lim_{H\to 0} J_{[4]} = \frac{e^3}{12\pi^3 H}E^2 e^{-\frac{\pi m^2}{|eE|}}, \tag{83}$$

which is exactly the same as the result obtained in Ref. [61] for the behavior of $J_{[2]}$, given by Eq. (80), in the Minkowski spacetime limit. It has been argued [61] that in the expanding dS the inverse of the Hubble constant H^{-1}, in fact, is equivalent to the finite time interval between switching on and off the electric field background in Minkowski spacetime. By this argument, we can see that the behavior (83) of the induced current in the Minkowski spacetime limit agrees with the electric current of the charged scalar particles produced by the Schwinger mechanism in Minkowski spacetime [40,41].

A comparison of the result of this article for the induced current $J_{[4]}$ [see Eq. (79)] to the result of Ref. [61] for the induced current $J_{[2]}$ [see Eq. (80)] is shown in Fig. 4. The figure is drawn for $\xi = 0$, and illustrates that, for the light scalars $\mu < 1$, the induced currents $J_{[4]}$ and $J_{[2]}$ differ considerably in the regime $\lambda \gtrsim \mu$. Subsequently, we set $\xi = 0$ and concentrate our attention on the regime $\lambda \gtrsim \mu$. We examine the behaviors of $J_{[4]}$ and $J_{[2]}$ in the two regimes: the infrared hyperconductivity with the criterion $\mu < \lambda \lesssim 1$ and the strong electric field $\lambda \gg \max(1, \mu, \xi)$.

B. Behaviors in the infrared hyperconductivity regime

In Ref. [61], it was pointed out that, in the infrared hyperconductivity regime $\mu < \lambda \lesssim 1$, the absolute value of the induced current $J_{[2]}$ monotonically increases with decreasing the electric field parameter λ, as shown in Fig. 4. In this regime, we can approximate expression (80) by

$$J_{[2]} \simeq \frac{eH^3}{8\pi^2}\left(\frac{6\lambda}{\lambda^2 + \mu^2}\right). \tag{84}$$

From Fig. 4, it is obvious that in the infrared hyperconductivity regime, the absolute value of the induced current $J_{[4]}$ reduces to zero at a certain value of the electric field parameter λ_* which depends on the mass parameter μ. Indeed, the sign of $J_{[4]}$ changes at λ_*. This figure also indicates that the absolute value of $J_{[4]}$ is a decreasing function of λ in the interval $\mu < \lambda < \lambda_*$ and is an increasing function for $\lambda > \lambda_*$. In the infrared hyperconductivity regime, we can approximate expression (79) by

$$J_{[4]} \simeq \frac{eH^3}{8\pi^2}\left(-\frac{7\lambda^3}{90\mu^4} - \frac{7\lambda}{9\mu^2} + \frac{6\lambda}{\lambda^2 + \mu^2}\right). \tag{85}$$

This expression has a zero at $\lambda_* \simeq 2.090\mu$. In the range of $\lambda > 2.090\mu$, the dominant contribution to $J_{[4]}$ comes from the first two terms in Eq. (85) which its absolute value increases with increasing λ. Thus, in this range, the infrared hyperconductivity phenomenon does not occur. On the other hand, in the interval $\mu < \lambda < 2.090\mu$, the dominant contribution to $J_{[4]}$ comes from the last term in Eq. (85) which increases with decreasing λ. Thus, in the interval $\mu < \lambda < 2.090\mu$, the infrared hyperconductivity phenomenon occurs. We therefore conclude that although there exist a period of the infrared hyperconductivity in our result for the induced current $J_{[4]}$, now this phenomenon occurs in the more restricted interval $\mu < \lambda < 2.090\mu$.

