# Cosmological evolution of Witten superconducting string networks

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We consider the evolution of current-carrying cosmic string networks described by the charge-velocity-dependent one-scale (CVOS) model beyond the linear equation-of-state regime, specifically focusing on the Witten superconducting model. We find that, generically, for almost chiral currents, the network evolution reduces dynamically to that of the linear case, which has been discussed in our previous work. However, the Witten model introduces a maximum critical current, which constrains the network scaling behavior during the radiation era when currents can grow and approach this limit. Unlike the linear model, only if the energy density in the critical current is comparable to the bare string tension will there be substantial backreaction on the network evolution, thus changing the observational predictions of superconducting strings from those expected from a Nambu-Goto network. During the matter era, if there are no external sources, then dynamical effects dilute these network currents, and they disappear at late times.

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## I. INTRODUCTION

In Ref. [1], Tom Kibble proposed that one-dimensional topological defects, dubbed cosmic strings, should form in many extensions of the standard model of particle physics. This has been confirmed explicitly in many phenomenological studies (see, for example, Refs. [2–4]). A detailed examination of most of these scenarios indicates that the strings should also be endowed with superconducting currents, as proposed originally by Witten [5], which have many interesting consequences (e.g., Refs. [6–13]). Beyond the original bosonic superconducting model, a variety of different mechanisms have also been proposed to generate string currents [5,14–16], making superconductivity even more ubiquitous.

The presence of a superconducting current flowing along the strings necessarily influences the corresponding cosmic string network evolution. Recently, we developed a charge-velocity-dependent one-scale (CVOS) model [17], which extends previous work [18–22] and offers an analytical approach to describing the most relevant statistical features of the network evolution. This approach includes phenomenological parameters that could, in the non-current-carrying case, be directly measured or statistically inferred from numerical simulations, thereby calibrating the models. In principle, such calibrated models can be used to evaluate the stochastic background of gravitational waves produced by a current-carrying network, much in the same way as is done for bare string networks [23,24], as well as other observational signatures. However, before one can make reliable predictions, there is a further bottleneck: network simulations for superconducting strings are not yet available with which to reliably estimate the CVOS model parameters, so in this case we need to survey a wider parameter range of possible physical consequences.

The CVOS analytic model requires a specific equation of state appropriate for each particular field theory model for current-carrying cosmic strings. Having reviewed first the general equation of state [17] for cosmic string network evolution, and then specialized to the simplest linear case [25], we now focus, in the present paper, on the original Witten model [5]—or, more precisely, the neutral Witten model, containing no long-range electromagnetic-like interactions [7,26].

This paper is organized as follows: In Sec. II, the CVOS model equations are reviewed, with a particular emphasis put on the various charge and current loss mechanisms, either in the process of loop formation or due to local curvature; we close the system by assuming that the framework is that of a radiation- or matter-dominated Friedman-Lemaître-Robertson-Walker (FLRW) background. The Witten model equation of state is implemented in Sec. III. It entails the

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existence of a critical current, leading to a leakage parameter dependence in the total charge.

The core of the article is Sec. IV, in which we study specific network dynamics during cosmological evolution. We demonstrate that for small currents, the evolution can be described by the previous linear model. However, for regimes with growing currents in the radiation era, we note the important role of the critical current in determining scaling solutions. We end with some conclusions and a discussion of the outcomes, their susceptibility to our underlying assumptions, and potential observational implications.

## II. GENERALIZED CVOS MODEL

In what follows, we make use of the recently developed CVOS model to describe the most relevant statistical properties of a network of superconducting cosmic strings, within the thin-string approximation; we refer the reader to Ref. [17], in which the relevant calculations are detailed.

### A. Relevant thermodynamical variables

Upon averaging over all the long strings, the current-carrying string network ends up being characterized by four macroscopic variables: namely, the root-mean-square (rms) velocity $v$ of the strings, the energy density $\rho$, the charge $Y$ and the 4-current amplitude $K$ (sometimes also called chirality, as it measures the spacelike or timelike character of the integrated current). These variables are originally defined in terms of the root mean squares of the timelike and spacelike currents—respectively, $Q^2 = \langle q^2 \rangle$ and $J^2 = \langle j^2 \rangle$—namely,

$$Y = \frac{1}{2}(Q^2 + J^2) \quad \text{and} \quad K = Q^2 - J^2, \tag{1}$$

so that $Q^2 = Y + K/2$ and $J^2 = Y - K/2$. Note that $Y$ being positive definite, Eq. (1) implies that the constraint

$$|K| \leq 2Y \tag{2}$$

should be satisfied at all times.

The Lorentz-invariant microscopic chirality $\kappa = q^2 - j^2$ is what enters the surface Lagrangian $f(\kappa)$ from which the equations of motion are derived. Averaging this quantity suggests its replacement with a macroscopic version, $F(K) = \langle f(\kappa) \rangle$, which, for simplicity, we assume to take the same form, as we discuss below.

The energy density $\rho$ can be split into two contributions: namely, that coming from the bare (without current) strings $\rho_0$, and that coming from the current itself. The Brownian assumption allows us to rewrite this energy density through two conformal characteristic lengths $L_C$ and $\xi_C$, leading to

$$\rho = \frac{\mu_0}{L_C^2 a^2} \quad \text{and} \quad \rho_0 = \frac{\mu_0}{\xi_C^2 a^2}, \tag{3}$$

where $\mu_0$ is the bare string tension, $L_C$ and $\xi_C$ correspond to the total (current-carrying) and bare string networks, respectively, and $a$ is the scale factor of the FLRW cosmological solution

$$ds^2 = a^2(\tau)(d\tau - d\mathbf{x}^2), \tag{4}$$

with $\tau$ being the conformal time. The relation between the characteristic lengths is found to be

$$\xi_C = \sqrt{F - 2Q^2 F'} L_C = W L_C, \tag{5}$$

thereby defining $W$ (with $F' \equiv dF/dK$). For later reference, we note that

$$\frac{\dot{W}}{W} = -\frac{1}{W^2}\left[\dot{Y}F' + \left(Y + \frac{K}{2}\right)\dot{K}F''\right], \tag{6}$$

with the overdot henceforth denoting conformal time derivatives.

