Fully nonlinear gravitational instabilities for expanding Newtonian universes with inhomogeneous pressure and entropy: Beyond the Tolman's solution

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Nonlinear gravitational instability is a crucial way to comprehend the clustering of matter and the formation of nonlinear structures in both the Universe and stellar systems. However, with the exception of a few exact particular solutions for pressureless matter, there are only some approximations and numerical and phenomenological approaches to study the nonlinear gravitational instability instead of mathematically rigorous analysis. We construct a family of particular solutions of the Euler-Poisson system that exhibits the nonlinear gravitational instability of matter with inhomogeneous pressure and entropy (i.e., the cold center and hot rim) in the expanding Newtonian universe. Despite the density perturbations being homogeneous, the pressure is not, resulting in significant nonlinear effects. By making use of our prior work on nonlinear analysis of a class of differential equations, we estimate that the growth rate of the density contrast is approximately $\sim\exp(t^{\frac{2}{3}})$, much faster than the growth rate anticipated by classical linear Jeans instability ($\sim t^{\frac{2}{3}}$). Our main motivation for constructing this family of solutions is to provide a family of reference solutions for conducting a fully nonlinear analysis of inhomogeneous perturbations of density contrast. We will present the general results in a mathematical article separately. Additionally, we emphasize that our model does not feature any shell-crossing singularities before mass accretion singularities since we are specifically interested in analyzing the mathematical mechanics of a pure mass accretion model, which poses limitations on the applicability of our model for understanding the realistic nonlinear structure formation.

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I. INTRODUCTION

Gravitational instability characterizes the mass accretions of self-gravitating systems, clustering of matter and helps us understand the formations of stellar systems and the nonlinear structures in the Universe. It traces back to Jeans [1] for Newtonian gravity in 1902 (thus called "Jeans instability"). However, it is worth noting Jeans' work is only in the linear regime since he linearized the Euler-Poisson system. It was generalized to general relativity by Lifshitz [2] and extended to the expanding universe by Bonnor [3], and later the linearized Jeans instability is widely applied (see Refs. [4,5]). However, the linear Jeans instability has some inconveniences. The first inconvenience comes from the linearization of the Euler-Poisson system. Due to the linearizations, the linear Jeans instability can be only applied to the case with small perturbations of the uniform density distribution [i.e., the density contrast $\varrho := (\rho - \mathring{\rho})/\mathring{\rho} < 1$] and only for a time before the perturbations growing large, since the larger perturbations will lead to larger deviations from the linearized scheme. With the accretions of the mass, the derivations of the linear Jeans instability will be completely spoiled since the increasing density leads to significant derivations from the linear regime. The second inconvenience is the growth rate of the density contrast predicted by the classical linearized version of the Jeans instability cannot yield the observed large inhomogeneities of the universe nowadays and formations of galaxies, because this growth rate ($\sim t^{\frac{2}{3}}$, see Refs. [3–6]) is too slow and thus is much less efficient (see also [4,5]). Therefore, it is urgent to study the fully nonlinear Jeans instability and, as pointed out by Rendall [7] in 2002, there are no results on Jeans instability available for the fully nonlinear case, and it becomes a long-standing open problem. The main goal of this paper is to construct a family of gravitationally unstable solutions with homogeneous density and inhomogeneous pressure and entropy distributions, which serves as a family of reference solutions for the fully nonlinear analysis of solutions with slightly inhomogeneous density. We will present this result separately for slightly inhomogeneous density in a mathematical article [8].

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On the other hand, although it is worthy to understand the fully nonlinear Jeans instability and the mathematical mechanics of a pure mass accretion model, it is important to acknowledge that the model presented in this article is idealized and simplified, lacking shell crossings and other singularities. In reality, things are not as simple as this. Usually, the evolution of density perturbations is believed to occur primarily in the linear regime because initial perturbations are small and require considerable time to grow. Then once the perturbation becomes of order unity, comparing with our model, the most cases are that the nonlinear approach may be of interest for only a short period due to shell-crossing causing violent relaxation and breaking down fluid approximations. In such cases, N-body simulations have been applied in modern cosmology.

In fact, the model presented in this article can approximate some local portion of a giant gas cloud. Its density is almost homogeneous in this portion and the center of it is very cold but the rim of the cloud is extremely hot. In addition, all the fluxes by the thermodynamics forces are negligibly small. We further idealize this model by assuming the cloud initially has homogeneous density; the initial temperature distribution of this cloud is spherical symmetric and proportional to the square of the radius of the position (the accurate descriptions in §II). The key result of this article is that, by using the mathematical tools developed in our previous paper [9] and taking the full nonlinear effects into account, the growth rates of the density contrast are at least of order $\sim\exp(t^{\frac{2}{3}})$ [see later (3.8) for detailed expressions] or even blow up at the finite time [see Eq. (3.9)]. This is much faster than that given by the classical linearized Jeans instability ($\sim t^{\frac{2}{3}}$, see Ref. [5], §6.3). It may contribute to the explanations of the observed large inhomogeneities of the Universe nowadays and the formations of galaxies. As Peebles [[10], Chap. 1, §4.B] pointed out, the exponential growth rates of the density contrast "often has been cited as what is wanted." Although he also claim the exponential growth rates is not possible for the Einstein–de Sitter model, in our current model, the exponential growth does happen.

Some nonlinear strategies involving approximations and numerical methods (e.g., the famous Zel'dovich solutions) have been discussed in several references (see Refs. [4,5,11,12]). Another famous exact solution describing the evolutions and collapses of the inhomogeneity is the Tolman solution (see Ref. [5], §6.4.1 and [13]) which gives an exact spherical symmetric dust solution, but it cannot be generalized to include the nonvanishing pressure effects, and the inconvenience of the parametric form of the solution is too complex to visualize the actual behaviors of it.

