

Multikink brane in Gauss-Bonnet gravity and its stability

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Einstein-Gauss-Bonnet gravity in high-dimensional spacetime is intriguing. Here, the properties of thick branes generated by a bulk scalar field in the five-dimensional Einstein-Gauss-Bonnet gravity were studied. With the help of the superpotential method, we obtain a series of multikink brane solutions. We also analyze the linear stability of the brane system under tensor perturbations and prove that they are stable. The massless graviton is shown to be localized near the brane and hence the four-dimensional Newtonian potential can be recovered. By comparing the properties of these thick branes under different superpotentials we find with some specific choice of superpotential the Gauss-Bonnet term can determine the scalar field are multikink or single kink.

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I. INTRODUCTION

Among several interesting descriptions of our Universe, an attractive one may be that our Universe acts as a four-dimensional hypersurface called 3-brane embedded in higher-dimensional spacetime [1–6]. At the end of the 20th century, the Arkani-Hamed-Dimopoulos-Dvali (ADD) model [5] and the Randall-Sundrum (RS) model [6] provided a new way to solve the hierarchical problem in the Standard Model of particle physics. With the development of higher-dimensional gravity theories, more and more works focused on the nature of the extra dimensions [7–11] and various potential observable effects of extra dimensions were proposed. This attracted the attention of physicists once again and opens a new era for studying extra dimensions.

In brane world scenarios, there are thin or thick branes in terms of their width along the extra dimensions. Both the branes are thin in ADD and RS models because the thickness of the brane has been neglected. Note that, in the thin brane model, the scalar curvature comes to be singular at the core of the brane because of the zero width [12]. To deal with this singularity, the Israel-Lanczos junction condition [13] should be introduced. Physically, a brane should have thickness and it emerges as an alternative to thin brane configuration. That

is to say, a brane should have nontrivial width along extra dimensions and there is no singularity problem as appeared in the thin braneworld models.

One way to obtain such nonsingular thick brane might be simply replacing the singular source term in the thin brane scenarios with a nonsingular source term. In 1983, Rubakov and Shaposhnikov first proposed the picture of the thick brane described by a domain wall along the extra dimension in five-dimensional flat spacetime [2]. They obtained the solution of a scalar field with a kinklike configuration. Such a solution connects the two vacua of the scalar potential and has a nontrivial topology. Inspired by the idea of the domain wall in five-dimensional spacetime, lots of literature focused on the properties of thick brane including gravity [14–22]. There were some models based on the nonminimal coupling between gravity and the scalar field [23–25]. Those nontrivial sources induced many new phenomena and abundant brane configurations. Furthermore, thick brane models [26–39] and thin brane models [40–42] were investigated in modified gravity theories such as $f(R)$ gravity. In particular, some brane solutions with a rich structure were obtained in [31,32].

When the spacetime dimension is higher than four, the Einstein-Hilbert action can be supplemented with higher-order curvature corrections which do not generate three or higher-order terms of equations of motion [43]. The gravity theory including the Gauss-Bonnet (GB) invariant term is a theory that satisfies the above-mentioned property, where the GB invariant arises as a correction in string theory [44–47] and is defined as follows:

$$\mathcal{R}_{\text{GB}} = R^{ABCD}R_{ABCD} - 4R^{AB}R_{AB} + R^2. \quad (1)$$

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The letters A, B, C, D in this paper are the indexes of the whole spacetime. In four-dimensional spacetime, the GB term is a topological term and acts as the boundary term that does not have influence on the classical field equations. When the spacetime dimension satisfies $D \geq 5$, the GB term is no longer topologically invariant and its influence will exit [46]. Recently, the GB term was applied to the investigation of inflation after the GW170817 event [48–52], cosmology [53–55], as well as black hole physics [56–59]. In addition, the entanglement wedge cross section was investigated in a five-dimensional anti-de Sitter-Vaidya spacetime with GB corrections [60]. It is also interesting to consider branes in the GB gravity. Thin brane models in the GB gravity were discussed in Refs. [61–65]. Besides the thin brane model, the thick brane models in the GB gravity with a bulk scalar field were widely investigated [54,66–73], and the brane model in the GB gravity was also applied to cosmology [74–86]. The domain wall solutions constructed by two scalar fields combining either in a kink-antikink or a trapping-bag configuration were found in five-dimensional GB gravity with one warped extra dimension [87]. Note that there is only one defect for the domain walls with kink or antikink; when multikinks are considered, the gravitating multi-defects can be obtained. This was originally investigated in Refs. [88,89]. Similar to the domain wall obtained in Ref. [2], the multikinklike configurations of a background scalar field will connect the corresponding multi-vacua of the scalar potential. Such novel multikink configurations will lead the thick brane possess a more richer inner structure. The corresponding inner structures will induce different effective potentials, and various novel Kaluza-Klein (KK) resonances will exist [90]. It has been proved that the dynamics of the KK resonances is closely related to the effective potentials [91]. Therefore, construction of thick branes described by multikink solutions with inner structures is very important. It is well-known that the effects of the GB invariant is absent in four-dimensional GB gravity, but one can investigate them in a higher-dimensional spacetime. In this paper we would like to construct a multikink brane solution in the higher-dimensional GB gravity and study the possible effects of the GB term on the brane structure. We also analyze the linear stability of the system under tensor perturbation and the localization of gravity.

The paper is organized as follows. In Sec. II, we introduce the method to solve the thick brane solutions in GB gravity. The system can be reduced to the first-order formulas by introducing a superpotential. In Sec. III, we construct thick branes with some superpotentials and a polynomial warp factor, respectively, and study the influences of the GB term on the thick brane. In Sec. IV, the linear stability of the brane system under the tensor perturbations and localization of gravity are analysed. Finally, a brief summary is given in Sec. V.

II. BRANE MODEL IN GB GRAVITY

In this section, we will introduce a new method to solve the thick brane in GB gravity. The brane model in D -dimensional GB gravity is described by the following action:

$$S = \int d^D x \sqrt{-g} \left[\frac{R}{2\kappa} + \alpha \mathcal{R}_{\text{GB}} + \mathcal{L}_m \right], \quad (2)$$

where $\kappa = 8\pi G_D = M_*^{2-D}$ with G_D the D -dimensional gravitational constant and M_* the D -dimensional fundamental mass scale, and α is the GB coupling constant with mass dimension $D - 4$. In this paper, we use the units $\kappa = c = \hbar = 1$. The Lagrangian density of the scalar field is given by

$$\mathcal{L}_m = -\frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - V(\phi), \quad (3)$$

where $V(\phi)$ is the scalar potential. Varying the action (2) with the metric and scalar field respectively, we can get the equations of motion as follows:

$$G_{AB} - 2\alpha\kappa Q_{AB} = \kappa T_{AB}, \quad (4)$$

$$g^{AB} \nabla_A \nabla_B \phi - \frac{\partial V(\phi)}{\partial \phi} = 0, \quad (5)$$

where $G_{AB} = R_{AB} - \frac{1}{2} R g_{AB}$ is the Einstein tensor and

$$Q_{AB} = \frac{1}{2} g_{AB} \mathcal{R}_{\text{GB}} - 2R R_{AB} + 4R_{AC} R^C_B + 4R_{ACBD} R^{CD} - 2R_{ACDE} R_B^{CDE} \quad (6)$$

is the Lanczos tensor [92]. The energy-momentum tensor of the scalar field reads

$$T_{AB} = g_{AB} \mathcal{L}_m + \partial_A \phi \partial_B \phi. \quad (7)$$

In this paper, we focus on the flat thick brane with Z_2 symmetry in a five-dimensional spacetime ($D = 5$). The metric is written as [7]

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (8)$$

where the warp factor $A(y)$ is an even function of the extradimensional coordinate y and $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric. The ordinary four-dimensional coordinate indexes μ, ν are from 0 to 3. The scalar curvature is

$$R = -4(5A'^2 + 2A''), \quad (9)$$

where the prime denotes the derivative with respect to the extradimensional coordinate y . The explicit equations of motion are

$$6(A'' + 2A'^2) - 48\alpha\kappa A'^2(A'' + A'^2) + \kappa(\phi'^2 + 2V) = 0, \quad (10a)$$

$$12A'^2 - 48\alpha\kappa A'^4 - \kappa(\phi'^2 - 2V) = 0, \quad (10b)$$

$$4A'\phi' + \phi'' - \frac{\partial V}{\partial \phi} = 0. \quad (10c)$$

It can be shown that there are only two independent equations in Eqs. (10a)–(10c) for the three functions $V(\phi)$, $\phi(y)$, and $A(y)$. The scalar potential $V(\phi)$ contains the contribution of the cosmological constant. In addition, the scalar field ϕ is assumed to be an odd function of the extradimensional coordinate y to localize the fermion zero mode on the brane [93].

To solve Eqs. (10a)–(10b), we introduce the so called superpotential used in supergravity to reduce the second-order field equations (10a)–(10b) to the first-order ones. Such a method has been used successfully in Refs. [94–100]. We first introduce a superpotential $W(\phi)$. The relation between the warp factor $A(y)$ and the superpotential $W(\phi)$ is given by

$$A'(y) = -\frac{1}{3}\kappa W(\phi), \quad (11)$$

where the superpotential $W(\phi)$ should be an odd function of ϕ since $A(y)$ and $\phi(y)$ are assumed to be even and odd, respectively. From Eq. (11), we have

$$A''(y) = -\frac{1}{3}\kappa W_\phi \phi'(y). \quad (12)$$

Substituting Eqs. (11) and (12) into Eqs. (10a)–(10c), we have

$$\begin{aligned} \frac{4}{3}\kappa^2 W^2 - 2\kappa W_\phi \phi' + \frac{16}{9}\alpha\kappa^4 W^2 W_\phi \phi' \\ - \frac{16}{27}\alpha\kappa^5 W^4 + \kappa(\phi'^2 + 2V) = 0, \end{aligned} \quad (13)$$

$$\frac{4}{3}\kappa^2 W^2 - \frac{16}{27}\alpha\kappa^5 W^4 - \kappa(\phi'^2 - 2V) = 0, \quad (14)$$

$$\phi'' - \frac{4}{3}\kappa W_\phi \phi' - V_\phi = 0, \quad (15)$$

where $W_\phi = \frac{dW(\phi)}{d\phi}$ and $V_\phi = \frac{dV(\phi)}{d\phi}$. After replacing the warp factor in terms of the relations (11) and (12), one can obtain a relation between the scalar field ϕ and the superpotential $W(\phi)$ by subtracting Eq. (13) from Eq. (14) as follows:

$$\phi' = \frac{1}{9}(9 - 8\alpha\kappa^3 W^2)W_\phi. \quad (16)$$

One can further obtain the scalar potential by substituting Eq. (16) into Eq. (14):

$$V = \frac{1}{162}(W_\phi^2(9 - 8\alpha\kappa^3 W^2)^2 + 12\kappa W^2(4\alpha\kappa^3 W^2 - 9)). \quad (17)$$

In our thick brane model, the background scalar field has a kinklike configuration and it will approach to a constant ϕ_∞ when the extradimensional coordinate $y \rightarrow \infty$. Thus the spacetime is an asymptotical AdS₅ spacetime, which is in accord with the RS brane model. The corresponding naked cosmological constant can be calculated from the scalar potential,

$$\Lambda_5 = \lim_{y \rightarrow \infty} 2\kappa V(\phi(y)) = \lim_{\phi \rightarrow \phi_\infty} 2\kappa V(\phi). \quad (18)$$

Now, the original field equations (10a)–(10c) have been replaced by the first-order formulas (11), (16), and (17). Once the superpotential $W(\phi)$ is given, we can solve all the functions for the brane solution with the help of the above first-order formulas.