### B. Charge and current leakage

There are two mechanisms by which the superconducting string network can lose energy: loop production and charge leakage.

One can phenomenologically describe charge and current losses through leakage by demanding that the larger the charge or current, the larger the loss (we generalize here the description proposed in Ref. [25]). This translates into

$$\left.\frac{dQ^2}{d\tau}\right|_{\text{leak}} = -A\frac{Q^2}{\xi_C} \quad \text{and} \quad \left.\frac{dJ^2}{d\tau}\right|_{\text{leak}} = -B\frac{J^2}{\xi_C}, \tag{7}$$

where $A$ and $B$ are called the charge and current leakage efficiencies, respectively, and the subscript "leak" indicates that we restrict attention to the part specifically due to leakage. Setting $A_\pm = A \pm B$, Eq. (7) implies the time evolution contribution for the variables $Y$ and $K$:

$$\left.\frac{dY}{d\tau}\right|_{\text{leak}} = -\frac{1}{2\xi_C}\left(A_+ Y + A_- \frac{K}{2}\right),$$
$$\left.\frac{dK}{d\tau}\right|_{\text{leak}} = -\frac{1}{\xi_C}\left(A_- Y + A_+ \frac{K}{2}\right). \tag{8}$$

The behaviors of the phenomenological parameters $A_\pm$ are not known precisely, so we cannot *a priori* set them to constants, though we have made this linear assumption previously [25]. We shall see below for the Witten model that, because the physically relevant equation of state entails a critical current, leakage should increase as the current approaches criticality. In any case, the specific form of the charge leakage efficiencies has to be inferred from field-theoretic studies of microscopic string behavior and future high-resolution numerical simulations of evolving random networks.

Since the non-current-carrying contribution of the network energy $\rho_0$ is, of course, insensitive to these losses, i.e.,

$$\left.\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau}\right|_{\mathrm{leak}} = 0, \tag{9}$$

it follows that Eq. (9), together with definition (3), implies that $\dot{\xi}_{\mathrm{c,leak}} = 0$, which, upon using Eq. (5), yields

$$\left.\frac{\dot{L}_{\mathrm{C}}}{L_{\mathrm{C}}}\right|_{\mathrm{leak}} = -\left.\frac{\dot{W}}{W}\right|_{\mathrm{leak}}. \tag{10}$$

Since $F$ depends only on the chirality, one has $\dot{F} = \dot{K}F'$, and one gets the restriction of Eq. (6) due to leakage by substituting Eq. (8) into Eq. (6), which provides the correction to the equation of motion of the characteristic length $L_{\mathrm{C}}$.

### C. Loop chopping

The energy loss due to the production of loops from long strings takes the form [27]

$$\left.\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau}\right|_{\mathrm{loops}} = -\frac{cv\rho_0}{\xi_{\mathrm{C}}}, \quad \left.\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right|_{\mathrm{loops}} = -g(Q,J)\frac{cv\rho}{\xi_{\mathrm{C}}}, \tag{11}$$

where the subscript "loops" means we restrict attention, in the calculation of the time derivative, to the contribution due to loop production on the long string densities.

In Eq. (11), $c$ is the loop chopping efficiency, which will be set to its Nambu-Goto value $c_{\mathrm{NG}} \sim 0.23$ in the forthcoming numerical calculation, and $g(Q,J)$ represents the modification of this bare chopping efficiency to account for the effects of the charge. Lacking numerical simulations for current-carrying strings, one can first assume the loop production not to be significantly modified by the inclusion of current effects; this assumption will have to be tested when current-carrying string network simulations become available.

Using the Brownian network properties [Eq. (3)] with the loop production equations (11) implies

$$\left.\frac{\dot{\xi}_{\mathrm{C}}}{\xi_{\mathrm{C}}}\right|_{\mathrm{loops}} = \frac{cv}{2\xi_{\mathrm{C}}} \quad \text{and} \quad \left.\frac{\dot{L}_{\mathrm{C}}}{L_{\mathrm{C}}}\right|_{\mathrm{loops}} = g\frac{cv}{2\xi_{\mathrm{C}}}. \tag{12}$$

Since Eq. (5) yields

$$\frac{\dot{\xi}_{\mathrm{C}}}{\xi_{\mathrm{C}}} = \frac{\dot{L}_{\mathrm{C}}}{L_{\mathrm{C}}} + \frac{\dot{W}}{W},$$

we obtain the relation

$$\left.\frac{\dot{W}}{W}\right|_{\mathrm{loops}} = -\frac{cv}{2\xi_{\mathrm{C}}}(g-1), \tag{13}$$

which means that if $g \neq 1$, the loop production affects the $Q^2$ and $J^2$ parameters. Here, we again assume that the loop production function $g$ has a linear dependence on $Q^2$ and $J^2$, which implies that

$$\left.\frac{\mathrm{d}Q^2}{\mathrm{d}\tau}\right|_{\mathrm{loops}} = -g_Q\frac{cv}{\xi_{\mathrm{C}}}Q^2,$$
$$\left.\frac{\mathrm{d}J^2}{\mathrm{d}\tau}\right|_{\mathrm{loops}} = -g_J\frac{cv}{\xi_{\mathrm{C}}}J^2, \tag{14}$$

where $g_Q$ and $g_J$ are some constants that tell us how much of the timelike and spacelike components of currents are lost due to loop production. One can demonstrate that these constants are related to the function $g$ in Eq. (12) by the following expressions:

$$g = 1 - g_Q\frac{F' + 2Q^2F''}{F - 2Q^2F'}Q^2 - g_J\frac{F' - 2Q^2F''}{F - 2Q^2F'}J^2. \tag{15}$$