In addition, the most important thing, compared with the Tolman solution, is that our method is robust. It can allow the presence of the pressure and can be generalized to more general cases by studying the corresponding dominant equations [e.g., the ordinary differential equation (ODE) (B1)] of the reference solutions first, then near every reference solutions, we can perturb the density contrast to obtain more general solution with inhomogeneous density as we will present in [8]. This paper is an example stating this idea and we will present this idea with other unstable models in future. Additionally, we emphasize that our model does not feature any shell-crossing singularities (see §VI for the proof) before mass accretion singularities since we are specifically interested in analyzing the mathematical mechanics of a pure mass accretion model. Therefore, it is important to recognize its limitations on the applicability of our model for understanding realistic nonlinear structure formation in the Universe, since most of the outcomes of nonlinear evolutions, as demonstrated by N-body simulations in modern cosmology, is the breakdown of the fluid approximation due to shell crossing, which goes beyond the scope of the calculations presented in this paper.

II. MODELS AND ASSUMPTIONS

We use the Newtonian universe as an approximation of a local universe. Under the following assumptions, although the universe has the homogeneous density, it is inhomogeneous for the distributions of the pressure and entropy. It is of course an ideal model due to these perfect assumptions, but we want to develop a method for fully nonlinear gravitational instability with effective pressure resisting the gravity. We will prove, in either case, the growth rate [at least $\sim\exp(t^{\frac{2}{3}})$, see Eq. (3.8)] of the density contrast due to the nonlinear Jeans instability is way faster than the one predicted by the classical linear Jeans instability ($\sim t^{\frac{2}{3}}$).

For the nonlinear version of the Jeans instability, we can not use Fourier analysis to solve the nonlinear differential equations derived from the Euler-Poisson system (see Ref. [6] for alternative non-Fourier based proof of the linear Jeans instability). In [9], we developed some preparing techniques for a class of the nonlinear ODEs and hyperbolic equations for the nonlinear Jeans instability. We intend to apply it to conclude the result of this article.

Let us now give the assumptions (and remarks for detailed meanings) of the model. According to the non-equilibrium thermodynamics (see, for example, [[14], Eqs. II.(5), II.(19), and III.(19)] or [15], §69), we assume the following:

(1) The fluids filled in the Newtonian universe are the ideal fluids and there is no chemical reactions, thus all the viscosity coefficients and chemical affinities of reactions vanish; All the phenomenological coefficients L_{ij} (see [14], Chap. IV for the definitions) are relatively small and negligible during the considered process, i.e., we assume all the $L_{ij}=0$, then all the fluxes in the entropy production vanish simultaneously with the thermodynamic forces (due to the facts that the entropy production satisfies

$\sigma = \sum_k J_k \cdot X_k$ and the linear phenomenological law gives $J_i = \sum_k L_{ik} X_k$). Therefore, every comoving parcel is adiabatic (isentropic).

(2) The initial entropy is distributed proportionally to $|\boldsymbol{x}|^2$, while the adiabatic comoving parcel ensures that the entropy remains proportional to the comoving position $|\boldsymbol{q}|^2$.

In other words, these assumptions means the Newtonian universe can be described by the following reduced Euler-Poisson system (we use the Einstein summation convention),

$$\partial_t \rho + \partial_i(\rho v^i) = 0, \tag{2.1}$$

$$\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0, \tag{2.2}$$

$$\partial_t S + v^i \partial_i S = 0, \tag{2.3}$$

$$\Delta\phi = \delta^{ij}\partial_i\partial_j\phi = 4\pi G\rho, \tag{2.4}$$

where ρ, v^i, p, ϕ, and S are the density, velocities, pressure of the fluids, gravitational potential and specific entropy, respectively. The equation of state is presumed by

$$p = K e^{\frac{S-S_0}{c_V}} \rho^\gamma + \mathfrak{p}, \quad \text{for } \gamma = \frac{4}{3} \quad \text{and} \quad K \geq 0. \tag{2.5}$$

where $\mathfrak{p} \in \mathbb{R}$ is a constant. The initial data at $t = t_0$ are given by

$$\mathring{\rho}(t_0) = \frac{\iota^3}{6\pi G t_0^2}, \qquad \mathring{v}^i(t_0, x^k) = \frac{2}{3t_0} x^i, \tag{2.6}$$

$$\mathring{\phi}(t_0, x^k) = \frac{2}{3}\pi G \mathring{\rho}(t_0)\delta_{ij}x^i x^j \quad \text{and}$$
$$S(t_0, x^k) = S_0 + c_V \ln(\kappa t_0^{-\frac{4}{3}} \delta_{kl} x^k x^l)^{\mathrm{sgn}(1-\iota^3)}, \tag{2.7}$$

where $\kappa > 0$ is a constant and sgn is a sign function,[1] and ι is a constant determined by

$$\iota := \iota(\tilde{K}) = \left(\frac{1}{2}\sqrt{1+18\tilde{K}} + \frac{1}{2}\right)^{\frac{1}{3}} - \left(\frac{1}{2}\sqrt{1+18\tilde{K}} - \frac{1}{2}\right)^{\frac{1}{3}} \in (0, 1] \quad \text{and} \quad \tilde{K} := \frac{K^3\kappa^3}{\pi G},$$

Before proceeding, about this model, let remark some notable facts:

(1) Note $\tilde{K}$ and ι are dimensionless constants depending on the molar mass of the fluids and the distributions of the entropy or temperature (see Appendix A for details).

(2) If $\tilde{K} = 0$ (equivalently, $\iota = 1$), then this model reduces to an isentropic case $S(t_0, x^k) \equiv S_0$ with a constant pressure $\mathfrak{p}$, thus the data and the solution (2.16) and (2.17) given below reduce to the classical Newtonian solutions for the homogeneous and isotropic Newtonian universe given in, e.g., [5], §1.2.3 or [4], §10.2. The results of this article reduce to the case of Tolman solutions.

(3) If $\iota \neq 1$, then the data of the entropy (2.7) implies the initial distribution of the temperature $\mathcal{T} \propto |\boldsymbol{x}|^2$ (see Appendix A for detailed explanations).