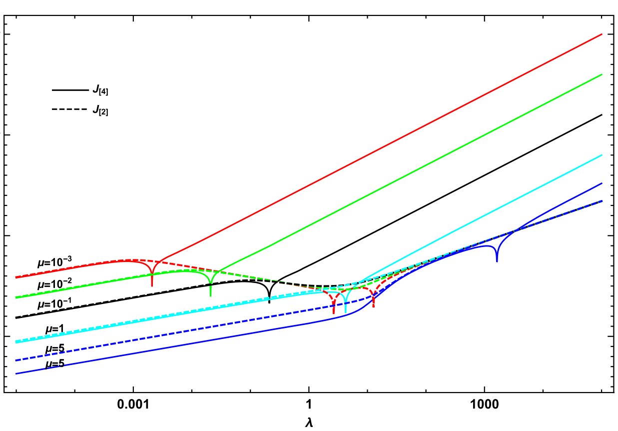

FIG. 4. Dependence of the absolute values of the normalized induced currents $|J_{[4]}|/eH^3$ (solid curves) and $|J_{[2]}|/eH^3$ (dashed curves) on the electric field parameter $\lambda = -eE/H^2$, for the case of minimal coupling $\xi = 0$. The lines correspond to different values of the scalar field mass parameter $\mu = m/H$ as indicated. Both axes have logarithmic scales.

C. Behaviors in the strong electric field regime

Figure 4 shows that although the absolute values of the two induced current $J_{[2]}$ and $J_{[4]}$ are increasing functions of λ in the strong electric field regime $\lambda \gg \max(1, \mu, \xi)$, the induced current $J_{[2]}$ does not depend on the scalar field mass, whereas the induced current $J_{[4]}$ is proportional to μ^{-4}. We find that, in the limit $\lambda \to \infty$ for a fixed μ, the induced current $J_{[4]}$ is given approximately by

$$J_{[4]} \simeq -\frac{eH^3}{8\pi^2}\left(\frac{7\lambda^3}{90\mu^4}\right) = \frac{7He^4}{720\pi^2}\left(\frac{E^3}{m^4}\right). \tag{86}$$

We observe that the behavior of the induced current $J_{[4]}$ in the Minkowski spacetime limit [see Eq. (83)] will be different from that in the strong electric field limit, see Eq. (86). Whereas, in [61], it was pointed out that the induced current $J_{[2]}$ has the same behavior in these two limits, indicated on the right side of Eq. (83). We note that in the Minkowski spacetime limit, the electric field strength E, the scalar field mass m, and the coupling constant ξ are regarded as fixed, and the Hubble constant H, tends to zero. Thus, in the Minkowski spacetime limit, although $\lambda = -eE/H^2$ and $\mu = m/H$ tend to infinity, the ratio λ/μ^2 remains finite. We also note that in the strong electric field regime, the Hubble constant H, the scalar field mass m, and the coupling constant ξ are regarded as fixed, and the electric field strength E, tends to infinity. Thus, in this regime, λ goes to infinity while μ is regarded as a fixed and finite value.

VI. CONCLUSIONS

The aim of the present research was to examine the induced energy-momentum tensor of a massive, complex scalar field coupled to the electromagnetic vector potential (7) which describes a uniform electric field background with a constant energy density in the conformal Poincaré patch of dS_4 where the metric takes the form (2). The dynamics of the complex scalar field is governed by the action (11). The energy-momentum tensor of the scalar field is not covariantly conserved in the presence of the electromagnetic field, as the consequence of the electromagnetic interactions, see Eq. (23). The nonconservation of the scalar field energy-momentum tensor is compatible with the nonconservation of the electromagnetic field energy-momentum tensor [see Eq. (24)], and hence, the total energy momentum tensor of the theory is covariantly conserved. We treated the classical gravitational field (2) and the classical electromagnetic field (7) as fixed field configurations which they are unaffected by the dynamics of the quantum complex scalar field in response to these backgrounds. We discussed canonical quantization of the scalar field. The normalized positive and negative frequency mode functions of the scalar field which behave like the positive and negative frequency Minkowski mode functions in the remote past are given by Eqs. (30) and (31). These mode functions determine the in-vacuum state of the scalar field.