Note that the solutions of nonlinear coupled differential equations are not unique, which will lead to two or more different background configurations with the same Lagrangian parameters and boundary conditions. In our setup, we force the scalar field $\phi(y)$ to be an odd function along the extradimensional coordinate y . Besides this assumption, we still add another condition and let the scalar field $\phi(y)$ to be a monotonic function of y . Thus, for such an assumption of the scalar field $\phi(y)$, it has an inverse function $y(\phi)$ and our solution is unique. With the help of the above assumption and boundary conditions, we can obtain the corresponding background solutions from the given superpotentials conveniently. We can write the inverse function $y(\phi)$ from Eq. (16) as follows:

$$y(\phi) = \int \frac{9}{(9 - 8\alpha\kappa^3 W^2)W_\phi} d\phi. \quad (19)$$

The warp factor can also be obtained as

$$A(y(\phi)) = \int \frac{3\kappa W}{(8\alpha\kappa^3 W^2 - 9)W_\phi} d\phi. \quad (20)$$

Furthermore, we introduce the conditions: $y(\phi = 0) = 0$ and $A(y = 0) = 0$. By specifying the suitable superpotential, one can obtain the thick brane solution from Eqs. (17), (19), and (20). The distribution of the thick brane can be described by the effective energy density along the extra dimension with respect to the static observer $u^A = (e^A, 0, 0, 0)$,

$$\rho = \tau_{MN}^{(\text{eff})} u^M u^N = -g^{00}(T_{00} + 2\alpha Q_{00}), \quad (21)$$

where $\tau_{MN}^{(\text{eff})} = T_{MN} + 2\alpha Q_{MN}$ is the effective energy-momentum tensor. It can also be expressed in terms of the superpotential as follows:

$$\rho = \frac{1}{81}[W_\phi^2(9 - 8\alpha\kappa^3 W^2)^2 + 6\kappa W^2(4\alpha\kappa^3 W^2 - 9)]. \quad (22)$$

Note that $\rho(|y| \rightarrow \infty)$ is nonvanishing since the above expression contains the contribution from the effective cosmological constant coming from the naked cosmological constant Λ_5 in (18) and the Lanczos tensor. In order to describe the shape of the brane better, we subtract the contribution of the effective cosmological constant, i.e., we make the following replacement

$$\rho(y) \rightarrow \rho(y) - \rho(|y| \rightarrow \infty). \quad (23)$$

Actually, there is a direct way to solve Eqs. (10a)–(10c). First, giving the potential $V(\phi)$ as a function of the scalar field ϕ , then the scalar field $\phi(y)$ and the warp factor $A(y)$ can be solved as functions of the extradimensional coordinate y . The scalar field $\phi(y)$ and the warp factor $A(y)$ contain the GB coupling α . In this way, the parameters in the potential $V(\phi)$ and the GB coupling α are independent. However, it is difficult to obtain the brane solution with this method as higher-order terms of the warp factor induced from the Lanczos tensor are contained in the equations of motion. Using the superpotential method the equations of motion can be downgraded in order, so that one can simplify the calculation.

In the superpotential method, it is the superpotential $W(\phi)$ but not the scalar potential $V(\phi)$ that is given independently. So the parameters in the superpotential $W(\phi)$ and the GB coupling α are independent. The scalar potential $V(\phi)$ is decided by the GB term and the superpotential $W(\phi)$ from Einstein equations. Therefore, the parameters in the potential $V(\phi)$ are no longer independent. In this case, one can study how the GB coupling α and the parameters in the superpotential $W(\phi)$ independently effect the solutions of the warp factor and scalar field. If the superpotential $W(\phi)$ was set as $W(\phi) = \phi$, the brane world solution can be solved analytically, and the potential $V(\phi)$ would be a ϕ^4 potential. This case has been studied in Ref. [101]. Here, we study the more general superpotential; since the analytical solutions would be more difficult to get we will use the numerical method to solve the equations of motion.

Besides the superpotential method for solving thick brane system, one can also obtain the thick brane solutions by using the relations between the scalar field as well as the scalar potential and the warp factor. This can be done conveniently with the following five-dimensional conformally flat metric

$$ds^2 = e^{2A(z)}[\eta_{\mu\nu}dx^\mu dx^\nu + dz^2], \quad (24)$$

where the warp factor $e^{2A(z)}$ is a function of the extradimensional coordinate z and the relation between z and y is given by $e^A dz = dy$. With a known warp factor, one can derive the expressions for the scalar potential and scalar field,

$$V(\phi(z)) = 12\alpha e^{-4A}(\partial_z A)^2[(\partial_z A)^2 + \partial_z^2 A] - \frac{3}{2\kappa} e^{-2A}[3(\partial_z A)^2 + \partial_z^2 A], \quad (25)$$

$$\phi(z) = \int dz \sqrt{\frac{3}{\kappa}} \sqrt{[1 - 8\alpha\kappa(\partial_z e^{-A})^2][(\partial_z A)^2 - \partial_z^2 A]}. \quad (26)$$

Therefore, we can obtain the thick brane solution by directly specifying the form of the warp factor. In Ref. [70], one analytical solution has been obtained, and we would like to extend this solution using other warp factors.

III. BRANE SOLUTIONS

In this section, we first consider various superpotentials for solving multikink branes. Then, we also construct a thick brane solution with a polynomial warp factor in the conformal coordinate. We will set $M_* = 1$ in the numerical calculations or plots in this paper.

A. Solutions with superpotential method

Once a superpotential is given, the warp factor, the scalar field, the energy density, and the scalar potential can be obtained numerically, just as discussed in Sec. II, and the property of the brane can be calculated. For some concrete examples, we consider the following three types of superpotentials:

$$W_1(\phi) = M_*^4(a\tilde{\phi} + b\tilde{\phi} \operatorname{sech}(\tilde{\phi})), \quad (27)$$

$$W_2(\phi) = M_*^4(a\tilde{\phi} + b \sinh(\tilde{\phi})), \quad (28)$$

$$W_3(\phi) = M_*^4(a\tilde{\phi} + b \sin(\tilde{\phi})), \quad (29)$$

where $\tilde{\phi} = \phi/(M_*^{3/2}v)$ is a dimensionless scalar field and a , b , and v are dimensionless parameters. Inspired by the sine-Gordon scalar potentials studied in Refs. [102,103], here we give three superpotentials containing the same polynomial and different nonpolynomial contributions. The three different nonpolynomial terms in these superpotentials are a finite term, a divergent term, and a periodic term, respectively. The different superpotentials induce different scalar potentials. With these three superpotentials the scalar potentials can have richer structures, which represent more complex interactions. We will show that multikink thick brane solutions can be obtained for all these superpotentials with different parameter spaces.

1. First superpotential

For the first superpotential (27), we obtain the scalar potential from (17), the function $y(\phi)$ from (19), and the warp factor from (20),

$$\begin{aligned}
V(\phi) &= \frac{2}{27} M_*^5 \tilde{\phi}^2 (b \operatorname{sech}(\tilde{\phi}) + a)^2 \\
&\quad \times (4\tilde{\alpha} \tilde{\phi}^2 (b \operatorname{sech}(\tilde{\phi}) + a)^2 - 9) \\
&\quad + \frac{1}{162} M_*^5 v^{-2} (9 - 8\tilde{\alpha} \tilde{\phi}^2 (b \operatorname{sech}(\tilde{\phi}) + a)^2)^2 \\
&\quad \times (b \operatorname{sech}(\tilde{\phi}) (1 - \tilde{\phi} \tanh(\tilde{\phi})) + a)^2, \quad (30)
\end{aligned}$$

$$\begin{aligned}
y(\phi) &= M_*^{-1} \int_0^{\tilde{\phi}} \frac{9v^2}{b \operatorname{sech}(\tilde{\phi}) - b\tilde{\phi} \tanh(\tilde{\phi}) \operatorname{sech}(\tilde{\phi}) + a} \\
&\quad \times \frac{1}{9 - 8\tilde{\alpha} (b\tilde{\phi} \operatorname{sech}(\tilde{\phi}) + a\tilde{\phi})^2} d\tilde{\phi}, \quad (31)
\end{aligned}$$

$$\begin{aligned}
A(y(\phi)) &= \int_0^{\tilde{\phi}} \frac{3v^2 (b\tilde{\phi} \operatorname{sech}(\tilde{\phi}) + a\tilde{\phi})}{b \operatorname{sech}(\tilde{\phi}) - b\tilde{\phi} \tanh(\tilde{\phi}) \operatorname{sech}(\tilde{\phi}) + a} \\
&\quad \times \frac{1}{8\tilde{\alpha} (b\tilde{\phi} \operatorname{sech}(\tilde{\phi}) + a\tilde{\phi})^2 - 9} d\tilde{\phi}. \quad (32)
\end{aligned}$$

We defined a dimensionless GB coupling constant $\tilde{\alpha} = \alpha/M_*$. Here, we also keep M_* in these expressions in order to check the dimensions of the quantities. The energy density $\rho(y)$ can be obtained from Eqs. (21), (22), and (23). We do not list it here.

The kinklike solution for the scalar field demands that the scalar field satisfies $\phi \rightarrow \text{const} = \phi_\infty$ when $y \rightarrow \infty$. That is to say, the integral function in Eq. (31) for $y(\phi)$ must approach infinity when $\phi \rightarrow \phi_\infty$. Thus, we get constraints on the parameters $\tilde{\alpha}$, a , and b that could support the existence of thick brane solutions. We list the constraints as follows:

- (i) when $\tilde{\alpha} > 0$, the restriction is $a \neq -b$;
- (ii) when $\tilde{\alpha} \leq 0$ and $b \geq 0$, the restriction is $-b < a < 0.2511b$;
- (iii) when $\tilde{\alpha} \leq 0$ and $b < 0$, the restriction is $0.2511b < a < -b$.

Then, specifying the suitable values of parameters a and b , we can obtain the single-kink, double-kink, and triple-kink thick brane solutions. See the corresponding results in Fig. 1 with the values of the parameters given in Table I. In this subsection, we set $v = 1$ for the scalar potential. From Fig. 1 we can see that the triple-kink scalar field configuration has a steplike energy density which is a new feature of the multikink scalar field.

2. Second superpotential

Next, we use the second superpotential (28) to construct thick brane solutions. The corresponding expression of the scalar potential is

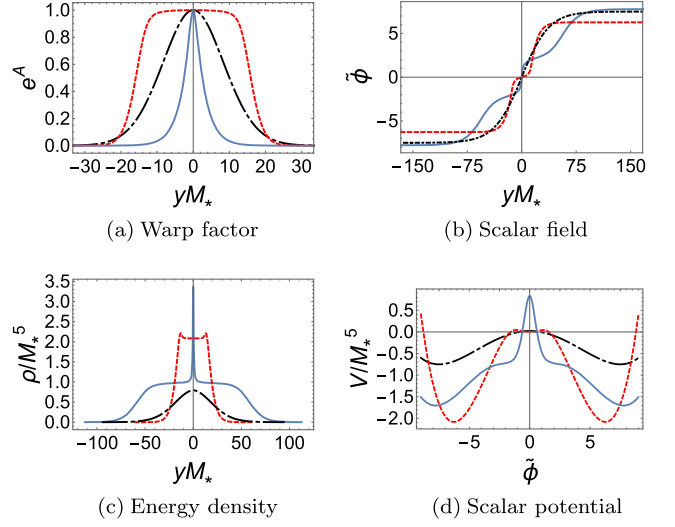


FIG. 1. The brane solutions for the first superpotential (27). The black dot-dashed lines, red dashed lines, and dark blue lines correspond to the single-kink, double-kink, and triple-kink scalar field solutions, respectively. The values of the parameters are listed in Table I.

$$\begin{aligned}
V(\phi) &= \frac{2M_*^5}{27} (a\tilde{\phi} + b \sinh(\tilde{\phi}))^2 (4\tilde{\alpha} (a\tilde{\phi} + b \sinh(\tilde{\phi}))^2 - 9) \\
&\quad + \frac{M_*^5}{162} v^{-2} (b \cosh(\tilde{\phi}) + a)^2 \\
&\quad \times (9 - 8\tilde{\alpha} (a\tilde{\phi} + b \sinh(\tilde{\phi}))^2)^2. \quad (33)
\end{aligned}$$

Substituting the superpotential (28) into Eqs. (19), (20), and (23), the function $y(\phi)$, the warp factor $A(y)$, and the energy density $\rho(y)$ can be obtained. We also need to restrict the parameters $\tilde{\alpha}$, a , and b with Eq. (19) which could support the existence of thick brane solutions just like we did in Sec. III A 1. The result is

- (i) when $\tilde{\alpha} > 0$, the restriction is $b \neq -a$;
- (ii) when $\tilde{\alpha} \leq 0$, the restriction is $b^2 < -ab$.