We note that this improved parametrization of the function $g$ offers some further clarity over that used in previous approaches [17,20,22,25]. Due to limited knowledge about the form of $g$ in the equation for $L_{\mathrm{C}}$, we assume that it should be linearly proportional to the timelike and spacelike components in the corresponding current loss functions.

Collecting timelike and spacelike components, one obtains expressions for the charge $Y$ and 4-current amplitude $K$ in the following forms:

$$\dot{Y}|_{\mathrm{loops}} = -\frac{cv}{2\xi_{\mathrm{C}}}\left(g_+Y + g_-\frac{K}{2}\right),$$
$$\dot{K}|_{\mathrm{loops}} = -\frac{cv}{\xi_{\mathrm{C}}}\left(g_-Y + g_+\frac{K}{2}\right), \tag{16}$$

where $g_\pm = g_Q \pm g_J$.

### D. Equations of motion

The relevant equations of motion for the quantities of cosmological interest—namely, the characteristic length of the network $\xi_{\mathrm{C}}$, the root mean square velocity $v$, together with the charge $Y$ and chirality $K$—can now be written after collecting all the modifications which must be added to Eq. (43) of Ref. [17] and discussed above. First, we set the characteristic length $L_{\mathrm{C}}$ to behave as $L_{\mathrm{C}} = \zeta(\tau)\tau$, so that a scaling solution will correspond to a constant value of the fraction $\zeta$. Taking into account the mechanisms described above, one finds

$$\dot{\zeta}\tau = \frac{n\zeta}{W^2}[v^2(F - F'K) - 2YF'] + \frac{cvg}{2W} - \frac{F'}{2W^3}\left(A_+ Y + A_- \frac{K}{2}\right) - \frac{F''}{W^3}\left(Y + \frac{K}{2}\right)\left(A_- Y + A_+ \frac{K}{2}\right) - \zeta, \quad (17)$$

where $g$ is given by Eq. (15). It turns out, however, that, using the relation (5) to express this current-carrying network correlation length in terms of the bare one $\xi_C$, and substituting into Eq. (17) all the results obtained in the previous section, including the time development of the charge $Y$ and chirality $K$, one finds that the bare correlation length provides a tremendous simplification to the equations describing the system evolution. Indeed, setting $\xi_C = \epsilon\tau$, this system becomes

$$\dot{\epsilon}\tau = \frac{1}{W^2}[n\epsilon v^2(F - F'K) - 2vkYF'] + \frac{1}{2}cv - \epsilon, \quad (18a)$$

$$\dot{v}\tau = \frac{1 - v^2}{W^2}\left\{\frac{k}{\epsilon}\left[F + 2\left(Y - \frac{K}{2}\right)F'\right] - 2vn(F - F'K)\right\}, \quad (18b)$$

$$\dot{Y}\tau = \left(\frac{vk}{\epsilon} - n\right)\frac{2YF' + (4Y^2 - K^2)F''}{F' + (2Y + K)F''} - \frac{cv}{2\epsilon}\left(g_+ Y + g_- \frac{K}{2}\right) - \frac{2A_+ Y + A_- K}{4\epsilon}, \quad (18c)$$

$$\dot{K}\tau = 2\left(\frac{vk}{\epsilon} - n\right)\frac{F'K}{F' + (2Y + K)F''} - \frac{cv}{\epsilon}\left(g_- Y + g_+ \frac{K}{2}\right) - \frac{2A_- Y + A_+ K}{2\epsilon}. \quad (18d)$$

Equation (18a) is indeed much simpler than Eq. (17), as announced. We have checked that one could solve either the entire system of Eq. (18), substituting Eq. (5) into the solution, or only the last three—i.e., Eqs. (18b), (18c), and (18d)—in terms of $L_C$ together with Eq. (17). Both solutions are numerically identical, as they should be.

A scaling solution is then achieved whenever the functions of time $\zeta(\tau)$, $v(\tau)$, $Y(\tau)$, and $K(\tau)$ simultaneously evolve towards constant solutions $\zeta_{SC}$, $v_{SC}$, $Y_{SC}$, and $K_{SC}$: the characteristic length is then a constant fraction of the Hubble scale.

In Eq. (18), the so-called momentum parameter $k$ is assumed to be a function of the rms velocity $v$ only, and is defined through [19]

$$k \equiv k(v) = \frac{2\sqrt{2}}{\pi}\frac{1 - 8v^6}{1 + 8v^6}(1 - v^2)\left(1 + 2\sqrt{2}v^3\right), \quad (19)$$

which is the same as was found to describe well the Nambu-Goto network simulations. In other words, we assume that the explicit form of this function, which comes mostly from curvature effects along the string worldsheets, is not affected by the presence of a nonzero charge or chirality. Note that there is, however, an implicit dependence, because their presence does impact the rms velocity.