(4) In the equation of state (2.5), the inclusion of the term $\mathfrak{p}$ does not alter the mathematical derivations; rather, it is included to increase the generality of the model.

(5) Note that ι satisfies an important identity (crucial in later derivations),

$$\iota^3 + 9\left(\frac{\tilde{K}}{6}\right)^{\frac{1}{3}}\iota - 1 = 0, \tag{2.8}$$

and $\iota(\tilde{K})$ is a decreasing function,[2] $\lim_{\tilde{K}\to 0}\iota(\tilde{K}) = 1$, and $\lim_{\tilde{K}\to+\infty}\iota(\tilde{K}) = 0$.

To simplify calculations, letting $s = (S - S_0)/c_V$ and along with the nondimensionalizations in Appendix A, we proceed with the dimensionless and normalized Euler-Poisson system:

$$\partial_t \rho + \partial_i(\rho v^i) = 0, \tag{2.9}$$

$$\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0, \tag{2.10}$$

$$\partial_t s + v^i \partial_i s = 0, \tag{2.11}$$

$$\Delta\phi = \delta^{ij}\partial_i\partial_j\phi = 4\pi\rho. \tag{2.12}$$

The equation of state becomes

$$p = K e^s \rho^{\frac{4}{3}} + \mathfrak{p}, \quad \text{for } K \geq 0. \tag{2.13}$$

The initial data at $t = 1$ is given by

[1]The sign function implies $\mathrm{sgn}(\alpha) = 1$ if $\alpha > 0$, and $\mathrm{sgn}(\alpha) = 0$ if $\alpha = 0$.

[2]Since one can verify that its derivative

$$\iota'(\tilde{K}) = \frac{3}{\sqrt[3]{2}\sqrt{18\tilde{K}+1}}\left(\frac{1}{\left(\sqrt{18\tilde{K}+1}+1\right)^{2/3}} - \frac{1}{\left(\sqrt{18\tilde{K}+1}-1\right)^{2/3}}\right) < 0.$$

$$\rho|_{t=1}=\frac{t^3}{6\pi},\quad v^i|_{t=1}=\frac{2}{3}x^i,\ \text{and}\ s|_{t=1}=\ln(\delta_{kl}x^kx^l)^{\mathrm{sgn}(1-t^3)}. \tag{2.14}$$

Let us try to find a homogeneous and expanding Newtonian solution in the meaning of a homogeneous density and the Hubble law dominated velocity field, we then obtain

$$\rho(t,x^k)=\mathring{\rho}(t),\qquad v^i(t,x^k)=\mathring{v}^i(t,x^k)=H(t)x^i. \tag{2.15}$$

There is an exact solution to the Euler-Poisson system (2.9)–(2.12) and the data (2.14) on $(t,x^k)\in[t_0,\infty)\times\mathbb{R}^3$,

$$\mathring{\rho}(t)=\frac{t^3}{6\pi t^2},\qquad \mathring{p}(t)=Kt^{-\frac{4}{3}}\delta_{kl}x^kx^l\mathring{\rho}^{\frac{4}{3}}+\mathfrak{p},$$
$$\mathring{v}^i(t,x^k)=\frac{2}{3t}x^i, \tag{2.16}$$

$$\mathring{\phi}(t,x^k)=\frac{2}{3}\pi\mathring{\rho}\delta_{ij}x^ix^j=\frac{t^3}{9t^2}\delta_{ij}x^ix^j,\quad\text{and}$$
$$\mathring{s}(t,x^k)=\ln(t^{-\frac{4}{3}}\delta_{kl}x^kx^l)^{\mathrm{sgn}(1-t^3)}, \tag{2.17}$$

III. MAIN RESULTS AND IDEAS

This article intends to conclude the nonlinear behavior of the homogeneous perturbations of the density contrast $\varrho:=(\rho-\mathring{\rho})/\mathring{\rho}$ by the following two steps and the main results are given by the following estimates (3.8) and (3.9) of the lower bounds of the growth rate of the density contrast.

Step 1: Let us assume that β and γ are two given positive constants and that the initial data (at $t=1$) of (2.9)–(2.12) have an homogeneous initial perturbations and are characterized by two positive parameters β and γ in the following ways:

$$\rho|_{t=1}=(1+\beta)\frac{t^3}{6\pi},\qquad v^i|_{t=1}=\left(\frac{2}{3}-\gamma\right)x^i\quad\text{and}$$
$$s|_{t=1}=\ln((1+\beta)^{\frac{2}{3}}\delta_{kl}x^kx^l)^{\mathrm{sgn}(1-t^3)}. \tag{3.1}$$

Then we will prove the solution of the Euler-Poisson system (2.9)–(2.12) becomes [we use notation $(\cdot)':=d(\cdot)/dt$]

$$\rho(t)=(1+f(t))\mathring{\rho}(t)=\frac{t^3(1+f(t))}{6\pi t^2}, \tag{3.2}$$

$$v^i(t,x^i)=\frac{2}{3t}x^i-\frac{f'(t)}{3(1+f(t))}x^i, \tag{3.3}$$

$$\phi(t,x^i)=\frac{2}{3}\pi\mathring{\rho}(1+f(t))|\boldsymbol{x}|^2=\frac{t^3(1+f(t))|\boldsymbol{x}|^2}{9t^2}, \tag{3.4}$$

$$s(t,x^k)=\ln(t^{-\frac{4}{3}}(1+f)^{\frac{2}{3}}\delta_{kl}x^kx^l)^{\mathrm{sgn}(1-t^3)}. \tag{3.5}$$

and the density contrast $\varrho(t)=f(t)$, where $|\boldsymbol{x}|^2:=\delta_{ij}x^ix^j$ and $f(t)$ is a solution of the following nonlinear ODE,

$$f''(t)+\frac{4}{3t}f'(t)-\frac{2}{3t^2}f(t)(1+f(t))-\frac{4(f'(t))^2}{3(1+f(t))}=0, \tag{3.6}$$

$$f|_{t=t_0}=\beta\quad\text{and}\quad f'|_{t=t_0}=3(1+\beta)\gamma. \tag{3.7}$$

Moreover, the pressure becomes $p(t)=\frac{Kt^4}{(6\pi)^{\frac{4}{3}}t^4}(1+f)^2\delta_{kl}x^kx^l$.

