Dissipative quintessence and its cosmological implications

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We consider a generalization of the quintessence type scalar field cosmological models, by adding a multiplicative dissipative term in the scalar field Lagrangian, which generally is represented in an exponential form. The generalized dissipative Klein-Gordon equation is obtained in a general covariant form in Riemann geometry, from the variational principle with the help of the Euler-Lagrange equations. The energy-momentum tensor of the dissipative scalar field is also obtained from the dissipative Lagrangian, and its properties are discussed in detail. Several applications of the general formalism are presented for the case of the cosmological Friedmann-Lemaître-Robertson-Walker metric. The generalized Friedmann equations in the presence of the dissipative scalar field are obtained for a specific form of dissipation, with the dissipation exponent represented as the time integral of the product of the Hubble function, and of a function describing the dissipative properties of the scalar field. For this case the Friedmann equations reduce to a system of differential-integral equations, which, by means of some appropriate transformation, can be represented in the redshift space as a first order dynamical system. Several cosmological models, corresponding to different choices of the dissipation function, and of the scalar field potential, are considered in detail. For the different values of the model parameters the evolution of the cosmological parameters (scale factor, Hubble function, deceleration parameter, the effective density and pressure of the scalar field, and the parameter of the dark energy equation of state, respectively) is considered in detail by using both analytical and numerical techniques. A comparison with the observational data for the Hubble function and with the predictions of the standard ΛCDM paradigm is presented for each dissipative scalar field model. In the large time limit the model describes an accelerating universe, with the effective negative pressure induced by the dissipative effects associated with the scalar field. Accelerated expansion in the absence of the scalar field potential is also possible, with the kinetic term dominating the expansionary evolution. The dissipative scalar field models describe well the data, with the model free parameters obtained by a trial and error method. The obtained results show the dissipative scalar field model offers an effective dynamical possibility for replacing the cosmological constant and for explaining the recent cosmological observational data.

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I. INTRODUCTION

The theory of general relativity [1,2] is extremely successful in explaining the gravitational phenomena at the level of the Solar System. A large number of observational and even experimental tests, including the high precision studies of the deflection of light, of the perihelion precession of the planet Mercury, of the Shapiro time delay effect, of the frame-dragging effect, and of the Nordtvedt effect in lunar motion, respectively, have confirmed the validity and the scientific soundness of the theory [3]. Recently, another of the theoretical predictions of general relativity was brilliantly confirmed by the experimental detection of the gravitational waves [4]. The gravitational wave studies open a new window into the universe, leading to a new perspective on the properties of the black holes and on the mass distribution of the massive compact astrophysical objects, for example, the neutron stars [5]. Very recently, the Event Horizon Telescope (EHT) was able to detect the shadow of the black hole in the center of the M87* galaxy [6,7], with the observations confirming the general relativistic black hole model. The shadow of a black hole is an important testing ground for the predictions of general relativity and of the modified theories of gravity.

However, the improvement of the observational techniques, and the extension of the observations on a much wider scale, led to the unexpected result that for gravitational systems much bigger than the Solar System, general

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relativity may not be able to provide an accurate description of their gravitational properties. This situation already appears at the galactic scale, and it becomes even more severe at cosmological scales. Hence, it seems that the theory of general relativity must face a number of very serious challenges, whose solutions may require a fundamental change in our view of gravity and of the physical properties of the large scale structures in the universe.

One of the major discoveries of the past few decades was related to the strong observational evidence indicating that presently the universe is in a state of accelerating expansion [8–15]. These results were obtained from the astrophysical observations of the distant type Ia supernovae, whose spatial distribution extends up to a redshift of $z \approx 2$. Surprising results also came from the high precision determinations of the temperature fluctuations of the Cosmic Microwave Background Radiation (CMBR), obtained by the Planck satellite [16,17]. Moreover, the stunning finding that the matter content of the universe consists of only 5% baryonic matter has also been decisively confirmed by multiple observations. Hence, the present day observational situation in cosmology convincingly indicates that 95% of the total composition of the universe resides in the form of two main (and mysterious) constituents, dark energy and dark matter, respectively. On the other hand, it is important to point out that a cosmic fluid, formed of normal matter, and obeying a perfect fluid type equation of state, cannot trigger and sustain the accelerated expansion of the universe [18].

A theoretical interpretation of the cosmological observations can be achieved through the reintroduction in the Einstein field equations of the cosmological constant Λ, first proposed by Einstein in 1917 [19], in order to obtain a static cosmological model. For the interesting history of the cosmological constant, of its rejections and returns, as well as of its many possible interpretations see Refs. [20–24]. The cosmological model, obtained by adding to the Einstein field equations the cosmological constant Λ, as well as a cold dark matter component, is called the ΛCDM model. Presently, the ΛCDM model represents one of the main theoretical instruments used for the comprehension of the cosmic dynamics, and for the interpretation of the observational data.

The ΛCDM paradigm gives very good fits to the observations. But it lacks a convincing theoretical foundation, which is first of all related to the interpretational problems related to the cosmological constant itself. This makes the physical basis of the ΛCDM model problematic. Moreover, the ΛCDM model is recently facing another major problem. Measurements of the Hubble constant in the early universe indicate a value of the order of $H_0 < 69$ km/s/Mpc, while local measurements give $H_0 > 71$ km s/Mpc [25]. The contradictions between the values obtained in the measurements of the Hubble constant are known as the Hubble tensions, and their extents depend on the used datasets.

Therefore, to obtain a theoretically consistent description of the universe, several approaches have been proposed, which try to solve the cosmological constant problem by assuming some alternative explanations of the cosmic dynamics, which could be described as the dark component, the dark gravity, and the dark coupling models [26].

One of the important alternatives to the ΛCDM models is represented by the dark components model [27–31]. In the framework of this approach one postulates that the basic constituents of the universe are the dark energy and the dark matter, respectively, whose physical properties could explain, at least at a phenomenological level, the cosmological observations. Many proposals for the physical nature of these two dark constituents have been considered and investigated in detail. Perhaps the simplest dark energy model can be obtained by using the quintessence type theories [32–36]. In the quintessence theory the cosmological evolution of the universe is fully determined and described by a single scalar field ϕ, in the presence of a self-interaction potential $V(\phi)$. The simplest gravitational action for the quintessence models is given by

$$S = \int \left[\frac{M_p^2}{2} R - \partial_\mu \phi \partial^\mu - V(\phi)\right] \sqrt{-g} d^4x, \tag{1}$$

where by R we have denoted the Ricci scalar, while M_p represents the Planck mass. The cosmological energy density and pressure of the quintessence scalar field are obtained as $\rho_Q = \dot{\phi}^2/2 + V(\phi)$ and $p_Q = \dot{\phi}^2/2 - V(\phi)$ [37], giving for the equation of state w of the quintessence field the expression $w = p_Q/\rho_Q = (\dot{\phi}^2/2 - V(\phi))/(\dot{\phi}^2/2 + V(\phi))$. Quintessence type cosmological models have been very successful in interpreting and explaining important characteristics of the cosmic evolution. For recent reviews on quintessence theories see Refs. [38,39]. In particular, quintessential cosmological models can solve the σ_8 tension by allowing the conformal coupling of a single dark energy scalar field to dark matter through a constant coupling [40]. The Hubble tension can be alleviated by considering a quintessence field that transitions from a matterlike to a cosmological constant behavior between the recombination and the present time [41]. The discrepancy between the local measurements of H_0 and that inferred from the cosmic microwave background observations can be reconciled by assuming the existence of an electroweak axion in the minimal supersymmetric standard model, with the axion energy density identified with the observed dark energy [42]. The best-fit of the dark energy parameters was used to reconstruct the quintessence Lagrangian in [43]. Because of the derived late phantom behavior of $w(z)$, the reconstructed quintessence models have a negative kinetic term. The possibility of alleviating both the H_0 and the σ_8 tensions simultaneously by means of the Albrecht-Skordis "quintessence" potential was considered in [44]. The quintessence field can reduce the size of the sound horizon r_s, while suppressing the power in matter density fluctuations before it dominates the present day energy density. For some recent works on the cosmological implications of the quintessence model see Refs. [45–52].

In [53] the coupled quintessence (CQ) model was proposed, in which the scalar field ϕ and the dark matter fluid interact with each other through a source term Q_ν, which appears in the conservation equations as

$$\nabla_\mu T^\mu_{\nu(\phi)} = -Q_\nu, \qquad \nabla_\mu T^\mu_{\nu(m)} = Q_\nu, \tag{2}$$

where $T^\mu_{\nu(\phi)}$ and $T^\mu_{\nu(m)}$ are the energy-momentum tensors of the scalar field and of the dark matter, respectively. It was also suggested [53] that the source term can be given by $Q_\nu = -\kappa\beta(\phi)T_{(m)}\nabla_\nu\phi$, where $T_{(m)}$ is the trace of the matter energy-momentum tensor, and $\beta(\phi)$ is the coupling function that determines the strength of the interaction.

Several other scalar field models have been explored from a cosmological perspective. In a class of string theories, depending on the form of the tachyon potential, the tachyon scalar field can act as a source of the dark energy [54–57]. The effective Lagrangian for the tachyon scalar field is given by

$$L = -V(\phi)\sqrt{1 + \partial_\mu\phi\partial^\mu\phi}, \tag{3}$$

where ϕ is the tachyon scalar field and $V(\phi)$ is its potential. The energy density and the pressure of the tachyon field are given by

$$\rho_T = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}, \qquad p_T = -V(\phi)\sqrt{1-\dot{\phi}^2}, \tag{4}$$

giving for the parameter of the equation of state w_T the expression

$$w_T = \frac{p_T}{\rho_T} = \dot{\phi}^2 - 1. \tag{5}$$

For recent studies on the tachyonic field cosmology see Refs. [58–60].

Another interesting scalar field theory that was introduced to explain the cosmological observations is the k-essence scalar field model of dark energy. The main characteristic of the model is the presence of a scalar field with a noncanonical kinetic energy term. The scalar field action for the k-essence is a function of the field ϕ and of $\chi = \dot{\phi}^2/2$, and it is given by [61–63],

$$S = \int p_{\rm DE}(\phi,\chi)\sqrt{-g}d^4x, \tag{6}$$

where the Lagrangian density corresponds to the pressure of a scalar field with a noncanonical kinetic term, given by

$$p_K = f(\phi)(-\chi + \chi^2), \tag{7}$$

The energy density of the k-essence field is given by

$$\rho_K = f(\phi)(-\chi + 3\chi^2). \tag{8}$$

For the parameter w_K of the equation of state of the k-essence field we obtain

$$w_K = \frac{\chi - 1}{3\chi - 1}. \tag{9}$$

For the cosmological applications, and implications, of the k-essence models, see Refs. [64–68], and references therein.

Finally, we mention the dilaton scalar field model, which is an attempt to solve the dark energy problem by using string theory [69–73]. For the dilation scalar the energy density and the pressure are given by

$$\rho_D = -\chi + 3ce^{\lambda\phi}\chi^2, \qquad p_D = -\chi + ce^{\lambda\phi}\chi^2, \tag{10}$$

where c and λ are constants. The parameter of the equation of state of the dilaton scalar field is given by

$$w_D = \frac{1 - ce^{\lambda\phi}}{1 - 3ce^{\lambda\phi}}. \tag{11}$$

There are also some other approaches to the cosmological phenomenology. For example, in the dark gravity approach it is assumed the gravitational interaction itself is modified on the galactic and cosmological scales. One possibility to modify gravity is to go beyond the Riemannian geometry of general relativity and to use more general geometries to describe gravity. In this direction theories in the presence of torsion [74–77], of nonmetricity [78–83], or in the Weitzenböck geometry [84,85] have been intensively investigated. The third theoretical avenue for explaining the cosmological phenomenology is the dark coupling approach, which assumes that ordinary matter can couple with geometry, through a curvature-matter coupling. The existence of such a coupling could explain the accelerated expansion of the universe, as well as the dark matter problem [86–90]. For reviews of the modified gravity theories see Refs. [91–96].

Decay processes play a central role in a wide range of phenomena, including nuclear fission or optical emission. Dissipation also appears in quantum systems, and it is a consequence of the dissipative interaction of the quantum system with its environment [97]. Energy decay is usually considered as a consequence of a thermodynamic system exchanging energy irreversibly with its environment, usually assumed to be a thermal bath. However, there are energy decay processes that cannot be explained by assuming a direct coupling to a thermal bath [98].

A class of dissipative effects, the bulk and shear viscous processes, has been extensively investigated in astrophysical and cosmological settings, and they are assumed to play an important role in the early evolution of the universe. A cosmic fluid with bulk viscous pressure, in the presence of the quintessence field can trigger the accelerated expansion phase of the universe [99]. The presence of bulk viscosity could also solve the coincidence problem of cosmology. The bulk viscous Chaplygin gas model was considered in [100]. A recent investigation of a unified cosmic fluid scenario in the presence of bulk viscosity, with the coefficient of the bulk viscosity having a power law evolution was carried out in [101]. Considering such a general bulk viscous scenario, the observational constraints using the latest cosmological datasets were obtained, and their behavior was analyzed at the level of both background solutions and cosmological perturbations. The observational analyses did show that a nonzero bulk viscous coefficient is always favored. Moreover, some of the bulk viscous models can weaken the current H_0 tension for some datasets. But from the Bayesian evidence analysis, it follows that the ΛCDM model is favored over the cosmological models with bulk viscosity.