We calculated the expectation values of all the components of the energy-momentum tensor in the in-vacuum state of the scalar field; the nonzero expectation values have been obtained in Eqs. (40), (42), (44), and (46). It is expected that these expectation values contain ultraviolet divergences, because the expectation values in the Hadamard in-vacuum state suffer from the same ultraviolet divergence properties as Minkowski spacetime. To render these in-vacuum expectation values finite, we employed the adiabatic regularization method. To adjust the set of appropriate counterterms, we treated the conformal scale factor and the electromagnetic vector potential as quantities of zero adiabatic order. As pointed out by Wald [18], in four dimensions, to obtain a renormalized energy-momentum tensor which is consistent with the Wald axioms, the subtraction counterterms should be expanded up to fourth adiabatic order. Under these assumptions, we then constructed the set of the appropriate adiabatic counterterms, which are given by Eqs. (57)–(60). The adiabatic regularization procedure was carried out by subtracting the counterterms from the corresponding unregularized in-vacuum expectation values. Thus, we have arrived at our final expressions for all the nonvanishing components of the induced energy-momentum tensor, which are given by Eqs. (61)–(63). This research has shown that all the off-diagonal components of the induced energy-momentum tensor are zero, and the components T_{22} and T_{33} are equal as consequences of the underlying symmetries of the backgrounds (2) and (6). Furthermore, since the electric field background (6) is not invariant under full symmetries of dS, this indeed violates the time reversal symmetry [51], and causes an electric current along its direction; we observe that T_{00}, T_{11}, and T_{22} are not equal to one another. The research has also shown that in the limit of zero electric field, our result for the induced energy-momentum tensor reduces to the form (65). The expression (65) accords with the result of computing the renormalized vacuum energy-momentum tensor of a real scalar field in dS_4 obtained in Refs. [11,38], except for the overall factor of 2 in Eq. (65). This factor of 2 is consistent with the complex scalar field as being made of two real scalar fields with the number of degrees of freedom doubling up. The absolute values of the expressions (61)–(63) as functions of the electric field parameter λ, for various values of the scalar field mass parameter μ, and two values of the coupling constant ξ are shown on the graphs in Figs. 1–3, respectively. These figures signal that the induced energy-momentum tensor is analytic and varies continuously with the parameters λ, μ, and ξ; this statement is consistent with the requirements discussed in [100]. For fixed values of μ and ξ, the absolute values of the nonvanishing components of the induced energy-momentum tensor are increasing functions of λ, but by excluding a neighborhood of the zero value points, this behavior is assured. For fixed values of λ and ξ, the absolute values of the nonvanishing components are decreasing functions of μ. For a fixed λ and μ, the nonvanishing components do not vary significantly with the parameter ξ in the range $0 \leq \xi \ll \lambda, \mu$. The qualitative behaviors shown in Figs. 1–3 can be given quantitative treatments by inspection of expressions (61)–(63) in the limiting regimes. The examination of the expressions (61)–(63) has shown that in the strong electric field regime $\lambda \gg \max(1, \mu, \xi)$, the components of the induced energy-momentum tensor can be approximated by Eq. (66). In the heavy scalar field regime $\mu \gg \max(1, \lambda, \xi)$, the components can be approximated according to Eq. (67). We also found that in the infrared regime, where $\mu \ll 1$, $\lambda \ll 1$, $\xi = 0$, the components can be approximated by Eqs. (69)–(72). To do a consistency check, we derived the trace anomaly of the induced energy-momentum tensor for the case of a free, massless, conformally invariant scalar field in dS_4; see Eq. (74). This investigation shows that our result (74) is in agreement with the earlier result of computing the trace anomaly [101] of a free, massless, conformally coupled real scalar field in dS_4, except for the overall factor of 2 in Eq. (74), as explained below Eq. (65).

One of the more significant findings to emerge from this research is that the nonconservation equation (23) implies the relation (77) between the induced energy-momentum tensor and the induced current. This relation in turn implies the renormalization condition for the induced current. We derived the expression (79) for the induced current of the scalar field by using the expressions (61), (73), and relation (77). This result for the induced current has been derived from the expressions which have been regularized by the counterterms expanded up to fourth adiabatic order.

