Motion of a spinning particle under the conservative piece of the self-force is Hamiltonian to first order in mass and spin

Francisco M. Blanco and Éanna É. Flanagan
Department of Physics, Cornell University, Ithaca, New York 14853, USA

(Received 22 February 2023; accepted 25 May 2023; published 9 June 2023)

We consider the motion of a point particle with spin in a stationary spacetime. We define, following Witzany *et al.* [Classical Quantum Gravity **36**, 075003 (2019)] and later Ramond [arXiv:2210.03866], a 12-dimensional Hamiltonian dynamical system whose orbits coincide with the solutions of the Mathisson-Papapetrou-Dixon equations of motion with the Tulczyjew-Dixon spin supplementary condition, to linear order in spin. We then perturb this system by adding the conservative pieces of the leading order gravitational self-force and self-torque sourced by the particle's mass and spin. We show that this perturbed system is Hamiltonian and derive expressions for the Hamiltonian function and symplectic form. This result extends a previous result for spinless point particles [F. M. Blanco and E. E. Flanagan, Phys. Rev. Lett. **130**, 051201 (2023)].

DOI: 10.1103/PhysRevD.107.124017

I. INTRODUCTION

In recent years, the detection of coalescences of binary black hole systems has started a new era of gravitational wave astronomy [1–3]. The coming years will bring many more detections with the next generation ground based detectors Cosmic Explorer [4] and the Einstein Telescope [5], the space based detector LISA [6], and potentially pulsar timing arrays [7]. The observation of gravitational waves requires precise waveform templates, which for binary coalescences can be obtained through a variety of different approximation methods valid in different regimes. Some of the techniques that have been used to understand the dynamics of black hole binaries are numerical relativity [8], the post-Newtonian approximation [9–12], the post-Minkowskian approximation [13] for which amplitude methods from quantum field theory are useful [14], the small mass ratio approximation [15,16], and the effective one-body framework which synthesizes information from the other approaches [17,18].

A theoretical issue that arises in the study of binary dynamics is whether the motion forms a Hamiltonian dynamical system when gravitational wave dissipation is turned off. This has been established to various orders in the post-Newtonian and post-Minkowskian approximations (see Ref. [19] and references therein). For nonspinning particles, it has also been established to first order in the small mass ratio approximation [20]. The small mass ratio approximation consists of an expansion in the ratio $\epsilon = \mu/M$ of the mass μ of the secondary object to the mass M of the primary object. The gravitational field of the secondary acts as a perturbation to the background geometry, which can be expanded in powers of ϵ. The interaction between the secondary and its own gravitational field gives rise to an acceleration with respect to the background geometry, described by the gravitational self-force [21,22]. The self-force itself can be divided into conservative and dissipative pieces. The former is derived from the time symmetric piece of the Green function while the latter comes from its time antisymmetric piece and is responsible for the dissipation that drives the slow inspiral. In previous work [20], we showed that the conservative piece of the first order self-force gives rise to Hamiltonian dynamics, and derived an explicit expression for the Hamiltonian. The goal of this paper is to extend that result to include the leading spin effects of the secondary.

The motion of a point particle with spin in general relativity, neglecting self-gravity, is described by the Mathisson-Papapetrou-Dixon equations [23–25]. A variety of Hamiltonian formulations of the dynamics in the test body limit have been given in [26–29]. Many of these formulate the dynamics as a constrained Hamiltonian system. We will follow an approach developed by Witzany *et al.* [26] and later generalized by Ramond [27] which yields an unconstrained Hamiltonian system on a 12-dimensional space. Going beyond the test body limit to include self-gravity and working to leading order in spin, the motion is described by a first order self-force which depends on mass and spin, and by a first order self-torque [30]. Specifically the self-force has terms of order $O(\mu^2)$, $O(S)$, and $O(\mu S)$, where μ is mass and S spin, and the self-torque scales as $O(\mu S)$. We will show that this dynamical system is also Hamiltonian, and we will derive the explicit form of the Hamiltonian.

The various spin-related conservative effects that arise in the dynamics of two-body systems are listed in Table I, which includes the scaling of the interaction energy and

TABLE I. A summary of different conservative effects in the dynamics of binary systems with spinning components in general relativity, in the small mass ratio limit $\mu \ll M$, where M is the mass of the primary and μ the mass of the secondary. The effects can be distinguished by the scaling of the associated interaction energy with the parameters of the system. The first column lists the effects and the second the interaction energies, where $\epsilon = \mu/M$ is the mass ratio and S is the spin of the secondary. The third column lists the total phase shift caused by the effect that is accumulated during the dissipative self-force driven inspiral through the relativistic regime. The second entry in each box is the result specialized to an order unity dimensionless spin parameter $S/\mu^2 \sim 1$ for the secondary. Some previous discussions of these effects in the literature are listed in the fourth column. The fifth column specifies if a Hamiltonian formulation of the effect was previously known and gives references. Finally the last column indicates which effects are included in the analysis of this paper.

Name of effect	Interaction energy	Accumulated phase shift in inspiral	Papers that discuss effect	Previously known Hamiltonian?	Included in this paper?
Geodesic motion	$\sim\mu$	$\sim\frac{1}{\epsilon}$		✓	✓
First order conservative self-force	$\sim\mu\epsilon$	~ 1	[31]	✓ [20]	✓
Second order conservative self-force	$\sim\mu\epsilon^2$	$\sim\epsilon$	[32–36]		
Leading spin-curvature coupling	$\sim\frac{S}{M}\sim\mu\epsilon$	$\sim\frac{S}{\mu^2}\sim 1$	[23–25,28,37–40]	✓ [26,27]	✓
Subleading spin-curvature coupling	$\sim\frac{S^2}{\mu M^2}\sim\mu\epsilon^2$	$\sim\frac{S^2}{\mu^3 M}\sim\epsilon$	[29]	✓ [29]	
First order conservative spin-induced self-force and self-torque	$\sim\frac{S\mu}{M^2}\sim\mu\epsilon^2$	$\sim\frac{S}{\mu M}\sim\epsilon$	[30,39,40]		✓

accumulated orbital phase with the parameters of the system. Effects which were previously known to admit Hamiltonian formulations include geodesic motion, the first order point-particle self-force [20], and the leading [26,27] and subleading [29] spin-curvature couplings. The effect for which we give a new Hamiltonian formulation is the self-interaction associated with spin, and it is listed in the last row. It consists of two different pieces which enter at the same order [30]. First, the regularized self-field of the secondary has a contribution of order $\sim\mu/M$, and the spin-curvature coupling force as well as the spin parallel transport get corrections due this metric perturbation. Second, there is a contribution to the regularized self-field of order $\sim S/M^2$, and the gradient of this field gives a correction to the self-force.