A problem less investigated in the physical literature is the possibility of a Lagrangian description of dissipative phenomena. In this respect one must make a clear distinction between physical (standard) and mathematical (nonstandard) Lagrangians. A physical Lagrangian is a Lagrangian function that can be represented as the difference between a kinetic energy term and a potential energy term. Other Lagrangians, which also give the correct equation of motion, but which cannot be represented as the difference of a kinetic and a potential term, are called mathematical, or nonstandard, Lagrangians. For example, the equation of motion of the damped oscillations, describing the motion of a single particle of mass m in an external field with potential $V(x)$ and in the presence of friction, can be obtained, via the Euler-Lagrange equations,

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0, \tag{12}$$

from the physical Lagrangian [102,103]

$$L=e^{\gamma t}\left(\frac{1}{2}m\dot{x}^2-V(x)\right), \tag{13}$$

where a dot denotes the derivative with respect to the time t and is given by

$$\ddot{x}+\gamma\dot{x}+\frac{1}{m}\frac{dV(x)}{dx}=0. \tag{14}$$

One can also construct a Hamiltonian for the damped oscillator in the standard way. By defining $p=\partial L\partial\dot{x}=me^{\gamma t}\dot{x}$, $\mathcal{H}=p\dot{x}-L$, one immediately obtains

$$\mathcal{H}=e^{-\gamma t}\frac{p^2}{2m}+e^{\gamma t}V(x),\quad \mathcal{H}=e^{\gamma t}\left(\frac{1}{2}m\dot{x}^2+V(x)\right). \tag{15}$$

It is important to note that $\mathcal{H}$ as defined above is not the energy of the system, which is still defined as $E=m\dot{x}^2/2+V(x)$, and satisfies the relation $dE/dt=F_d\dot{x}$, where $F_d=-m\gamma\dot{x}$ is the dissipative force. Therefore, $\mathcal{H}=e^{\gamma t}E$, and it cannot be interpreted as the energy of the system [104]. It is also interesting to point out that the equation $\ddot{x}+k\dot{x}=0$ can be derived from the physical (standard) Lagrangian $L=e^{kt}\dot{x}^2/2$, as well as from the nonphysical Lagrangians $L=1/(e^{2kt}\dot{x}+e^{kt})$, $L=\dot{x}\ln|\dot{x}|-kx$, or $L=(\dot{x}^{\nu}+e^{-\nu kt})^{1/\nu}$, respectively [102].

From a mathematical point of view, the Klein-Gordon equation describing the cosmological evolution of a scalar field in a Friedmann-Lemaître-Robertson-Walker (FLRW) geometry belongs to the general class of equations of the form

$$\ddot{x}+F(t)\dot{x}+g'(x)=0, \tag{16}$$

where F and g are arbitrary functions of time. Equation (16) can be derived from the dissipative physical Lagrangian, $=0$,

$$L=e^{\int F(t)dt}\left(\frac{1}{2}\dot{x}^2-g(x)\right), \tag{17}$$

with the use of the Euler-Lagrange equations, by taking into account that $\partial L/\partial\dot{x}=\dot{x}e^{\int F(t)dt}$, $d(\partial L/\partial\dot{x})/dt=e^{\int F(t)dt}[\ddot{x}+F(t)\dot{x}]$, and $\partial L/\partial x=-g'(x)e^{\int F(t)dt}$, respectively.

In the Minkowskian space a natural dissipative extension of the scalar field and of the Klein-Gordon equation can be considered by adopting for the Lagrangian density the expression [105]

$$L_\phi=e^{k_\alpha x^\alpha}\left(\frac{1}{2}\partial_\mu(\phi)\partial^\mu\phi-V(\phi)\right), \tag{18}$$

where k_α are constants. From the Euler-Lagrange equations, $\partial L/\partial\phi-\partial_\mu(\partial L/\partial\phi_{,\mu})=0$, where $\phi_{,\mu}=\partial\phi/\partial x^\mu$, we obtain the equation of motion of the dissipative scalar field as

$$\partial_\mu\partial^\mu\phi+k_\mu\partial^\mu\phi+V'(\phi)=0. \tag{19}$$

In the above equation, by analogy with the equations of motion for the damped oscillators, one could interpret the term $k_\mu\partial^\mu\phi$ as corresponding to a dissipative friction term.

It is the goal of the present paper to extend and formulate the variational formulation of the dissipative scalar field by using a fully covariant approach in the Riemannian geometric framework, and to formulate the dissipative

Klein-Gordon equation for the scalar field in a general covariant form. The dissipation is introduced in the Lagrangian via a dissipation exponent Γ, assumed to be, in general, a function of the metric tensor, of the scalar field, and of the coordinates of the base spacetime manifold. By using the analogy with the simple damped harmonic oscillator, the dissipative Lagrangian is obtained then by multiplying the Lagrangian of the "ideal" scalar field with the exponential of Γ, so that the new Lagrangian is constructed as the product of e^{Γ} and the standard Lagrange function of the scalar field. The generalized Klein-Gordon equations are obtained in a fully covariant form for the case of the dissipation exponent having various functional forms. Particular cases of the dissipative Klein-Gordon equation are also discussed in detail.

Once the Lagrangian density L_{ϕ} of the scalar field is known, the basic physical properties of the field can be obtained from the energy-momentum tensor, which can be straightforwardly obtained from L_{ϕ} through variation with respect to the metric. We obtain the general form of the energy-momentum tensor of the dissipative scalar field, which involves the presence of a new tensor, the dissipation tensor, which gives a new, significant, and important contribution both to the energy density and to the pressure of ϕ.

As a particular example of the general formalism we consider the case in which the dissipation exponent can be expressed as the invariant integral of the divergence of a four-vector u^{λ} and of an arbitrary function $Q(x^{\mu})$, which we call the dissipation function, having the mathematical representation given by $\Gamma(g_{\alpha\beta}, x^{\mu}) = \int_{\Omega} \nabla_{\lambda} u^{\lambda} Q(x^{\mu}) \sqrt{-g} d^4x$. This case is analyzed in detail, and the dissipative Klein-Gordon equation, as well as the corresponding energy-momentum tensor, is obtained in a covariant form.

We extensively apply the obtained results to generalize, and extend, the standard cosmological scalar field models, which have been successfully used to explain the recent acceleration of the universe. To do this, we restrict our analysis to the case of the flat, isotropic, and homogeneous FLRW geometries. For the specific cosmological applications, we assume that the dissipation exponent can be expressed as the integral of the product of the Hubble function $H(t)$ and of the dissipation function $Q(t)$, in the form $\Gamma(t) = 3\int H(t) Q(t) dt$. We obtain the generalized dissipative Klein-Gordon equation, as well as the corresponding energy-momentum tensor for the field in a cosmological setting. With the help of these quantities, the generalized Friedmann equation describing the cosmological evolution in the presence of the dissipative scalar field is obtained.

To test the cosmological viability of the dissipative scalar field model, we consider several explicit cosmological models, corresponding to various choices of the dissipation function Q. First, the existence of the de Sitter type solution for this model is proven. Then, several classes of cosmological models, corresponding to a constant Q and to a redshift dependent dissipation, are considered in detail. Models in which the kinetic term and the potential term of the field can be neglected are investigated numerically. In each case a comparison with the observational data for the Hubble function and with the standard ΛCDM model are performed, and it is shown that the models give a good description of the data. The obtained results indicate that the dissipative scalar field model can be considered as a viable extension of the standard quintessence type cosmological models. This model also offers a firm theoretical foundation, via its variational principle, to different classes of scalar field cosmologies, and allows the possibility of their rigorous generalization.

The present paper is organized as follows. In Sec. II, after briefly reviewing the basic theory of the ideal cosmological scalar fields, we introduce the dissipative scalar field in the FLRW geometry via the variational principle. The generalized Klein-Gordon equations are obtained for a dissipation exponent given by $\Gamma(t) = 3\int H(t) Q(\phi(t), t) dt$, with several particular cases considered. The covariant form of the Klein-Gordon equation is obtained, and a particular case is investigated in detail. The Einstein and the generalized Friedmann equations are obtained in Sec. III. Simple cosmological applications of the dissipative scalar field model are investigated in Sec. IV, by considering some simple forms of the scalar field, and by assuming a constant Q. Comparisons with the observational data and the standard ΛCDM model are also performed. A cosmological model with a dynamic, redshift dependent dissipation function Q is analyzed in Sec. V. Finally, we discuss and conclude our results in Sec. VI.

In the present paper we use the Landau-Lifshitz [106] sign and geometric conventions.

II. THE DISSIPATIVE KLEIN-GORDON EQUATION

In this section we will introduce the basic variational formalism for the description of the dissipative scalar fields. After briefly reviewing the nondissipative case, as well as its cosmological applications, we proceed to the systematic presentation of the various forms of the dissipative Klein-Gordon equation and of their cosmological formulations.

A. Nondissipative (ideal) scalar fields

In a Riemannian geometry, the action for an ideal scalar field with self-interaction potential $V(\varphi)$ is given by

$$S_{\phi} = \int L_{\phi} d^4x = \int \left[\frac{1}{2} g^{\alpha\beta} \frac{\partial\phi}{\partial x^{\alpha}} \frac{\partial\phi}{\partial x^{\beta}} - V(\phi)\right] \sqrt{-g} d^4x, \quad (20)$$

where $g^{\alpha\beta}$ are the components of the metric tensor and $-g$ is its determinant.

The Euler-Lagrange equations, giving the minimum of the action, are

$$\frac{\partial}{\partial x^\alpha}\frac{\partial L_\phi}{\partial \phi_{,\alpha}} - \frac{\partial L_\phi}{\partial \phi} = 0. \tag{21}$$

Therefore, we obtain

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial \phi}{\partial x^\beta}\right) + \frac{dV(\phi)}{d\phi}\sqrt{-g} = 0. \tag{22}$$

But, it is easy to check that [106]

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial \phi}{\partial x^\beta}\right) = \nabla_\alpha\nabla^\alpha\phi = \Box\phi, \tag{23}$$

where by ∇_α we have denoted the covariant derivative with respect to the metric and $\Box = \nabla_\alpha\nabla^\alpha$ is the d'Alembert operator. Hence, we obtain the covariant Klein-Gordon equation in Riemann geometry, describing the dynamics of an ideal, nondissipative scalar field, as given by

$$\Box\phi + \frac{dV(\phi)}{d\phi} = 0. \tag{24}$$

The energy-momentum tensor is defined generally by the relation [106]

$$T_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta[\sqrt{-g}L(g_{\alpha\beta},\phi)]}{\delta g^{\alpha\beta}}, \tag{25}$$

where $L(g_{\alpha\beta},\phi)$ is any physical Lagrangian function, which is assumed to be independent of the derivatives of the metric tensor. Hence, for the energy-momentum tensor of the ideal scalar field we obtain

$${}^{(\phi)}T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \left(\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right)g_{\alpha\beta}. \tag{26}$$

With the use of the Klein-Gordon equation (24), one could immediately check that the energy-momentum tensor of the ideal scalar field satisfies the conservation condition $\nabla_\beta({}^{(\phi)}T^\beta_\alpha) = 0$.

$T_{\alpha\beta}$ can be recast in the standard form of a perfect fluid,

$$T_{\alpha\beta} = (\rho_\phi + p_\phi)U_\alpha U_\beta - p_\phi g_{\alpha\beta}, \tag{27}$$

where we have introduced the energy density ρ_ϕ and the pressure p_ϕ of the scalar field, defined as

$$\rho_\phi = \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + V(\phi),$$
$$p_\phi = \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi), \tag{28}$$

and the effective four-velocity of the field U_α, given by

$$U_\alpha = \frac{\phi_{,\alpha}}{\sqrt{g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}}, \tag{29}$$

respectively.

1. Application: The case of the FLRW geometry

The standard flat, isotropic, and homogeneous cosmological FLRW metric is given by

$$ds^2 = c^2dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{30}$$

where $a(t)$ is the scale factor. Then, we have $\sqrt{-g} = a^3(t)$. Furthermore, we assume $\phi = \phi(t)$. An important observational quantity, the Hubble parameter, is defined as $H(t) = \dot{a}(t)/a(t)$.

The Lagrangian of the time-dependent ideal scalar field is given by

$$L_\phi = a^3\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{31}$$

Thus, Eq. (21) gives

$$\frac{1}{a^3}\frac{d}{dt}\left(a^3\frac{d\phi}{dt}\right) + \frac{dV(\phi)}{d\phi} = 0 \tag{32}$$

or, equivalently,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{33}$$

The energy density and the pressure of the cosmological scalar field become

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{34}$$

B. The dissipative scalar field

We consider now the variational formulation of the dissipative Klein-Gordon equation. We begin our analysis with the simple case of the cosmological scalar fields, which are further generalized to a full covariant formalism.