In Ref. [61], the induced current of the massive, minimally coupled $\xi = 0$, scalar field has been evaluated by subtracting the counterterm expanded up to second adiabatic order; see Eq. (80). In the discussion of Eq. (83), we remarked that our result for the induced current in the Minkowski spacetime limit agrees with the electric current of the charged scalar particles produced by the Schwinger mechanism in Minkowski spacetime [40,41]. A comparison of the result of this article for the induced current $J_{[4]}$ [see Eq. (79)] to the result of Ref. [61] for the induced current $J_{[2]}$ [see Eq. (80)] is shown in Fig. 4. The figure is drawn for $\xi = 0$, and illustrates that, for the light scalars $\mu < 1$, the induced currents $J_{[4]}$ and $J_{[2]}$ differ considerably in the regime $\lambda \gtrsim \mu$. From Fig. 4, it is obvious that in the infrared hyperconductivity regime $\mu < \lambda \lesssim 1$, the absolute value of the induced current $J_{[4]}$ reduces to zero at a certain value of the electric field parameter λ_* which depends on the mass parameter μ. We found that in the infrared hyperconductivity regime, the induced current $J_{[4]}$ can be approximated by Eq. (85) which has a zero at $\lambda_* \simeq 2.090\mu$. In the interval $\mu < \lambda < 2.090\mu$, the dominant contribution to $J_{[4]}$ comes from the last term in Eq. (85) which increases with decreasing λ. Thus, the infrared hyperconductivity phenomenon occurs in the interval $\mu < \lambda < 2.090\mu$. We therefore conclude that although there exist a period of the infrared hyperconductivity in our result for the induced current $J_{[4]}$, now this phenomenon occurs in the more restricted interval $\mu < \lambda < 2.090\mu$. Figure 4 also shows that the absolute value of the induced current $J_{[4]}$ is an increasing function of λ in the strong electric field regime $\lambda \gg \max(1, \mu, \xi)$, but decreases as μ^{-4} with increasing μ. We saw that in the limit $\lambda \to \infty$ for a fixed μ, the induced current $J_{[4]}$ is given approximately by Eq. (86). The results of this investigation show that the behavior of the induced current $J_{[4]}$ in the Minkowski spacetime limit [see Eq. (83)] will be different from that in the strong electric field limit, see Eq. (86).

This would be a fruitful area for further work. A natural progression of this work is to analyze the backreaction of the induce energy-momentum tensor on the gravitational field of dS_4 and the backreaction of the induce current $J_{[4]}$ on the electromagnetic field. This is also an issue for future research to explore the induced energy-momentum tensor of a Dirac field in the context of our discussion.

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APPENDIX: EVALUATION OF THE MOMENTUM INTEGRALS OVER THE MODE FUNCTIONS

In the Appendix, we present further supplementary data associated with the calculation of the expressions (36)–(38). It is convenient at this stage to switch to the three-dimensional spherical momentum space, and then, we can perform the entire three-dimensional integrals in three-dimensional spherical coordinates. We introduce a transformation from the Cartesian momentum coordinates (k_x, k_y, k_z) to the spherical momentum coordinates (k, θ, ϕ) by equations

$$k_x = k\cos\theta, \quad k_y = k\sin\theta\cos\phi, \quad k_z = k\sin\theta\sin\phi. \tag{A1}$$

In these coordinates, the variable $r = k_x/k$ is given by $r = \cos\theta$ with range $-1 \le r \le 1$. The integration measure is then $d^3k = k^2 \sin\theta d\phi d\theta dk$. It is useful to convert the variables of integration from the momentum k to the dimensionless physical momentum $p = -k\tau$, and from the angle θ to the variable r. Thus, the previous expression for the integration measure may be rewritten as

$$d^3k = -H^3\Omega^3(\tau) d\phi dr p^2 dp. \tag{A2}$$

Accordingly, we use a dimensionless physical momentum cutoff Λ, which is related to the momentum cutoff K as $\Lambda = -K\tau$. To begin the evaluation of (36)–(38), we change the integration measure according to (A2) and substitute the expression (30) for $U_{\mathbf{k}}(x)$. After integration over azimuth angle ϕ and some simplifications, the expressions (36)–(38) reduce to