## III. WITTEN EQUATION OF STATE

To specify a particular type of superconducting cosmic string, we need to define its equation of state. The Witten model, originally proposed in Ref. [5], or rather its neutral version [26] (to ensure long-range electromagnetic-like effects to be negligible [7]), was found to be accurately characterized by the following averaged equation of state [28]:

$$F_{\rm mag}(K) = 1 - \frac{1}{2}\frac{K}{1 - \alpha K} \qquad \text{for } K \leq 0, \quad (20a)$$

$$F_{\rm elec}(K) = 1 + \frac{\ln(1 - 2\alpha K)}{4\alpha} \qquad \text{for } K \geq 0, \quad (20b)$$

where the model-dependent dimensionless parameter $\alpha$ is given by $\alpha = (m_H/m_\sigma)^2$, with $m_\sigma$ being the vacuum mass of the current-generating condensate and $m_H$ that of the string-forming Higgs field. Generically, $\alpha$ is expected to be much larger than unity, and some degree of fine-tuning would be required to obtain $\mathcal{O}(1)$. In any case, it must lie in the range $1 < \alpha < \infty$, to which we will restrict our attention below. Note that the coefficient in Eq. (20b) differs in an irrelevant way from that in Ref. [17]: this is to ensure the small-$K$ behavior to be identical for both functions up to second order.

Shown in Fig. 1 is a representation of the equation of state proposed in Eq. (20), which also implies the following restriction on the 4-current amplitude value [26]:

$$-\frac{1}{3\alpha} < K < \frac{1 - e^{-4\alpha}}{2\alpha}. \quad (21)$$

Here, the first constraint stems from the requirement that the longitudinal velocity of perturbations propagating along the worldsheet be positive in the magnetic case with $K \leq 0$—i.e., using $F_{\rm mag}$ [Eq. (20a)]. The second constraint merely expresses $F_{\rm elec} \geq 0$ [Eq. (20b)]. These limits lead to $K \to 0$ as $\alpha \to \infty$, meaning that if the current-carrier mass is vanishingly small compared to that of the Higgs field, the contribution of the current becomes negligible and one recovers a Nambu-Goto string. Note also that for small chiralities, Eq. (20) can be expanded as $F \sim 1 - \frac{1}{2}K - \frac{1}{2}\alpha K^2$ in both cases.

It is important to emphasize that the equation of state (20) is applicable only provided the current remains in the range given by Eq. (21). If, for some reason (due to initial conditions or dynamical evolution), the chirality came to exceed this range, one should include possible

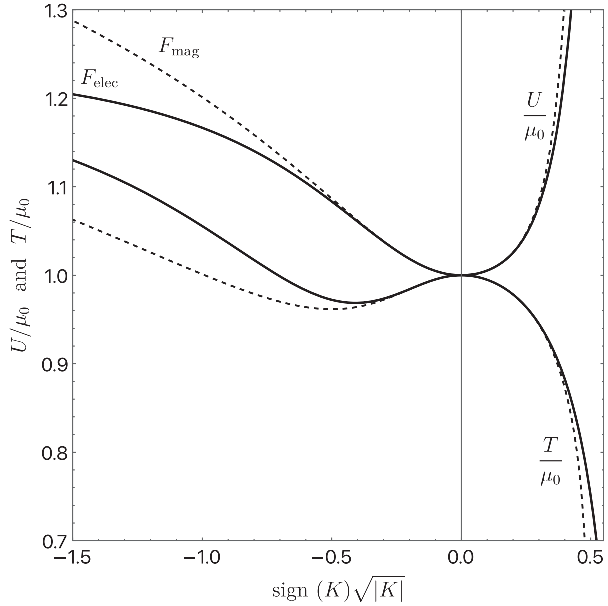


FIG. 1. Equation of state for the Witten model using Eq. (20) with $\alpha = 2$. The constraints on the state parameter $K$ are clearly seen: on the magnetic side with $K \leq 0$, below the critical value $-1/(3\alpha)$, the energy per unit length $U$ and the tension $T$ are both decreasing, implying the longitudinal perturbation velocity $c_{\rm L}^2 = -{\rm d}T/{\rm d}U < 0$, while on the electric regime with $K \geq 0$, the ever-decreasing tension eventually reaches a point for which $T < 0$, and hence a transverse perturbation velocity $c_{\rm T}^2 = T/U < 0$: both cases lead to instabilities and are thus dynamically excluded.

electromagnetic corrections [29] that might change the equation of state or, most probably, lead to additional charge-loss mechanisms. (We briefly discuss one such example in Sec. IV B.) However, in this work we restrict our attention to macroscopic variables, and we can anticipate that, on average, the equation of state (20) holds. In that case, and for the relevant $\alpha \gg 1$, the string network on average should tend towards chiral conditions—i.e, $K_{\rm sc} = 0$. The largest deviation from the chiral case happens when $\alpha \simeq 1$, implying $-0.33 \lesssim K_{\rm sc} \lesssim 0.86$. This $\alpha \sim 1$ regime is actually the most interesting case, because it is where the backreaction on the underlying string is most significant, to the extent that it may even lead to stable vorton solutions.

The equation of state (20), which implies the existence of a maximum chirality $K_{\rm cr} \sim \mathcal{O}(1/\alpha)$ from Eq. (21), will also have a dramatic effect on the evolution of the string currents. In principle, a critical current in $K$ may not seem to impose a limit on the total current $Y$ because of the Lorentz invariance along a straight string, which allows $Y$ to be boosted up to arbitrarily large values. However, in the realistic context of an expanding background, we work in a special cosmological frame in which the overall Brownian string network is at rest (like the CMB). The actual rms current $Y$ consists of random contributions of correlation length, which, for the present Witten model, consists of both positive- and negative-chirality currents limited in magnitude by the critical current $|K| \lesssim K_{\rm cr}$. For this reason, we can expect the averaged total current to obey approximately the same limit $Y \lesssim K_{\rm cr}$, and in fact, the cumulative stochastic current will be considerably smaller and depend on the ratio of the correlation length of the current to that of the string and other factors. This limit would probably have to be determined quantitatively using numerical simulations, but we can be confident that the average current will not much exceed the critical current $K_{\rm cr}$.