For extreme mass ratio inspirals with near maximal spin $S \sim \mu^2$, the accumulated phase shift over an inspiral due to the spin related self-interaction scales as $\mu/M \ll 1$, as shown in Table I. Hence the Hamiltonian we derive will not be relevant for computations of waveforms for LISA, for which post-1-adiabatic waveforms which include all the $O(1)$ effects in the accumulated phase will suffice. However, it may yield useful information for the effective one body framework [17,18] (since our calculation validates the Hamiltonian dynamics assumption) and thereby aid waveform modeling for comparable mass binary systems for LIGO. We also note that our analysis does not include second order point-particle self-force effects, and subleading spin-curvature effects, even though they enter at the same order as the spin self-interaction effect when $S \sim \mu^2$ as shown in Table I. It would be interesting to extend our analysis to include these effects.

The organization of this paper is as follows. In Sec. II we review the dynamics of a test spinning particle up to linear order in spin, given by the Mathisson-Papapetrou-Dixon equations. We specialize to the Tulczyjew-Dixon spin supplementary condition and review the Hamiltonian formulation of the resulting dynamical system. The existence of two Casimir invariants makes the Poisson brackets degenerate. By passing to the submanifold of the phase space on which the Casimirs are constant, we obtain a true Hamiltonian dynamical system with nondegenerate Poisson brackets, following Witzany *et al.* [26] and Ramond [27]. In Sec. III we define pseudo-Hamiltonian dynamical systems and review a general result in the theory of these systems that gives sufficient conditions for a pseudo-Hamiltonian system to be Hamiltonian [20]. We derive in Sec. IV a pseudo-Hamiltonian formulation of the dynamics of a spinning point particle including self-force effects. This is obtained by replacing the metric in the test-particle Hamiltonian by an effective metric, which includes perturbations proportional to the particle's mass and spin. Last, in Sec. V we apply the result from Sec. III to obtain a Hamiltonian description of the motion of a spinning particle.

Throughout this paper we use geometric units with $G = c = 1$.

II. HAMILTONIAN DESCRIPTION OF THE MOTION OF A SPINNING TEST PARTICLE

The motion of an extended body in general relativity, neglecting self-gravity, can be reduced to the motion of a point particle of mass μ endowed with a series of mass and current multipole moments [41–44]. If we restrict ourselves to the pole-dipole approximation, where only the mass and spin are included, the dynamics are given by the well-known Mathisson-Papapetrou-Dixon (MPD) equations [23–25]

$$\nabla_{\vec{u}} p_\mu = -\frac{1}{2} R_{\mu\nu\alpha\beta} u^\nu S^{\alpha\beta}, \tag{1a}$$

$$\nabla_{\vec{u}} S^{\alpha\beta} = 2p^{[\alpha} u^{\beta]}. \tag{1b}$$

Here

$$\frac{dx^\mu}{d\tau} = u^\mu \tag{2}$$

is the 4-velocity of the particle, $S^{\alpha\beta}$ is its spin tensor, p_μ is its 4-momentum, $\nabla_{\vec{u}} = u^\alpha \nabla_\alpha$ is the covariant derivative with respect to proper time τ, and $R_{\mu\nu\alpha\beta}$ is the Riemann tensor. The set of Eqs. (1) and (2) comprises 14 equations for 17 independent unknowns $x^\mu(\tau)$, $u^\mu(\tau)$, $p_\mu(\tau)$, and $S^{\alpha\beta}(\tau)$. Hence the dynamical system is not yet completely specified. This incompleteness arises because of the freedom to choose different definitions of the center-of-mass worldline $x^\mu(\tau)$ of the extended body [37,38]. A definition can be chosen by imposing a so-called spin supplementary condition of the form

$$S^{\alpha\beta} V_\beta = 0 \tag{3}$$

for some timelike vector V_β.

In this paper, we use the Tulczyjew-Dixon spin supplementary condition [25,42]

$$S^{\alpha\beta} p_\beta = 0, \tag{4}$$

which reduces the MPD equations to [40]

$$\frac{dx^\mu}{d\tau} = \frac{1}{\mu} g^{\mu\nu} p_\nu - \frac{1}{2\mu^3} R_{\sigma\rho\alpha\beta} S^{\sigma\mu} p^\rho S^{\alpha\beta} + O(S^4), \tag{5a}$$

$$\nabla_{\vec{u}} p_\mu = -\frac{1}{2\mu} R_{\mu\nu\alpha\beta} S^{\alpha\beta} p^\nu + O(S^3), \tag{5b}$$

$$\nabla_{\vec{u}} S^{\alpha\beta} = -\frac{1}{\mu} u^{[\alpha} S^{\beta]\sigma} u^\rho S^{\mu\nu} R_{\sigma\rho\mu\nu} + O(S^4). \tag{5c}$$

Here we have dropped terms cubic and higher order in spin and have defined the particle mass

$$\mu = \sqrt{-g^{\alpha\beta} p_\alpha p_\beta}. \tag{6}$$

The dynamics to leading order in spin is obtained by dropping all of the terms quadratic in spin in Eq. (5) and keeping only the term in the momentum evolution equation (5b) which is linear in spin [27,40]. This term corresponds to the leading spin-curvature coupling effect listed in Table I. In this approximation the MPD equations (5) reduce to

$$\frac{dx^\mu}{d\tau} = \frac{1}{\mu} g^{\mu\nu} p_\nu, \tag{7a}$$

$$\nabla_{\vec{u}} p_\mu = -\frac{1}{2\mu} R_{\mu\nu\alpha\beta} S^{\alpha\beta} p^\nu, \tag{7b}$$

$$\nabla_{\vec{u}} S^{\alpha\beta} = 0. \tag{7c}$$

We note that the spin supplementary condition (4) is not preserved by the dynamics (7). This arises because we are working to linear order in spin. In this paper we shall adopt Eq. (7) as the definition of the dynamical system we are working with, even though this definition is formally inconsistent with the spin supplementary condition from which it was derived. The inconsistency is higher order in spin and so can be safely ignored for our purposes.