$$\langle \mathrm{in}|T_{00}|\mathrm{in}\rangle = \Omega^2 \frac{H^4}{8\pi^2}\int_{-1}^{+1} dr \left\{2\mathcal{I}_1 - 4\lambda r\mathcal{I}_2 + \left(\frac{1}{4} - \gamma^2 - 18\bar{\xi} + \lambda^2 r^2\right)\mathcal{I}_3 + 2\Im[\mathcal{I}_4] - 2\Re[(6\xi - 1 - i\lambda r)\mathcal{I}_5] + \mathcal{I}_6\right\}, \tag{A3}$$

$$\langle \mathrm{in}|T_{11}|\mathrm{in}\rangle = \Omega^2 \frac{H^4}{8\pi^2}\int_{-1}^{+1} dr \{2r^2\mathcal{I}_1 - 4\lambda r\mathcal{I}_2 + [(4\xi + 1)\lambda^2 + (4\xi - 1)\mu^2 - (4\xi - 1)\lambda^2 r^2 + (6\xi - 1)(8\xi - 1)]\mathcal{I}_3 - 2\Im[(4\xi - 1)\mathcal{I}_4] + 2\Re[(1 - 6\xi + i\lambda r - 4i\lambda r\xi)\mathcal{I}_5] - (4\xi - 1)\mathcal{I}_6\}, \tag{A4}$$

and

$$\langle \mathrm{in}|T_{22}|\mathrm{in}\rangle = \langle \mathrm{in}|T_{33}|\mathrm{in}\rangle = \Omega^2 \frac{H^4}{8\pi^2}\int_{-1}^{+1} dr\{(1-r^2)\mathcal{I}_1 + [(4\xi-1)(\lambda^2+\mu^2-\lambda^2 r^2) + (6\xi-1)(8\xi-1)]\mathcal{I}_3 - 2\Im[(4\xi-1)\mathcal{I}_4] + 2\Re[(1-6\xi+i\lambda r-4i\lambda r\xi)\mathcal{I}_5] - (4\xi-1)\mathcal{I}_6\}, \tag{A5}$$

where $\Im$ and $\Re$ stand for the imaginary and real parts of any expressions, respectively. In these expressions, the momentum integrals over the Whittaker functions have been defined by

$$\mathcal{I}_1 = e^{\pi\lambda r}\int_0^\Lambda dp p^3 |W_{\kappa,\gamma}(-2ip)|^2, \tag{A6}$$

$$\mathcal{I}_2 = e^{\pi\lambda r}\int_0^\Lambda dp p^2 |W_{\kappa,\gamma}(-2ip)|^2, \tag{A7}$$

$$\mathcal{I}_3 = e^{\pi\lambda r}\int_0^\Lambda dp p |W_{\kappa,\gamma}(-2ip)|^2, \tag{A8}$$

$$\mathcal{I}_4 = \left(\frac{1}{4}-\gamma^2-\lambda^2 r^2+i\lambda r\right)e^{\pi\lambda r}\int_0^\Lambda dp p^2 W_{\kappa-1,\gamma}(-2ip)W_{-\kappa,\gamma}(2ip), \tag{A9}$$

$$\mathcal{I}_5 = \left(\frac{1}{4}-\gamma^2-\lambda^2 r^2+i\lambda r\right)e^{\pi\lambda r}\int_0^\Lambda dp p W_{\kappa-1,\gamma}(-2ip)W_{-\kappa,\gamma}(2ip), \tag{A10}$$

$$\mathcal{I}_6 = \left|\frac{1}{4}-\gamma^2-\lambda^2 r^2+i\lambda r\right|^2 e^{\pi\lambda r}\int_0^\Lambda dp p W_{\kappa-1,\gamma}(-2ip)W_{-\kappa-1,\gamma}(2ip). \tag{A11}$$