Step 2: In Step 1, we have represented the perturbation solution in terms of functions $f(t)$ and its derivative $f_0(t):=f'(t)$. To understand the behaviors of the perturbation solution, especially the growth rates of the density contrast ϱ, we have to know the detailed behaviors of the functions f and f_0. In fact, the behaviors of f and f_0 can be acquired by solving the ODE (3.6) and (3.7) which has been well studied in our companion article [9]. We list the conclusions of the solutions to the ODE (3.6) and (3.7) in Appendix B and using it, we conclude the density contrast has the lower bound estimate, for $t\in(1,t_m)$,

$$\varrho(t)=f(t)>\exp\left(\frac{3(\ln(1+\beta)+3\gamma)t^{\frac{2}{3}}+2(\ln(1+\beta)-\frac{9}{2}\gamma)t^{-1}}{5}\right)-1. \tag{3.8}$$

In addition, by Theorem B.1, if further the initial data satisfies $\gamma>1/3$, we have an improved lower bound estimate on the growth rate of ϱ, for $t\in(1,t_m)$,

$$\varrho(t)=f(t)>\frac{1+\beta}{(1-3\gamma+3\gamma t^{-\frac{1}{3}})^3}-1. \tag{3.9}$$

The lower bound of ϱ blows up at $t^\star=(1-\frac{1}{3\gamma})^{-3}>t_0=1$. These lower bounds give an estimate of the growth rates of the density contrast ϱ.

In the rest of this article, we will only need to elaborate Step 1, i.e., solving the Euler-Poisson system (2.9)–(2.12) under the perturbed data (3.1) and further the perturbation equations.

IV. EQUATIONS OF PERTURBATIONS

Let us first decompose the variables (ρ,v^i,p,ϕ) to the exact background solution (2.16) and (2.17) and the perturbed parts, and define a density contrast ϱ,

$$\rho = \mathring{\rho} + \tilde{\rho}, \qquad v^i = \mathring{v}^i + \tilde{v}^i, \qquad \phi = \mathring{\phi} + \tilde{\phi}, \qquad s = \mathring{s} + \tilde{s}, \tag{4.1}$$

$$p = \mathring{p} + \tilde{p}, \quad \text{and} \quad \varrho := \frac{\tilde{\rho}}{\mathring{\rho}}. \tag{4.2}$$

Next we introduce the Lagrangian coordinates q^k defined by $x^k = a(t)q^k$ where[3] $a(1) := 1$, and the time derivatives are obtained at q^k (i.e., the material derivatives). We also denote

$$D_t := \partial_t|_{q^k} = \partial_t|_{x^k} + \mathring{v}^i \partial_i = \partial_t|_{x^k} + Hx^j \partial_j, \tag{4.3}$$

$$D_i := a(t)\partial_i. \tag{4.4}$$

Note in the next, we will slightly abuse the notations and do not distinguish the variables in terms of the Eulerian x^i and Lagrangian coordinate q^i, that is, we abuse, e.g., $\tilde{v}^k(t, x^i)$ and $\tilde{v}^k(t, q^i)$ for the simplicity of the expressions and readers should be clear according to the contexts.

Let us review how to reexpress the Euler-Poisson system (2.9)–(2.12) in terms of the perturbation variables $(\varrho, \tilde{v}^i, \tilde{s}, \tilde{\phi})$ given by (4.1) and (4.2). First, let us consider the conservation of mass. Substituting the decomposition (4.1) and (4.2) into Eq. (2.9), using the Hubble laws (2.15) and applying the Lagrangian coordinates (4.3)–(4.4), the conservation of mass (2.9) becomes

$$D_t\tilde{\rho} + 3H\tilde{\rho} + \mathring{\rho}a^{-1}D_i\tilde{v}^i + \tilde{\rho}a^{-1}D_i\tilde{v}^i + \tilde{v}^i a^{-1}D_i\tilde{\rho} = 0.$$

Using (4.2) and $\partial_t\mathring{\rho} + 3H\mathring{\rho} = 0$, and after straightforward calculations, we obtain

$$D_t\varrho + (1+\varrho)a^{-1}D_i\tilde{v}^i + \tilde{v}^i a^{-1}D_i\varrho = 0. \tag{4.5}$$

Second, by (4.2) and (4.4), the Poisson equation in terms of the Lagrangian coordinate q^i becomes

$$\delta^{ij}D_iD_j\tilde{\phi} = 4\pi a^2\mathring{\rho}\varrho = \frac{2\iota^3}{3t^{\frac{2}{3}}}\varrho. \tag{4.6}$$

In the end, we turn to the balance of momentum (2.10). By subtracting the background (2.16) and (2.17) from (2.10), and in terms of Lagrangian coordinates, using (4.3) and (4.4), with the help of (4.2), the balance of momentum (2.10) becomes

$$D_t\tilde{v}^i + H\tilde{v}^i + \tilde{v}^j a^{-1}D_j\tilde{v}^i + \frac{a^{-1}D^i(\tilde{p}/\mathring{\rho})}{1+\varrho} - \frac{2K\iota}{(6\pi)^{\frac{1}{3}}t^{\frac{4}{3}}}\frac{\varrho}{1+\varrho}q^i + a^{-1}D^i\tilde{\phi} = 0. \tag{4.7}$$

Then we consider the conservation of the entropy. In terms of Lagarangian coordinates, direct calculations that imply (2.11) become

$$D_t\tilde{s} + \mathrm{sgn}(1-\iota^3)\cdot\frac{2\delta_{ij}q^i\tilde{v}^j}{a\delta_{kl}q^kq^l} + \tilde{v}^i a^{-1}D_i\tilde{s} = 0. \tag{4.8}$$