1. Dissipation in the FLRW geometry

The Lagrangian of a dissipative scalar field in a FLRW type geometry can be taken as

$$L_\phi = a^3e^{3\int H(t)Q(t)dt}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{35}$$

In the following we will call the function $\Gamma(t) = 3\int H(t)Q(t)dt$ *the dissipation exponent*, where $Q = Q(t)$ is *the dissipation function*. Then, the Euler-Lagrange equation giving the minimum of the action constructed with the help of the Lagrangian (35) takes the form

$$\left[3a^2\dot{a}\frac{d\phi}{dt}+3a^3H(t)Q(t)\frac{d\phi}{dt}+a^3\frac{d^2\phi}{dt^2}+\frac{dV(\phi)}{d\phi}a^3\right]\times e^{3\int H(t)Q(t)dt}=0, \tag{36}$$

giving, for $a\neq 0$,

$$\ddot{\phi}(t)+3H(t)(1+Q(t))\dot{\phi}(t)+\frac{dV(\phi)}{d\phi}=0. \tag{37}$$

Hence, the function $Q(t)$ acts as a novel dissipative term in the cosmological Klein-Gordon equation.

a. Scalar field dependent dissipation function. Let us assume now that $Q=Q(\phi(t))$. In the following by a prime we denote the derivative with respect to ϕ or, more generally, with the argument of the function. In this case the Euler-Lagrange equation leads to the evolution equation of the dissipative scalar field as given by

$$\ddot{\phi}+3H(1+Q(\phi(t)))\dot{\phi}+V'(\phi)-3\int H(t)Q'(\phi(t))dt\left(\frac{1}{2}\dot{\varphi}^2-V(\phi)\right)=0. \tag{38}$$

b. Hamiltonian formulation for the time dependent dissipation function. We introduce now the generalized momentum, defined, in the case of the Lagrangian (35), as

$$P_\phi=\frac{\partial L_\phi}{\partial\dot{\phi}}=a^3e^{3\int H(t)Q(t)dt}\dot{\phi}, \tag{39}$$

which allows one to introduce the generalized, time dependent Hamiltonian function $\mathcal{H}$ as

$$\mathcal{H}=P_\phi\dot{\phi}-L_\phi=a^3\left(\frac{1}{2}\dot{\phi}^2+V(\phi)\right)e^{3\int H(t)Q(t)dt}, \tag{40}$$

as well as the generalized effective energy density of the scalar field ρ_ϕ, defined according to

$$\rho_\phi^{(\mathrm{eff})}=\left(\frac{1}{2}\dot{\phi}^2+V(\phi)\right)e^{3\int H(t)Q(t)dt}=\rho_\phi e^{3\int H(t)Q(t)dt}. \tag{41}$$

These expressions for the Hamiltonian and energy density are also valid for the case of the scalar field dependent dissipation function, $Q=Q(\phi(t))$.

c. Potentials depending on the time derivative of the scalar field only. We consider now scalar field models in which the potential depends on the time derivatives of the scalar field only, $V=V(\dot{\phi})$. Then the Euler-Lagrange equation takes the form

$$\frac{d}{dt}\left[a^3e^{3\int H(t)Q(t)dt}\left(\dot{\phi}-\frac{dV(\dot{\phi})}{d\dot{\phi}}\right)\right]=0, \tag{42}$$

and it admits the first integral

$$\dot{\phi}-\frac{dV(\dot{\phi})}{d\dot{\phi}}=\frac{C}{a^3}e^{-3\int H(t)Q(t)dt}, \tag{43}$$

where C is an arbitrary constant of integration. In the particular case

$$V(\dot{\phi})=(1-\alpha)\frac{\dot{\phi}^2}{2}, \tag{44}$$

the dissipative scalar field satisfies the first order differential equation, given by

$$\dot{\phi}=\frac{C}{\alpha}\frac{e^{-3\int H(t)Q(t)dt}}{a^3}. \tag{45}$$

C. Covariant formulation of the dissipative Klein-Gordon equation

We introduce now the Lagrangian of the dissipative scalar field in the general covariant form as

$$S_\phi=\int L_\phi\sqrt{-g}d^4x=\int e^{\Gamma(g_{\alpha\beta},\phi,x^\alpha)}\left[\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-V(\phi)\right]\sqrt{-g}d^4x, \tag{46}$$

where the dissipation exponent $\Gamma(g_{\alpha\beta},\phi,x^\alpha)$ is an arbitrary scalar function of the metric tensor, of the scalar field, and of the coordinates. A particular, and useful, representation of the dissipation function is given by the expression

$$\Gamma(g_{\alpha\beta},\phi,x^\alpha)=\int\nabla_\lambda u^\lambda Q(\phi,x^\mu)\sqrt{-g}d^4x, \tag{47}$$

with u^λ denoting the velocity four-vector of the cosmological fluid. In the case of the FLRW geometry, in the comoving frame $u^\lambda=(1,0,0,0)$, and $\nabla_\lambda u^\lambda=(1/\sqrt{-g})\partial_\alpha(\sqrt{-g}u^\alpha)=(1/a^3)\frac{d}{dt}a^3=3H$. In an arbitrary coordinate system [106],

$$\nabla_\lambda u^\lambda=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}(\sqrt{-g}u^\lambda)=\frac{\partial u^\lambda}{\partial x^\lambda}+u^\lambda\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\lambda}\sqrt{-g}. \tag{48}$$

The Euler-Lagrange equations corresponding to the action (46) are given by

$$\nabla_\alpha(\sqrt{-g}g^{\alpha\beta}e^{\Gamma(g_{\alpha\beta},\phi,x^\alpha)}\nabla_\beta\phi) + \left[\frac{dV(\phi)}{d\phi} - \frac{\partial\Gamma(g_{\alpha\beta},\phi,x^\alpha)}{\partial\phi}\right]e^{\Gamma(g_{\alpha\beta},\phi,x^\mu)}\sqrt{-g} = 0 \tag{49}$$

or, equivalently,

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\beta}\right) + \sqrt{-g}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\beta}\frac{\partial\Gamma(g_{\alpha\beta},\phi,x^\alpha)}{\partial x^\alpha} + \left[\frac{dV(\phi)}{d\phi} - \frac{\partial\Gamma(g_{\alpha\beta},\phi,x^\alpha)}{\partial\phi}\right]\sqrt{-g} = 0, \tag{50}$$

finally giving the dissipative covariant Klein-Gordon equation as

$$\Box\phi + g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\Gamma(g_{\alpha\beta},\phi,x^\alpha) + V'(\phi) - \frac{\partial\Gamma(g_{\alpha\beta},\phi,x^\alpha)}{\partial\phi} = 0. \tag{51}$$

D. The energy-momentum tensor of the dissipative scalar field

By taking into account that $\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$, and that the Lagrangian density of the scalar field does not depend on the derivatives of the metric, we obtain for the energy-momentum tensor the general expression

$$^{(\phi)}T_{\alpha\beta} = 2\frac{\delta L_\phi}{\delta g^{\alpha\beta}} - L_\phi g_{\alpha\beta}, \tag{52}$$

from which one obtains the energy-momentum tensor of the dissipative scalar field as

$$^{(\phi)}T_{\alpha\beta} = e^{\Gamma(g_{\alpha\beta},\phi,x^\alpha)}\left[\nabla_\alpha\phi\nabla_\beta\phi + (\Theta_{\alpha\beta} - g_{\alpha\beta}) \times \left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right)\right], \tag{53}$$

where we have denoted

$$\Theta_{\alpha\beta}(g_{\mu\nu},\phi,x^\mu) = 2\frac{\delta\Gamma(g_{\alpha\beta},\phi,x^\mu)}{\delta g^{\alpha\beta}}. \tag{54}$$

We may call the tensor $\Theta_{\alpha\beta}(g,\phi,x^\mu)$ the dissipation tensor of the scalar field. For $\Theta_{\alpha\beta} = 0$ and $\Gamma = 0$, we recover the energy-momentum tensor of the ideal scalar field.

With the help of the energy density and pressure of the ideal scalar field, the energy-momentum tensor of the dissipative scalar field can be written as

$$^{(\phi)}T_{\alpha\beta} = e^{\Gamma(g_{\alpha\beta},\phi,x^\alpha)}[(\rho_\phi + p_\phi)U_\alpha U_\beta + (\Theta_{\alpha\beta} - g_{\alpha\beta}p_\phi)]. \tag{55}$$

1. The particular case $\Gamma(g_{\alpha\beta},x^\alpha) = \int\nabla_\lambda u^\lambda Q(x^\mu)\sqrt{-g}d^4x$

For the particular case of the dissipation exponent given by Eq. (47), for the variation with respect to the metric of $\Gamma(g_{\alpha\beta},x^\mu)$ we obtain

$$\begin{aligned}\frac{\delta\Gamma(g_{\alpha\beta},x^\mu)}{\delta g^{\alpha\beta}} &= \frac{\delta}{\delta g^{\alpha\beta}}\int\nabla_\lambda u^\lambda Q(x^\mu)\sqrt{-g}d^4x \\ &= \int\frac{\delta}{\delta g^{\alpha\beta}}[\nabla_\lambda u^\lambda Q(x^\mu)\sqrt{-g}]d^4x \\ &= \int\left[\frac{\delta(\nabla_\lambda u^\lambda)}{\delta g^{\alpha\beta}}\sqrt{-g} + \nabla_\lambda u^\lambda\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}}\right]Q(x^\mu)d^4x \\ &= \int\left[\frac{\delta(\nabla_\lambda u^\lambda)}{\delta g^{\alpha\beta}} + \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}}\nabla_\lambda u^\lambda\right] \\ &\quad\times Q(x^\mu)\sqrt{-g}d^4x.\end{aligned} \tag{56}$$

With the use of Eq. (48), we obtain

$$\frac{\delta(\nabla_\lambda u^\lambda)}{\delta g^{\alpha\beta}} = \frac{\partial}{\partial x^\lambda}\frac{\delta u^\lambda}{\delta g^{\alpha\beta}} + \frac{\delta}{\delta g^{\alpha\beta}}\left(u^\lambda\frac{\partial\ln\sqrt{-g}}{\partial x^\lambda}\right). \tag{57}$$

The variation of the four-velocity with respect to the metric is given by

$$\delta u^\lambda = \frac{1}{2}u^\lambda u_\alpha u_\beta\delta g^{\alpha\beta}, \tag{58}$$

which can be obtained from the relations $\delta g^{\alpha\beta}u_\alpha u_\beta = 2u_\mu\delta u^\mu = 2u_\mu(u^\mu\delta g^{\alpha\beta}u_\alpha u_\beta/2)$, respectively, where we have also used the condition of the normalization of the four-velocity, $g^{\alpha\beta}u_\alpha u_\beta = 1$. Thus, we immediately find

$$\frac{\delta u^\lambda}{\delta g^{\alpha\beta}} = \frac{1}{2}u^\lambda u_\alpha u_\beta. \tag{59}$$

Hence, Eq. (57) becomes

$$\begin{aligned}\frac{\delta(\nabla_\lambda u^\lambda)}{\delta g^{\alpha\beta}} &= \frac{\partial}{\partial x^\lambda}\frac{\delta u^\lambda}{\delta g^{\alpha\beta}} + \frac{\delta u^\lambda}{\delta g^{\alpha\beta}}\frac{\partial\ln\sqrt{-g}}{\partial x^\lambda} + u^\lambda\frac{\partial}{\partial x^\lambda}\frac{\delta\ln\sqrt{-g}}{\delta g^{\alpha\beta}} \\ &= \frac{1}{2}\frac{\partial u^\lambda}{\partial x^\lambda}u_\alpha u_\beta + \frac{1}{2}u^\lambda\frac{\partial}{\partial x^\lambda}(u_\alpha u_\beta) \\ &\quad + \frac{1}{2}u^\lambda u_\alpha u_\beta\frac{\partial\ln\sqrt{-g}}{\partial x^\lambda} - \frac{1}{2}u^\lambda\frac{\partial}{\partial x^\lambda}g_{\alpha\beta} \\ &= \frac{1}{2}\left(\frac{\partial u^\lambda}{\partial x^\lambda} + u^\lambda\frac{\partial\ln\sqrt{-g}}{\partial x^\lambda}\right)u_\alpha u_\beta + \frac{1}{2}u^\lambda\frac{\partial(u_\alpha u_\beta - g_{\alpha\beta})}{\partial x^\lambda} \\ &= \frac{1}{2}\nabla_\lambda u^\lambda u_\alpha u_\beta + \frac{1}{2}u^\lambda\frac{\partial}{\partial x^\lambda}(u_\alpha u_\beta - g_{\alpha\beta}),\end{aligned} \tag{60}$$

where we have used the relation $\delta\ln\sqrt{-g}/\delta g^{\alpha\beta} = (1/\sqrt{-g})\delta\sqrt{-g}/\delta g^{\alpha\beta}$.

Consequently,

$$\frac{\delta(\nabla_\lambda u^\lambda)}{\delta g^{\alpha\beta}}+\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\alpha\beta}}\nabla_\lambda u^\lambda =\frac{1}{2}\left[\nabla_\lambda u^\lambda h_{\alpha\beta}+u^\lambda\frac{\partial}{\partial x^\lambda}h_{\alpha\beta}\right], \tag{61}$$

where we have introduced the projection operator $h_{\alpha\beta}$, defined according to $h_{\alpha\beta}=u_\alpha u_\beta-g_{\alpha\beta}$. Hence, for the variation of the dissipative exponent of this particular case, defining the dissipation tensor of the scalar field, we obtain

$$\Theta_{\alpha\beta}=\int\left[\nabla_\lambda u^\lambda h_{\alpha\beta}+u^\lambda\frac{\partial}{\partial x^\lambda}h_{\alpha\beta}\right]Q(x^\mu)\sqrt{-g}d^4x. \tag{62}$$

Therefore, for a dissipation exponent $\Gamma=\int\nabla_\lambda u^\lambda Q(x^\mu)\sqrt{-g}d^4x$, depending on the metric, a vector field u^λ, and the coordinates only, the energy-momentum tensor of the dissipative scalar field can be written as

$$^{(\phi)}T_{\alpha\beta}=e^{\int\nabla_\lambda u^\lambda Q(x^\mu)\sqrt{-g}d^4x}\times\left[\nabla_\alpha\phi\nabla_\beta\phi+\left(\int\left[\nabla_\lambda u^\lambda h_{\alpha\beta}+u^\lambda\frac{\partial}{\partial x^\lambda}h_{\alpha\beta}\right]\times Q(x^\mu)\sqrt{-g}d^4x-g_{\alpha\beta}\right)\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-V(\phi)\right)\right]. \tag{63}$$

a. Dissipative energy-momentum tensor in FLRW geometry. For an FLRW universe, the components of the energy-momentum tensor of the dissipative scalar field become

$$^{(\phi)}T_0^0=e^{\Gamma(g_{\alpha\beta},t)}\left[(1+\Theta_0^0)\frac{\dot{\phi}^2}{2}+(1-\Theta_0^0)V(\phi)\right]=\rho_\phi^{(\mathrm{eff})}, \tag{64}$$

$$-^{(\phi)}T_i^i=e^{\Gamma(g_{\alpha\beta},t)}(1-\Theta_i^i)\left(\frac{1}{2}\dot{\phi}^2-V(\phi)\right)=p_\phi^{(\mathrm{eff})}\delta_i^i,\qquad i=1,2,3. \tag{65}$$

In the particular case of the dissipation exponent given by Eq. (47), by taking into account that in the FLRW geometry $h_{00}=0$, we obtain for the 00 component of the energy-momentum tensor the expression

$$^{(\phi)}T_0^0=e^{3\int H(t)Q(t)dt}\left(\frac{1}{2}\dot{\phi}^2+V(\phi)\right)=\rho_\phi^{(\mathrm{eff})}. \tag{66}$$

Equation (66) also gives the Hamiltonian constraint for the Lagrangian of the dissipative scalar field.