Let Γ_s denote the phase space consisting of the bundle over spacetime with coordinates $(x^\mu, p_\nu, S^{\alpha\beta})$. As is well known, there exist a Hamiltonian function and a Poisson bracket structure on Γ_s that give rise to the dynamical system (7) [26–29,45,46]. The Poisson brackets are

$$\{x^\mu, x^\nu\} = 0, \tag{8a}$$

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \tag{8b}$$

$$\{p_\mu, p_\nu\} = -\frac{1}{2} R_{\mu\nu\alpha\beta} S^{\alpha\beta}, \tag{8c}$$

$$\{x^\mu, S^{\alpha\beta}\} = 0, \tag{8d}$$

$$\{S^{\alpha\beta}, p_\mu\} = -\Gamma^\alpha_{\mu\rho} S^{\rho\beta} - \Gamma^\beta_{\mu\rho} S^{\alpha\rho}, \tag{8e}$$

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = 2g^{\mu[\beta} S^{\alpha]\nu} - 2g^{\nu[\beta} S^{\alpha]\mu}, \tag{8f}$$

and the Hamiltonian H_0 is

$$H_0(x, p, S) = -\sqrt{-g^{\mu\nu} p_\mu p_\nu}. \tag{9}$$

It will be convenient to make a change of coordinates on phase space to simplify the form (8) of the Poisson brackets [47]. We choose an arbitrary orthonormal basis $\vec{e}_\Lambda = e^\alpha_\Lambda \partial_\alpha$ for $0 \le \Lambda \le 3$, with $\vec{e}_\Lambda \cdot \vec{e}_\Sigma = \eta_{\Lambda\Sigma}$, the Minkowski metric with signature $(-1, 1, 1, 1)$. We use uppercase Greek indices for orthonormal basis indices and lowercase Greek indices for spacetime indices. We define the dual basis $\mathbf{e}^\Lambda = e^\Lambda_\mu dx^\mu$ by $e^\Lambda_\mu e^\mu_\Sigma = \delta^\Lambda_\Sigma$, and the components of the spin connection by

$$\omega_{\alpha\Lambda\Sigma} = e_{\Lambda\rho} \nabla_\alpha e^\rho_\Sigma. \tag{10}$$

We define new phase space coordinates $(x^\alpha, \pi_\alpha, S^{\Lambda\Pi})$ by

$$\pi_\alpha = p_\alpha - \frac{1}{2}\omega_{\alpha\Lambda\Sigma} e^\Lambda_{\ \mu} e^\Sigma_{\ \nu} S^{\mu\nu}, \tag{11a}$$

$$S^{\Lambda\Sigma} = e^\Lambda_{\ \mu} e^\Sigma_{\ \nu} S^{\mu\nu}. \tag{11b}$$

In these new coordinates the only nonvanishing Poisson brackets are

$$\{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \tag{12a}$$

$$\{S^{\Theta\Pi}, S^{\Gamma\Lambda}\} = 2\eta^{\Theta[\Lambda} S^{\Gamma]\Pi} - 2\eta^{\Pi[\Lambda} S^{\Gamma]\Theta}. \tag{12b}$$

Substituting the coordinate change (11) into the Hamiltonian (9) and linearizing in spin gives the form of the Hamiltonian in these coordinates:

$$H_0(x,\pi,S) = -\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu} + \frac{g^{\mu\nu}\pi_\mu\omega_{\nu\Theta\Pi}S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}}. \tag{13}$$

It will also be convenient to define a new mass parameter m related to the norm of the new momentum 4-vector

$$m = \sqrt{-g^{\alpha\beta}\pi_\alpha\pi_\beta}, \tag{14}$$

which is related to our previously defined mass (6) by $m = \mu + O(S)$. In the following sections we will expand the Hamiltonian of the system in powers of m and S, by counting factors of π_μ and $S^{\Lambda\Pi}$. Using this counting the first term in the Hamiltonian (13) is $O(m)$ while the second one is $O(S)$.

Although the Hamiltonian function (13) and Poisson structure (12) give rise to the dynamical system (7) on Γ_s, the dynamical system is not Hamiltonian since the Poisson structure (12) is degenerate. The degeneracy is due to the existence of two Casimir invariants[1] [26,27]

$$S_*^2 = \frac{1}{8}\epsilon_{\Gamma\Sigma\Xi\Pi} S^{\Gamma\Sigma} S^{\Xi\Pi}, \tag{15a}$$

$$S_\circ^2 = \frac{1}{2}\eta_{\Gamma\Sigma}\eta_{\Xi\Pi} S^{\Gamma\Xi} S^{\Sigma\Pi}, \tag{15b}$$

which satisfy

$$\{S_*^2, F\} = \{S_\circ^2, F\} = 0 \tag{16}$$

for any function F on phase space. Denoting by $y^{\mathcal{A}}$ abstract coordinates on Γ_s, the Poisson structure can be written as a tensor $\Omega^{\mathcal{AB}}$, and its degeneracy implies that a symplectic form $\Omega_{\mathcal{AB}}$ satisfying $\Omega_{\mathcal{AB}}\Omega^{\mathcal{BC}} = \delta^{\mathcal{A}}_{\mathcal{C}}$ does not exist. Thus, Γ_s is a Poisson manifold but not a symplectic manifold.

We can overcome this difficulty and obtain a true Hamiltonian description of the dynamics as follows, following [26,27]. Fix values $S_\circ^2$ and S_*^2 of the Casimirs, and consider the corresponding submanifold Γ of Γ_s. Denote by Q^A abstract coordinates on Γ and by $y^{\mathcal{A}} = y^{\mathcal{A}}(Q^B)$ the embedding map. There exists an invertible Poisson structure Ω^{AB} on Γ whose pushforward

$$\Omega^{\mathcal{AB}} = \frac{\partial y^{\mathcal{A}}}{\partial Q^A}\frac{\partial y^{\mathcal{B}}}{\partial Q^B}\Omega^{AB} \tag{17}$$

to Γ_s coincides with the Poisson structure (12). It follows that the dynamical vector field $v^{\mathcal{A}} = \Omega^{\mathcal{AB}}\partial_{\mathcal{B}} H_0$ on Γ_s is the pushforward $v^A \partial y^{\mathcal{A}}/\partial Q^A$ of the Hamiltonian vector field $v^A = \Omega^{AB}\partial_B \bar{H}_0$ on Γ, where $\bar{H}_0$ is the pullback of H_0 to Γ (below we will drop the bar). Thus, the dynamics restricted to Γ is Hamiltonian and Γ is a symplectic manifold.