With this superpotential, we can also find thick brane solutions with single-kink and double-kink scalar field configurations. We show three different solutions with double-kink configurations in Fig. 2, for which the parameters are given in Table II.

TABLE I. Parameters used in Fig. 1.

Solution	Lines	$\tilde{\alpha}$	a	b
Solution_a1	Black dot-dashed lines	0.500	0.200	0.010
Solution_a2	Red dashed lines	0.180	0.400	-0.375
Solution_a3	Dark blue lines	0.220	0.290	1.000

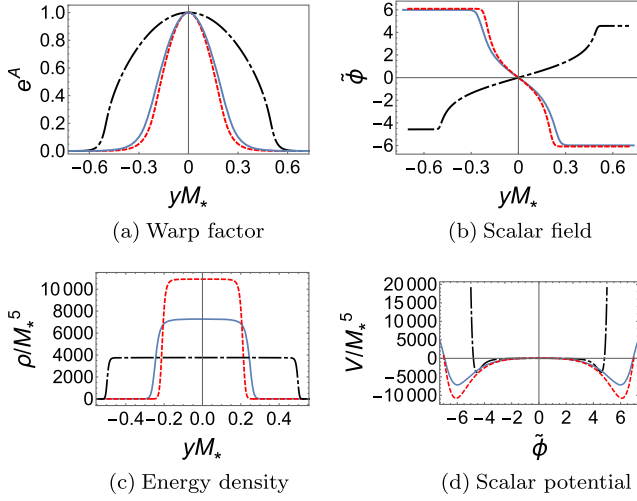


FIG. 2. The brane solutions for the second superpotential (28). All three lines correspond to the double-kink solutions. The values of the parameters are listed in Table II.

3. Third superpotential

Finally, we use the third superpotential (29) to derive the thick brane solutions. With the above superpotential, we get the scalar potential as follows:

$$\begin{aligned}
 V(\phi) = & \frac{2M_*^5}{27} (a\tilde{\phi} + b \sin(\tilde{\phi}))^2 (4\tilde{\alpha}(a\tilde{\phi} + b \sin(\tilde{\phi}))^2 - 9) \\
 & + \frac{M_*^5}{162} v^{-2} (a + b \cos(\tilde{\phi}))^2 \\
 & \times (9 - 8\tilde{\alpha}(a\tilde{\phi} + b \sin(\tilde{\phi}))^2)^2. \quad (34)
 \end{aligned}$$

Other functions can also be obtained with the above superpotential. For thick brane solutions, the constraints for the parameters that support the existence of thick brane solutions are

- (i) when $\tilde{\alpha} > 0$, then $a \neq \pm b$;
- (ii) when $\tilde{\alpha} \leq 0$, then $-b^2 < ab < b^2$.

The superpotential (29) has rich structure due to the combination of the linear term $\tilde{\phi}$ and the periodic function $\sin(\tilde{\phi})$. For such a superpotential, we find that any number of kinks for a background scalar field can be constructed with some suitable parameters. For instance, the number of kinks approaches to infinity when the parameters satisfy $a > -b > 0$ and $\tilde{\alpha} \rightarrow +0$. This is a new result of the GB term and the superpotential (29). We show some multikink brane solutions in Fig. 3.

TABLE II. Parameters used in Fig. 2.

Solution	Lines	$\tilde{\alpha}$	a	b
Solution_b1	Black dot-dashed lines	0.0001	2.00	2.00
Solution_b2	Red dashed lines	-0.003	-11.0	0.050
Solution_b3	Dark blue lines	-0.003	-10.0	0.05

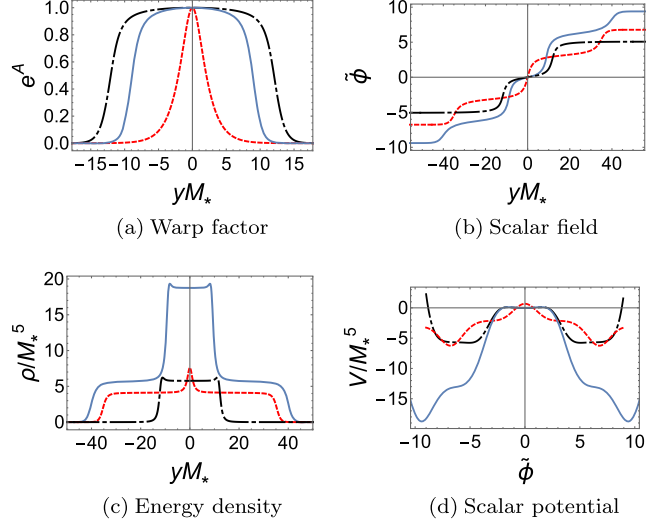


FIG. 3. The brane solutions for the third superpotential (29). The black dot-dashed lines, red dashed lines, and dark blue lines correspond to the double-kink, triple-kink, and 4-kink scalar field solutions, respectively. The values of the parameters are listed in Table III.

Similarly, we can also find other brane world solutions for a given superpotential with our method given in the last section. This method is powerful in the GB brane model and can be generalized to brane models in other gravity theories.

From the above three typical examples of multikink brane solutions, we see that the warp factors with even kink solutions seem fatter than the ones with odd kink solutions. The energy densities and scalar potentials of multikink scalar fields are obviously different for different numbers of kinks. In these solutions we cannot distinguish the number of kinks from the warp factor. This is because the influence of the multikink scalar field to the warp factor is too small. With some suitable superpotentials, the influence to the warp factor can be seen. We also find that the GB term plays an important role in finding multikink scalar field solutions. When $\tilde{\alpha} = 0$ the scalar field in these three cases could not be multikink scalar fields, and are at most single-kink scalar fields. When $\tilde{\alpha} \neq 0$ the scalar fields could be multikink scalar fields. Thus, the GB term could lead to richer brane structure. Note that the superpotentials contain polynomial and nonpolynomial contributions and can be seen as three extensions of the superpotential in Ref. [101]. With these contributions from the nonpolynomial and the GB term, the multikink scalar fields can be obtained.

B. Solution with a polynomial-expanded warp factor

In the last subsection, we introduced the superpotential method to solve the thick brane system and shown that we can obtain the thick brane solutions with multikink background scalar fields by choosing some suitable superpotentials. In this subsection, we focus on the second

method introduced at the end of Sec. II. As an example, we adopt the following polynomial-expanded warp factor with respect to the extradimensional coordinate z :

$$e^{A(z)} = \sqrt{\frac{s_1(z)}{s_2(z)}}, \quad (35)$$

$$s_1(z) = \sum_{i=0}^n (kz)^{2i}, \quad (36)$$

$$s_2(z) = \sum_{i=0}^{n+1} (kz)^{2i}, \quad (37)$$

where n is a nonnegative integer and k is a positive real parameter with mass dimension. The scalar potential and the scalar field can be solved from (25) and (26), respectively. It can be seen that $\tilde{\alpha} \leq \frac{1}{8} \frac{M_*^2}{k^2}$ can guarantee the scalar field is real. We plot the brane solutions in Fig. 4 and Fig. 5 by choosing different values of n and $\tilde{\alpha}$, respectively. Since the warp factor and energy density are independent of $\tilde{\alpha}$, we do not repeat their plots in Fig. 5. With the increase of the parameter n , the warp factor shown in Fig. 4(a) becomes much wider with a platform around $z = 0$, and the scalar field shown in Fig. 4(b) changes to double-kink from single-kink when $n \geq 1$. The energy density in Fig. 4(c) shows that the brane splits into two sub-branes. When the parameter n increases to infinity, the warp factor, the scalar field, the energy density, and the scalar potential still have the similar shapes as the case of a finite n . From Fig. 5 we see that the scalar field and scalar potential are stretched with the decrease of the parameter $\tilde{\alpha}$.

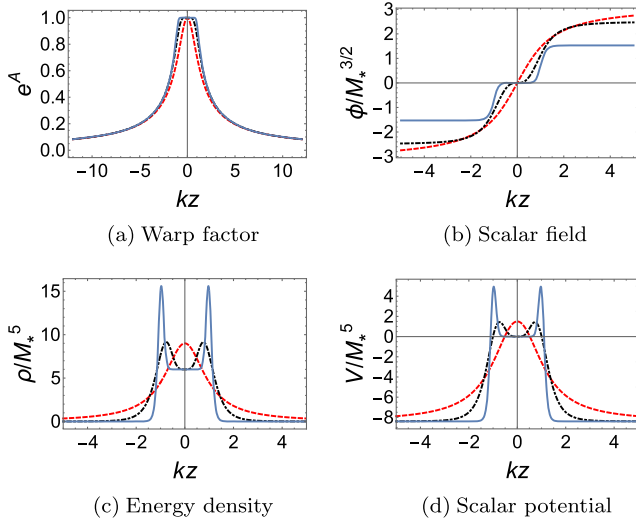


FIG. 4. The brane solutions for the warp factor (35) with different values of n . The red dashed lines, black dot-dashed lines, and dark blue lines correspond to $n = 0$, $n = 2$, and $n = 9$, respectively. Here we let $\tilde{\alpha} = -0.1$.

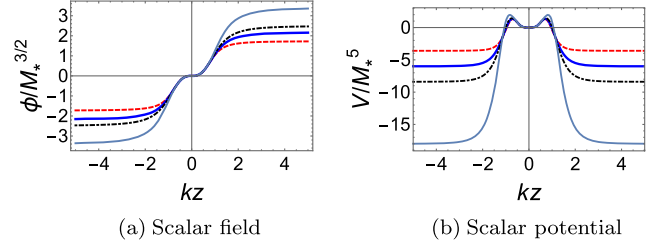


FIG. 5. The brane solutions for the warp factor (35) with different values of $\tilde{\alpha}$. The red dashed lines, blue solid lines, black dot-dashed lines, and dark blue lines correspond to $\tilde{\alpha} = 0.1$, $\tilde{\alpha} = 0$, $\tilde{\alpha} = -0.1$, and $\tilde{\alpha} = -0.5$, respectively. Here we let $n = 2$.

IV. STABILITY OF THE SYSTEM AND LOCALIZATION OF GRAVITY

So far, we have obtained four kinds of thick brane solutions. It has been shown that the four-dimensional Newtonian potential can be recovered. Note that the four-dimensional Newtonian potential is generated by the massless KK graviton and an additional correction will be induced by the massive KK gravitons. Therefore, to check that whether the four-dimensional Newtonian potential can be recovered, we need to study the properties of the brane system under four-dimensional tensor perturbations of the background metric. The total metric is given by

$$ds^2 = e^{2A(z)}[(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dz^2], \quad (38)$$

where $h_{\mu\nu} = h_{\mu\nu}(x^\mu, z)$ is a transverse and traceless tensor perturbation of the background metric (24). The transverse and traceless gauge conditions are given by $\partial^\mu h_{\mu\nu} = 0 = h^\mu_\mu$.