E. The dissipative Klein-Gordon equation with $e^{k_\mu x^\mu}$ type dissipation

We consider now an alternative Lagrangian for the description of the dissipative scalar field, given by

$$S_\phi=\int L_\phi\sqrt{-g}d^4x =\int e^{k_\mu x^\mu}\left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi-V(\phi)\right]\sqrt{-g}d^4x, \tag{67}$$

where k_α are the components of a constant four-vector and x^α, $\alpha=0,1,2,3$, are the coordinates on the spacetime manifold. For this form of dissipation the Euler-Lagrange equations take the form

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}e^{k_\mu x^\mu}\frac{\partial\phi}{\partial x^\beta}\right)+\frac{dV(\phi)}{d\phi}e^{k_\mu x^\mu}\sqrt{-g}=0, \tag{68}$$

giving the following dissipative Klein-Gordon equation:

$$\Box\phi+g^{\alpha\beta}k_\alpha\nabla_\beta\phi+\frac{dV(\phi)}{d\phi}=0. \tag{69}$$

The energy-momentum tensor of the dissipative scalar field becomes

$$(\phi)T_{\alpha\beta}=e^{k_\mu x^\mu}\left[\nabla_\alpha\phi\nabla_\beta\phi\right. \tag{70}$$

$$\left.-\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-V(\phi)\right)(g_{\alpha\beta}-k_\alpha x_\beta)\right]. \tag{71}$$

In the case of the FLRW geometry, with $k_\alpha=(k_0,0,0,0)$, the action of the dissipative scalar field is given by

$$L_\phi=a^3e^{3k_0t}\left(\frac{1}{2}\dot{\phi}^2-V(\phi)\right), \tag{72}$$

leading to the dissipative Klein-Gordon equation

$$\ddot{\phi}+(3H+k_0)\dot{\phi}+V'(\phi)=0. \tag{73}$$

The components of the energy-momentum tensor of this type of dissipative scalar fields are obtained as

$$^{(\phi)}T_0^0=e^{k_0t}\left[\frac{1}{2}(1+k_0t)\dot{\phi}^2+(1-k_0t)V(\phi)\right]=\rho_\phi^{(\mathrm{eff})} \tag{74}$$

and

$$
{}^{(\phi)}T_i^i = -e^{k_0 t}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) = -p_\phi^{(\mathrm{eff})}, \quad i = 1,2,3, \tag{75}
$$

respectively.

F. Potentials depending on the gradient of the scalar field

Finally, we consider the covariant formulation of the dissipative Klein-Gordon equation in the presence of a potential that depends on the magnitude of the gradients of the scalar field X, $V = V(X)$, with $X = \nabla_\alpha\phi\nabla^\alpha\phi$. The action of the dissipative scalar field system is given by

$$
S_\phi = \int e^{\Gamma(g_{\alpha\beta},x^\mu,\phi)}\left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(X)\right]\sqrt{-g}d^4x, \tag{76}
$$

leading to the Euler-Lagrange equation

$$
\begin{aligned}
&\left(1-\frac{dV(X)}{dX}\right)\Box\phi + g^{\alpha\beta}\nabla_\beta\phi\nabla_\alpha\Gamma(g_{\alpha\beta},\phi,x^\mu)\\
&\quad - g^{\alpha\beta}\nabla_\beta\phi\nabla_\alpha\frac{dV(X)}{dX} - \left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(X)\right]\\
&\quad \times\frac{\partial\Gamma(g_{\alpha\beta},\phi,x^\mu)}{\partial\phi} = 0.
\end{aligned} \tag{77}
$$

The energy-momentum tensor of the scalar field in the presence of scalar field dependent potentials is obtained in the form

$$
\begin{aligned}
{}^{(\phi)}T_{\alpha\beta} = e^{\Gamma(g_{\alpha\beta},\phi,x^\mu)}\Bigg[&\left(1-2\frac{dV(X)}{dX}\right)\nabla_\alpha\phi\nabla_\beta\phi\\
&+\left(2\frac{\delta\Gamma(g_{\alpha\beta},\phi,x^\mu)}{\delta g^{\alpha\beta}} - g_{\alpha\beta}\right)\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(X)\right)\Bigg].
\end{aligned} \tag{78}
$$

III. THE EINSTEIN GRAVITATIONAL FIELD EQUATIONS

In the following we consider a gravitational model, containing, besides the gravitational term, a nonminimally coupled dissipative scalar field, with Lagrangian density L_ϕ, and an ordinary matter term, described by the Lagrangian L_m. Hence, the action of the present theory can be generally written down as

$$
\begin{aligned}
S &= \int_\Omega\left[-\frac{c^4}{16\pi G}R(g) + L_\phi + L_m\right]\sqrt{-g}d^4x\\
&= \int_\Omega\Bigg[-\frac{c^4}{16\pi G}R(g) + e^{\Gamma(g_{\alpha\beta},x^\alpha,\phi,\partial_\alpha\phi)}\left(\frac{1}{2}g^{\mu\nu}\frac{\partial\phi(x^\alpha)}{\partial x^\mu}\frac{\partial\phi(x^\alpha)}{\partial x^\nu}\right.\\
&\qquad \left. - V(\phi(x^\alpha))\right) + L_m\Bigg]\sqrt{-g}d^4x.
\end{aligned} \tag{79}
$$

The variation of the Ricci scalar is obtained in the following form:

$$
\begin{aligned}
\delta(R\sqrt{-g})R &= \delta(R_{\mu\nu}g^{\mu\nu}\sqrt{-g})\\
&= \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\right)\sqrt{-g}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}.
\end{aligned} \tag{80}
$$

The term $g^{\mu\nu}\delta R_{\mu\nu}$ can be written as $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda w^\lambda$, where $w^\lambda = g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu}$, with $\Gamma^\lambda_{\mu\nu}$ denoting the Christoffel symbols associated with the Riemannian metric g. In the standard approaches to general relativity, the boundary term $g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}$ is canceled out with the use of the Gauss theorem,

$$
\begin{aligned}
\int_\Omega g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d^4x &= \int_\Omega \nabla_\lambda w^\lambda\sqrt{-g}d^4x\\
&= \int_{\partial\Omega} w^\lambda dS_\lambda,
\end{aligned} \tag{81}
$$

where dS_λ is the element of integration over the hypersurface surrounding the four-volume element $d\Omega$, under the assumption that the variations of the field cancel at the integration limits. Hence, the gravitational field equations in the presence of a dissipative scalar field and a vanishing boundary term take the form

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}\left({}^{(\phi)}T_{\mu\nu} + {}^{(m)}T_{\mu\nu}\right), \tag{82}
$$

where the energy-momentum tensor of the dissipative scalar field is given by Eq. (53), while ${}^{(m)}T_{\mu\nu}$ is the energy-momentum tensor of ordinary matter, defined as ${}^{(m)}T_{\mu\nu} = (2/\sqrt{-g})\delta(\sqrt{-g}L_m)/\delta g^{\mu\nu}$. The variation of the action (79) with respect to the scalar field ϕ gives the equation of motion of the scalar field, Eq. (51), respectively.

A. The generalized Friedmann equations

We will consider in the following the case of a dissipative scalar field with a dissipation exponent given by $\Gamma(t) = 3\int H(t)Q(t)dt$. Then the effective density $\rho_\phi^{(\mathrm{eff})}$ of the dissipative scalar field (the Hamiltonian constraint) is given by Eq. (41).

We also assume that in the comoving frame the energy-momentum tensor of the scalar field is given by ${}^{(\phi)}T_0^0 = \rho_\phi^{(\mathrm{eff})}$ and ${}^{(\phi)}T_i^i = -p_\phi^{(\mathrm{eff})}\delta_i^i$, $i = 1, 2, 3$. For the adopted form of the dissipation exponent the effective energy of the scalar field is given by Eq. (66).

To determine the form of the effective pressure $p_\phi^{(\mathrm{eff})}$ of the dissipative scalar field, we impose the condition of the conservation of the effective quantities in the cosmological background, which can be formulated as

$$\dot{\rho}_\phi^{(\mathrm{eff})} + 3H(\rho_\phi^{(\mathrm{eff})} + p_\phi^{(\mathrm{eff})}) = 0. \tag{83}$$

Equivalently, Eq. (83) can be written as

$$\dot{\rho}_\phi + 3H(1+Q)\rho_\phi + 3Hp_\phi^{(\mathrm{eff})}e^{-\Gamma(t)} = 0 \tag{84}$$

or

$$\dot{\phi}\,\ddot{\phi} + \dot{\phi}V'(\phi) + 3H(1+Q)\frac{\dot{\phi}^2}{2} + 3H(1+Q)V(\phi) + 3Hp_\phi^{(\mathrm{eff})}e^{-\Gamma(t)} = 0. \tag{85}$$

With the use of the dissipative Klein-Gordon equation (37) we can now fix the effective form of the pressure of the dissipative scalar field as

$$p_\phi^{(\mathrm{eff})} = (1+Q)p_\phi e^{\Gamma(t)}. \tag{86}$$

It is easy to check that with this form of $p_\phi^{(\mathrm{eff})}$, Eq. (83) is equivalent to the dissipative Klein-Gordon equation (37).

Hence, for the flat FLRW metric (30), the Friedmann equations in the presence of a dissipative scalar field, with the dissipative exponent $\Gamma(t) = 3\int H(t)Q(t)dt$, take the form

$$3H^2 = \frac{8\pi G}{c^2}(\rho_\phi^{(\mathrm{eff})} + \rho_m c^2) = \frac{8\pi G}{c^2}\left[\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)e^{3\int H(t)Q(t)dt} + \rho_m c^2\right], \tag{87}$$

$$2\dot{H} + 3H^2 = -\frac{8\pi G}{c^2}(p_\phi^{(\mathrm{eff})} + p_m) = -\frac{8\pi G}{c^2}\left[(1+Q)\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \times e^{3\int H(t)Q(t)dt} + p_m\right], \tag{88}$$

which must be considered together with the dissipative Klein-Gordon equation, Eq. (37). By eliminating the term $3H^2$ between Eqs. (87) and (88) we obtain the time evolution of the Hubble function as

$$2\dot{H} = -\frac{8\pi G}{c^2}\left[\left(1+\frac{Q}{2}\right) - QV(\phi)\right]e^{3\int H(t)Q(t)dt} - \frac{8\pi G}{c^2}(\rho_m c^2 + p_m). \tag{89}$$

For $Q = 0$ we recover the basic equations describing the standard quintessence cosmological models.

Once $V(\phi)$, $Q(t)$, and the equation of state of the cosmological matter $p_m = p_m(\rho_m)$ are known, the system of Eqs. (87)–(89) and (37) represents a system of differential-integral equations for the unknowns (H, ϕ, ρ_m).

From the Friedmann equations (87) and (88) we can obtain the generalized conservation equation

$$\frac{d}{dt}(a^3\rho_\phi^{(\mathrm{eff})}) + \frac{da^3}{dt}p_\phi^{(\mathrm{eff})} + \frac{d}{dt}(a^3\rho_m c^2) + \frac{da^3}{dt}p_m = 0. \tag{90}$$

Since we have already assumed that the effective dissipative scalar field is conserved, it follows that the matter energy density is also conserved, and hence no energy-matter transfer can take place between the dissipative scalar field and the normal baryonic matter. Hence, the baryonic matter content of the universe satisfies the conservation equation,

$$\dot{\rho}_m + 3H\left(\rho_m + \frac{p_m}{c^2}\right) = 0. \tag{91}$$

A useful cosmological observational quantity, the deceleration parameter q, having the definition

$$q = \frac{d}{dt}\frac{1}{H} - 1 = -\frac{\dot{H}}{H^2} - 1, \tag{92}$$

is obtained as

$$q = \frac{1}{2}\left[1 + 3\frac{(1+Q(t))(\frac{1}{2}\dot{\phi}^2 - V(\phi))e^{3\int H(t)Q(t)dt} + p_m}{(\frac{1}{2}\dot{\phi}^2 + V(\phi))e^{3\int H(t)Q(t)dt} + \rho_m c^2}\right]. \tag{93}$$

We can also introduce the parameter w of the equation of state of the dark energy, which is given by

$$w = \frac{p_\phi^{(\mathrm{eff})}}{\rho_\phi^{(\mathrm{eff})}} = (1+Q)\frac{p_\phi}{\rho_\phi}. \tag{94}$$

1. Dimensionless form of the generalized Friedmann equations

To simplify the mathematical expressions of the Friedmann equations, we define a set of dimensionless variables $(\tau, h, \Phi, U, r_m, P_m)$, defined according to

$$t = \frac{1}{H_0}\tau, \quad H = H_0 h, \quad \phi = \sqrt{\frac{3c^2}{8\pi G}}\Phi, \quad V = \frac{3H_0^2c^2}{8\pi G}U, \quad \rho_m = \frac{3H_0^2}{8\pi G}r_m, \quad p_m = \frac{3H_0^2c^2}{8\pi G}P_m, \tag{95}$$

where H_0 is the present day value of the Hubble function.