We will restrict attention to submanifolds Γ of Γ_s for which

$$S_\circ^2 \geq 0, \qquad S_*^2 = 0. \tag{18}$$

The conditions (18) will be satisfied at some point along the trajectory if there exists a timelike vector f_β for which

$$S^{\alpha\beta} f_\beta = 0 \tag{19}$$

at that point [27]. Then the conditions (18) will be satisfied at all points along the trajectory, since the quantities $S_\circ^2$ and S_*^2 are conserved by the dynamics by Eq. (16). Our spin supplementary condition (4) satisfies the criterion (19). Although this condition is not preserved by the dynamical evolution (7), as noted above, if we choose initial data that satisfy the spin supplementary condition, then the conditions (18) will be satisfied all along the trajectory. Thus, without loss of any physical generality, we can restrict attention to submanifolds Γ satisfying (18) [26,27].

We now review the construction of the nondegenerate Poisson structure Ω^{AB} on Γ [27]. We define coordinates $\{x^\mu, \pi_\mu, \sigma, \rho_\sigma, \varsigma, \rho_\varsigma\}$ on Γ by the relations

$$S^{23} = X\cos\sigma, \tag{20a}$$

$$S^{31} = X\sin\sigma, \tag{20b}$$

$$S^{12} = \rho_\sigma, \tag{20c}$$

$$S^{01} = Y\rho_\sigma \sin\varsigma\cos\sigma + Y\rho_\varsigma\cos\varsigma\sin\sigma + XZ\cos\sigma, \tag{20d}$$

$$S^{02} = Y\rho_\sigma\sin\varsigma\sin\sigma - Y\rho_\varsigma\cos\varsigma\cos\sigma + XZ\sin\sigma, \tag{20e}$$

$$S^{03} = Z\rho_\sigma - XY\sin\varsigma, \tag{20f}$$

[1]The quantities S_*^2 and $S_\circ^2$ are intended to be interpreted the same way as the square of a 4-momentum; that is, they can have either sign despite the notation as a square.

with

$$X=\sqrt{\rho_\varsigma^2-\rho_\sigma^2},\qquad Y=\sqrt{1-\frac{S_\circ^2}{\rho_\varsigma^2}-\frac{S_*^2}{\rho_\varsigma^4}},\qquad Z=\frac{S_*^2}{\rho_\varsigma^2}. \tag{21}$$

We define a Poisson structure by

$$\{\sigma,\rho_\sigma\}=1, \tag{22a}$$

$$\{\varsigma,\rho_\varsigma\}=1, \tag{22b}$$

$$\{x^\mu,\pi_\nu\}=\delta^\mu_\nu, \tag{22c}$$

with all other brackets vanishing. This is equivalent to the symplectic form $\mathbf{\Omega}=d\rho_\sigma\wedge d\sigma+d\rho_\varsigma\wedge d\varsigma+d\pi_\mu\wedge dx^\mu$. One can check that the pushforward of the Poisson structure (22) using the embedding (20) and (21) gives the Poisson structure (12).

To summarize, the Hamiltonian system on the 12-dimensional phase space Γ is given by the Poisson brackets (22) and by the Hamiltonian (13) expressed in terms of the coordinates $\{x^\mu,\pi_\mu,\sigma,\rho_\sigma,\varsigma,\rho_\varsigma\}$ using the map (20) and (21).

III. GENERAL RESULT FOR PSEUDO-HAMILTONIAN DYNAMICAL SYSTEMS

In this section we define a class of dynamical systems called pseudo-Hamiltonian dynamical systems, and we review a general result for these systems [20] which will be the foundation for the result of this paper derived in Sec. V below. A *pseudo-Hamiltonian* dynamical system (see [20] for details) consists of a phase space Γ, a closed, nondegenerate two-form Ω_{AB}, and a smooth pseudo-Hamiltonian function $\mathcal{H}:\Gamma\times\Gamma\to\mathbf{R}$, for which the dynamics are given by integral curves of the vector field

$$v^A=\Omega^{AB}\frac{\partial}{\partial Q^B}\mathcal{H}(Q,Q')|_{Q'=Q}. \tag{23}$$

We now specialize to pseudo-Hamiltonian systems which are perturbations of Hamiltonian systems, with symplectic form and pseudo-Hamiltonian

$$\Omega_{AB}=\Omega_{0AB}, \tag{24a}$$

$$\mathcal{H}(Q,Q')=H_0(Q)+\varepsilon\mathcal{H}_1(Q,Q')+O(\varepsilon^2). \tag{24b}$$

Here ε is a formal expansion parameter. The pseudo-Hamiltonian perturbation $\mathcal{H}_1$ is defined in terms of a function $G:\Gamma\times\Gamma\to\mathbf{R}$ via

$$\mathcal{H}_1(Q,Q')=\int_{-\infty}^{\infty}d\tau'\tilde{G}(0,Q,\tau',Q'), \tag{25}$$

where we have defined

$$\tilde{G}(\tau,Q,\tau',Q')=G[\varphi_\tau(Q),\varphi_{\tau'}(Q')]. \tag{26}$$

Here $\varphi_\tau\colon\Gamma\to\Gamma$ is the Hamiltonian flow associated with the zeroth order Hamiltonian system that takes any point τ units along the corresponding integral curve. Writing Q^A for abstract coordinates on Γ, the flow satisfies the relations

$$\varphi_\tau\circ\varphi_{\tau'}=\varphi_{\tau+\tau'}, \tag{27a}$$

$$\left.\frac{d}{d\tau}\right|_{\tau=0}\varphi_\tau^A(Q)=\Omega_0^{AB}\partial_BH_0. \tag{27b}$$