The integrals in Eqs. (A6)–(A11) are of the same kind of those momentum integrals over the Whittaker functions which encountered in the calculation of the induced current of a scalar field in two- [60] and four-dimensional [61] de Sitter spacetimes. The first step in the evaluation of the integrals (A6)–(A11) by the method that explained in Ref. [61], is to plug the Mellin-Barnes representation (28) for the Whittaker function W and make use of the theorem of residues. It is rather straightforward, but rather lengthy, to show that our final results are

$$\begin{aligned}\mathcal{I}_1 &= \frac{\Lambda^4}{4} + \frac{\lambda r}{3}\Lambda^3 + \frac{1}{16}(4\gamma^2+12\lambda^2 r^2-1)\Lambda^2 + \frac{\lambda r}{8}(12\gamma^2+20\lambda^2 r^2-7)\Lambda + \frac{1}{128}(27+48\gamma^4+560\lambda^4 r^4-120\gamma^2 \\ &\quad + 480\gamma^2\lambda^2 r^2 - 520\lambda^2 r^2)\log(2\Lambda) - \frac{7\gamma^4}{32} + \frac{83\gamma^2}{64} - \frac{533}{96}\lambda^4 r^4 + \frac{1607}{192}\lambda^2 r^2 - \frac{59}{16}\gamma^2\lambda^2 r^2 - \frac{351}{512} \\ &\quad - \left(\frac{55\gamma^2}{24} + \frac{35\lambda^2 r^2}{8} - \frac{355}{96}\right)\frac{\gamma\lambda r}{\sin(2\pi\gamma)}(\cos(2\pi\gamma)+e^{2\pi\lambda r}) - \frac{i}{256\sin(2\pi\gamma)}(27+48\gamma^4+560\lambda^4 r^4-120\gamma^2+480\gamma^2\lambda^2 r^2 \\ &\quad - 520\lambda^2 r^2)\left[\pi\sin(2\pi\gamma) + (e^{2\pi\lambda r}+e^{-2i\pi\gamma})\psi\left(\frac{1}{2}-\gamma+i\lambda r\right) - (e^{2\pi\lambda r}+e^{2i\pi\gamma})\psi\left(\frac{1}{2}+\gamma+i\lambda r\right)\right],\end{aligned} \tag{A12}$$

$$\begin{aligned}\mathcal{I}_2 &= \frac{\Lambda^3}{3} + \frac{\lambda r}{2}\Lambda^2 + \frac{1}{8}(4\gamma^2+12\lambda^2 r^2-1)\Lambda + \frac{\lambda r}{8}(12\gamma^2+20\lambda^2 r^2-7)\log(2\Lambda) - \frac{37}{12}\lambda^3 r^3 + \frac{95}{48}\lambda r - \frac{5\gamma^2}{4}\lambda r \\ &\quad - (4\gamma^2+15\lambda^2 r^2-4)\frac{\gamma}{6\sin(2\pi\gamma)}(\cos(2\pi\gamma)+e^{2\pi\lambda r}) - \frac{i\lambda r}{16\sin(2\pi\gamma)}(12\gamma^2+20\lambda^2 r^2-7) \\ &\quad \times\left[\pi\sin(2\pi\gamma) + (e^{2\pi\lambda r}+e^{-2i\pi\gamma})\psi\left(\frac{1}{2}-\gamma+i\lambda r\right) - (e^{2\pi\lambda r}+e^{2i\pi\gamma})\psi\left(\frac{1}{2}+\gamma+i\lambda r\right)\right],\end{aligned} \tag{A13}$$

$$\mathcal{I}_3 = \frac{\Lambda^2}{2} + \lambda r\Lambda + \frac{1}{8}(4\gamma^2 + 12\lambda^2 r^2 - 1)\log(2\Lambda) - \frac{\gamma^2}{4} - \frac{7\lambda^2 r^2}{4} + \frac{5}{16} - \frac{3\gamma\lambda r}{2\sin(2\pi\gamma)}(\cos(2\pi\gamma) + e^{2\pi\lambda r}) - \frac{i}{16\sin(2\pi\gamma)}(4\gamma^2 + 12\lambda^2 r^2 - 1)\left[\pi\sin(2\pi\gamma) + (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right) - (e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right)\right], \tag{A14}$$