The dimensionless matter density can also be written as $r_m = \rho_m/\rho_c = \Omega_m$, where $\rho_c = 3H_0^2/8\pi G$ is the critical density, while Ω_m denotes the density parameter of the baryonic matter.

Then the Friedman, the Klein-Gordon, and the energy balance equations take the form

$$h^2 = \left[\frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 + U(\Phi)\right] e^{3\int h(\tau)Q(\tau)d\tau} + r_m, \quad (96)$$

$$2\frac{dh}{d\tau} + 3h^2 = -3\left\{\left[(1+Q)\left(\frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 - U(\Phi)\right)\right] \times e^{3\int h(\tau)Q(\tau)d\tau} + P_m\right\}, \quad (97)$$

$$\frac{d^2\Phi}{d\tau^2} + 3h(1+Q)\frac{d\Phi}{d\tau} + U'(\Phi) = 0, \quad (98)$$

$$\frac{dr_m}{d\tau} + 3h(r_m + P_m) = 0. \quad (99)$$

Moreover, we introduce the substitution

$$u(\tau) = \int h(\tau)Q(\tau)d\tau, \quad (100)$$

giving $u'(\tau) = Q(\tau)h(\tau)$. Then the system of the Friedmann-Klein-Gordon equations of the dissipative quintessence cosmology can be formulated as a second order differential system, given by

$$\left(\frac{du}{d\tau}\right)^2 = Q^2\left\{\left[\frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 + U(\Phi)\right]e^{3u} + r_m\right\}, \quad (101)$$

$$\frac{2}{Q}\frac{d^2u}{d\tau^2} - \frac{2}{Q^2}\frac{du}{d\tau}\frac{dQ}{d\tau} + \frac{3}{Q^2}\left(\frac{du}{d\tau}\right)^2 = -3\left\{\left[\frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 - U(\Phi)\right]e^{3u} + P_m\right\}, \quad (102)$$

$$\frac{d^2\Phi}{d\tau^2} + 3\left(1 + \frac{1}{Q}\right)\frac{du}{d\tau}\frac{d\Phi}{d\tau} + U'(\Phi) = 0, \quad (103)$$

$$\frac{dr_m}{d\tau} + 3h(r_m + P_m) = 0. \quad (104)$$

2. The generalized Friedmann equations in the redshift space

To allow a straightforward comparison between the theoretical predictions and cosmological observations we introduce, instead of the time variable, the redshift z, defined as $1/a = 1 + z$.

Then the system of equations describing the cosmological evolution in the presence of a dissipative scalar field takes the form

$$\frac{du(z)}{dz} = -\frac{Q(z)}{1+z}, \quad (105)$$

$$\frac{d\Phi(z)}{dz} = -\frac{v(z)}{(1+z)h(z)}, \quad (106)$$

$$h^2(z) = \left[\frac{1}{2}(1+z)^2h^2(z)\left(\frac{d\Phi(z)}{dz}\right)^2 + U(\Phi)\right]e^{3u(z)} + r_m(z), \quad (107)$$

$$-2(1+z)h(z)\frac{dh(z)}{dz} + 3h^2(z) = -3\left\{\left[(1+Q(z))\left(\frac{1}{2}\left(1+z\right)^2h^2(z)\left(\frac{d\Phi}{dz}\right)^2 - U(\Phi)\right)\right] \times e^{3u(z)}\right\}, \quad (108)$$

$$-(1+z)h(z)\frac{dv(z)}{dz} + 3h(z)(1+Q(z))v(z) + U'(\Phi) = 0, \quad (109)$$

$$-(1+z)\frac{dr_m(z)}{dz} + 3r_m(z) = 0, \quad (110)$$

where we have denoted $v = d\Phi/d\tau$ and we have assumed $P_m = 0$.

By eliminating $h^2(z)$ between Eqs. (107) and (108), we obtain for $h(z)$ the following differential equation:

$$h(z)\frac{dh(z)}{dz} = \left[\frac{3}{2}(1+z)h^2(z)\left(1 + \frac{Q(z)}{2}\right)\left(\frac{d\Phi(z)}{dz}\right)^2 - \frac{3}{2}\frac{Q(z)}{1+z}U(\Phi)\right]e^{3u(z)} + \frac{3}{2}\frac{r_m}{1+z}. \quad (111)$$

Equations (105)–(110) represent a system of first order ordinary differential equations with the unknowns (u, Φ, v, h, r_m), with the solution satisfying the constraint (107). To solve the system, the functional form of the functions $Q(z)$ and $U(\phi)$ must be provided. The system must be integrated with the initial conditions $u(0) = u_0$, $\Phi(0) = \Phi_0$, $v(0) = v_0$, $h(0) = 1$, and $r_m(0) = r_{m0}$, respectively. Equation (110) can be immediately integrated to give for the matter density parameter the expression

$$r_m(z) = \Omega_m(z) = \Omega_{m0}(1+z)^3, \quad (112)$$

where Ω_{m0} is the present day matter density parameter.

IV. SIMPLE COSMOLOGICAL MODELS WITH DISSIPATIVE SCALAR FIELD

In the present section we will investigate the cosmological implications of the dissipative scalar field models by considering some simple analytical forms of the dissipation function $Q(\tau)$. We will consider the effects of dissipation only on the late cosmological evolution, and hence we will neglect the effects of the matter pressure in the field equations (105)–(110), which are the basic equations describing the expansionary dynamics of the universe for the dissipation exponent given by $\Gamma(\tau) = 3\int H(\tau)Q(\tau)d\tau$.

To test the cosmological viability of the dissipative scalar field model we will compare its theoretical predictions with the standard ΛCDM model, and with a set of observational data for the Hubble function.

In a three component universe, consisting of baryonic matter, dark matter, and dark energy, respectively, the Hubble function of the ΛCDM model is given by

$$H = H_0\sqrt{\frac{\Omega_m^{(cr)}}{a^3} + \Omega_\Lambda} = H_0\sqrt{\Omega_m^{(cr)}(1+z)^3 + \Omega_\Lambda}, \qquad (113)$$

where $\Omega_m^{(cr)} = \Omega_b^{(cr)} + \Omega_{\rm DM}^{(cr}$, and $\Omega_b^{(cr)} = \rho_b/\rho_{cr}$, $\Omega_{\rm DM} = \rho_{\rm DM}/\rho_{cr}$, and $\Omega_\Lambda = \Lambda/\rho_{cr}$ denote the density parameters of the baryonic matter, dark matter, and dark energy, respectively. In the ΛCDM model the deceleration parameter is given by the relation

$$q(z) = \frac{3(1+z)^3\Omega_m}{2[\Omega_\Lambda + (1+z)^3\Omega_m]} - 1. \qquad (114)$$

In the following we adopt for the matter and dark energy density parameters of the ΛCDM model the values $\Omega_{\rm DM} = 0.2589$, $\Omega_b = 0.0486$, and $\Omega_\Lambda = 0.6911$, respectively [17]. Then total matter density parameter $\Omega_m = \Omega_{\rm DM} + \Omega_b$ has the value $\Omega_m = 0.3089$. The present day value of the deceleration parameter is given by $q(0) = -0.5381$, indicating that presently the universe is in an accelerating phase.

A. The de Sitter solution

As a first example of a cosmological model with dissipative scalar field we will consider the case for which the Hubble function is a constant, $h = h_0 = \text{const}$, corresponding to an exponential expansion of the universe, and with a deceleration parameter $q = -1$. Moreover, we assume a vacuum universe, with $r_m = 0$. Then, by adding Eqs. (96) and (97) we obtain the relation

$$h_0^2 \frac{2+Q}{1+Q} e^{-\Gamma(\tau)} = 2U(\Phi). \qquad (115)$$

This equation is identically satisfied for $Q = -2$, and $U(\Phi) = 0$. The Klein-Gordon equation becomes

$$\frac{d^2\Phi}{d\tau^2} - 3h_0\frac{d\Phi}{d\tau} = 0, \qquad (116)$$

with the general solution given by

$$\Phi(\tau) = \frac{C_1}{3h_0}e^{3h_0\tau} + C_2, \qquad (117)$$

where C_1 and C_2 are arbitrary constants of integration. We can take $C_2 = 0$ without any loss of generality. For the dissipation exponent we obtain the expression $\Gamma(\tau) = -6h_0\tau$. Hence, in this simple model, the exponential expansion of the universe is triggered by the exponential increase of the scalar field, downsized by the decrease of the dissipation exponent.

An alternative approach for obtaining de Sitter type solutions is based on directly solving the Friedmann constraint equation (96) for a constant h and vanishing matter energy density. Then we obtain the differential equation,

$$h_0^2 e^{-\Gamma(\tau)} = \frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 + U(\Phi), \qquad (118)$$

which, once the field potential and the dissipation exponent are known, can be directly integrated to give the evolution of the scalar field $\Phi(\tau)$. For $U(\Phi) = 0$, we obtain

$$\Phi(\tau) = \sqrt{2}h_0\int e^{-\Gamma(\tau)/2}d\tau + \text{const}. \qquad (119)$$

For $\Gamma(\tau) = -6h_0\tau$, and taking the additive integration constant as zero, we obtain $\Phi(\tau) = (\sqrt{2}/3)e^{3h_0\tau}$, which allows us to fix the integration constant C_1 in Eq. (117) as $C_1 = \sqrt{2}h_0$.

1. de Sitter type expansion with constant dissipation function $Q_0 \neq -2$

Let us assume now that the dissipation function Q takes constant values at least on a finite time interval, so that $Q = Q_0 = \text{const} \neq -2$. Then, we obtain

$$U(\Phi(\tau)) = \frac{h_0^2}{2}\frac{2+Q_0}{1+Q_0}e^{-3h_0Q_0\tau}, \qquad Q_0 \neq -2. \qquad (120)$$

In the limit of large τ, the scalar field potential tends to zero, $\lim_{\tau\to\infty}U(\Phi(\tau)) = 0$.

The Klein-Gordon equation takes the form

$$\frac{d}{d\tau}\left(\frac{d\Phi}{d\tau}\right)^2 + 6h_0(1+Q_0)\left(\frac{d\Phi}{d\tau}\right)^2 - 3h_0^3\frac{2+Q_0}{1+Q_0}e^{-3h_0Q_0\tau} = 0. \qquad (121)$$

A first integration leads to

$$\frac{d\Phi}{d\tau} = \sqrt{C_3 - \frac{(2+Q_0)h_0^2}{(1+Q_0)[1+6h_0(1+Q_0)]}e^{-3h_0Q_0\tau}}, \qquad (122)$$

where C_3 is an arbitrary integration constant, leading to the time evolution of the scalar field as given by

$$\Phi(\tau) = \frac{2\sqrt{C_3}}{3h_0Q_0}\left[\sqrt{1-\frac{h_0^2(2+Q_0)e^{-3h_0Q_0t}}{C_3(1+Q_0)[6h_0(1+Q_0)+1]}} - \tanh^{-1}\left(\sqrt{1-\frac{h_0^2(2+Q_0)e^{-3h_0Q_0t}}{C_3(1+Q_0)[6h_0(1+Q_0)+1]}}\right)\right], \tag{123}$$

where an integration constant has been set to zero. In this model the dependence of the potential on the scalar field is given in a parametric form, $U = U(\tau)$, $\Phi = \Phi(\tau)$. Both positive and negative values of Q_0 are possible, and the evolution of the scalar field, and of its potential, is basically determined by the dissipation constant. Hence, depending on the numerical value of Q_0, a large number of exponentially expanding vacuum cosmological models can be obtained. In the limit of large times, as one can see from Eq. (122), $\Phi(\tau) \approx \sqrt{C_3}\tau$, and, even in the absence of the potential of the scalar field, the de Sitter expansionary phase is triggered by the time derivative of the scalar field.

B. Models with dynamical Hubble function

In the present subsection we will consider two simple cosmological models in the presence of a dissipative scalar field and of a matter component. We will consider two classes of models, under the assumptions that either the scalar field potential or its kinetic term can be neglected. A comparison with the observational data, and with the standard ΛCDM model will also be performed.

1. Models with vanishing scalar field potential

We consider now a cosmological model in the presence of a dissipative scalar field and of ordinary pressureless matter, in which we give up the assumption of the global constancy of the Hubble function. For simplicity, we assume that the dissipation function Q is a constant, $Q = Q_0 = \text{const}$, and that the potential of the scalar field vanishes, $U(\Phi) = 0$. Then, the evolution of the matter density is given by Eq. (112). From the Klein-Gordon equation (98) we obtain for the time derivative of the scalar field the expression

$$\dot{\Phi} = \dot{\Phi}_0 a^{-3(1+Q_0)}, \tag{124}$$

where $\dot{\Phi}_0$ is an arbitrary constant of integration. For the dissipation exponent we obtain $\Gamma = 3Q_0\int h(\tau)d\tau = 3Q_0 \ln a$. By combining Eqs. (96) and (97), we obtain the cosmological evolution equation of the model as

$$\frac{dh}{d\tau} = -\frac{3}{2}\left(1+\frac{Q_0}{2}\right)\dot{\Phi}^2 e^{\Gamma} - \frac{3}{2}r_m. \tag{125}$$

In the redshift space we obtain the following differential equation for $h(z)$:

$$h(z)\frac{dh(z)}{dz} = \frac{3}{2}\left(1+\frac{Q_0}{2}\right)\dot{\Phi}_0^2(1+z)^{5+3Q_0} + \frac{3}{2}\Omega_{m0}(1+z)^2. \tag{126}$$

Equation (126) must be integrated with the initial condition $h(0) = 1$, after fixing the numerical values of the parameters $(Q_0, \dot{\Phi}_0, \Omega_{m0})$.