The linear perturbation equation is

$$\delta G_\mu^\nu - \epsilon \delta Q_\mu^\nu = \kappa \delta T_\mu^\nu, \quad (39)$$

where $\epsilon = 2\tilde{\alpha}\kappa M_*$. The above equation can be expanded as

$$\begin{aligned} & (1 + 2\epsilon \bar{R})\delta R_\mu^\nu - \frac{1}{2}\bar{R}h_\mu^\nu - 2\epsilon \left[\frac{1}{4}\bar{R}_{GB}h_\mu^\nu \right. \\ & + 2(\delta R_{\mu\beta}\bar{R}^{\nu\beta} + \bar{R}_{\mu\beta}\delta R^{\nu\beta}) + 2(\delta R_{AB}\bar{R}_\mu^{\nu B} + \bar{R}_{AB}\delta R_\mu^{\nu B}) \\ & \left. - (\delta R_{\mu ABC}\bar{R}^{\nu ABC} + \bar{R}_{\mu ABC}\delta R^{\nu ABC}) \right] = \kappa \delta T_\mu^\nu. \end{aligned} \quad (40)$$

The quantities with a bar and with a delta in Eq. (40) correspond to the background and the perturbation, respectively. With the help of the equations of motion of the background, the equation for the tensor perturbation under the transverse and traceless gauges can be derived as follows [70]:

$$\begin{aligned} & (1 - 4\epsilon e^{-2A}\partial_z^2 A)\square^{(4)}h_\mu^\nu + [1 - 4\epsilon e^{-2A}(\partial_z A)^2]\partial_z^2 h_\mu^\nu \\ & + [3\partial_z A - 4\epsilon e^{-2A}(2(\partial_z A)\partial_z^2 A + (\partial_z A)^3)]\partial_z h_\mu^\nu = 0, \end{aligned} \quad (41)$$

where $\square^{(4)} = \eta^{\mu\nu} \partial_\mu \partial_\nu$ denotes the four-dimensional d'Alembert operator on the brane. The above equation can also be given in the physical coordinate y by the coordinate transformation $dy = e^A dz$. The result is

$$\square^{(4)} h_\mu^\nu + F^2(y) h_\mu^{\nu\prime\prime} + H(y) h_\mu^{\nu\prime} = 0, \quad (42)$$

where the prime denotes the derivative respect to physical coordinate y and

$$F(y) = e^A \sqrt{\frac{(1 - 4\epsilon A^2)}{1 - 4\epsilon(A^2 + A'')}}}, \quad (43)$$

$$H(y) = e^{2A} \frac{4(A' - 2\epsilon A' A'' - 4\epsilon A'^3)}{1 - 4\epsilon(A^2 + A'')}. \quad (44)$$

Then, by considering the KK decomposition of the tensor perturbation $h_\mu^\nu(x^\mu, y) = \chi_\mu^\nu(x^\mu) \varphi(y)$, we obtain the four-dimensional Klein-Gordon equation of the four-dimensional KK graviton $\chi_\mu^\nu(x^\mu)$ with mass m and the equation of the extradimensional part $\varphi(y)$,

$$(\square^{(4)} - m^2) \chi_\mu^\nu(x^\mu) = 0, \quad (45)$$

$$F^2(y) \varphi''(y) + H(y) \varphi'(y) + m^2 \varphi(y) = 0. \quad (46)$$

Using another coordinate transformation $dy = F(y(w)) dw$, equation (46) can be rewritten in the coordinate w as

$$\partial_w^2 \varphi(w) + \mathcal{K}(w) \partial_w \varphi(w) + m^2 \varphi(w) = 0, \quad (47)$$

where $\mathcal{K}(w) = \frac{H - \partial_w F}{F}$. Finally, by making the field transformation $\varphi(w) = f(w) \xi(w)$ with the function $f(w)$ satisfying

$$\frac{\partial_w f(w)}{f(w)} = \frac{1}{2} \left(\frac{\partial_w F(w)}{F(w)} - \frac{H(w)}{F(w)} \right), \quad (48)$$

we can transform the equation for $\varphi(w)$ as the following Schrödinger-like equation

$$(-\partial_w^2 + V_{\text{eff}}(w)) \xi(w) = m^2 \xi(w), \quad (49)$$

where the effective potential in the coordinate w is given by

$$V_{\text{eff}}(w) = \frac{2F(\partial_w H - \partial_w^2 F) - 4H \partial_w F + 3(\partial_w F)^2 + H^2}{4F^2}. \quad (50)$$

The Schrödinger-like equation (49) can be expressed as the following form

$$Q^\dagger Q \xi(w) = m^2 \xi(w), \quad (51)$$

where the operators Q and Q^\dagger are given by

$$Q = \partial_w + P(w), \quad Q^\dagger = -\partial_w + P(w), \quad (52)$$

$$P(w) = \frac{1}{2} \left[\frac{\partial_w F}{F} - \frac{H}{F} \right]. \quad (53)$$

Equation (51) guarantees $m^2 \geq 0$ and there is no tachyonic graviton. So, it is proved that the system is stable under the linear tensor perturbations. The tensor zero mode can be obtained by solving Eq. (51) with $m^2 = 0$,

$$\xi_0(w) = N_0 e^{\pm \int P(w) dw}, \quad (54)$$

where N_0 is the normalization constant. A localized graviton zero mode should satisfy the normalization condition $\int_{-\infty}^{\infty} \xi_0^2(w) dw = 1$.

Next, we use the relation $\frac{\partial w}{\partial y} = \frac{1}{F}$ to write the expressions of the effective potential V_{eff} and the tensor zero mode ξ_0 in the physical coordinate y ,

$$V_{\text{eff}}(w(y)) = \frac{1}{4} \left(\frac{H^2(y)}{F^2(y)} - \frac{4H(y)F'(y)}{F(y)} + F'^2(y) \right) + \frac{1}{4} (2H'(y) - 2F(y)F''(y)), \quad (55)$$

$$\xi_0(w(y)) = N_0 e^{\int \mathcal{P}_\pm(y) dy}, \quad (56)$$

where the functions $\mathcal{P}_\pm(y)$ are given by

$$\mathcal{P}_\pm(y) = \pm \frac{F(y)F'(y) - H(y)}{2F(y)^2}. \quad (57)$$

For an asymptotic AdS₅ spacetime described by the metric (8), the warp factor $e^{A(y)}$ tends to $A(y) \rightarrow -k|y|$ when $|y| \rightarrow \infty$. Thus, the behavior of the effective potential at infinity is

$$\lim_{|y| \rightarrow \infty} V_{\text{eff}} \rightarrow \frac{15}{4} k^2 e^{-2k|y|}, \quad (58)$$

which shows that it will tend to zero when $|y| \rightarrow \infty$. The asymptotic behavior of the factor $\int \mathcal{P}_\pm dy$ of the zero mode at infinity is

$$\lim_{|y| \rightarrow \infty} \int \mathcal{P}_\pm(y) dy = \pm \left(\frac{3k}{2} |y| + C_0 \right), \quad (59)$$

where C_0 is an integration constant. The asymptotic behavior (59) shows that the following condition

$$\int_{-\infty}^{\infty} \xi_0(y)^2 dy = 1 \quad (60)$$

can be satisfied with a finite normalization constant N_0 for the case of \mathcal{P}_- in Eq. (59). Therefore, it is proved that the

corresponding zero mode $\xi_0(w(y)) = N_0 e^{\int \mathcal{P}_-(y) dy}$ is localized on the brane and the four-dimensional Newtonian potential can be recovered.

Next, we focus on the profiles of the effective potentials and the corresponding configurations of the tensor zero modes on the thick brane solutions obtained in Sec. III.

Note that, the effective potential (55) would diverge when the function $F(y)$ given in Eq. (43) goes to zero or the function $H(y)$ given in Eq. (44) diverges at certain finite extra-dimension coordinate $y = y_0$, which is equivalent to

$$(4\epsilon A'^2 - 1)(4\epsilon(A'^2 + A'') - 1) = 0. \quad (61)$$

The divergence of the effective potential means that there exists a nonsmooth tensor zero mode in the smooth thick brane.

Finally, we give the shapes of the effective potentials and tensor zero modes of our four kinds of thick brane solutions. Figures 6–8 show the effective potentials and tensor zero modes of the thick brane solutions generated by the superpotentials (27), (28), and (29), respectively. It can be seen that the distribution of each effective potential is consistent with the energy density of the corresponding thick brane. With the specific superpotential $W(\phi)$, the effective potential V_{eff} can have a richer structure. For instance, in Fig. 7 one can see that there are several wells in the effective potential. Figures 9 and 10 show the effective potential and tensor zero mode of the thick brane solution (35) for different values of n and $\tilde{\alpha}$, respectively. Here, we obtain the discontinuous effective potential and nonsmooth

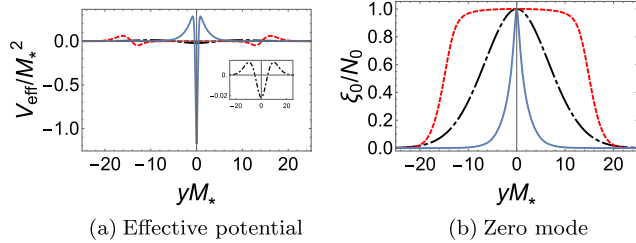


FIG. 6. The effective potential and zero mode for the first brane solution given in Sec. III A with the superpotential (27). The parameters are given in Table I.

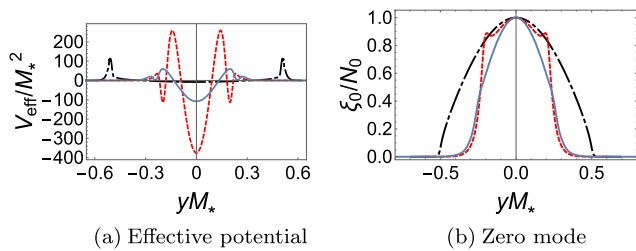


FIG. 7. The effective potential and zero mode for the second brane solution in Sec. III A with the superpotential (28). The parameters are given in Table II.

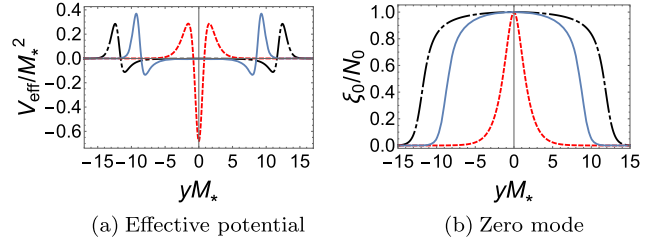


FIG. 8. The effective potential and zero mode for the third brane solution in Sec. III A with the superpotential (29). The parameters are given in Table III.

tensor zero mode for a smooth thick brane when n or $-\tilde{\alpha}$ has a large value. This is a new feature that did not find in general relativity. Although the profiles of these effective potentials and tensor zero modes are quite different, the localization condition (60) are satisfied and so all the zero modes we obtained can be localized on the brane and the four-dimensional Newtonian potential can be recovered.

We have discussed the localization of the KK gravitons and the corresponding tachyon stability. Next, we further discuss the ghost problem of the tensor perturbations. It has been proved that the linear perturbation equations can be derived via the Hamiltonian variation principle in terms of the quadratic actions [39,104,105]. Therefore, the coefficients of

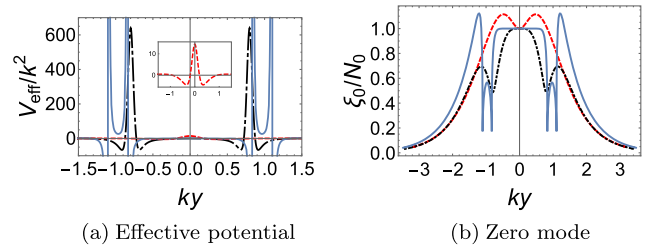


FIG. 9. The effective potential and zero mode for the brane solution given in Sec. III B with the given warp factor (35) for different values of n . The parameters are set to $\tilde{\alpha} = -0.1$ and $n = (0, 2, 9)$ for the red dashed, black dot-dashed, and dark blue lines, respectively.