Gathering the above equations (4.5)–(4.8) together, the Euler-Poisson system (2.9)–(2.12), in terms of the Lagrangian coordinates, becomes

$$D_t\varrho + \frac{(1+\varrho)}{a}D_i\tilde{v}^i + \frac{\tilde{v}^i}{a}D_i\varrho = 0, \tag{4.9}$$

$$D_t\tilde{v}^i + H\tilde{v}^i + \tilde{v}^j a^{-1}D_j\tilde{v}^i + \frac{a^{-1}D^i(\tilde{p}/\mathring{\rho})}{1+\varrho} - \frac{2K\iota}{(6\pi)^{\frac{1}{3}}t^{\frac{4}{3}}}\frac{\varrho}{1+\varrho}q^i + a^{-1}D^i\tilde{\phi} = 0, \tag{4.10}$$

$$D_t\tilde{s} + \mathrm{sgn}(1-\iota^3)\cdot\frac{2\delta_{ij}q^i\tilde{v}^j}{a\delta_{kl}q^kq^l} + \tilde{v}^i a^{-1}D_i\tilde{s} = 0, \tag{4.11}$$

$$\delta^{ij}D_iD_j\tilde{\phi} = \frac{2\iota^3}{3t^{\frac{2}{3}}}\varrho. \tag{4.12}$$

V. PERTURBATION SOLUTIONS

In this section, let us focus on solving Eqs. (4.9)–(4.12) under the initial perturbation (3.1). In this assumption, we have that the density contrast is independent of the spatial variables and the velocity satisfies the Hubble's law, thus we assume the forms of ϱ and $\tilde{v}^i$ by

$$\varrho(t, q^k) \equiv \varrho(t) \quad \text{and} \quad \tilde{v}^i(t, q^k) = \tilde{H}(t)q^i, \tag{5.1}$$

where the function $\tilde{H}(t)$ is to be determined variable. In this case, by (4.12), direct calculations imply

$$\tilde{\phi} = \frac{2}{3}\pi a^2\mathring{\rho}\varrho|\boldsymbol{q}|^2 = \frac{\iota^3\varrho|\boldsymbol{q}|^2}{9t^{\frac{2}{3}}} \quad \text{and}$$
$$\delta^{ij}D_i\tilde{\phi} = \frac{4}{3}\pi a^2\mathring{\rho}\varrho q^j = \frac{2\iota^3\varrho}{9t^{\frac{2}{3}}}q^j, \tag{5.2}$$

where $|\boldsymbol{q}|^2 := \delta_{ij}q^iq^j$.

Now our task becomes solving $\varrho(t)$ and $\tilde{H}(t)$ from Eqs. (4.9) and (4.10). Taking the divergence of $\tilde{v}^i$ leads to $D_i\tilde{v}^i = 3\tilde{H}(t)$, noting (5.1) yields $D_i\varrho = 0$, (4.9) implies

[3]In fact, $a(t) = a(1)t^{\frac{2}{3}} = t^{\frac{2}{3}}$ provided $a(1) = 1$, since by the Hubble law (2.15) and the Lagrangian coordinates $x^k = a(t)q^k$, we obtain $H(t) := \frac{\dot{a}(t)}{a(t)}$. Then by $H = \frac{2}{3t}$ [see Eqs. (2.15) and (2.16)], we can solve $a(t)$.

$$\tilde{H}(t) = -\frac{t^{\frac{2}{3}}}{3}(\ln(1+\varrho(t)))'$$

$$\Rightarrow \quad \tilde{v}^i = -\frac{t^{\frac{2}{3}}}{3}q^i(\ln(1+\varrho(t)))' = -\frac{t^{\frac{2}{3}}\varrho'(t)}{3(1+\varrho(t))}q^i. \quad (5.3)$$

By the data (2.14) and (3.1), we have the data $\tilde{s}|_{t=1} = \ln(1+\beta)^{\frac{2}{3}\mathrm{sgn}(1-\imath^3)}$, which is a homogeneous perturbation of the entropy, thus we assume $\tilde{s}$ is also independent of x^i and is homogeneous, then $D_i\tilde{s} = 0$. Using the velocity (5.3), direct integrating (4.11) imply

$$\tilde{s}(t) = \ln[(1+\varrho(t))^{\frac{2}{3}\mathrm{sgn}(1-\imath^3)}].$$

Next, we solve $\varrho(t)$. Noting, if $\imath \neq 1$, then $\tilde{p} = K\delta_{kl}q^kq^l\mathring{\rho}^{\frac{4}{3}}[(1+\varrho)^2 - 1] = K\delta_{kl}q^kq^l\mathring{\rho}^{\frac{4}{3}}\varrho(\varrho+2)$, and substituting (5.2)–(5.3) into (4.10), straightforward calculations imply

$$\tilde{H}' + \frac{2}{3t}\tilde{H} + \tilde{H}^2a^{-1} + \frac{2K\imath\varrho}{(6\pi)^{\frac{1}{3}}t^{\frac{4}{3}}} + \frac{2\imath^3\varrho}{9t^{\frac{4}{3}}} = 0.$$

Then the crucial step is using the identity (2.8), i.e., $\imath^3 + \frac{9K}{(6\pi)^{\frac{1}{3}}}\imath = 1$ in terms of the dimensionless K, which leads to

$$\tilde{H}' + \frac{2}{3t}\tilde{H} + \tilde{H}^2a^{-1} + \frac{2\varrho}{9t^{\frac{4}{3}}} = 0. \quad (5.4)$$

If $\imath = 1$, then $K = 0$ and $p = 0$. We also arrive at (5.4).