The variations of the Hubble function and of the deceleration parameter are presented as a function of the redshift z in Fig. 1. The cosmological parameters corresponding to the ΛCDM model are also shown, together with a set of observational data for the Hubble function, as compiled in [107].

As one can see from Fig. 1, this simple dissipative cosmological model gives a good description of the observational data and coincides with the predictions of the ΛCDM model for a large range of redshifts. The Hubble function can be expressed in an exact form as

$$h(z) = \sqrt{1+\frac{\dot{\Phi}_0^2}{2}[(1+z)^{3(2+Q_0)}-1] + \Omega_{m0}[(1+z)^3-1]}, \tag{127}$$

while the deceleration parameter can be obtained in the form

$$q(z) = \frac{(1+z)}{h(z)}\frac{dh(z)}{dz} - 1 = \frac{(z+1)[3\Omega_{m0}(1+z)^2 + \frac{3}{2}\dot{\Phi}_0^2(2+Q_0)(1+z)^{3Q_0+5}]}{2\{1+\frac{1}{2}\dot{\Phi}_0^2[(1+z)^{3(2+Q_0)}-1] + \Omega_{m0}[(1+z)^3-1]\}} - 1. \tag{128}$$

The parameter of the equation of state of the dissipative scalar field takes the form

$$w = (1+Q_0) = \text{const}. \tag{129}$$

The best fit with the observational data is provided for $Q_0 = -1.29$, which gives $w = -0.29$. The variations of the effective density $\rho_\phi^{(\text{eff})}$ of the scalar field,and of its effective pressure $p_\rho^{(\text{eff})}$ are represented in Fig. 2.

For the best fit values $Q_0 = -1.29$, the effective density and pressure of the scalar field are constants, with the pressure taking small negative values. Such a dissipative

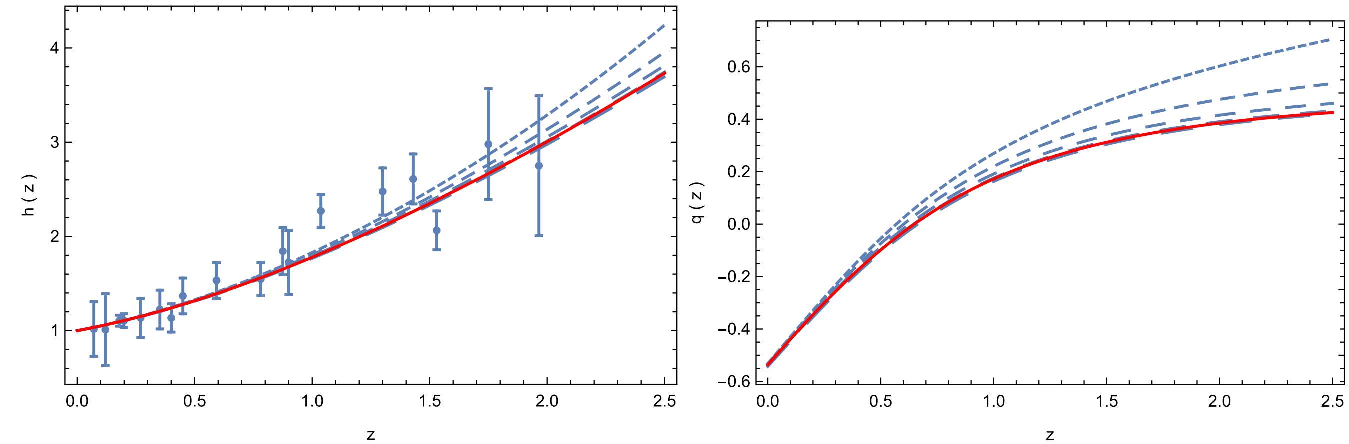


FIG. 1. Variation of the dimensionless Hubble function $h(z)$ (left panel) and of the deceleration parameter $q(z)$ (right panel), in the dissipative scalar field cosmological model with $U(\Phi) = 0$, for $\dot{\Phi}_0 = 0.12$, for $\Omega_{m0} = 0.30$, and for different values of Q_0: $Q_0 = -0.29$ (dotted curve), $Q_0 = -0.49$ (short dashed curve), $Q_0 = -0.69$ (dashed curve), $Q_0 = -0.89$ (long dashed curve), and $Q_0 = -1.29$ (ultralong dashed curve), respectively. The predictions of the ΛCDM model are represented by the red solid curve, while the observational data are represented together with their error bars.

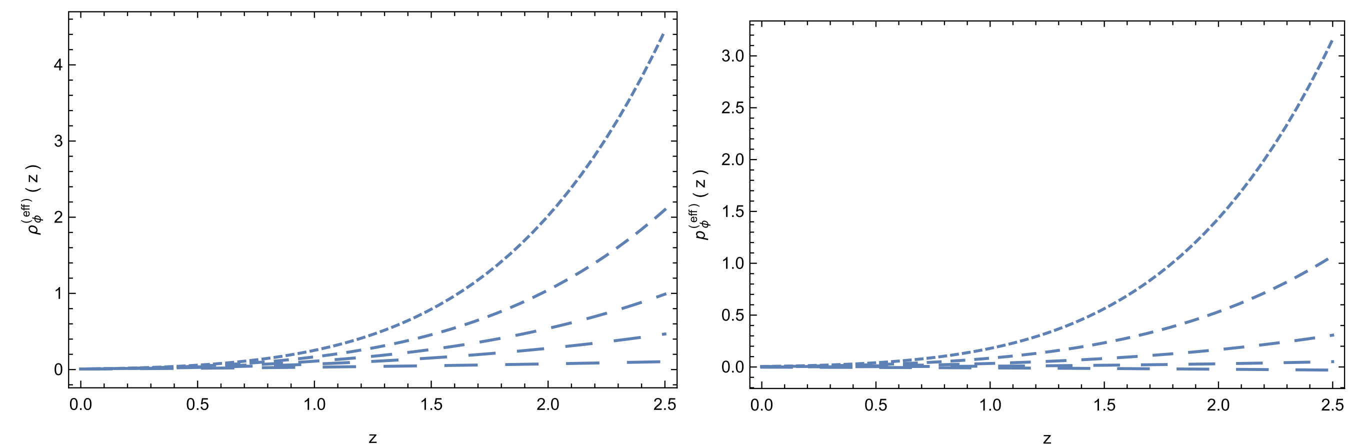


FIG. 2. Variation of the effective energy density $\rho_\phi^{(\mathrm{eff})}$ of the scalar field (left panel) and of the effective pressure $p_\rho^{(\mathrm{eff})}$ (right panel), in the dissipative scalar field cosmological model with $U(\Phi) = 0$, for $\dot{\Phi}_0 = 0.12$, $\Omega_{m0} = 0.30$, and for different values of Q_0: $Q_0 = -0.29$ (dotted curve), $Q_0 = -0.49$ (short dashed curve), $Q_0 = -0.69$ (dashed curve), $Q_0 = -0.89$ (long dashed curve), and $Q_0 = -1.29$ (ultralong dashed curve), respectively.

scalar field behaves like a cosmological constant even in the absence of the potential term.

For the sake of completeness, we will also consider one more parameter for the present cosmological model, which allows testing its viability, namely, the $Om(z)$ diagnostic, with

$$Om(z) = \frac{h^2(z) - 1}{(1+z)^3 - 1}. \tag{130}$$

For the ΛCDM model, the function $Om(z)$ is a constant, equal to the present day matter density Ω_{m0}. The variation of the $Om(z)$ function is represented in Fig. 3.

For $Q_0 = -1.29$, the behavior of the $Om(z)$ function in this dissipative scalar field model is very close to its behavior in the standard ΛCDM paradigm.

2. Dissipative scalar field models with negligible kinetic term

We consider now the case in which the potential term dominates the effective energy density and pressure of the scalar field; that is, $U(\Phi)$ satisfies the condition $U(\Phi) \gg \dot{\Phi}^2/2$. For simplicity, we assume that the scalar field potential is given by the expression

$$U(\Phi) = \frac{m}{2}\Phi^2, \tag{131}$$

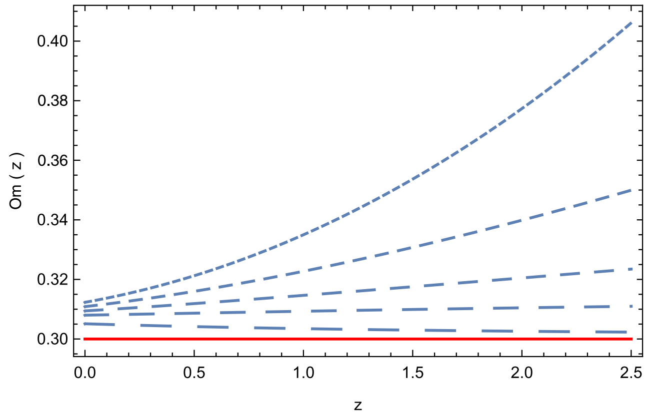


FIG. 3. Variation of the $Om(z)$ diagnostic function in the dissipative scalar field cosmological model with $U(\Phi) = 0$, for $\dot{\Phi}_0 = 0.12$, $\Omega_{m0} = 0.30$, and for different values of Q_0: $Q_0 = -0.29$ (dotted curve), $Q_0 = -0.49$ (short dashed curve), $Q_0 = -0.69$ (dashed curve), $Q_0 = -0.89$ (long dashed curve), and $Q_0 = -1.29$ (ultralong dashed curve), respectively. The solid red curve represents the $Om(z)$ function in the ΛCDM cosmology.

where m is a constant. In the following we also neglect the matter pressure, taking $P_m = 0$, and assume that the dissipation function is a constant. Then, in the redshift space, the system of equations describing the evolution of the scalar field and of the Hubble function takes the form

$$\frac{d\Phi(z)}{dz} = -\frac{u(z)}{(1+z)h(z)}, \tag{132}$$

$$(1+z)h(z)\frac{du(z)}{dz} - 3h(z)(1+Q_0)u(z) - m\Phi(z) = 0, \tag{133}$$

$$h(z)\frac{dh(z)}{dz} = -\frac{3}{4}Q_0 m\Phi^2(z)(1+z)^{3Q_0-1} + \frac{3}{2}\Omega_{m0}(1+z)^2. \tag{134}$$

The system of equations (132)–(134) must be solved with the initial conditions $\Phi(0) = \Phi_0$, $u(0) = u_0$, and $h(0) = 1$, respectively.

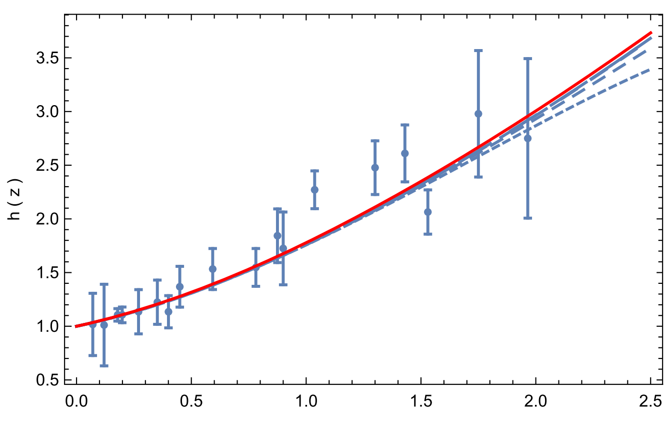
The redshift evolutions of the Hubble function and of the deceleration parameter of the dissipative scalar field model with a negligible kinetic term are represented in Fig. 4. As one can see from the two panels of Fig. 4, with the values of Q_0 moving into the negative range, the concordance with the cosmological data and the ΛCDM model becomes better and better for both $h(z)$ and $q(z)$. For $Q_0 = -0.45$, both the Hubble function and the deceleration parameter are basically visually indistinguishable from the predictions of the standard cosmological paradigm.

The variation of the scalar field potential and the $Om(z)$ diagnostic function are presented in Fig. 5. The scalar field potential is roughly a constant, almost exactly mimicking a cosmological constant. The redshift variation of the Φ^2 type potential is (almost) exactly compensated by the dissipation exponent, resulting in an almost constant contribution to the Friedmann equations. However, the cosmological evolution, even accelerated, is not exactly of the de Sitter type. The $Om(z)$ function also tends toward its ΛCDM value, and thus this cosmological parameter is well recovered in the dissipative scalar field cosmology.

The parameter of the equation of state of the dissipative quintessence type dark energy is given by

$$w = -(1+Q_0) \approx -0.55, \tag{135}$$

if one uses the best empirical approximation of Q_0. This constant negative equation of state is different from the equation of state of the quintessence fields with negligible kinetic terms, which is $w = -1$.



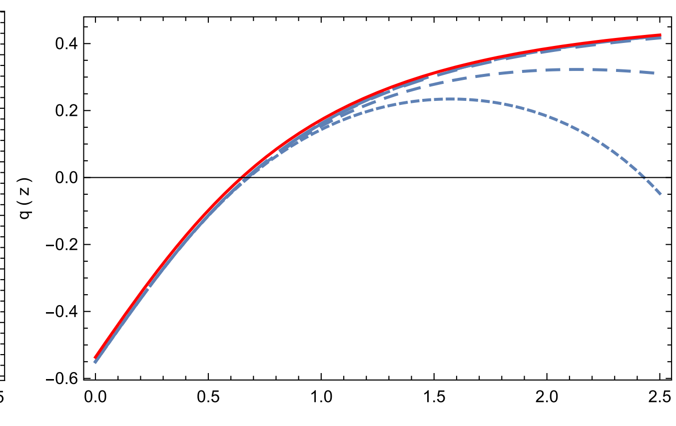


FIG. 4. Variation of the dimensionless Hubble function $h(z)$ (left panel) and of the deceleration parameter $q(z)$ (right panel), in the dissipative scalar field cosmological model with a negligibly kinetic term, and $U(\Phi) = m\Phi^2/2$, for $\Phi(0) = 0.11$, $u(0) = 0.30$, $\Omega_{m0} = 0.30$, $m = 0.12$, and for different values of Q_0: $Q_0 = 0.45$ (dotted curve), $Q_0 = 0.35$ (short dashed curve), $Q_0 = 0.15$ (dashed curve), $Q_0 = -0.15$ (long dashed curve), and $Q_0 = -0.45$ (ultralong dashed curve), respectively. The predictions of the ΛCDM model are represented by the red solid curve, while the observational data are represented together with their error bars.