The function G is assumed to satisfy the conditions

$$G(Q,Q')=G(Q',Q), \tag{28a}$$

$$\tilde{G}(\tau,Q,\tau',Q')\to0\quad\text{as }\tau\text{ or }\tau'\to\pm\infty. \tag{28b}$$

In Ref. [20] we showed that any pseudo-Hamiltonian dynamical system of the form (23) and (24) can be recast as a Hamiltonian system, with Hamiltonian and symplectic form

$$\tilde{H}(Q)=H_0(Q)+\varepsilon\tilde{H}_1(Q)+O(\varepsilon^2), \tag{29a}$$

$$\tilde{\Omega}_{AB}=\Omega^0_{AB}+\varepsilon\tilde{\Omega}^1_{AB}+O(\varepsilon^2). \tag{29b}$$

Here the perturbation to the Hamiltonian is

$$\tilde{H}_1(Q)=\int d\tau'\tilde{G}(0,Q,\tau',Q), \tag{30}$$

and the perturbation to the symplectic form is

$$\tilde{\Omega}^1_{AB}(Q)=\left[\frac{\partial}{\partial Q^A}\frac{\partial}{\partial Q^{B'}}\int d\tau d\tau'\chi(\tau,\tau')\tilde{G}(\tau,Q,\tau',Q')\right]_{Q'=Q}, \tag{31}$$

where $\chi(\tau,\tau')=[\mathrm{sgn}(\tau)-\mathrm{sgn}(\tau')]/2$.

A more convenient representation of the Hamiltonian system (29) can be obtained by performing a linearized diffeomorphism on phase space [20]. Under such a diffeomorphism parametrized by a vector field ξ^A, the perturbations to the Hamiltonian and symplectic form transform as

$$\tilde{H}_1\to H_1=\tilde{H}_1+\mathcal{L}_\xi H_0, \tag{32a}$$

$$\tilde{\Omega}_{1AB}\to\Omega_{1AB}=\tilde{\Omega}_{1AB}+(\mathcal{L}_\xi\Omega_0)_{AB}. \tag{32b}$$

We choose the linearized diffeomorphism to be

$$\xi^A=\frac{1}{2}\Omega_0^{AB}\left[\frac{\partial}{\partial Q^{B'}}\int d\tau\int d\tau'\chi\tilde{G}(\tau,Q,\tau',Q')\right]_{Q'=Q}. \tag{33}$$

This yields for the new symplectic form perturbation

$$\Omega_{1AB} = 0, \tag{34}$$

and the new Hamiltonian

$$H(Q) = H_0(Q) + \varepsilon H_1(Q) + O(\varepsilon^2), \tag{35}$$

with

$$H_1(Q) = \frac{1}{2}\int d\tau' \tilde{G}(0, Q, \tau', Q), \tag{36}$$

which differs from (30) by a factor of 1/2.

IV. PSEUDO-HAMILTONIAN DESCRIPTION OF THE MOTION OF A SELF-GRAVITATING SPINNING PARTICLE

In this section we cast the motion of a spinning particle including the leading order self-force and self-torque as a pseudo-Hamiltonian dynamical system of the type discussed in the previous section. This will allow us to use the general result discussed there to deduce that the motion is Hamiltonian.

We start by reviewing the similar pseudo-Hamiltonian formulation of the motion of a spinless point particle including the leading order self-force [20]. For the zeroth order geodesic motion we use phase space coordinates $Q^A = (x^\mu, p_\mu)$ with symplectic form $\Omega_0 = dp_\mu \wedge dx^\mu$ and Hamiltonian $H_0 = -\sqrt{-g^{\mu\nu}(x) p_\mu p_\nu}$. For the first order motion, consider a particle at location $x^{\mu'}$ with initial 4-momentum $p_{\mu'}$. Writing $Q' = (x', p')$, we denote by $\varphi_{\tau'}(Q') = [x^{\bar\mu}(\tau'), p_{\bar\mu}(\tau')]$ the geodesic with initial data Q', where τ' is proper time. From this geodesic we can compute the Lorenz gauge metric perturbation

$$h_R^{\mu\nu}(x, Q') = \frac{1}{\sqrt{-g^{\mu'\nu'} p_{\mu'} p_{\nu'}}} \times \int d\tau' G_R^{\mu\nu\bar\mu\bar\nu}[x, \bar{x}(\tau')] p_{\bar\mu}(\tau') p_{\bar\nu}(\tau'). \tag{37}$$

Here the symmetric Green function $G_R^{\mu\nu\bar\mu\bar\nu}$ is the retarded Green function regularized according to the Detweiler-Whiting prescription [15,48]. The forced motion of the particle is then equivalent at linear order to geodesic motion in the metric $g_{\mu\nu} + h_{R\,\mu\nu}$, where Q' is held fixed when evaluating the geodesic equation and then evaluated at $Q' = Q$ [16,48]. We can therefore obtain a pseudo-Hamiltonian description of the dynamics by replacing the metric $g_{\mu\nu}(x)$ in the Hamiltonian with $g_{\mu\nu}(x) + h_{R\,\mu\nu}(x, Q')$ and expanding to linear order. We can also specialize to including just the conservative piece of the self-force, by replacing the regularized retarded Green function $G_R^{\mu\nu\bar\mu\bar\nu}$ with the average $G^{\mu\nu\bar\mu\bar\nu}$ of the retarded and advanced Green functions, regularized in the same way, and replacing the metric perturbation $h_{R\,\mu\nu}$ with its conservative piece $h_{\mu\nu}$.

Turn now to the corresponding story for spinning point particles. For the zeroth order motion we use phase space coordinates $Q^A = (x^\mu, \pi_\mu, \sigma, \rho_\sigma, \varsigma, \rho_\varsigma)$ on Γ defined in Eq. (20), with symplectic form (22) and Hamiltonian (13). This motion is described by the equations of motion (7) and is zeroth order in self-gravity, but contains effects first order in spin.