In order to implement this limit phenomenologically, we can consider what would happen when $Y > Y_{\rm cr}$, which we assume is of order $Y_{\rm cr} \approx 2K_{\rm cr}$ because of the constraint in Eq. (2), representing an average current in which some random microscopic regions must have chiral currents $K$ that exceed the critical value. In that case, we anticipate an additional enhancement of the charge leakage, along the lines of the discussion in Ref. [29], and in agreement with the results of instabilities observed in numerical simulations in both the electric (charge loss) and magnetic (current unwinding) regimes (see, e.g., Refs. [30,31]). One may use the electric regime bound in Eq. (21), in our numerical calculations below, to assume $Y_{\rm cr} = 2/(3\alpha)$ for the sake of definiteness.

Whenever $Y \gtrsim Y_{\rm cr}$, one expects a rapid escape of particles and energy from a localized string region due to the current unwinding or charge emission. We can model this enhancement of current or charge loss by a modification of the charge leakage parameter $A_\pm$ in Eq. (8) to become $Y$ dependent as follows:

$$A_\pm(Y) = \frac{A_{\rm const}}{1 - {\rm e}^{-(Y - Y_{\rm cr})^2}}. \tag{22}$$

The nonlinear leakage coefficient $A_{\rm const} \geq 0$ will ensure that the average current $Y$ is constrained by the critical current cutoff [Eq. (21)] at all times. As we will see, this eliminates some of the large-current scaling regimes found previously for the linear model, unless the critical current itself is sufficiently large, $\alpha \sim 1$.

To estimate the charge leakage dependence on the charge amplitude $A(Y)$, one can utilize the formalism introduced in Ref. [32] and applied to certain chiral cosmic string loop variations in Refs. [33,34]. Nevertheless, a thorough investigation into the charge leakage function of infinite strings of the CVOS model requires additional research in this area, which we leave for future study.

## IV. NETWORK EVOLUTION

Having defined the thermodynamical equations of motion and the relevant equation of state, we now are in a position to analyze the various cases for the macroscopic variables $L_{\rm C}$, $v$, $Y$, and $K$. We first consider in what follows, Sec. IV A, the situation for which the charge leakage is negligible, before taking it into account in a phenomenological way in Sec. IV B.

### A. Neglecting charge leakage

We begin by investigating the case when the phenomenological parameter describing charge leakage mechanisms is absent—i.e., we set $A_\pm = 0$. A realistic current-carrying string network should lose some energy into radiation, but the actual amount is unknown given the lack of relevant simulation information, so we have to consider the possibility of negligible losses.

When the equation of state is of the linear kind—in practice, when $F \to 1 - \frac{1}{2}K$, which is the small-current (chiral) limit—it has been shown in Ref. [25] that this leads to a "frozen" network with $Y \to 1$, while the rms velocity $v$ and the correlation length ratio $\zeta$ decrease as a power law with time during the radiation era. As we will see below, this behavior does not persist in the full Witten model unless there is a large critical current. In the matter-dominated era, the network behaves in a Nambu-Goto way. Figure 2 shows a typical numerical solution of the set of Eqs. (18) with $A_\pm = 0$, and using the Witten equation of state [Eq. (20)] showing the aforementioned time developments.

When the initial conditions for the network, set deep into the radiation era, are close to chirality—i.e., setting $K_{\rm ini} \ll 1$, as is expected from the phase transition and the Kibble current-forming mechanism—we find that there is no visible difference between this model and the linear case, and this is true for any value of $\alpha > 1$. If the initial value of the chirality $K_{\rm ini}$ is not negligible and positive (or, respectively, negative), the saturating value of the charge is $Y_{\rm max} < 1$ (respectively, $Y_{\rm max} > 1$); it is exemplified in Fig. 2.

In Ref. [25], it was found that the charge saturation regime was leading to $Y_{\rm max} \to 1$, a condition entirely depending on the linearity of the equation of state. Deviations of the charge saturation value $Y$ from unity for the nonlinear equation of state discussed in the present work are due to the explicit appearance of $K$ in the relation [Eq. (5)] between bare string energy and energy of the current [Eq. (3)]. This effect increases with $\alpha$, so in the limit of large mass difference, the current $K \to 0$, and the saturation current $Y_{\rm max} \sim 1$. After the radiation-to-matter transition, all solutions tend toward the currentless Nambu-Goto case.

For negligible charge losses, we conclude that the network behavior is not substantially modified by using the Witten equation of state instead of the linear one. Nevertheless, there is one caveat to be pointed out. For large $\alpha$, even if the saturation current is small, with $Y \ll 1$, one could still be in a regime with $\alpha Y > 1$, which would seem to exceed expectations on the existence of a critical current for the string. Clearly, having $\alpha \gg 1$ and $Y \sim 1$ seems unphysical (i.e., inconsistent with the realistic equation of state we are studying). We revisit this point in Sec. IV B.