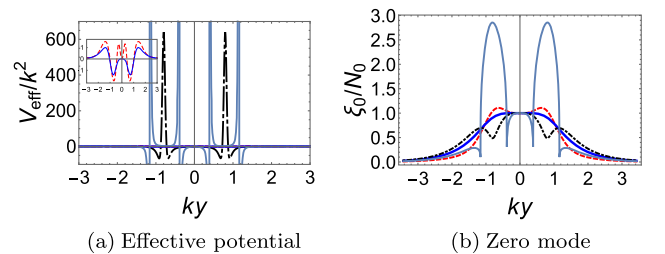


FIG. 10. The effective potential and zero mode for the brane solution given in Sec. III B with the given warp factor (35) for different values of $\tilde{\alpha}$. The parameters are set to $n = 2$ and $\tilde{\alpha} = (0.1, 0, -0.1, -0.5)$ for the red dashed, blue solid, black dot-dashed, and dark blue lines, respectively.

the kinetic term for the quadratic action of $h_{\mu\nu}$ can be recast in terms of the equation of motion (41). That is to say, the kinetic terms of the quadratic action will have the following forms:

$$-K_1(z)h^{\mu\nu}\partial_t^2 h_{\mu\nu} + K_1(z)h^{\mu\nu}\partial_x^2 h_{\mu\nu} + K_2(z)h^{\mu\nu}\partial_z^2 h_{\mu\nu}, \quad (62)$$

where

$$K_1(z) = 1 - 8\tilde{\alpha}\kappa e^{-2A}\partial_z^2 A, \quad (63)$$

$$K_2(z) = 1 - 8\tilde{\alpha}\kappa e^{-2A}(\partial_z A)^2. \quad (64)$$

Therefore, the no-ghost conditions for the KK tensor modes are

$$K_1 > 0 \quad \text{and} \quad K_2 > 0. \quad (65)$$

It can be seen that the above conditions are satisfied in GR with $\tilde{\alpha} = 0$, and there is no ghost in GR. Next we will give the constraint on the GB coupling parameter $\tilde{\alpha}$. It is noted that the value of function $e^{-2A}(\partial_z A)^2$ is always positive definite, for which we can get the following inequality

$$\tilde{\alpha} < \frac{e^{2A(z)}}{8\kappa(\partial_z A)^2}. \quad (66)$$

For the function $e^{-2A}\partial_z^2 A$, its value can be positive or negative, therefore, we can get another two conditions

$$\tilde{\alpha} > \frac{e^{2A(z)}}{8\kappa\partial_z^2 A}, \quad \text{if } \partial_z^2 A < 0, \quad (67)$$

$$\tilde{\alpha} < \frac{e^{2A(z)}}{8\kappa\partial_z^2 A}, \quad \text{if } \partial_z^2 A > 0. \quad (68)$$

To check the ghost instability of our model, we define the following two functions:

$$\Omega_1 = \frac{e^{2A(z)}}{8\kappa\partial_z^2 A}, \quad (69)$$

$$\Omega_2 = \frac{e^{2A(z)}}{8\kappa(\partial_z A)^2}. \quad (70)$$

Then we can check whether the conditions (66), (67), and (68) are satisfied for our solutions. For the thick brane models given in case III A 1, we give the result in Fig. 11. It can be seen that the values of the functions Ω_1 and Ω_2 along the extra dimension will gradually tend towards the values of $\tilde{\alpha}$, for which the functions K_1 and K_2 are positive definite and there are no ghost instabilities for the tensor perturbations. For the thick brane models that are given in case of III A 2, we give the result in Fig. 12 and we show

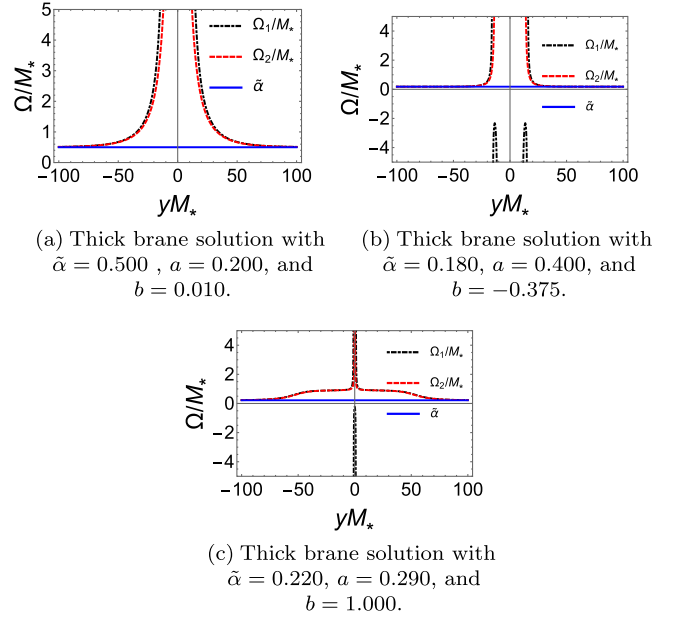


FIG. 11. Comparison between the functions (Ω_1, Ω_2) and values of $\tilde{\alpha}$ for the thick brane solution of case III A 1.

that there are no ghost instabilities for the solutions in subfigures 12(a), 12(b), and 12(c). For the thick brane models that are given in case of III A 3, we give the results in Fig. 13 and prove that there are no ghost instabilities for these solutions.

We should note that, it is hard to figure out what are the exact effects of the GB term on the properties of thick brane for the thick brane solutions listed in Tables I, II, and III.

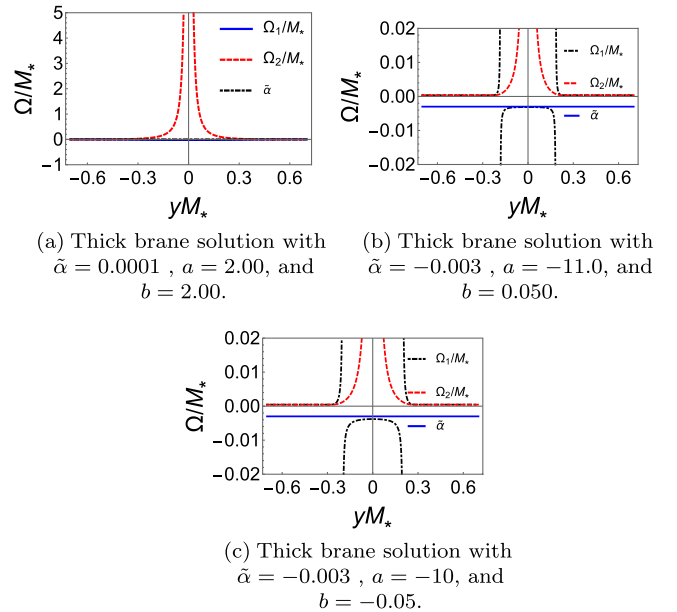


FIG. 12. Comparison between the functions (Ω_1, Ω_2) and values of $\tilde{\alpha}$ for the thick brane solution of case III A 2.

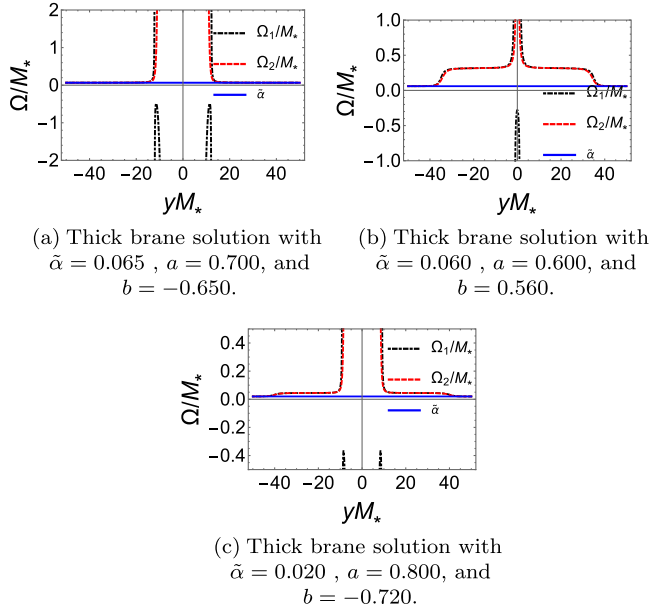


FIG. 13. Comparison between the functions (Ω_1, Ω_2) and values of $\tilde{\alpha}$ for the thick brane solution of case III A 3.

However, the solutions listed in Tables IV and V are derived in terms of the directly given warp factors. Therefore, for a determined warp factor $A(z)$, the conditions (66), (67), and (68) can give the constraint of the GB coupling parameter $\tilde{\alpha}$ directly. As concrete examples we have checked the corresponding constraints obtained from the

TABLE III. Parameters used in Fig. 3.

Solution	Lines	$\tilde{\alpha}$	a	b
Solution_c1	Black dot-dashed lines	0.065	0.700	-0.650
Solution_c2	Red dashed lines	0.060	0.600	0.560
Solution_c3	Dark blue lines	0.020	0.800	-0.720

TABLE IV. Parameters used in Fig. 4.

Solution	Lines	$\tilde{\alpha}$	n
Solution_d1	Red dashed lines	-0.1	0
Solution_d2	Black dot-dashed lines	-0.1	1
Solution_d3	Dark blue lines	-0.1	2

TABLE V. Parameters used in Fig. 5.

Solution	Lines	$\tilde{\alpha}$	n
Solution_e1	Red dashed lines	0.1	2
Solution_e2	Blue solid lines	0	2
Solution_e3	Black dot-dashed lines	-0.1	2
Solution_e4	Dark blue lines	-0.5	2

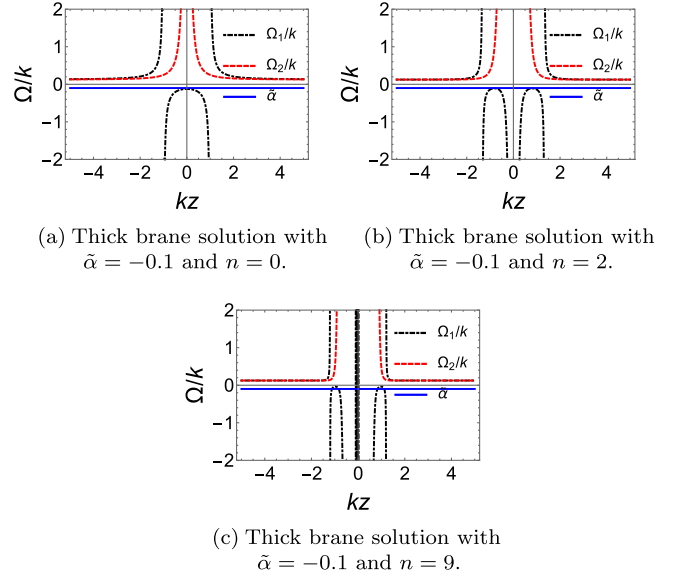


FIG. 14. Comparison between the functions (Ω_1, Ω_2) and values of $\tilde{\alpha}$ for the thick brane solutions in Fig. 4.

warp factors (35) and we give the results in Figs. 14–16. For the solutions listed in Table V, the value of $\tilde{\alpha}$ should satisfy

$$-0.1 < \tilde{\alpha} < 0.1 \quad (71)$$

in terms of the constraints (66), (67), and (68). We find that when we choose $\tilde{\alpha} = -0.5$, the tensor perturbation will have the ghost instability. We also choose different values of $\tilde{\alpha}$ as follows:

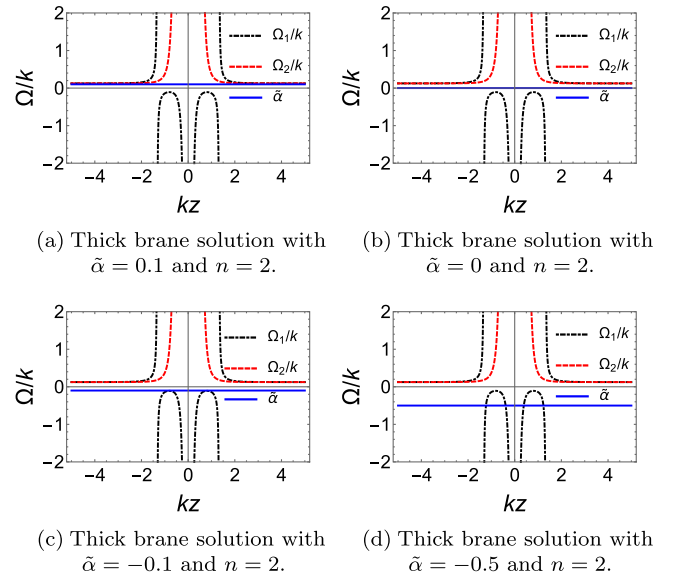


FIG. 15. Comparison between the functions (Ω_1, Ω_2) and values of $\tilde{\alpha}$ for the thick brane solutions in Fig. 5.