For the first order motion, consider a particle at location $x^{\mu'}$ with initial 4-momentum $p_{\mu'}$ and initial spin $S^{\Lambda'\Sigma'}$ [here the spin variable $S^{\Lambda'\Sigma'}$ should be understood to be a shorthand for the four variables $\sigma, \rho_\sigma, \varsigma, \rho_\varsigma$ defined in Eq. (20)]. Writing $Q' = (x^{\mu'}, \pi_{\mu'}, S^{\Lambda'\Sigma'})$, we denote by $\varphi_{\tau'}(Q') = [x^{\bar\mu}(\tau'), \pi_{\bar\mu}(\tau'), S^{\bar\Lambda\bar\Sigma}(\tau')]$ the solution to the zeroth order motion and spin evolution (7), where τ' is proper time. We can compute from this zeroth order motion a metric perturbation as follows. Inserting the stress energy tensor of a spinning point particle given by Eq. (9) of Ref. [30] into the linearized Einstein equation gives the Lorenz gauge metric perturbation

$$h_R^{\mu\nu}(x, Q') = \int d\tau' G_R^{\mu\nu\bar\mu\bar\nu}[x, \bar{x}(\tau')] \frac{p_{\bar\mu}(\tau') p_{\bar\nu}(\tau')}{\sqrt{-g^{\bar\lambda\bar\sigma} p_{\bar\lambda} p_{\bar\sigma}}} - \int d\tau' \nabla_{\bar\rho} G_R^{\mu\nu\bar\mu\bar\nu}[x, \bar{x}(\tau')] \frac{p_{\bar\mu}(\tau') e_{\bar\Lambda\bar\nu} e^{\bar\rho}_{\bar\Sigma} S^{\bar\Lambda\bar\Sigma}}{\sqrt{-g^{\bar\lambda\bar\sigma} p_{\bar\lambda} p_{\bar\sigma}}}. \tag{38}$$

Here barred indices indicate quantities that are evaluated at $x^{\bar\mu}(\tau')$, and $\nabla_{\bar\rho}$ acts only on the second argument of the Green function. The factor of $\sqrt{-\vec{p}^2}$ could be evaluated either at x^μ or at $x^{\bar\mu}$, since it is conserved by the dynamics (7); we choose the latter for later convenience. Now it is known that the self-forced and self-torqued motion of the spinning particle is given at linear order by evaluating the equations of motion (7) in the metric $g_{\mu\nu} + h_{R\,\mu\nu}$, where Q' is held fixed when evaluating the equations and then evaluated at $Q' = Q$ [30]. It follows that we can obtain a pseudo-Hamiltonian description of the dynamics by making the replacements

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + h_{R\,\mu\nu}(x, Q'), \tag{39a}$$

$$e^\mu_\Lambda \to e^\mu_\Lambda - \frac{1}{2} e^\nu_\Lambda g^{\sigma\mu} h_{R\nu\sigma}(x, Q') \tag{39b}$$

in the Hamiltonian (13) and expanding to linear order. Here the perturbation to the orthonormal basis is chosen to maintain orthonormality. Note that in order to apply the result of Sec. III, we must use the form (13) and (22) of the dynamical system for which the symplectic form is constant and so not modified by the substitutions (39), rather than the original form (8) and (9). This is because the result requires that the symplectic form be unperturbed; cf. Eq. (24a).

To complete the pseudo-Hamiltonian formulation of the dynamics, we need to write the pseudo-Hamiltonian in terms of the phase space variables $(x^\mu, \pi_\nu, S_{\Lambda\Sigma})$. We start by writing the metric perturbation (38) in terms of the new momentum variable (11a) and expanding to linear order in spin, which gives

$$h_R^{\alpha\beta}(x, Q') = h_{R(m)}^{\alpha\beta}(x, Q') + h_{R(S)}^{\alpha\beta}(x, Q'), \tag{40}$$

where

$$h_{R(m)}^{\alpha\beta}(x, Q') = \int d\tau' G^{\alpha\beta\bar{\mu}\bar{\nu}}[x, \bar{x}(\tau')] \frac{\pi_{\bar{\mu}}(\tau')\pi_{\bar{\nu}}(\tau')}{\sqrt{-g^{\bar{\rho}\bar{\sigma}}\pi_{\bar{\rho}}\pi_{\bar{\sigma}}}}, \tag{41a}$$

$$\begin{aligned} h_{R(S)}^{\alpha\beta}(x, Q') = & \int d\tau' G^{\alpha\beta\bar{\mu}\bar{\nu}}[x, \bar{x}(\tau')] \frac{\pi_{\bar{\mu}}\omega_{\bar{\nu}\bar{\Theta}\bar{\Pi}} S^{\bar{\Theta}\bar{\Pi}}}{\sqrt{-g^{\bar{\rho}\bar{\sigma}}\pi_{\bar{\rho}}\pi_{\bar{\sigma}}}} \\ & - \int d\tau' \nabla_{\bar{\rho}} G^{\alpha\beta\bar{\mu}\bar{\nu}}[x, \bar{x}(\tau')] \frac{\pi_{\bar{\mu}}(\tau') e_{\bar{\Theta}\bar{\nu}} e_{\bar{\Pi}}^{\bar{\rho}} S^{\bar{\Theta}\bar{\Pi}}(\tau')}{\sqrt{-g^{\bar{\rho}\bar{\sigma}}\pi_{\bar{\rho}}\pi_{\bar{\sigma}}}} \\ & + \frac{1}{2}\int d\tau' G^{\alpha\beta\bar{\mu}\bar{\nu}}[x, \bar{x}(\tau')] \frac{\pi_{\bar{\mu}}\pi_{\bar{\nu}}\pi^{\bar{\rho}}\omega_{\bar{\rho}\bar{\Lambda}\bar{\Theta}} S^{\bar{\Lambda}\bar{\Theta}}}{[-g^{\bar{\sigma}\bar{\lambda}}\pi_{\bar{\sigma}}\pi_{\bar{\lambda}}]^{3/2}}. \end{aligned} \tag{41b}$$