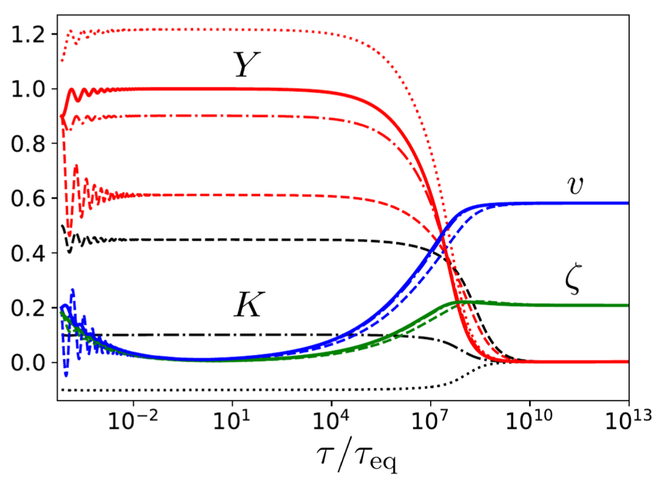


FIG. 2. Time evolution for the no-leakage case $A = 0$ of the velocity $v$, charge $Y$, chirality $K$, and characteristic length ratio $\zeta$ as functions of conformal time, beginning deep in the radiation and ending in the matter era. We chose the loop-chopping parameter value to coincide with the Nambu-Goto case—i.e., $c = 0.23$—and we maximize the effect of the equation of state by setting $\alpha = 1$. Shown here are the cases for initial conditions set with initial values $K_{\rm ini} = 0$ (full lines), $K_{\rm ini} = 0.5$ (dashed lines), $K_{\rm ini} = 0.1$ (dash-dotted lines), and $K_{\rm ini} = -0.1$ (dotted lines). For $K_{\rm ini} = 0$, and more generally for initial conditions close to the chiral case, one recovers the linear regime frozen solution with the saturation point (constant value of $Y$) depending on the scaling value of $K$ in a manner independent of the value of $\alpha > 1$.

### B. Charge leakage with critical current

Having discussed the modification due to the Witten equation of state on the frozen network, we introduce the leakage and its charge dependence [Eq. (22)]. We first consider the chopping efficiency to be independent of the charge and current, so that $g \to 1$, i.e., assuming negligible biases $g_Q \ll 1$ and $g_J \ll 1$, and then including their contribution.

#### *1. The unbiased case*

Let us turn our attention to the impact of the critical current [Eq. (21)] by considering the nonlinear charge leakage term [Eq. (22)]. We start by noting that even for large critical currents—i.e., for $\alpha \sim 1$—the chiral case $K \approx 0$ qualitatively yields the same large-current scaling behavior which was observed previously for the linear model—i.e., where backreaction from the current has a significant influence on network evolution [25]. This is illustrated in Fig. 3.

For critical currents $K_{\rm cr}$ well below the string energy density (with $\alpha \gg 1$), we can expect that there exist initial conditions during the radiation era for which it should be possible for the strings to acquire currents approaching the critical value. Their behavior would, however, closely mimic that of relativistic Nambu-Goto strings, since the

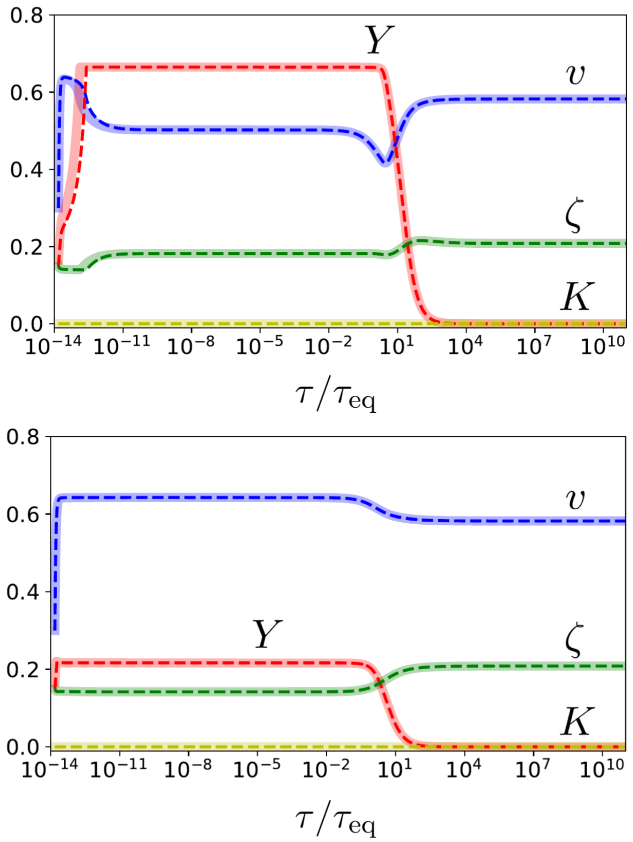


FIG. 3. Time evolution of the velocity $v$, charge $Y$, chirality $K$, and $\zeta$ through the radiation and matter epochs, using parameter values $c = 0.23$, $g_\pm = 0$, $A_- = 0$, and $A_+$ given by Eqs. (22) or (23) with $A_{\mathrm{const}} = 10^{-6}$. Evolving using either the Witten [Eq. (20)] (dashed lines) or the linear equation of state (solid line) for $\alpha = 1$ (upper panel) and $\alpha = 3$ (lower panel) leads to indistinguishable evolutions in this case.

relevant differences should be of order $\alpha^{-1}$. In all the cases, as in the linear equation-of-state case, these currents will quickly be diluted in the matter-dominated era and become negligible. This is again shown in Fig. 3.