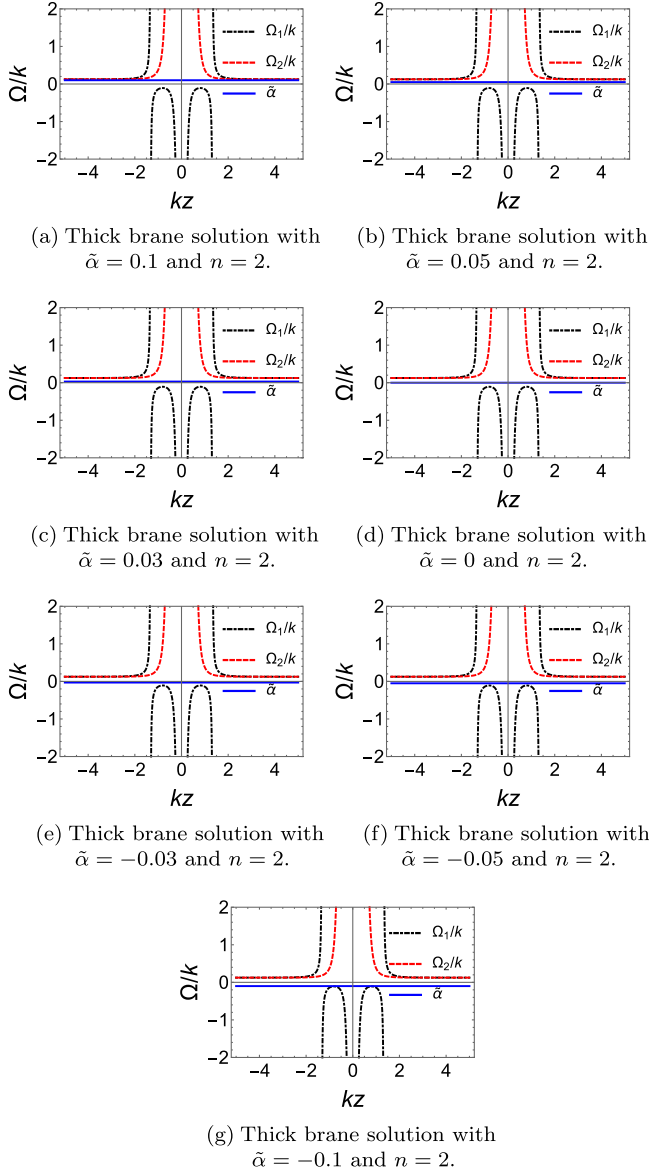


FIG. 16. Comparison between the functions (Ω_1, Ω_2) and values of $\tilde{\alpha}$ for the thick brane solutions with an determined warp factor.

$$\tilde{\alpha} = (-0.1, -0.05, -0.03, 0, 0.03, 0.05, 0.01) \quad (72)$$

and give the results in Fig. 16, our results confirm that the correctness of constraints (66), (67), and (68). We will give a summary about the ghost instability of the four-dimensional tensor perturbation in following Table VI.

V. SCALAR PERTURBATION

In this section, we investigate the properties of scalar perturbations of our model. Reference [70] studied the full perturbations and provided the corresponding results about the localization of the tensor modes, vector modes, and

scalar modes. Here, we give a brief summary about the scalar perturbations and discuss the corresponding localization and stability of our models.

The complete form of metric with the scalar perturbation is

$$\delta g_{MN} = e^{2A(z)} \begin{pmatrix} 2\eta_{\mu\nu}\psi + 2\partial_\mu\partial_\nu E & \partial_\mu C \\ \partial_\mu C & 2\xi \end{pmatrix}. \quad (73)$$

It can be seen that the scalar perturbations are parametrized by the four scalar functions ψ, ξ, C, E . However, the above scalar perturbation functions are not gauge invariant [70] under an infinitesimal transformation as follows:

$$x^M \rightarrow \tilde{x}^M = x^M + \epsilon^M, \quad (74)$$

where $\epsilon_M = a^2(y)(\epsilon_\mu, \epsilon_z)$ and the gauge functions are $\epsilon_\mu = \partial_\mu \epsilon + \zeta_\mu$ with $\partial_\mu \zeta^\mu = 0$. In Ref. [70], the authors have constructed the gauge-invariant scalars as follows:

$$\Psi = \psi - \partial_z A (\partial_z E + C), \quad (75)$$

$$\Xi = \xi + \frac{1}{e^{A(z)}} \partial_z [e^{A(z)} (C + \partial_z E)], \quad (76)$$

$$X = \chi - \partial_z \phi (\partial_z E + C). \quad (77)$$

The equations of motion for these gauge-invariant scalars can also be obtained, see the details in Ref. [70].

Using the rescaled Ψ in terms of

$$\Phi = \frac{e^{3A(z)/2} q}{\partial_z \phi} \Psi, \quad (78)$$

the master equation of motion for the scalar perturbation is

$$-\partial_z^2 \Phi + u \partial_z^2 \left(\frac{1}{u} \right) \Phi - \left(1 - \frac{\partial_z q}{\partial_z A q} \right) \partial_\alpha \partial^\alpha \Phi = 0, \quad (79)$$

where

$$q = 1 - \frac{4\epsilon(\partial_z A)^2}{e^{2A(z)}} \quad (80)$$

and

$$u(z) = \frac{e^{3A(z)/2} \partial_z \phi}{\partial_z A}. \quad (81)$$

It has been proved that the equation obeyed by Φ is also obeyed by the appropriately re-scaled Ξ [70].

TABLE VI. Stabilities of our thick brane models.

Thick brane solution	K_1	K_2	Θ	V_s	Stability
Solution_a1	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_a2	Positive definite	Positive definite	Positive definite	Not positive definite	Unstable
Solution_a3	Positive definite	Positive definite	Not positive definite	Positive definite	Unstable
Solution_b1	Positive definite	Positive definite	Not positive definite	Positive definite	Unstable
Solution_b2	Positive definite	Positive definite	Positive definite	Not positive definite	Unstable
Solution_b3	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_c1	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_c2	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_c3	Positive definite	Positive definite	Positive definite	Not positive definite	Unstable
Solution_d1	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_d2	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_d3	<0	Positive definite	Positive definite	Positive definite	Unstable
Solution_e1	Positive definite	Positive definite	Not positive definite	Positive definite	Unstable
Solution_e2	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_e3	Positive definite	Positive definite	Positive definite	Positive definite	Stable
Solution_e4	<0	Positive definite	Positive definite	Positive definite	Unstable

Applying the following KK decomposition

$$\Phi = \sum_n^\infty s_n(x^\mu)\theta_n(z), \quad (82)$$

and assuming the four-dimensional part $s_n(x^\mu)$ satisfies the four-dimensional Klein-Gordon equation as follows:

$$\partial_\alpha \partial^\alpha s_n(x^\mu) = m_n^2 s_n(x^\mu), \quad (83)$$

one can derive following equation:

$$-\partial_z^2 \theta_n(z) + \left[u \partial_z^2 \left(\frac{1}{u} \right) \right] \theta_n(z) = m_n^2 \left(1 - \frac{\partial_z q}{\partial_z A q} \right) \theta_n(z). \quad (84)$$

Here, m_n is the four-dimensional observed effective mass of the scalar perturbation s_n . It can be seen that when the coupling constant $\alpha = 0$, we have $q = 1$, and Eq. (84) reduce to the GR case [106–108]

$$-\partial_z^2 \theta_n(z) + u \partial_z^2 \left(\frac{1}{u} \right) \theta_n(z) = m_n^2 \theta_n(z). \quad (85)$$

Note that, the localization of massless scalar perturbation has been ruled out [106–108] and there is no tachyon instability in GR. To check that whether there are tachyon instabilities for the scalar perturbations of our thick brane solutions, we define the following functions:

$$\Theta = 1 - \frac{\partial_z q}{\partial_z A q}, \quad (86)$$

$$V_s = u \partial_z^2 \left(\frac{1}{u} \right). \quad (87)$$

Clearly, one positive definite Θ and one positive definite V_s will rule out the existence of the solution with an imaginary m_n [107]. We have checked all our solutions and give the

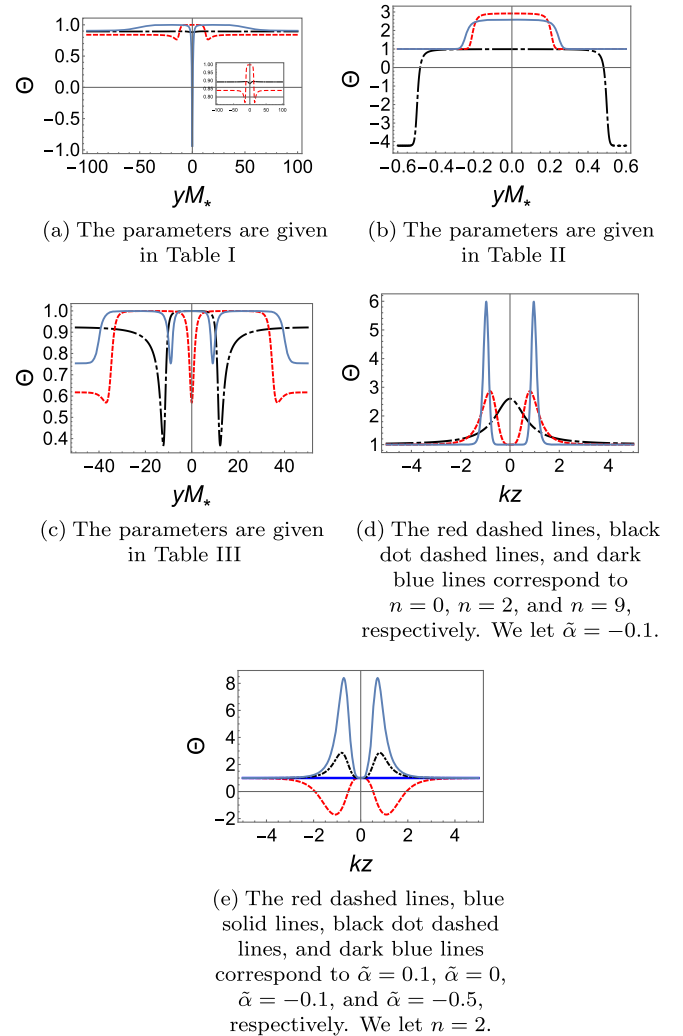


FIG. 17. Profiles of function Θ along the extra dimension for all the thick brane models.

corresponding results in Figs. 17 and 18. Combining the results about ghost instabilities of tensor perturbation and tachyon instabilities of scalar perturbations, we finally give a summary about the instabilities of our thick brane solutions in Table. VI.

So far, we have studied the properties of thick branes in five-dimensional GB gravity. We propose the numerical method to solve the thick brane solutions, the localization of massless KK graviton and whether the four-dimensional Newtonian potential can be recovered are still studied, the ghost instability of four-dimensional tensor perturbation and tachyon instability of scalar perturbations are also discussed. Here, we briefly summarize how the GB term affects the properties of the thick brane. For simplicity, we only consider the solutions in Table V to discuss this.

When the GB coupling parameter $\tilde{\alpha}$ goes from the negative value to the positive value, the profile of scalar field will gradually transform from the red dashed line to the dark blue line as shown in Fig. 5. Our further study shows that such

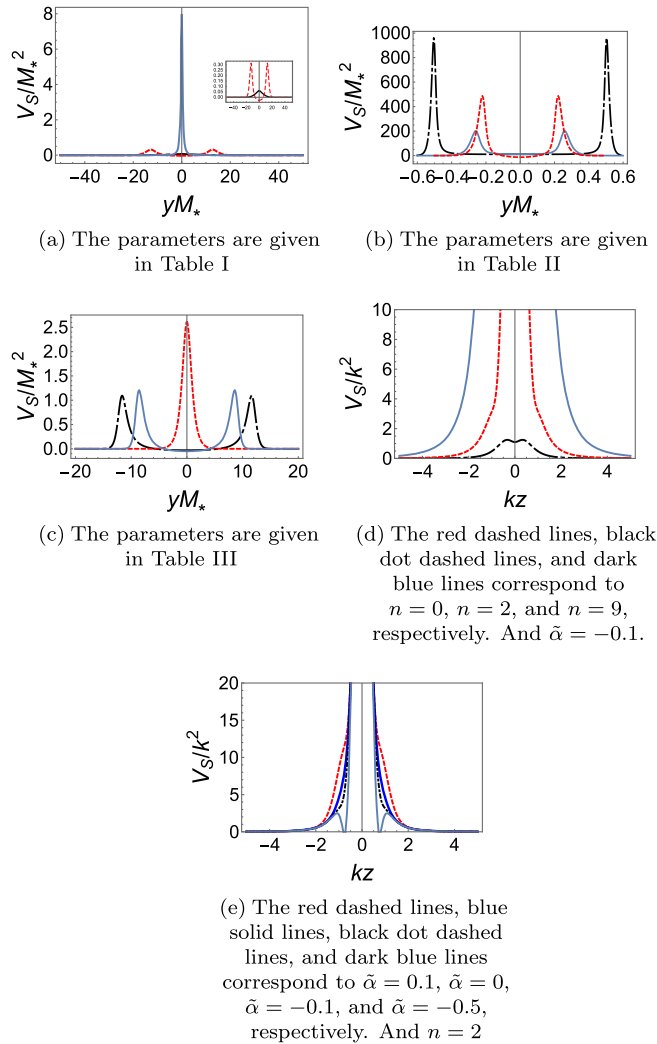


FIG. 18. Profiles of function V_s along the extra dimension for all the thick brane models.

change will induce the ghost instability of tensor perturbation and the tachyon instability of scalar perturbation, see the details in Figs. 15 and 17(e). We find that the thick brane solution will have the ghost stability of tensor perturbation but not have the tachyon instability of scalar perturbations with the negative $\tilde{\alpha}$. When the coupling parameter $\tilde{\alpha}$ is vanishing, the thick brane will be stable and there are no any instabilities. When the coupling parameter $\tilde{\alpha}$ becomes positive, both the tachyon instabilities of scalar perturbations and the ghost instability of tensor perturbation will exist.