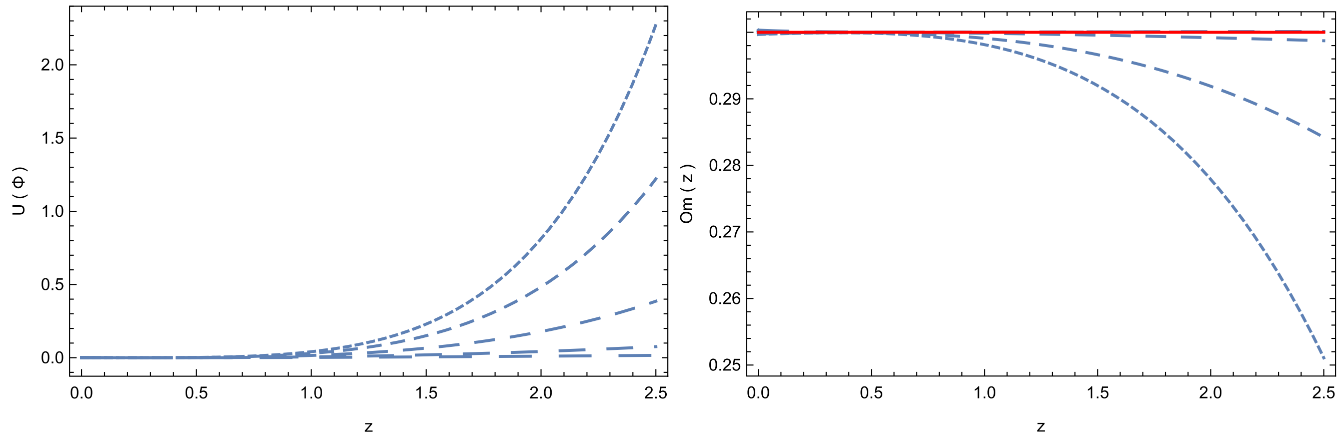


FIG. 5. Variation of the scalar field potential $U(\Phi) = m\Phi^2/2$ (left panel) and of the $Om(z)$ function (right panel), in the dissipative scalar field cosmological model with a negligibly kinetic term, for $\Phi(0) = 0.11$, $u(0) = 0.30$, $\Omega_{m0} = 0.30$, $m = 0.12$, and for different values of Q_0: $Q_0 = 0.45$ (dotted curve), $Q_0 = 0.35$ (short dashed curve), $Q_0 = 0.15$ (dashed curve), $Q_0 = -0.15$ (long dashed curve), and $Q_0 = -0.45$ (ultralong dashed curve), respectively. The predictions of the ΛCDM model are represented by the red solid curve.

V. COSMOLOGICAL MODELS WITH DYNAMICAL DISSIPATION FUNCTION

We consider now a more general class of cosmological models, in which the dissipation function is dynamical. For simplicity, we adopt for Q a simple functional representation as

$$Q(z) = Q_0(1+z)^{\alpha}, \tag{136}$$

where Q_0 and α are constants. For the potential of the scalar field we still adopt the simple quadratic form (131), and we also keep the kinetic term of the field in the mathematical formalism. The system of equations to be solved is Eqs. (105)–(110), together with a set of appropriately chosen initial conditions. By taking into account the explicit form of $Q(z)$, Eq. (105) can be integrated to give

$$u(z) = -\frac{Q_0}{\alpha}(1+z)^{\alpha}. \tag{137}$$

Then, the equations describing the cosmological evolution of the universe in the presence of a dissipative scalar field with a dynamic dissipation function take the form

$$\frac{d\Phi(z)}{dz} = -\frac{v(z)}{(1+z)h(z)}, \tag{138}$$

$$h(z)\frac{dh(z)}{dz} = \frac{3}{4}\left\{[2+Q_0(1+z)^{\alpha}](1+z)h^2[z]\left(\frac{d\Phi}{dz}\right)^2 - mQ_0(1+z)^{\alpha-1}\Phi^2(z)\right\}e^{-\frac{3Q_0}{\alpha}(1+z)^{\alpha}} + \frac{3}{2}\Omega_{m0}(1+z)^2, \tag{139}$$

$$(1+z)h(z)\frac{dv(z)}{dz} - 3h(z)[1+Q_0(1+z)^{\alpha}]v(z) - m\Phi(z) = 0. \tag{140}$$

The system of equations (138)–(140) must be integrated with the initial conditions $\Phi(0) = \Phi_0$, $v(0) = v_0$, and $h(0) = 1$, once the numerical values of the parameters (Q_0, α, m) have been specified.

For the sake of comparison we also present the cosmological evolution in the presence of the ideal quintessence field with quadratic potential, with $\Gamma = 0$, that is, in the absence of any dissipative phenomena. The results of the numerical integration of the ideal quintessence field equations are represented by an orange curve.

The redshift variations of the Hubble function and of the deceleration parameter are represented in Fig. 6, for a constant Q_0 and different values of α. The numerical results show a relatively strong dependence on the numerical values of the parameter α, but for $\alpha = -0.60$, the predictions of the dissipative scalar field cosmological model, and of the ΛCDM model basically coincide for both the Hubble function and the deceleration parameter. For low redshifts, up to $z \approx 1.5$, the cosmological evolution is basically independent of the numerical values of α, and the concordance with the ΛCDM model is very good, at least for the rescaled Hubble function $h(z)$. The model can also reproduce very well the predictions of the ΛCDM model for the deceleration parameter.

The variations of the effective energy density of the dissipative scalar field, as well as the behavior of the effective pressure for the quadratic field potential, are represented in Fig. 7. For the best fit values of the model with the cosmological observations both the energy density and the pressure become approximately constant in the considered range of z, and hence they mimic a cosmological constant.

The parameter $w(z)$ of the equation of state of the scalar field is given by

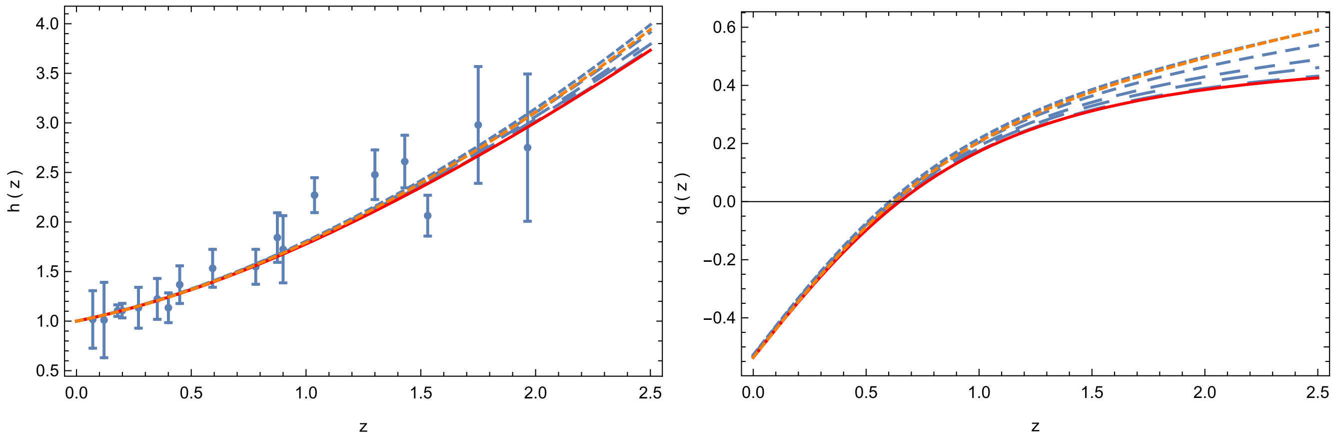


FIG. 6. Variation of the dimensionless Hubble function $h(z)$ (left panel) and of the deceleration parameter $q(z)$ (right panel), in the dissipative scalar field cosmological model with dynamical dissipation function, and quadratic scalar field potential $U(\Phi) = m\Phi^2/2$, for $\Phi(0) = 0.11$, $v(0) = 0.30$, $\Omega_{m0} = 0.3089$, $m = 0.12$, $Q_0 = -0.89$, and for different values of α: $\alpha = -1.29$ (dotted curve), $\alpha = -1.15$ (short dashed curve), $\alpha = -0.98$ (dashed curve), $\alpha = -0.85$ (long dashed curve), and $\alpha = -0.60$ (ultralong dashed curve), respectively. The predictions of the ΛCDM model are represented by the red solid curve, while the observational data are given together with their error bars. The evolution of the cosmological parameters of the ideal quintessence field with quadratic potential, with $\Gamma = 0$, is represented for $m = 0.682$, $\Phi(0) = 0.19$, and $v(0) = 0.01$ by the orange curve.

$$w(z) = \frac{[1 + Q_0(1+z)^\alpha][(1+z)^2 h^2(z)(\frac{d\Phi(z)}{dz})^2 - m\Phi^2(z)]}{[(1+z)^2 h^2(z)(\frac{d\Phi(z)}{dz})^2 + m\Phi^2(z)]}. \tag{141}$$

The variation of the functions $w(z)$ and $Om(z)$ are represented in Fig. 8. It is interesting to note that even the parameter of the equation of state of the scalar field is positive for all considered redshift range, and the model still can explain satisfactorily the observational data and gives almost the same predictions as the ΛCDM model. The behavior of the $Om(z)$ function is strongly dependent on the numerical values of α, but for $\alpha = -0.60$ it approaches significantly the ΛCDM value.

On the other hand, as one can see from Figs. 6, 7, and 8, by using a different set of values for the potential parameter m and for the initial conditions $\Phi(0)$ and $v(0)$, the ideal quintessence field model with quadratic potential can also give a good description of the observational data for the Hubble function, and of the ΛCDM model. However, significant differences do appear in the behaviors of the energy density and pressure of the ideal and dissipative scalar field, as well as in the parameter of the equation of state of the dark energy.

Hence, at least in principle, it is possible to construct ideal quintessence models that mimic their dissipative counterparts at the background evolution level by adopting different values for the potential parameters, and for the

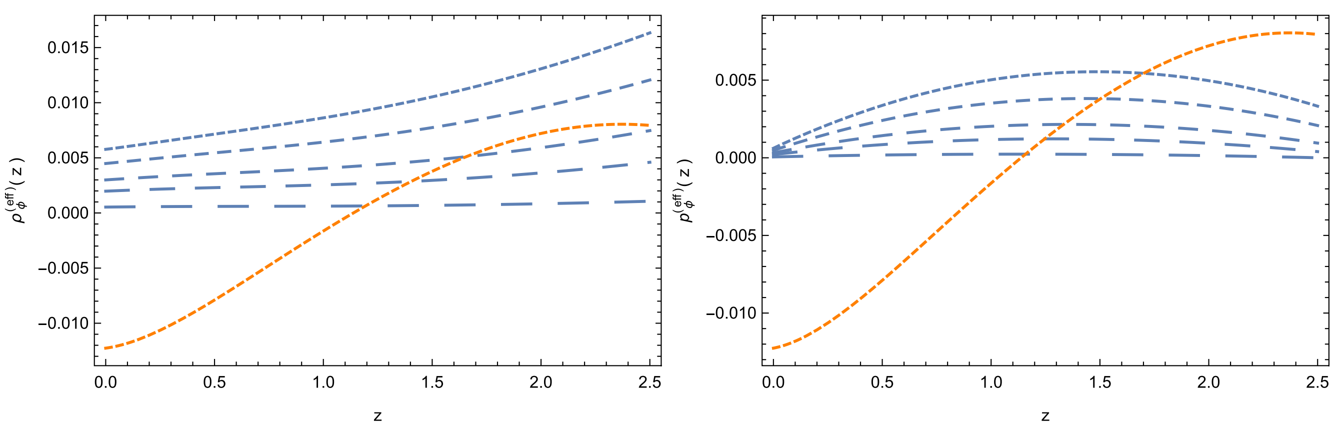


FIG. 7. Variation of the effective energy density of the scalar field (left panel) and of the effective pressure (right panel), in the dissipative scalar field cosmological model with a dynamical dissipation function, for $\Phi(0) = 0.11$, $v(0) = 0.30$, $\Omega_{m0} = 0.3089$, $m = 0.12$, $Q_0 = -0.89$, and for different values of α: $\alpha = -1.29$ (dotted curve), $\alpha = -1.15$ (short dashed curve), $\alpha = -0.98$ (dashed curve), $\alpha = -0.85$ (long dashed curve), and $\alpha = -0.60$ (ultralong dashed curve), respectively. The evolution of the cosmological parameters of the ideal quintessence field with quadratic potential, with $\Gamma = 0$, is represented for $m = 0.682$, $\Phi(0) = 0.19$, and $v(0) = 0.01$ by the orange curve.