Below we will need the metric perturbation $h_R^{\mu\nu}$ accurate to $O(m)$ and $O(S)$, and we can neglect $O(m^2)$, $O(mS)$, and $O(S^2)$ contributions. Hence in the expressions (41) it is sufficient to use the geodesic worldline rather than the solution to Eq. (7) which incorporates $O(S)$ corrections to the worldline. We next make the replacements (39) in the Hamiltonian (13). This yields the pseudo-Hamiltonian

$$\begin{aligned} \mathcal{H}^R(Q, Q') = & -\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu} + \frac{g^{\mu\nu}\pi_\mu\omega_{\nu\Theta\Pi}S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} - \frac{h_R^{\mu\nu}(x, Q')\pi_\mu\pi_\nu}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} - \frac{h_R^{\mu\nu}(x, Q')\pi_\mu\omega_{\nu\Theta\Pi}S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} + \frac{g^{\mu\nu}\pi_\mu e_\Theta^\alpha e_\Pi^\beta h_{\nu[\alpha;\beta]}^R S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} \\ & - \frac{1}{4}\frac{h_R^{\mu\nu}(x, Q')\pi_\mu\pi_\nu\pi^\rho\omega_{\rho\Lambda\Theta}S^{\Lambda\Theta}}{[-g^{\sigma\lambda}\pi_\sigma\pi_\lambda]^{3/2}}, \end{aligned} \tag{42}$$

where we used that the perturbation to the spin connection is $\delta\omega_{\mu\Lambda\Pi} = e_\Lambda^\alpha e_\Pi^\beta h_{\mu[\alpha;\beta]}$.

As an aside, we can verify as follows that the pseudo-Hamiltonian (42) with symplectic form (22) gives the correct dynamics for a spinning particle under the effect of the first order gravitational self-force. Using Eq. (23) we obtain for the equations of motion $\nabla_{\vec{u}}u^\mu = a^\mu$ and $\nabla_{\vec{u}}S^{\mu\nu} = N^{\mu\nu}$, where the self-acceleration a^μ and self-torque $N^{\mu\nu}$ are given by

$$\begin{aligned} a^\mu = & -\frac{1}{2}[g^{\mu\lambda} + u^\mu u^\lambda][2h_{\lambda\rho;\sigma}^R - h_{\rho\sigma;\lambda}^R]u^\rho u^\sigma \\ & - \frac{1}{2m}R^\mu{}_{\alpha\beta\gamma}\left[1 - \frac{1}{2}h_{\rho\gamma}^{R(m)}u^\rho u^\gamma\right]u^\alpha S^{\beta\gamma} \\ & + \frac{1}{2m}[g^{\mu\nu} + u^\mu u^\nu][2h_{\nu(\alpha;\beta)\gamma}^{R(m)} - h_{\alpha\beta;\nu\gamma}^{R(m)}]u^\alpha S^{\beta\gamma}, \end{aligned} \tag{43a}$$

$$N^{\mu\nu} = u^{(\rho}S^{\sigma)[\mu}g^{\mu]\lambda}[2h_{\lambda\rho;\sigma}^{R(m)} - h_{\rho\sigma;\lambda}^{R(m)}], \tag{43b}$$

and where the metric perturbation $h_{\mu\nu}^R$ has been evaluated at $Q' = Q$ after the derivatives have been taken. These equations agree with those of Ref. [30]. They can also be obtained by making the substitutions (39) in the equations of motion (7). As discussed in the Introduction, we keep only terms of order $O(m^2)$, $O(S)$, and $O(mS)$ in the self-force, and $O(mS)$ in the self-torque, which explains why we have replaced $h_R^{\mu\nu}$ with $h_{R(m)}^{\mu\nu}$ [cf. Eq. (41a)] in some of the terms in (43).

V. HAMILTONIAN FORMULATION OF THE CONSERVATIVE MOTION OF A SELF-GRAVITATING SPINNING PARTICLE

In this section we show that the motion of a spinning point particle under the action of the first order conservative self-force is Hamiltonian, by combining the pseudo-Hamiltonian formulation of the motion derived in Sec. IV with the general result of Sec. III. To do this we need to read off the function $G(Q, Q')$ on phase space defined by Eqs. (25) and (26), and to verify that it satisfies the required properties (28).

We start by specializing to the conservative sector of the dynamics. As described after Eq. (37) above in the nonspinning case, this is achieved by replacing in the pseudo-Hamiltonian (42) the regularized retarded Green function $G_R^{\mu\nu\bar{\mu}\bar{\nu}}$ with the average $G^{\mu\nu\bar{\mu}\bar{\nu}}$ of the retarded and advanced Green functions, regularized in the same way, and replacing the metric perturbation $h_{R\mu\nu}$ with its conservative piece $h_{\mu\nu}$. Note that this Green function obeys the symmetry property

$$G^{\mu\nu\alpha'\beta'}(x, x') = G^{\alpha'\beta'\mu\nu}(x', x). \tag{44a}$$

Next, by comparing the pseudo-Hamiltonian given by Eqs. (40) and (42) with the general form given by Eqs. (24b), (25), and (26), we obtain for the function $G(Q, Q')$ on phase space

$$\begin{aligned} G(Q,Q') = \frac{1}{4} N N' [& -2\pi_\mu \pi_\nu \pi_{\rho'} \pi_{\sigma'} G^{\mu\nu\rho'\sigma'}(x,x') - 2\pi_\mu \pi_\nu S^{\Theta'\Pi'} \pi_{\rho'} \omega_{\sigma'\Theta'\Pi'} G^{\mu\nu\rho'\sigma'}(x,x') - 2\pi_{\rho'}\pi_{\sigma'} S^{\Theta\Pi} \pi_\mu \omega_{\nu\Theta\Pi} G^{\mu\nu\rho'\sigma'}(x,x') \\ & + 2\pi_\mu \pi_\nu \pi_{\rho'} e_{\Theta'\sigma'} e^{\lambda'}_{\Pi'} S^{\Theta'\Pi'} \nabla_{\lambda'} G^{\mu\nu\rho'\sigma'}(x,x') + 2\pi_{\rho'}\pi_{\sigma'}\pi_\mu e_{\Theta\nu} e^{\lambda}_{\Pi} S^{\Theta\Pi} \nabla_\lambda G^{\mu\nu\rho'\sigma'}(x,x') \\ & - N'^2 \pi_\mu \pi_\nu \pi_{\alpha'} \pi_{\beta'} \pi_{\rho'} \omega^{\rho'}_{\Theta'\Pi'} S^{\Theta'\Pi'} G^{\mu\nu\alpha'\beta'}(x,x') - N^2 \pi_{\alpha'} \pi_{\beta'} \pi_\mu \pi_\nu \pi_\rho \omega^{\rho}_{\Theta\Pi} S^{\Theta\Pi} G^{\mu\nu\alpha'\beta'}(x,x')], \end{aligned} \tag{45}$$

where

$$N = \frac{1}{\sqrt{-g^{\alpha\beta}\pi_\alpha \pi_\beta}}, \qquad N' = \frac{1}{\sqrt{-g^{\rho'\sigma'}\pi_{\rho'}\pi_{\sigma'}}}. \tag{46}$$

Because of the symmetry property (44a) of the Green function, the function (45) satisfies the required symmetry property (28a). It also satisfies the required asymptotic conditions (28b) for the reasons discussed in Ref. [20].