VI. CONCLUSIONS

It is well known that the GB term is topologically invariant and does not affect the equations of motion in four-dimensional spacetime. However, when the spacetime dimension satisfies $D \geq 5$, the GB term is no longer topologically invariant and its influence should be considered. Thus, in this paper we investigated the property of a thick brane in the five-dimensional Einstein GB gravity and showed that how the GB term affects the property of a thick brane. We introduced two methods for solving the thick brane solutions. In the first method, all the variables including the warp factor A , the scalar potential V , and the extradimensional coordinate y are considered as functions of the scalar field ϕ . Then we can write the brane solution formally under the help of an auxiliary superpotential $W(\phi)$, which is given by Eqs. (17), (19), and (20). Note that, in this method, we have an assumption, i.e., the scalar field $\phi(y)$ is a monotonic function of the extradimensional coordinate y . It was shown that this novel method works well for solving thick brane solutions with various superpotentials. In the second method, the expressions for the scalar field and scalar potential were obtained in terms of the warp factor, and one can directly obtain the thick brane solution by specifying a warp factor.

By choosing three special kinds of superpotentials, we first derived three types of thick brane solutions. We gave the constraints for the parameters in the superpotentials (27), (28), and (29) that could support the existence of a thick brane solution. We showed that one can construct multikink thick brane solutions with a suitable superpotential. We then obtained the thick brane solution by specifying a polynomial-expanded warp factor.

We showed that the massless four-dimensional tensor zero modes can be localized on the thick branes and the four-dimensional Newtonian potential can be recovered. The effective potentials in these brane models for the KK gravitons have the rich structures such as multiwell, which would lead to massive resonant KK gravitons [25,26,32,33]. Furthermore, it was found that one can obtain a discontinuous effective potential and nonsmooth tensor zero mode in the smooth thick brane solution, which was not found in general relativity. This singularity is originated from the zero point of the function $F(y)$ (43) or the divergence of the function $H(y)$ given in Eq. (44) in the effective potential (50).

The singularity condition is given by Eq. (61), from which it is clear that this singularity comes from the GB term for some parameter space and there is no singularity in general relativity. Such a situation was also found in the $f(R)$ brane model, see Ref. [26] for the details. The inner structure of the effective potential with singularities will also support a series of resonant KK gravitons. Moreover, we also studied the ghost instability of four-dimensional tensor perturbation and the tachyon instability of scalar perturbation. We found that these thick brane with rich inner structures might be unstable due to the existence of the ghost and tachyon. Our results indicate that deviation of $\tilde{\alpha}$ from zero too far in both the positive and negative directions can lead to instability of the thick brane, and only the thick brane solutions with GB coupling parameter $\tilde{\alpha}$ near zero could be stable.

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- [1] K. Akama, An early proposal of 'Brane World', Lect. Notes Phys. **176**, 267 (1982).
 - [2] V. A. Rubakov and M. E. Shaposhnikov, Do we live inside a domain wall?, *Phys. Lett.* **125B**, 136 (1983).
 - [3] Th. Kaluza, Zum Unitätsproblem der Physik, *Int. J. Mod. Phys. D* **27**, 1870001 (2018).
 - [4] O. Klein, Quantum theory and 5-dimensional relativity theory, *Z. Phys.* **37**, 895 (1926).
 - [5] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, The hierarchy problem and new dimensions at a millimeter, *Phys. Lett. B* **429**, 263 (1998).
 - [6] L. Randall and R. Sundrum, A Large Mass Hierarchy from a Small Extra Dimension, *Phys. Rev. Lett.* **83**, 3370 (1999).
 - [7] L. Randall and R. Sundrum, An Alternative to Compactification, *Phys. Rev. Lett.* **83**, 4690 (1999).
 - [8] H.-C. Cheng, 2009 TASI Lecture—Introduction to Extra Dimensions, [arXiv:1003.1162](https://arxiv.org/abs/1003.1162).
 - [9] T. G. Rizzo, Introduction to extra dimensions, *AIP Conf. Proc.* **1256**, 27 (2010).
 - [10] Y.-X. Liu, *Introduction to Extra Dimensions and Thick Braneworlds*, Memorial Volume for Yi-Shi Duan (World Scientific, 2018), pp. 211–275.
 - [11] R. Maartens and K. Koyama, Brane-World gravity, *Living Rev. Relativity* **13**, 5 (2010).
 - [12] M. Gogberashvili, Hierarchy problem in the shell-Universe model, *Int. J. Mod. Phys. D* **11**, 1635 (2002).
 - [13] W. Israel, Singular hypersurfaces and thin shells in general relativity, *Nuovo Cimento B* **48**, 463 (1967).
 - [14] O. DeWolfe, D. Z. Freedman, S. S. Gubser, and A. Karch, Modeling the fifth-dimension with scalars and gravity, *Phys. Rev. D* **62**, 046008 (2000).
 - [15] K. A. Bronnikov and B. E. Meierovich, A general thick brane supported by a scalar field, *Gravitation Cosmol.* **9**, 313 (2003).
 - [16] D. Bazeia, A. R. Gomes, L. Losano, and R. Menezes, Braneworld models of scalar fields with generalized dynamics, *Phys. Lett. B* **671**, 402 (2009).
 - [17] M. Toharia, M. Trodden, and E. J. West, Scalar kinks in warped extra dimensions, *Phys. Rev. D* **82**, 025009 (2010).
 - [18] D. Bazeia, C. Furtado, and A. R. Gomes, Brane structure from scalar field in warped space-time, *J. Cosmol. Astropart. Phys.* **02** (2004) 002.
 - [19] Oleg O. Novikov, A. A. Andrianov, and V. A. Andrianov, Thick branes from self-gravitating scalar fields, *AIP Conf. Proc.* **1606**, 313 (2015).
 - [20] D. Bazeia, A. S. Lobão Jr, and R. Menezes, Thick brane models in generalized theories of gravity, *Phys. Lett. B* **743**, 98 (2015).
 - [21] Q.-Y. Xie, Q.-M. Fu, T.-T. Sui, L. Zhao, and Y. Zhong, First-order formalism and thick branes in mimetic gravity, *Symmetry* **13**, 1345 (2021).
 - [22] J.-J. Wan, Z.-Q. Cui, W.-B. Feng, and Y.-X. Liu, Smooth braneworld in 6-dimensional asymptotically AdS space-time, *J. High Energy Phys.* **05** (2021) 017.
 - [23] K. Nozari and S. D. Sadatian, Bouncing universe with a nonminimally coupled scalar field on a moving domain wall, *Phys. Lett. B* **676**, 1 (2009).
 - [24] Y.-X. Liu, F.-W. Chen, H. Guo, and X.-N. Zhou, Non-minimal coupling branes, *J. High Energy Phys.* **05** (2012) 108.
 - [25] H. Guo, Y.-X. Liu, Z.-H. Zhao, and F.-W. Chen, Thick branes with a non-minimally coupled bulk-scalar field, *Phys. Rev. D* **85**, 124033 (2012).
 - [26] Z.-G. Xu, Y. Zhong, H. Yu, and Y.-X. Liu, The structure of $f(R)$ -brane model, *Eur. Phys. J. C* **75**, 368 (2015).
 - [27] B.-M. Gu, Y.-P. Zhang, H. Yu, and Y.-X. Liu, Full linear perturbations and localization of gravity on $f(R, T)$ brane, *Eur. Phys. J. C* **77**, 115 (2017).