Figure 3 makes clear that the Witten model [Eq. (20)] essentially coincides with the much simpler linear model for small chiralities whenever $g_\pm = 0$ and $A_- = 0$. This is by no means a trivial result, as it well known that such an approximation is not adequate when it comes to describing microscopic stability issues [26,28]: despite the apparent differences between the linear regime and the Witten model at small chiralities, they both produce the same thermodynamical behavior. For this reason, the key new ingredient from the Witten model—which is the critical current—can be equally introduced in the linear model to the same effect. This still leaves open the question about the detailed nonlinear implementation of charge leakage near the critical current $Y \sim K_{\mathrm{cr}}$, which we have only approximately treated in Eq. (22). However, we point out that our results are not qualitatively changed if a different charge leakage function is chosen—for example,

$$A_\pm = \frac{A_{\mathrm{const}}}{|Y - Y_{\mathrm{cr}}|}, \tag{23}$$

where we assume the same critical value $Y_{\mathrm{cr}} = 2/(3\alpha)$ as before. The network current behavior for this inverse form is indistinguishable from that shown in Fig. 3: the exact functional form of $A(Y)$ appears to be mostly irrelevant, provided there exists a critical current above which the leakage becomes overwhelming ($A \to \infty$). The detailed behavior of this charge leakage function should be inferred from numerical simulations, but phenomenologically the correction factor has a negligible influence at small currents (where we can use the simple linear model), though there are more detailed differences as we approach the critical current. The form of the charge leakage term, representing a complex nonlinear process, will depend, in principle, on the parameters of a specific superconducting cosmic string model—see, e.g., Refs. [35,36].

### *2. The biased case*

Another nontrivial parameter that keeps the system of Eq. (18) chiral, $K_{\mathrm{sc}} = 0$, is $g_+$. A nonvanishing $g_+ \neq 0$ can arise from various underlying mechanisms through which loops naturally form with more or less charge than the long strings. One possible such mechanism stems from the fact that the current momentum along the strings may smooth the loops [37], which would correspond to $g_+ < 0$. On the other hand, colliding strings create a bubble of electromagnetic radiation that can carry away some amount of charge [30], and this would lead to $g_+ > 0$. Again, numerical simulations of specific models will be required to decide which mechanism prevails. Naturally, it is even possible that they cancel one another, leading to the $|g| \ll 1$ case discussed above.

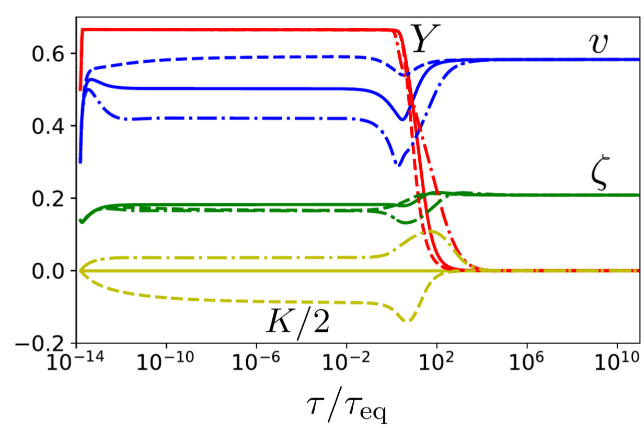


FIG. 4. Time evolution of the velocity $v$, charge $Y$, chirality $K$, and characteristic length ratio $\zeta$ for the Witten model with $\alpha = 1$, $c = 0.23$, $A_- = 0$, $g_+ = 0$, and $A_+$ is given by Eq. (22), when $A_{\mathrm{const}} = 10^{-6}$. The parameter $g_- = 0$ corresponds to solid lines, $g_- = -0.2$ for dash-dotted lines, and $g_- = 0.2$ for dashed lines.

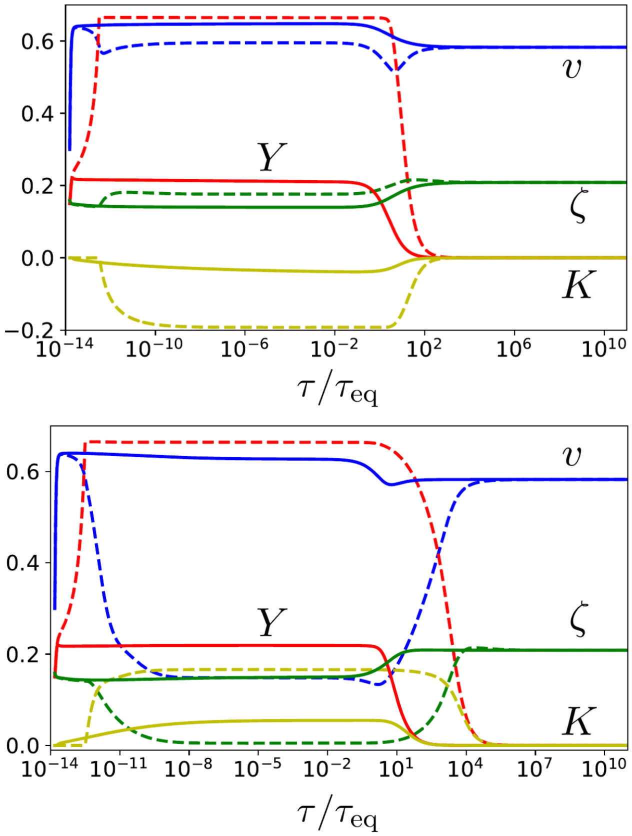


FIG. 5. Time evolution of the velocity $v$, charge $Y$, chirality $K$, and $\zeta$ through the radiation and matter epochs, using parameter values $c = 0.23$, $g_\pm = 0$, and $A_- = 0.1A_+$ for the upper panel ($A_- = -0.1A_+$ for the lower panel), while $A_+$ is given by the function in Eq. (22) with $A_{\rm const} = 10^{-6}$. The evolution is driven by the Witten equation of state (20) with $\alpha = 1$ (dashed lines) and $\alpha = 3$ (solid lines).