Noting that (5.3) yields

$$\tilde{H}' = -\frac{2}{9}\frac{t^{-\frac{1}{3}}\varrho'}{1+\varrho} - \frac{t^{\frac{2}{3}}}{3}\frac{\varrho''}{1+\varrho} + \frac{t^{\frac{2}{3}}}{3}\frac{(\varrho')^2}{(1+\varrho)^2},$$

and substituting it and (5.3) into (5.4), we arrive at[4]

$$\varrho''(t) + \frac{4}{3t}\varrho'(t) - \frac{2}{3t^2}\varrho(t)(1+\varrho(t)) - \frac{4}{3}\frac{(\varrho'(t))^2}{(1+\varrho(t))} = 0. \quad (5.5)$$

By the data (3.1) and the definition of the perturbed variables (4.1) and (4.2), we obtain the data of $\varrho|_{t=1}$,

$$\varrho|_{t=1} = \beta. \quad (5.6)$$

Since (5.5) is a second order ODE, to solve this equation, we have to know the initial data of $\varrho'|_{t=1}$. In order to find this data and solve the Euler-Poisson system (4.9)–(4.12), we note that (4.9) must hold at the initial time $t = 1$. Thus, with the help of (3.1) and the data $\tilde{v}^i|_{t=1} = (v^i - \mathring{v}^i)|_{t=1} = -\gamma q^i$ [by (2.14) and (3.1), and noting $a(1) = 1$], noting $D_i\varrho = 0$, we have

$$\varrho'|_{t=1} = -\left(\frac{1+\varrho}{a}D_i\tilde{v}^i\right)\Big|_{t=1} - \left(\frac{\tilde{v}^i}{a}D_i\varrho\right)\Big|_{t=1} = 3(1+\beta)\gamma. \quad (5.7)$$

Using the ODE (5.5) and the data (5.6) and (5.7), with the help of Theorem B.1 (see Appendix B), we obtain $\varrho(t) = f(t)$, where $f(t)$ is given by Theorem B.1. Further, we conclude the solutions (3.2)–(3.5). This completes Step 1. In summary, we obtain a family of solutions depending on two parameters β and γ.

VI. CONCLUSIONS AND DISCUSSIONS

Let us firstly compare the nonlinear growth rates (3.8)–(3.9) of the density contrast with that ($\sim t^{\frac{2}{3}}$) predicted by the classical linear version of the Jeans instability. By using $t^{-1} \leq t^{\frac{2}{3}}$ for $\ln(1+\beta) - \frac{9}{2}\gamma < 0$ and $t^{-1} > 0$ for $\ln(1+\beta) - \frac{9}{2}\gamma \geq 0$, (3.8) implies

$$\varrho(t) > \exp(At^{\frac{2}{3}}) - 1 = At^{\frac{2}{3}} + \mathrm{O}(t^{\frac{4}{3}}), \quad (6.1)$$

where $A \coloneqq \min\{\ln(1+\beta), \frac{3}{5}(\ln(1+\beta) + 3\gamma)\}$ is a constant and $\mathrm{O}(t^{\frac{4}{3}})$ means the remainder terms are at least of order $t^{\frac{4}{3}}$. We note the growth rate ($\sim t^{\frac{2}{3}}$) by the classical Jeans instability is just the first order approximation of the lower bound estimate (6.1) if expanding $\exp(At^{\frac{2}{3}})$ with respect to $t^{\frac{2}{3}}$. The nonlinear effects indeed significantly boost the growth of the density contrast as ϱ grows larger. According to the Taylor expansion (6.1), we see that, when t is small enough, it consists of the result of the classical linearized Jeans instability. From these improved faster growth rates due to the nonlinear effects, we can see that indeed the classical linearized Jeans instability can only be applied to the small initial perturbation of the density and only work for a short time before the density grows large enough. However, the nonlinear method proposed by this article does not require the initial perturbations small and it works for a long time before the Euler-Poisson system breaks down.

The method of this article relies on the nonlinear analysis of a type of nonlinear differential equations, which is mathematically rigorous without approximations and numerical calculations. This method is robust and systematic, and if we also use the ideas and method (the Cauchy problem of the Fuchsian formulation of a second order hyperbolic equation which allows certain pressure) from our previous paper [9], §3, it is possible to study the general cases of the nonlinear Jeans instability, at least for the case with nonvanishing pressure and small inhomogeneous perturbations.

[4]Note the same equation has also been obtained in [16] for spherical collapse model.

Although this model of the Universe in the present paper is a simplified model, it can still capture some of the main nonlinear effects on the local Newtonian universe. Thus this result helps us have a better understanding of the formation of the nonlinear structures in the universe and stellar systems. The lower bound estimates (3.8) and (3.9) of the growth rates of the density contrast are far more efficient than the one predicted by the classical linear Jeans instability. Therefore, it may be possible that these nonlinear results of the growth rates can contribute to the explanations of the observed large inhomogeneities of the Universe nowadays and the formations of galaxies. Due to these much larger and more efficient growth rates of the density contrast, we may not require substantial initial perturbations of the density contrast at the early times of the Universe and weaken the constraints on the initial spectrum of perturbations.

In the end, we would like to note that there are no shell-crossing singularities[5] present in our model. This is evident when observing the velocity field (3.3) in spherical coordinates,

$$v(t,r) = \left(\frac{2}{3t} - \frac{f'(t)}{3(1+f(t))}\right) r,$$

where $r \coloneqq |\boldsymbol{x}|$. This velocity field implies

$$\partial_r v = \frac{2}{3t} - \frac{f'(t)}{3(1+f(t))}.$$

Using relations in the companion article [[8], Eqs. (3.18) and (3.89)], i.e.,

$$\frac{(f')^2}{(1+f)^2} = \frac{\chi f}{Bt^2} \quad \text{and} \quad \frac{\chi(t)}{B} = 4 + \frac{\mathfrak{G}(t)}{B}$$
$$\Rightarrow \frac{f'}{1+f} = \frac{2}{t}\sqrt{f}\left(1 + \frac{\mathfrak{G}}{4B}\right)^{\frac{1}{2}},$$

we conclude

$$\lim_{t\to t_m} \partial_r v = \lim_{t\to t_m} \frac{2}{3t}\left[1 - \sqrt{f}\left(1 + \frac{\mathfrak{G}}{4B}\right)^{\frac{1}{2}}\right] = -\infty \quad \text{and}$$
$$|\partial_r v| < \infty \quad \text{for } t \in [1, t_m),$$

since $\lim \sqrt{f}/t = \infty$ by Theorem B.1.(3) and $\lim_{t\to t_m} \mathfrak{G}(t) = 0$ (by [[8] Proposition B.4]). This result implies the characteristic curves generated by the velocity field do not intersect for $t \in [1, t_m)$ until $t = t_m$ (the mass accretion singularities lie on the whole $\{t_m\} \times \mathbb{R}^3$) and it means there is no shell crossing singularities for $t \in [1, t_m)$.