- [28] X.-N. Zhou, Y.-Z. Du, H. Yu, and Y.-X. Liu, Localization of gravitino field on $f(R)$ thick branes, *Sci. China Phys. Mech. Astron.* **61**, 110411 (2018).
- [29] Y. Zhong, K. Yang, and Y.-X. Liu, Linearization of a warped $f(R)$ theory in the higher-order frame II: The equation of motion approach, *Phys. Rev. D* **97**, 044032 (2018).
- [30] B.-M. Gu, Y.-X. Liu, and Y. Zhong, Stable Palatini $f(R)$ braneworld, *Phys. Rev. D* **98**, 024027 (2018).
- [31] Z.-Q. Cui, Z.-C. Lin, J.-J. Wan, Y.-X. Liu, and L. Zhao, Tensor perturbations and thick branes in higher-dimensional $f(R)$ gravity, *J. High Energy Phys.* **12** (2020) 130.
- [32] J. Chen, W.-D. Guo, and Y.-X. Liu, Thick branes with inner structure in mimetic $f(R)$ gravity, *Eur. Phys. J. C* **81**, 709 (2021).
- [33] H. Yu, Y. Zhong, B.-M. Gu, and Y.-X. Liu, Gravitational resonances on $f(R)$ -brane, *Eur. Phys. J. C* **76**, 195 (2016).
- [34] J. L. Rosa, A. S. Lobão, and D. Bazeia, Impact of compact-like and asymmetric configurations of thick branes on the scalar-tensor representation of $f(R, T)$ gravity, *Eur. Phys. J. C* **82**, 191 (2022).
- [35] J. L. Rosa, M. A. Marques, D. Bazeia, and S. N. Lobo, Thick branes in the scalar-tensor representation of $f(R, T)$ gravity, *Eur. Phys. J. C* **81**, 981 (2021).
- [36] H. Guo, L.-L. Wang, C.-E. Fu, and Q.-Y. Xie, Gravity and Matter on a Pure Geometric Thick Polynomial $f(R)$ Brane, *Phys. Rev. D* **107**, 104017 (2023).
- [37] V. Dzhunushaliev, V. Folomeev, G. Nurtayeva, and S. D. Odintsov, Thick branes in higher-dimensional $f(R)$ gravity, *Int. J. Geom. Methods Mod. Phys.* **17**, 2050036 (2020).
- [38] K. Nozari and N. Sadeghnezhad, Braneworld mimetic $f(R)$ gravity, *Int. J. Geom. Methods Mod. Phys.* **16**, 1950042 (2019).
- [39] Y. Zhong and Y. X. Liu, Pure geometric thick $f(R)$ -branes: Stability and localization of gravity, *Eur. Phys. J. C* **76**, 321 (2016).
- [40] N. Banerjee and T. Paul, Inflationary scenario from higher curvature warped spacetime, *Eur. Phys. J. C* **77**, 672 (2017).
- [41] E. Elizalde, S. D. Odintsov, T. Paul, and D. Sáez-Chillón Gómez, Inflationary universe in $F(R)$ gravity with anti-symmetric tensor fields and their suppression during its evolution, *Phys. Rev. D* **99**, 063506 (2019).
- [42] I. Banerjee, T. Paul, and S. SenGupta, Bouncing cosmology in a curved braneworld, *J. Cosmol. Astropart. Phys.* **02** (2021) 041.
- [43] D. Lovelock, The Einstein tensor and its generalizations, *J. Math. Phys. (N.Y.)* **12**, 498 (1971).
- [44] J. T. Wheeler, Symmetric solutions to the Gauss-Bonnet extended Einstein equations, *Nucl. Phys.* **B268**, 737 (1986).
- [45] D. G. Boulware and S. Deser, String Generated Gravity Models, *Phys. Rev. Lett.* **55**, 2656 (1985).
- [46] B. Zwiebach, Curvature squared terms and string theories, *Phys. Lett.* **156B**, 315 (1985).
- [47] D. J. Gross and J. H. Sloan, The quartic effective action for the heterotic string, *Nucl. Phys.* **B291**, 41 (1987).
- [48] S. D. Odintsov, V. K. Oikonomou, and F. P. Fronimos, Rectifying Einstein-Gauss-Bonnet inflation in view of GW170817, *Nucl. Phys.* **B958**, 115135 (2020).
- [49] V. K. Oikonomou, A refined Einstein-Gauss-Bonnet inflationary theoretical framework, *Classical Quantum Gravity* **38**, 195025 (2021).
- [50] S. D. Odintsov, V. K. Oikonomou, and F. P. Fronimos, Non-minimally coupled Einstein-Gauss-Bonnet inflation phenomenology in view of GW170817, *Ann. Phys. (Amsterdam)* **420**, 168250 (2020).
- [51] V. K. Oikonomou and F. P. Fronimos, A nearly massless graviton in Einstein-Gauss-Bonnet inflation with linear coupling implies constant-roll for the scalar field, *Europhys. Lett.* **131**, 30001 (2020).
- [52] E. Elizalde, S. D. Odintsov, V. K. Oikonomou, and T. Paul, Extended matter bounce scenario in ghost free $f(R, \mathcal{G})$ gravity compatible with GW170817, *Nucl. Phys.* **B954**, 114984 (2020).
- [53] D. Chirkov and S. A. Pavluchenko, Some aspects of the cosmological dynamics in Einstein-Gauss-Bonnet gravity, *Mod. Phys. Lett. A* **36**, 2150092 (2021).
- [54] S. D. Odintsov, V. K. Oikonomou, and F. P. Fronimos, Late-time cosmology of scalar-coupled $f(R, \mathcal{G})$ gravity, *Classical Quantum Gravity* **38**, 075009 (2021).
- [55] M. F. Shamir, Bouncing universe in $f(G, T)$ gravity, *Phys. Dark Universe* **32**, 100794 (2021).
- [56] P. K. Yerra and C. Bhamidipati, Topology of black hole thermodynamics in Gauss-Bonnet gravity, *Phys. Rev. D* **105**, 104053 (2022).
- [57] G.-Q. Li, J.-X. Mo, and Y.-W. Zhuang, Corrections to Hawking radiation and Bekenstein-Hawking entropy of novel four-dimensional black holes in Gauss-Bonnet gravity, *Gen. Relativ. Gravit.* **53**, 97 (2021).
- [58] B. Chen, P.-C. Li, Y. Tian, and C.-Y. Zhang, Holographic turbulence in Einstein-Gauss-Bonnet gravity at large D , *J. High Energy Phys.* **01** (2019) 156.
- [59] C. Hu, X.-X. Zeng, and X.-M. Liu, Phase transition and critical phenomenon of AdS black holes in Einstein-Gauss-Bonnet gravity, *Sci. China Phys. Mech. Astron.* **56**, 1652 (2013).
- [60] Y.-Z. Li, C.-Y. Zhang, and X.-M. Kuang, Entanglement wedge cross-section with Gauss-Bonnet corrections and thermal quench, *Sci. China Phys. Mech. Astron.* **64**, 120413 (2021).
- [61] J. E. Kim and H. M. Lee, Gravity in the Einstein-Gauss-Bonnet theory with the Randall-Sundrum background, *Nucl. Phys.* **B602**, 346 (2001).
- [62] J. E. Kim, B. Kyaee, and H. M. Lee, Various modified solutions of the Randall-Sundrum model with the Gauss-Bonnet interaction, *Nucl. Phys.* **B582**, 296 (2000).
- [63] I. P. Neupane, Completely localized gravity with higher curvature terms, *Classical Quantum Gravity* **19**, 5507 (2002).
- [64] K. A. Meissner and M. Olechowski, Brane localization of gravity in higher derivative theory, *Phys. Rev. D* **65**, 064017 (2002).
- [65] K. Aoyanagi and K. Maeda, Creation of a brane world with Gauss-Bonnet term, *Phys. Rev. D* **70**, 123506 (2004).
- [66] O. Corradini and Z. Kakushadze, Localized gravity and higher curvature terms, *Phys. Lett. B* **494**, 302 (2000).
- [67] I. P. Neupane, Consistency of higher derivative gravity in the brane background, *J. High Energy Phys.* **09** (2000) 040.

- [68] A. A. Andrianov, V. A. Andrianov, and O. O. Novikov, Gravity effects on thick brane formation from scalar field dynamics, *Eur. Phys. J. C* **73**, 2675 (2013).
- [69] G. Germán, A. Herrera-Aguilar, D. Malagón-Morejón, I. Quiros, and R. da Rocha, Study of field fluctuations and their localization in a thick braneworld generated by gravity nonminimally coupled to a scalar field with the Gauss-Bonnet term, *Phys. Rev. D* **89**, 026004 (2014).
- [70] M. Giovannini, Thick branes and Gauss-Bonnet self-interactions, *Phys. Rev. D* **64**, 124004 (2001).
- [71] K. Farakos and P. Pasipoularides, Gauss-Bonnet gravity, brane world models, and non-minimal coupling, *Phys. Rev. D* **75**, 024018 (2007).
- [72] H. A. Alfredo, M. M. Dagoberto, R. M. L. Refugio, and I. Quiros, Thick braneworlds generated by a non-minimally coupled scalar field and a Gauss-Bonnet term: Conditions for localization of gravity, *Classical Quantum Gravity* **29**, 035012 (2012).
- [73] M. Dias, J. M. Hoff da Silva, and R. da Rocha, Thick braneworlds and the Gibbons-Kallos-Linde no-go theorem in the Gauss-Bonnet framework, *Europhys. Lett.* **110**, 20004 (2015).
- [74] C. Charmousis and J. F. Dufaux, General Gauss-Bonnet brane cosmology, *Classical Quantum Gravity* **19**, 4671 (2002).
- [75] R. A. Brown, Brane universes with Gauss-Bonnet-induced-gravity, *Gen. Relativ. Gravit.* **39**, 477 (2007).
- [76] B. Abdesselam and N. Mohammedi, Brane world cosmology with Gauss-Bonnet interaction, *Phys. Rev. D* **65**, 084018 (2002).
- [77] G. L. Alberghi and A. Tronconi, Gauss-Bonnet brane cosmology with radion stabilization, *Phys. Rev. D* **73**, 027702 (2006).
- [78] K. Nozari, F. Kiani, and N. Rashidi, Gauss-Bonnet braneworld cosmology with modified induced gravity on the brane, *Adv. High Energy Phys.* **2013**, 968016 (2013).
- [79] K. Konya, Gauss-Bonnet brane-world cosmology without $Z(2)$ -symmetry, *Classical Quantum Gravity* **24**, 2761 (2007).
- [80] N. Okada and S. Okada, Simple inflationary models in Gauss-Bonnet brane-world cosmology, *Classical Quantum Gravity* **33**, 125034 (2016).
- [81] R. Herrera and N. Videla, Intermediate inflation in Gauss-Bonnet braneworld, *Eur. Phys. J. C* **67**, 499 (2010).
- [82] I. V. Fomin, Cosmological inflation with Einstein-Gauss-Bonnet gravity, *Phys. Part. Nucl.* **49**, 525 (2018).
- [83] S. Nojiri, S. D. Odintsov, and S. Ogushi, Cosmological and black hole brane world universes in higher derivative gravity, *Phys. Rev. D* **65**, 023521 (2002).
- [84] S. Nojiri, S. D. Odintsov, and S. Ogushi, Friedmann-Robertson-Walker brane cosmological equations from the five-dimensional bulk (A)dS black hole, *Int. J. Mod. Phys. A* **17**, 4809 (2002).
- [85] J. E. Lidsey, S. Nojiri, and S. D. Odintsov, Brane world cosmology in (anti)-de Sitter Einstein-Gauss-Bonnet-Maxwell gravity, *J. High Energy Phys.* **06** (2002) 026.
- [86] R. Díaz, F. Gómez, and M. Pinilla, De Sitter brane-world solution in 5-dimensional Einstein-Gauss-Bonnet gravity, *Gen. Relativ. Gravit.* **52**, 86 (2020).
- [87] M. Giovannini, Kink-anti-kink, trapping bags and five-dimensional Gauss-Bonnet gravity, *Phys. Rev. D* **74**, 087505 (2006).
- [88] M. Giovannini, Gravitating multidefects from higher dimensions, *Phys. Rev. D* **75**, 064023 (2007).
- [89] M. Giovannini, Non-topological gravitating defects in five-dimensional anti-de Sitter space, *Classical Quantum Gravity* **23**, L73 (2006).
- [90] Y. Zhong, Y. P. Zhang, W. D. Guo, and Y. X. Liu, Gravitational resonances in mimetic thick branes, *J. High Energy Phys.* **04** (2019) 154.
- [91] Q. Tan, Y. P. Zhang, W. D. Guo, J. Chen, C. C. Zhu, and Y. X. Liu, Evolution of scalar field resonances in a braneworld, *Eur. Phys. J. C* **83**, 84 (2023).
- [92] D. Lovelock, The 4-dimensionality of space and the Einstein tensor, *J. Math. Phys. (N.Y.)* **13**, 874 (1972).
- [93] Y.-X. Liu, J. Yang, Z.-H. Zhao, C.-E. Fu, and Y.-S. Duan, Fermion localization and resonances on a de Sitter thick brane, *Phys. Rev. D* **80**, 065019 (2009).
- [94] D. Bazeia, F. A. Brito, and J. R. S. Nascimento, Supergravity brane worlds and tachyon potentials, *Phys. Rev. D* **68**, 085007 (2003).
- [95] F. Brito, M. Cvetič, and S. Yoon, From a thick to a thin supergravity domain wall, *Phys. Rev. D* **64**, 064021 (2001).
- [96] V. I. Afonso, D. Bazeia, and L. Losano, First-order formalism for bent brane, *Phys. Lett. B* **634**, 526 (2006).
- [97] K. Skenderis and P. K. Townsend, Gravitational stability and renormalization-group flow, *Phys. Lett. B* **468**, 46 (1999).
- [98] D. Z. Freedman, C. Núñez, M. Schnabl, and K. Skenderis, Fake supergravity and domain wall stability, *Phys. Rev. D* **69**, 104027 (2004).
- [99] V. H. Hamity and D. E. Barraco, First order formalism of $f(R)$ gravity, *Gen. Relativ. Gravit.* **25**, 461 (1993).
- [100] B. Julia and S. Silva, On first order formulations of supergravities, *J. High Energy Phys.* **01** (2000) 026.
- [101] I. Low and A. Zee, Naked singularity and Gauss-Bonnet term in brane world scenarios, *Nucl. Phys.* **B585**, 395 (2000).
- [102] R. Koley and S. Kar, Scalar kinks and fermion localisation in warped spacetimes, *Classical Quantum Gravity* **22**, 753 (2005).
- [103] Y.-X. Liu, L.-D. Zhang, L.-J. Zhang, and Y.-S. Duan, Fermions on thick branes in background of Sine-Gordon kinks, *Phys. Rev. D* **78**, 065025 (2008).
- [104] Y. Zhong and Y. X. Liu, Linearization of thick K-branes, *Phys. Rev. D* **88**, 024017 (2013).
- [105] Y. Zhong and Y.-X. Liu, Pure geometric thick $f(R)$ -branes: Stability and localization of gravity, *Eur. Phys. J. C* **76**, 321 (2016).
- [106] M. Giovannini, Gauge invariant fluctuations of scalar branes, *Phys. Rev. D* **64**, 064023 (2001).
- [107] S. Kobayashi, K. Koyama, and J. Soda, Thick brane worlds and their stability, *Phys. Rev. D* **65**, 064014 (2002).
- [108] M. Giovannini, Localization of metric fluctuations on scalar branes, *Phys. Rev. D* **65**, 064008 (2002).