The result of the string network evolution with $g_+ \neq 0$ is similar to cases shown in Fig. 3. It happens because for a particular scaling regime (in our case, the radiation-dominated epoch), there is a degeneracy between $g_+$ and $A_+$ parameters: we can always tune parameters $g_+$ and $A_+$ so that the scaling values of a string network evolution are unchanged. Hence, the nontrivial value of the parameter $g_+$ does not introduce anything new, since the current is nontrivial only in a radiation-dominated epoch.

Up to this point, we did not find any difference between the full equation of state and its linear approximation. This is due to the fact that for the range of parameters we considered, no nontrivial integrated chirality ensues. We now consider such a more general situation for which the bias parameters $g_-$ and $A_-$ are non-negligible.

We just saw that the parameters $A_+$ and $g_+$ lead to some amount of degeneracy in the scaling solution. A similar situation occurs between $A_-$ and $g_-$: one can always change $A_-$ and $g_-$ to keep scaling variables the same for a particular expansion rate. In Fig. 4, we demonstrate the evolution of string networks for different $g_-$ values. We can see that the deviation of $Q$ leads to a nontrivial chirality in the radiation-dominated epoch.

Varying the parameter $A_-$, we obtain similar behavior—i.e., the string acquires nontrivial chirality—which we demonstrate in Fig. 5. Also, we show that an increase of $\alpha$ leads to a decrease of the chirality variable, bringing dynamical variables of the model with the complete equation of state closer to the linear approximation.

## V. CONCLUSIONS

We performed an exhaustive numerical exploration of the solutions of the CVOS model [17] describing macroscopic quantities (rms velocity, correlation length, current amplitude, and charge) as functions of time for a current-carrying cosmic string network, adopting as the underlying microscopic description the equation of state derived [7,26,28] in the framework of the so-called Witten superconducting cosmic string field theory [5]. It has been previously shown that, at the microscopic level, approximating the equation of state by its linear expansion, even for vanishingly small currents, is not a valid procedure, as perturbation velocities do not satisfy the correct relation [26], which in turn may yield significant differences for vorton [6] stability [38] and their cosmological consequences [39]. Having studied in detail the linear equation of state in a previous work [25], it was therefore of interest to understand if the microscopic situation extended to the macroscopic one, also including, for the first time, a simple phenomenological modeling of a critical current.

The general model includes several phenomenological parameters that, in principle, should be calibrated from high-resolution field theory network simulations yet to be undertaken, though efforts towards this goal are ongoing in our team. Two such parameters happen to be crucial for the CVOS model: namely, the charge leakage $A_+$, representing all possible mechanisms through which the long string network under consideration can lose charge, and the skew $g_+$ between the charge contained on long strings and on loops. One can argue that because there exist different mechanisms potentially increasing or decreasing $g_+$, its value may be rather small (resulting from the cancellation of competing effects). Similarly, at small currents, we do not know in detail how charge may be dissipated from the network, so the parameter $A_+$ in this regime is also largely unknown. For this reason, nothing is preventing either $g_+$ or $A_+$ from having a negligible effect on network evolution, while currents and charges remain well below critical.

We also studied the bias parameter $A_-$, which can be nontrivial if the leakage mechanism is not symmetric for timelike and spacelike current components. Similarly, the current carried away due to loop production $g_-$ might have a preferred channel that distorts the symmetry between timelike and spacelike parts. In both cases, the string network deviates from chirality and the evolution becomes distinct from the linear approximation, with the deviation, estimated to be of order $\alpha^{-1}$, becoming vanishing small for a large-enough $\alpha$ value.

We have studied all possible cases for relevant parameters and found that, for negligible charge leakage ($A_\pm \ll 1$) and close to chiral initial conditions ($K_{\rm ini} \approx 0$), one always recovers the linear solution, whatever the value of the bias. The same result holds for non-negligible charge losses and skew parameter $g_+$, but vanishingly small biases ($A_-, g_- \approx 0$). Finally, when all parameters have non-negligible values, we found that the general trends are the same as those found in the linear model—i.e., the network approaches one of two possible scaling solutions, charged or Nambu-Goto with different scaling values, with the choice dependent on the network's initial conditions. When the mass of the current-carrier becomes much smaller than the string-forming Higgs field ($\alpha \gg 1$), however, one again recovers the linear situation (provided the current remains subcritical), so it appears to be a valid approximation for these cosmological considerations.

Scenarios with a large current (frozen) network in the radiation era may have observational implications that are very different from those of simpler Nambu-Goto networks. However, how easy it is to realize such scenarios remains somewhat unclear, since the evolution of charges and currents on the network depends on their equation of state and initial conditions (and also, in the case of CVOS modeling, on the corresponding model parameters). In the specific case studied in the present work, for large $\alpha \gg 1$, the small relative critical current will set a low upper limit on $Y$, thus preventing much backreaction from the current on the network evolution. Unless the microscopic parameters are tuned such that the critical current has a comparable energy to the underlying string, then the general expectation is that the network behavior will be close to that of Nambu-Goto. In this context, the most interesting case is when $\alpha \sim 1$, since then $Y$ can grow much larger and there can be significant backreaction; this is also the regime suitable for vorton formation [40,41]. A more detailed exploration of this limit is left for subsequent work.

In a broader context, our analysis shows that the evolution of superconducting string networks with equations of state beyond the linear case (specifically, in the present analysis, the Witten model) can, at least for most physically realistic cases reduce, on a macroscopic scale, to the model with a linear equation of state. The conclusion is that the CVOS model offers a useful quantitative tool with which to start a realistic exploration of the observational constraints on superconducting cosmic string networks. We leave this task for a forthcoming analysis.

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