In summary, this model gives mathematically rigorous and physically decent nonlinear estimates on the growth rates [at least $\sim \exp(t^{\frac{2}{3}})$] of the density contrast ϱ on the local portion of the Universe characterized by the Newtonian expanding universe. The mathematical tools and methods have the potential for general cases of the nonlinear version of the Jeans instability. Based on these tools and methods developed by our prior article [9], the fully nonlinear Jeans instabilities both for the Newtonian universe and general relativity are in progress.

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APPENDIX A: NONDIMENSIONALIZATIONS AND PARAMETERS

First note the dimensions of variables in this article are

$$[x^i] = L, \qquad [t] = T(\text{Time}), \qquad [\mathcal{T}] = \mathbb{T}(\text{Temperature}), \qquad [s] = 1, \qquad [p] = \frac{M}{LT^2}, \qquad [\rho] = \frac{M}{L^3},$$
$$[G] = \frac{L^3}{MT^2}, \qquad [\phi] = \frac{L^2}{T^2}, \qquad [v^i] = \frac{L}{T}, \qquad [K_T] = \frac{L^3}{T^2 M^{\frac{1}{3}}}.$$

To introduce dimensionless variables, we define

$$\rho = \rho_T \hat{\rho}, \qquad K = K_T \hat{K}, \quad \text{and} \quad \mathcal{T} = \mathcal{T}_T \hat{\mathcal{T}}$$

where ρ_T, K_T, and $\mathcal{T}_T$ are the typical values for the density, the coefficient of the equation of state and the temperature. Then let

[5] See, for instance, [17,18] for related cases.

$$\hat{p} = \frac{p}{K_T \rho_T^{\frac{4}{3}}}, \qquad \hat{x}^i = \sqrt{\frac{G\rho_T}{K_T \rho_T^{\frac{1}{3}}}} x^i = \rho_T^{\frac{1}{3}} \sqrt{\frac{G}{K_T}} x^i, \qquad \hat{t} = \sqrt{G\rho_T} t, \qquad \hat{\phi} = \frac{1}{K_T \rho_T^{\frac{1}{3}}} \phi,$$

$$v^i = \sqrt{K_T \rho_T^{\frac{1}{3}}} \hat{v}^i \quad \text{and} \quad \kappa = \frac{G^{\frac{1}{3}}}{K_T} \hat{\kappa}.$$

Then $[\hat{K}] = [\hat{p}] = [\hat{v}^i] = [\hat{\rho}] = [\hat{x}^i] = [\hat{t}] = [\hat{\mathcal{T}}] = [\hat{\kappa}] = 1$. Thus all our dynamical variables and coordinates are dimensionless and the three constants ρ_T, K_T, and $\mathcal{T}_T$ can be used to fix the length, time, and temperature scales by using units so that

$$\rho_T = \frac{1}{t_0^2 G}, \qquad K_T = \frac{G^{\frac{1}{3}}}{\kappa}, \quad \text{and} \quad \mathcal{T}_T = 1$$

where t_0 is the initial time. In this case, we have $\hat{t} = t/t_0$ and $\hat{\kappa} = 1$.

We claim $\tilde{K}$ is dimensionless quantity and further so is ι, since

$$\tilde{K} = \frac{K^3 \kappa^3}{\pi G} = \frac{K_T^3 \hat{K}^3 G \hat{\kappa}^3}{K_T^3 \pi G} = \frac{\hat{K}^3 \hat{\kappa}^3}{\pi}.$$

Recalling [[19], Chap. II] if we assume the specific heat c_V (at constant volume) is a constant, for polytropic changes, we have $\hat{\mathcal{T}} = \Theta_{\gamma'} \hat{\rho}^{\gamma'-1}$ and $p = R\mathcal{T}\rho$ (note $[R] = \frac{L^2}{T^2 \mathbb{T}}$) where $\Theta_{\gamma'}$ is the polytropic temperature and $\gamma' = \frac{c_p - c}{c_V - c}$ is the polytropic exponent and $R = \bar{R}/\mathcal{M}$ is the specific gas constant and $\bar{R} = 8.314\ \mathrm{J/(mol \cdot K)}$ is the universal gas constant and $\mathcal{M}$ is the molar mass. We then arrive at $p = R\mathcal{T}_T \Theta_{\gamma'} \rho_T^{1-\gamma'} \rho^{\gamma'}$. In view of the expression of the entropy,

$$S = S_0 + c_V[\ln \Theta_{\gamma'} + (\gamma' - \gamma) \ln \hat{\rho}] \quad \Rightarrow \quad \Theta_{\gamma'} = e^{\frac{S-S_0}{c_V}} \hat{\rho}^{\gamma-\gamma'}, \tag{A1}$$

where $\gamma = \frac{c_p}{c_V} = 1 + \frac{R}{c_V}$. We reexpress $p = K e^{\frac{S-S_0}{c_V}} \rho^\gamma$, where $K = \bar{R}\mathcal{T}_T \rho_T^{1-\gamma}/\mathcal{M}$. If $\gamma = 4/3$, the nondimensionalized equation of state becomes $\hat{p} = \hat{K} e^s \hat{\rho}^{\frac{4}{3}}$, where denoting $s = (S - S_0)/c_V$,
$\hat{K} = \bar{R}\mathcal{T}_T/(\mathcal{M} K_T \rho_T^{\frac{1}{3}})$. By (A1), we have

$$e^s = \hat{\mathcal{T}} \hat{\rho}^{-\frac{1}{3}}$$

Thus, the initial condition of entropy (2.14) implies the initial distribution of the temperature $\mathcal{T} \propto |\boldsymbol{x}|^2$.