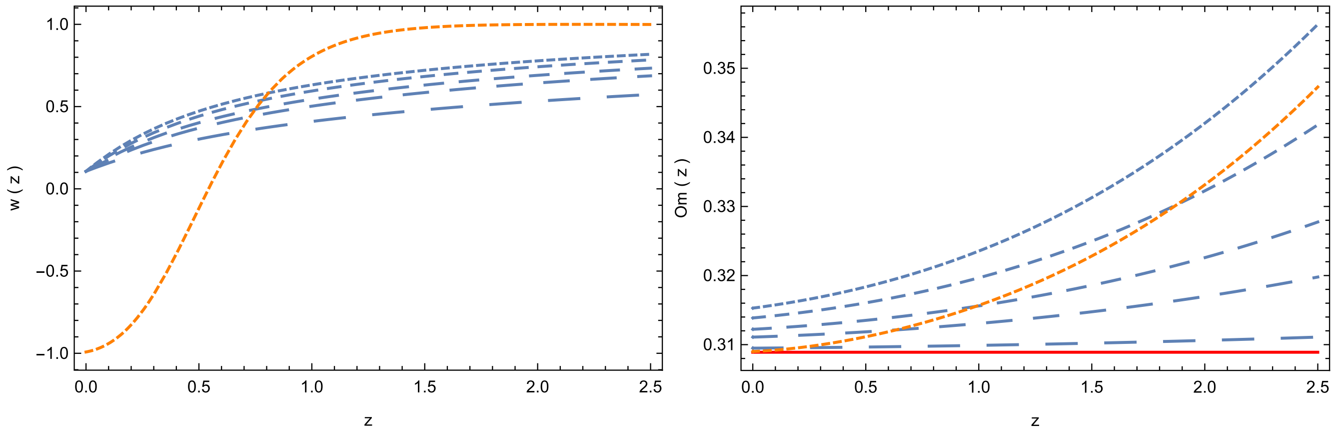


FIG. 8. Variation of the equation of state $w(z)$ (left panel) and of the $Om(z)$ function (right panel), in the dissipative scalar field cosmological model with a dynamical dissipation function, for $\Phi(0) = 0.11$, $v(0) = 0.30$, $\Omega_{m0} = 0.3089$, $m = 0.12$, $Q_0 = -0.89$, and for different values of α: $\alpha = -1.29$ (dotted curve), $\alpha = -1.15$ (short dashed curve), $\alpha = -0.98$ (dashed curve), $\alpha = -0.85$ (long dashed curve), and $\alpha = -0.60$ (ultralong dashed curve), respectively. The prediction of the ΛCDM model for the $Om(z)$ function is represented by the red solid line. The evolution of the cosmological parameters of the ideal quintessence field with quadratic potential, with $\Gamma = 0$, is represented for $m = 0.682$, $\Phi(0) = 0.19$, and $v(0) = 0.01$ by the orange curve.

initial conditions of the scalar field. The opposite situation may also be possible, with dissipative scalar field models giving an equivalent effective description of ideal quintessential field models. However, a rigorous statistical analysis of the observational datasets (Hubble, Pantheon, etc.) may still allow one to clearly discriminate between ideal and dissipative quintessence field models, due to their very different predictions for the parameter of the dark energy equation of state.

Nevertheless, important differences may appear at the perturbative level between ideal and dissipative quintessence models. In [108] it was shown, after performing a dynamical system analysis of the background and perturbation equations in the ΛCDM cosmology and in the quintessence models with an exponential potential, that in the case of quintessence the perturbations drastically modify the properties and stability of the background evolution. It turns out that in the quintessence model there is one and only one stable point. The behavior of this stable point leads either to an exponentially increasing matter clustering, not detected in cosmological observations, or to a physically not interesting Laplacian instability. Hence, the quintessence cosmological models may be in a severe disadvantage as compared to the standard ΛCDM model. Some of these problems may be solvable in the dissipative quintessence scenario, which, for example, may limit the exponential increase of the matter clustering via the dissipation of the scalar field energy.

VI. DISCUSSIONS AND FINAL REMARKS

In the present paper we have considered the cosmological implications of a dissipative scalar field, whose theoretical description can be obtained from a variational principle, inspired by the case of the simple damped harmonic oscillator. In performing such a generalization and extension of the scalar field models we assume that dissipation may be a general property of physical systems, and its presence should be unavoidable in any natural process. It is interesting to note that at very low temperatures the superfluid component of the liquid helium behaves as an irrotational ideal fluid, flowing without friction [72,73]. However, once a critical velocity v_c is reached, dissipation sets in, and the flow is not frictionless anymore. In the standard physical interpretation of this process, it is assumed that dissipation in the superfluid flow is due to the creation, motion, and evolution of the superfluid quantized vortices in the liquid [72,73]. Dissipation can generally be attributed to the interaction of the given physical system with an external (thermal, for example) bath or to the interaction with another physical system. The interaction between dark energy and dark matter may provide a possible physical mechanism for the presence of the dissipative effects of the two basic components of the universe.

Various forms of the dissipative Klein-Gordon equation have been investigated, mostly from a mathematical point of view. The dissipative Klein-Gordon equations are usually strongly nonlinear partial differential equations. An equation of the form

$$\Box u + u = -g(\partial_t u)^2, \tag{142}$$

where g is a constant, and $\Box u = \partial_t^2 - \partial_x^2$, was investigated in [109], where it was shown that the solution of the nonlinear equation has an additional logarithmic time decay in comparison with the free evolution. The dissipative one-dimensional Klein-Gordon equation

$$u_t - u_{xx} + b(x)u_t + f(u) + h(\nabla u) = 0, \quad (143)$$

where f, g, h, b are arbitrary functions, was studied in [110]. A particular dissipative nonlinear Klein-Gordon equation of the form

$$u_{tt} - u_{xx} + \alpha u - \beta u^3 = 0, \quad (144)$$

with α and β constants, plays an important role in many fields of physics, such as in the study of the liquid helium, dislocations in crystals, the Bloch wall motion, ferromagnetic materials, the unified theory of elementary particles, Josephson array, charge density waves, and the propagation of magnetic flux on a Josephson line. (See Refs. [110] and references therein.) A nonlinear dissipative Klein-Gordon equation, given by

$$u_{tt} - \Delta u + u + \gamma u_t = |u|^{p-1} \quad (145)$$

was studied from a mathematical point of view in [111]. Hence, a large number of dissipative Klein-Gordon type equations have been proposed and investigated in detail in both mathematical and physical literature. However, most of these equations have been proposed on a phenomenological basis, as mostly empirical models for the description of some physical processes.

In dealing with the dissipation problem, in the present work we have introduced a comprehensive description of the dissipative scalar fields, based on a variational principle, which was inspired by the mathematics of one of the simplest possible dissipative systems, the damped harmonic oscillator. Dissipative processes can also be described by variational principles, even that these principles are not as commonly used as the variational principles for conservative systems. However, the Lagrangians for dissipative systems are almost as simple as those for conservative systems, and, with the use of the Euler-Lagrange equations, they allow a direct and systematic derivation of the equation of motion, as well as to obtain the basic physical properties and characteristics of the dissipative systems.

In the present approach to the scalar field physics we have introduced theoretical models in which the ordinary Lagrangian of the field is multiplied by an arbitrary function of the coordinates, of the metric, and of the scalar field. The Euler-Lagrange equations straightforwardly lead to various dissipative formulations and extensions of the Klein-Gordon equation, whose forms depend now on the dissipation exponent, and function. In a Riemannian geometry, the variational mathematical formalism allows one to obtain the dissipative Klein-Gordon equations in an explicitly covariant form. The main goal of the present study was, besides introducing the theoretical formalism, to explore the implications of the dissipative scalar fields in cosmology. Scalar fields have already been extensively used as successful dark energy models, which can mimic/replace the cosmological constant, and thus provide powerful alternatives to the standard ΛCDM paradigm. To develop some cosmological applications, we have considered dissipative scalar field models leading to the generalized Klein-Gordon equation of the form $\ddot{\phi} + 3H(1+Q)\dot{\phi} + V'(\phi) = 0$, which was also considered previously in the framework of warm inflationary cosmological models, but without being derived from a variational principle [112,113]. This dissipative Klein-Gordon equation can be derived from the standard Lagrangian $L_\phi = e^{3\int H(t)Q(t)dt}\rho_\phi$.

The variational principle allows not only the systematic introduction of the dissipation in scalar field models but also obtains the effective energy density and pressure that can be associated with the scalar field. The effective energy of the field can be obtained as the effective Hamiltonian derived in the standard way from the field Lagrangian. On the other hand, to obtain the effective pressure of the field we have imposed the cosmological conservation of the effective quantities. Generally, the Friedmann equations imply the conservation of the total matter-field content of the universe. By imposing the independent conservation laws for matter and field we have neglected the possibility of any interaction between scalar field and cosmological matter, even that such a possibility cannot be ruled out *a priori*.

The generalized conservation equation, with the effects of the matter ignored, uniquely determines the effective pressure of the dissipative field in the form $p_\phi^{(\mathrm{eff})} = (1+Q)(\dot{\phi}^2/2 - V(\phi))e^{3\int H(t)Q(t)dt}$. This effective field pressure and the effective density $\rho_\phi^{(\mathrm{eff})} = (\dot{\phi}^2/2 - V(\phi)) \times e^{3\int H(t)Q(t)dt}$ are the physical quantities that appear in the generalized Friedmann equations that describe the cosmological dynamics. From a mathematical point of view, the Friedmann equations become differential-integral equations, with the inclusion of the dissipative effects leading to a significant increase in the mathematical problem of the cosmological evolution. However, the cosmological problem is still solvable relatively straightforwardly for the considered dissipation exponent, since by means of simple mathematical transformations, one can reformulate the Friedmann-Klein-Gordon system in the redshift space as a first order differential dynamical system, whose solutions can be obtained easily numerically. We have examined in detail several cosmological models, which were obtained for different choices of the dissipation function, and of the scalar field potential. From the point of view of the dissipation function, we have considered models with constant Q and with Q a particular function of the redshift. For the scalar field potential we have also adopted two forms only, $V(\phi) = 0$ and $V(\phi) = m\phi^2/2$, respectively.

From a cosmological point of view, the most significant change in the modeling of dark energy comes from the expression of the effective pressure. First of all, successful cosmological models without the presence of the potential can easily be constructed by assuming that the dissipation function satisfies the condition $1 + Q < 0$. With this choice the kinetic term of the pressure becomes positive in the second Friedmann equation, and an effective negative pressure of the form $p_\phi^{(\mathrm{eff})} = ((1+Q)\dot{\phi}^2/2)e^{3\int H(t)Q(t)dt}$ can effectively trigger, and control, the accelerated expansion of the universe, thus playing the role of the cosmological constant and of the dark energy. On the other hand, for this model, the sign of the kinetic term in the effective energy of the field has the correct sign. Hence, no self-interacting potential is necessary for a dissipative scalar field to accelerate the universe, the role of the potential being taken over by the dissipation function. On the other hand, while many fundamental physical models do exist for the scalar field potential, to the best knowledge of the present author, no theoretical models for the dissipation exponent have been considered in the framework of the fundamental theories of elementary particle physics.

It is important to point out that, even at low redshifts $z < 2$, the predictions of the dissipative quintessence model do coincide with the predictions of the ΛCDM model, and with the observational data, some significant deviations may appear at higher redshifts $z > 2.5$. For standard quintessence models, the deviations from the evolution of the ΛCDM are bounded to be below the 10% level at 95% confidence at redshifts below $z = 1.5$ [114]. It would be interesting to investigate if the inclusion of the dissipative processes of the scalar field could significantly change this bound. On the other hand, in the present models the dissipation function can be taken as an increasing function of the redshift (a decreasing function of time), and thus, at enough high redshifts, due to the presence of the function e^Γ in the expressions of ρ_ϕ and p_ϕ, in the early universe the contributions of the scalar field energy density and pressure become negligible, and the universe is matter dominated, with a decelerating evolution. Hence, generally, we expect that the dissipative quintessence evolution takes place in three phases. In the first phase, at low redshifts $z < 2$, the model (almost) coincides with ΛCDM and describes the present day accelerating evolution. At intermediate redshifts, in the (approximate) range $2 < z < 5$, the dissipative quintessential cosmological expansion may differ, even significantly, from the ΛCDM evolution. However, at $z > 5$, both models become matter dominated, and thus their large redshift dynamics coincides again. Hence, the early matter dominated cosmological phase is recovered in a large redshift limit in the present model, due to the presence of the dissipation function in the expressions of the basic quantities describing the quintessence field, and without the necessity of introducing any supplementary assumptions, for example, a change of the potential or specific initial conditions. Moreover, there are no restrictions on the scalar field potential, since the inclusion of a proper dissipation function in the scalar field equations would automatically recover the early matter dominated era.

To confront this theoretical model with the observations we have considered several simple models, obtained by assuming some simple forms for the dissipation function and for the scalar field potential. All the considered cases have been compared with a (limited) set of observational data for the Hubble function and with the predictions of the ΛCDM model. The generalized Friedmann equations have been solved numerically, with the initial conditions chosen for the scalar field and its derivative so that the models come as close as possible to the observations and to the ΛCDM model. I would like to point out that no fitting was used to obtain, and fix, the free parameters of the models, but the results have been obtained by the trial and search method. As a general conclusion of these investigations one can say that the dissipative scalar field model, in its various versions, can give a good description of the observational cosmological data and succeeds in reproducing the predictions of the ΛCDM model. Of course, a detailed analysis of a larger number of cosmological data is necessary, before one could give a fair estimate of the potential of the dissipative scalar field cosmological models. And deep investigations into the origin and physical mechanisms of dissipation at both classical and quantum levels are also necessary.

By taking into account the results of the present work, the dissipative scalar field cosmological models could become an attractive physical alternative to the standard ΛCDM model concerning the theoretical interpretation and the explanation of the observational data. It may also give a new vision, and a better comprehension of the complex, and unexpected, dynamical processes that take place in the universe.

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