$$\mathcal{I}_4 = -\frac{i}{16}(4\gamma^2 + 4\lambda^2 r^2 - 4i\lambda r - 1)\Lambda^2 - \frac{1}{8}(4\gamma^2 + 4\lambda^2 r^2 - 4i\lambda r - 1)(1 + 2i\lambda r)\Lambda - \frac{3i}{128}(4\gamma^2 + 4\lambda^2 r^2 - 4i\lambda r - 1)(4\gamma^2 + 20\lambda^2 r^2 - 20i\lambda r - 9)\log(2\Lambda) + \frac{79}{32}i\lambda^4 r^4 + \frac{47}{8}\lambda^3 r^3 - \frac{409}{64}i\lambda^2 r^2 + \frac{39\gamma^2}{16}i\lambda^2 r^2 - \frac{109}{32}\lambda r + \frac{21\gamma^2}{8}\lambda r + \frac{7i}{32}\gamma^4 - \frac{83i}{64}\gamma^2 + \frac{351i}{512} + \frac{\gamma}{4\sin(2\pi\gamma)}\left(4\gamma^2 + \frac{15}{2}i\lambda^3 r^3 + 15\lambda^2 r^2 + \frac{13}{2}i\gamma^2\lambda r - \frac{97}{8}i\lambda r - 4\right)(\cos(2\pi\gamma) + e^{2\pi\lambda r}) - \frac{3}{256\sin(2\pi\gamma)}(4\gamma^2 + 4\lambda^2 r^2 - 4i\lambda r - 1)(4\gamma^2 + 20\lambda^2 r^2 - 20i\lambda r - 9)\left[\pi\sin(2\pi\gamma) + (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right) - (e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right)\right], \tag{A15}$$

$$\mathcal{I}_5 = -\frac{i}{8}(4\gamma^2 + 4\lambda^2 r^2 - 4i\lambda r - 1)\Lambda - \frac{1}{8}(4\gamma^2 + 4\lambda^2 r^2 - 4i\lambda r - 1)(1 + 2i\lambda r)\log(2\Lambda) - 2\gamma^2 - 10\lambda^2 r^2 - \frac{16}{3}i\lambda^3 r^3 - 4i\gamma^2\lambda r + \frac{20}{3}i\lambda r + \frac{3}{2} - \frac{i\gamma}{3\sin(2\pi\gamma)}(8\gamma^2 + 12\lambda^2 r^2 - 18i\lambda r - 8)(\cos(2\pi\gamma) + e^{2\pi\lambda r}) + \frac{i}{16\sin(2\pi\gamma)}(4\gamma^2 + 4\lambda^2 r^2 - 4i\lambda r - 1)(1 + 2i\lambda r)\left[\pi\sin(2\pi\gamma) + (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right) - (e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right)\right], \tag{A16}$$

$$\mathcal{I}_6 = \frac{1}{64}(16\gamma^4 - 8\gamma^2 + 16\lambda^4 r^4 + 32\gamma^2\lambda^2 r^2 + 8\lambda^2 r^2 + 1)\log(2\Lambda) - \frac{3\gamma^4}{16} + \frac{7\gamma^2}{32} - \frac{25}{48}\lambda^4 r^4 - \frac{7}{8}\gamma^2\lambda^2 r^2 - \frac{29}{96}\lambda^2 r^2 - \frac{11}{256} - \frac{\gamma}{48\sin(2\pi\gamma)}(16\gamma^4 - 8\gamma^2 + 16\lambda^4 r^4 + 32\gamma^2\lambda^2 r^2 + 8\lambda^2 r^2 + 1)(12\lambda^3 r^3 + 20\gamma^2\lambda r + 7\lambda r)(\cos(2\pi\gamma) + e^{2\pi\lambda r}) - \frac{i}{128\sin(2\pi\gamma)}(16\gamma^4 - 8\gamma^2 + 16\lambda^4 r^4 + 32\gamma^2\lambda^2 r^2 + 8\lambda^2 r^2 + 1)\left[\pi\sin(2\pi\gamma) + (e^{2\pi\lambda r} + e^{-2i\pi\gamma})\psi\left(\frac{1}{2} - \gamma + i\lambda r\right) - (e^{2\pi\lambda r} + e^{2i\pi\gamma})\psi\left(\frac{1}{2} + \gamma + i\lambda r\right)\right], \tag{A17}$$

where we use $\log(z)$ to denote the natural logarithm function, and $\psi(z)$ to denote the digamma function. By substituting Eqs. (A12)–(A17) into expressions (A3)–(A5) and performing the integrals over r, we obtain the results (40), (42), and (44), respectively.

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