To simplify notation, we will drop the "hat" from the hatted variables throughout main body of this article.

APPENDIX B: MATHEMATICAL PREPARATION OF A NONLINEAR ODE

The main mathematical tool is a type of ODEs developed in our previous article [9], §2. For readers' convenience, we quote the results without proofs in this appendix and readers can find detailed proof in [9], §2. We consider the solutions $f(t)$ to the following type ODE,

$$f''(t) + \frac{a}{t} f'(t) - \frac{b}{t^2} f(t)(1 + f(t)) - \frac{c(f'(t))^2}{1 + f(t)} = 0, \tag{B1}$$

$$f(t_0) = \beta > 0 \quad \text{and} \quad f'(t_0) = \beta_0 > 0, \tag{B2}$$

where $\beta, \beta_0 > 0$ are positive constants and

$$a > 1, \qquad b > 0, \quad \text{and} \quad 1 < c < 3/2. \tag{B3}$$

From now on, to simplify the notations, we denote

$$\triangle := \sqrt{(1-a)^2 + 4b} > -\bar{a}, \qquad \bar{a} = 1 - a < 0,$$
$$\bar{c} = 1 - c < 0$$

and introduce constants $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$, $\mathtt{D}$, and $\mathtt{E}$ depending on the initial data β and β_0 to (B1)–(B2) and parameters a, b, and c,

$$\mathtt{A} := \frac{t_0^{-\frac{\bar{a}-\triangle}{2}}}{\triangle} \left(\frac{t_0 \beta_0}{(1+\beta)^2} - \frac{\bar{a} + \triangle}{2} \frac{\beta}{1+\beta} \right),$$

$$\mathtt{B} := \frac{t_0^{-\frac{\bar{a}+\triangle}{2}}}{\triangle} \left(\frac{\bar{a} - \triangle}{2} \frac{\beta}{1+\beta} - \frac{t_0 \beta_0}{(1+\beta)^2} \right) < 0,$$

$$\mathtt{C} := \frac{2}{2 + \bar{a} + \triangle} \left(\ln(1+\beta) + \frac{\bar{a} + \triangle}{2b} \frac{t_0 \beta_0}{1+\beta} \right) t_0^{-\frac{\bar{a}+\triangle}{2}} > 0,$$

$$\mathtt{D} := \frac{\bar{a} + \triangle}{2 + \bar{a} + \triangle} \left(\ln(1+\beta) - \frac{1}{b} \frac{t_0 \beta_0}{1+\beta} \right) t_0,$$

$$\mathtt{E} := \frac{\bar{c} \beta_0 t_0^{1-\bar{a}}}{\bar{a}(1+\beta)} > 0.$$

We define the following two critical times $t_\star$ and $t^\star$.

(1) Let $\mathcal{R} := \{t_r > t_0 | \mathtt{A} t_r^{\frac{\bar{a}-\triangle}{2}} + \mathtt{B} t_r^{\frac{\bar{a}+\triangle}{2}} + 1 = 0\}$ and define $t_\star := \min \mathcal{R}$.
(2) If $t_0^{\bar{a}} > \mathtt{E}^{-1}$, we define $t^\star := (t_0^{\bar{a}} - \mathtt{E}^{-1})^{1/\bar{a}} \in (0, \infty)$, i.e., $t = t^\star$ solves $1 - \mathtt{E} t_0^{\bar{a}} + \mathtt{E} t^{\bar{a}} = 0$.

We are now in a position to state the main theorem on ODE (B1) and (B2) and the proof can be found in [9], §2.

Theorem B.1. Suppose constants $\mathscr{a}$, $\mathscr{b}$, and $\mathscr{c}$ are defined by (B3), $t_\star$ and $t^\star$ are defined above and the initial data $\beta, \beta_0 > 0$, then

(1) $t_\star \in [0, \infty)$ exists and $t_\star > t_0$.

(2) There is a constant $t_m \in [t_\star, \infty]$, such that there is a unique solution $f \in C^2([t_0, t_m))$ to the equation (B1) and (B2), and

$$\lim_{t\to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t\to t_m} f_0(t) = +\infty.$$

(3) f satisfies upper and lower bound estimates,

$$1 + f(t) > \exp(\mathtt{C}t^{\frac{\bar{\mathscr{a}}+\triangle}{2}} + \mathtt{D}t^{-1}) \quad \text{for } t \in (t_0, t_m),$$
$$1 + f(t) < (\mathtt{A}t^{\frac{\bar{\mathscr{a}}-\triangle}{2}} + \mathtt{B}t^{\frac{\bar{\mathscr{a}}+\triangle}{2}} + 1)^{-1} \quad \text{for } t \in (t_0, t_\star).$$

Furthermore, if the initial data satisfies $\beta_0 > \bar{\mathscr{a}}(1+\beta)/(\bar{\mathscr{c}}t_0)$, then

(4) $t_\star$ and $t^\star$ exist and finite, and $t_0 < t_\star < t^\star < \infty$.

(5) There is a finite time $t_m \in [t_\star, t^\star)$, such that there is a solution $f \in C^2([t_0, t_m))$ to Eq. (B1) with the initial data (B2), and

$$\lim_{t\to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t\to t_m} f_0(t) = +\infty.$$

(6) The solution f has improved lower bound estimates, for $t \in (t_0, t_m)$,

$$(1+\beta)(1 - \mathtt{E}t_0^{\bar{\mathscr{a}}} + \mathtt{E}t^{\bar{\mathscr{a}}})^{1/\bar{\mathscr{c}}} < 1 + f(t).$$

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