It now follows from the result reviewed in Sec. III that the dynamical system (43) admits a Hamiltonian description. The Hamiltonian function is given by Eqs. (13), (26), (35), (36), and (45), and the symplectic form is given by (22), in phase space coordinates given by Eq. (33).

VI. CONCLUSIONS

We have shown that the conservative dynamics of the two-body problem in general relativity is Hamiltonian to the first subleading order in the mass and spin of the secondary. This result may yield useful information for the effective one body framework and thereby aid waveform modeling for comparable mass binary systems for LIGO. Extending this result to the second-order $O(m^2)$ point-particle self-force is a direction that is currently being explored.

ACKNOWLEDGMENTS

We thank Vojtěch Witzany and Paul Ramond for helpful discussions and references. This research was supported in part by NSF Grant No. PHY-2110463.

[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Phys. Rev. Lett. **116**, 061102 (2016).
[2] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Phys. Rev. Lett. **116**, 241103 (2016).
[3] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Phys. Rev. X **6**, 041015 (2016).
[4] M. Evans *et al.*, arXiv:2109.09882.
[5] M. Punturo *et al.*, Classical Quantum Gravity **27**, 194002 (2010).
[6] P. Amaro-Seoane *et al.*, arXiv:1702.00786.
[7] J. Antoniadis *et al.*, Mon. Not. R. Astron. Soc. **510**, 4873 (2022).
[8] L. Lehner and F. Pretorius, Annu. Rev. Astron. Astrophys. **52**, 661 (2014).
[9] L. Blanchet, Living Rev. Relativity **17**, 2 (2014).
[10] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic* (Cambridge University Press, Cambridge, England, 2014).
[11] M. Levi, Rep. Prog. Phys. **83**, 075901 (2020).
[12] R. A. Porto, Phys. Rep. **633**, 1 (2016).
[13] T. Damour, Phys. Rev. D **94**, 104015 (2016).
[14] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Phys. Rev. Lett. **126**, 171601 (2021).
[15] E. Poisson, A. Pound, and I. Vega, Living Rev. Relativity **14**, 7 (2011).
[16] L. Barack and A. Pound, Rep. Prog. Phys. **82**, 016904 (2019).
[17] T. Damour, Fundam. Theor. Phys. **177**, 111 (2014).
[18] A. Taracchini *et al.*, Phys. Rev. D **89**, 061502 (2014).
[19] G. Schäfer and P. Jaranowski, Living Rev. Relativity **21**, 7 (2018).
[20] F. M. Blanco and E. E. Flanagan, Phys. Rev. Lett. **130**, 051201 (2023).
[21] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).
[22] T. C. Quinn and R. M. Wald, Phys. Rev. D **56**, 3381 (1997).
[23] M. Mathisson, Acta Phys. Pol. **6**, 163 (1937).
[24] A. Papapetrou, Proc. R. Soc. A **209**, 248 (1951).
[25] W. G. Dixon, Proc. R. Soc. A **314**, 499 (1970).
[26] V. Witzany, J. Steinhoff, and G. Lukes-Gerakopoulos, Classical Quantum Gravity **36**, 075003 (2019).
[27] P. Ramond, arXiv:2210.03866.
[28] E. Barausse, E. Racine, and A. Buonanno, Phys. Rev. D **80**, 104025 (2009); **85**, 069904(E) (2012).
[29] J. Vines, D. Kunst, J. Steinhoff, and T. Hinderer, Phys. Rev. D **93**, 103008 (2016).
[30] J. Mathews, A. Pound, and B. Wardell, Phys. Rev. D **105**, 084031 (2022).
[31] R. Fujita, S. Isoyama, A. Le Tiec, H. Nakano, N. Sago, and T. Tanaka, Classical Quantum Gravity **34**, 134001 (2017).

[32] A. Pound, Phys. Rev. D **90**, 084039 (2014).
[33] A. Pound, B. Wardell, N. Warburton, and J. Miller, Phys. Rev. Lett. **124**, 021101 (2020).
[34] S. E. Gralla, Phys. Rev. D **85**, 124011 (2012).
[35] A. Pound, Phys. Rev. Lett. **109**, 051101 (2012).
[36] D. Bini and T. Damour, Phys. Rev. D **93**, 104040 (2016).
[37] O. Semerák, Mon. Not. R. Astron. Soc. **308**, 863 (1999).
[38] K. Kyrian and O. Semerák, Mon. Not. R. Astron. Soc. **382**, 1922 (2007).
[39] L. V. Drummond and S. A. Hughes, Phys. Rev. D **105**, 124041 (2022).
[40] L. V. Drummond and S. A. Hughes, Phys. Rev. D **105**, 124040 (2022).
[41] W. G. Dixon, Phil. Trans. R. Soc. A **277**, 59 (1974).
[42] W. Tulczyjew, Acta Phys. Pol. **18**, 393 (1959).
[43] G. Lukes-Gerakopoulos, J. Seyrich, and D. Kunst, Phys. Rev. D **90**, 104019 (2014).
[44] A. I. Harte, Fundam. Theor. Phys. **179**, 327 (2015).
[45] J. W. van Holten, Int. J. Geom. Methods Mod. Phys. **13**, 1640002 (2016).
[46] G. d'Ambrosi, S. S. Kumar, and J. van Holten, Phys. Lett. B **743**, 478 (2015).
[47] G. E. Tauber, Int. J. Theor. Phys. **27**, 335 (1988).
[48] S. L. Detweiler and B. F. Whiting, Phys. Rev. D **67**, 024025 (2003).